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Coming Events

March 9-10, 2007: Follow-up Workshop on Statistics in Genomics and Proteomics

Organizer

Antónia Turkman (University of Lisbon).

Aims

This meeting follows the event "Workshop on Statistics in Genomics and Proteomics" which took place in Estoril from 5 to 8 of October 2005.

The objective of this "Follow-up Meeting" is to assess the impact of the thematic term "Statistics in Genomics and Proteomics", one and a half years after the event. We aim at a rather informal atmosphere to incite the discussion about the different perspectives on the development of the subject in Portugal and elsewhere.

Simon Tavaré, Sophie Schbath and Wolfgang Urfer will give talks on themes related with the themes of the Workshop. There will be round table discussions with the objective of identifying new (and perhaps not so new) important statistical issues on genomics, proteomics and other "omics", as well as strengthening and/or establishing further collaboration on these themes. There will be a contributed paper session where we incite participants to bring their work in progress, to gather ideas for further development and to promote a healthy flow of discussion.

The event will be held at Hotel Quinta das Lágrimas, Coimbra.

Programme

March 9 (Friday)

16:00 - 17:00 Simon Tavaré

17:00 - 17:30 Coffee-Break

17:30 – 18:30 Chris Cannings, *Theory of some regular discrete dynamical systems on networks*

18:30 – 20:00 Round-Table Discussion

20:00 Dinner

March 10 (Saturday)

9:00 - 11:00 Contributed Papers (Work in Progress)

11:00 - 11:30 Coffee-Break

11:30 - 12:30 Sophie Schbath, *On the assessment of exceptional motifs in biological networks*

12:30 Lunch

14:00 - 15:00 Wolfgang Urfer

15:00 - 16:30 Round-Table Discussion and Close-up

For more information about the event, see

wsgp.deio.fc.ul.pt/fup.html

April 10-14, 2007: Workshop on Mathematical Control Theory and Finance

Organizers

Manuel Guerra (Chairman) and Maria do Rosário Grossinho (Technical Univ. of Lisbon), Fátima Silva Leite (Univ. of Coimbra), Eugénio Rocha and Delfim Torres (Univ. of Aveiro).

Aims

The "high tech" character of modern business has increased the need for advanced methods. These rely heavily on mathematical techniques and seem indispensable for competitiveness of modern enterprises. It became essential for the financial analyst to possess a high level of mathematical skills.

On the other hand, the complex challenges posed by the problems and models relevant to finance has, for a long time, been an important source of new research topics for mathematicians.

The use of techniques from stochastic optimal control constitutes a well established and important branch of mathematical finance. Other branches of control theory have, until now, found comparatively less application in financial problems. Deterministic and stochastic control theories have, to some extent, developed as different branches of mathematics. However, there are many points of contact between them and in recent years the exchange of ideas between these fields as intensified.

We strongly believe there is ample opportunity for fruitful collaboration between specialists of deterministic and stochastic control theory and specialists on finance, both from academic and business background. It is this kind of collaboration that we would like to foster by organizing a workshop with the participation of some of the leading specialists in control theory and its applications to finance.

Thus, the Workshop aims

To provide a state-of-the-art knowledge in the fields of deterministic and stochastic control theory and its applications to mathematical finance;

To increase the mutual knowledge of the mathematical tools developed by the control theory community and the issues and models relevant to financial applications;

To promote the development of interdisciplinary collaborations between researchers from different areas;

To increase interaction between the research community and the business sector.

Contributions are welcome in the fields of deterministic control theory, stochastic control theory, problems, methods and applications in finance, related to control theory.

Tutorials

Italo C. Dolcetta (Univ. of Rome "La Sapienza", Italy)
Deterministic optimal control

Bronislaw Jakubczyk (Univ. of Warsaw, Poland)
Nonlinear control theory

Ioannis Karatzas (Columbia Univ., USA)
Stochastic differential equations

Dmitry Kramkov (Carnegie Mellon, USA)
Mathematical problems in finance

Nizar Touzi (CRES, France)
Stochastic optimal control

Plenary Lectures

Andrei Agrachev (SISSA-International School for Advanced Studies, Italy)

Ole E. Barndorff-Nielsen (Univ. of Aarhus, Denmark)

Eugene A. Feinberg (Univ. of New York at Stony Brook, USA)

Jean-Paul Gauthier (Univ. of Dijon, France)
Harmonic Analysis on Moore Groups for Pattern Recognition in Support-Vector-Machine Context

Ioannis Karatzas (Columbia Univ., USA)

Terry Lyons (Univ. of Oxford, UK)
Inverting the signature of a path – extensions of a theory of Chen

Goran Peskir (Univ. of Manchester, UK)
Optimal Prediction Problems

Andrei Sarychev (Univ. of Florence, Italy)
Existence and Lipschitzian regularity for relaxed minimizers

Albert Shiryaev (Steklov Institute, Russia)
On the Minimax Quickest Detection of a Change of the Drift of the Brownian Motion

Vladimir Zakalyukin (Univ. of Liverpool, UK and Moscow Univ., Russia)
Entropy estimations in motion planning

Xun-Yu Zhou (Chinese Univ. of Hong Kong)
Behavioral Portfolio Selection: Single Period vs Continuous Time

The event will be held at the Instituto Superior de Economia e Gestão, the School of Economics and Management of the Technical University of Lisbon.

For more information about the event, see

srv-ceoc.mat.ua.pt/conf/wmctf2007

April 13-19, 2007: The 60th Study Group Mathematics with Industry 2007

Organizers

Gonçalo Xufre Silva (ACMat/ISEL), José Carlos Quadrado (ISEL), José Francisco Rodrigues (CMAF-UL), Leonel Linhares da Rocha (ACMat/ISEL), Pedro Freitas (Dep. Math., FMH/TU Lisbon, GFM-UL), Tiago Charters de Azevedo (ACMat/ISEL).

Aims

The purpose of these meetings is to streghthen the links between Mathematics and Industry by using Mathematics to tackle industrial problems which are proposed by industrial partners.

This meeting is part of the series of European Study Groups and will count with the participation of several European experts with a large experience in this type of events.

More information on study groups and related aspects is available at the International Study Groups website, the Smith Institute and the European Consortium for Mathematics in Industry.

How it works

At the beginning of the week, a representative from each invited company presents their industrial problem to the participating mathematicians. The academic participants then allocate themselves to the groups who work full-time on each problem over the next three days. By the last day, each group of mathematicians must ensure that their ideas are developed enough to collate them into a final presentation to the other study-group participants. The collaboration does not necessarily end at this point as a report will be provided by us for each of the problems considered. This will also function as a formal record of the work done for the company, and provides the possibility of encouraging further research, leading to new links between industry and academia.

Training session

“Mathematical modelling and analysis of industrial problems” This will consist of short courses on mathematical modelling and problem solving, aiming at familiarising participants with the type of situation that will be met at a typical study group. The course will be based on case studies from previous study groups and students will work in groups on these problems under the guidance of invited experts who have a wide experience of tackling industrial problems.

It will be a shorter version of the very successful ECMI modelling weeks which are run for students each year.

The event will be held at the Instituto Superior de Engenharia de Lisboa.

For more information about the event, see

pwp.net.ipl.pt/dem.isel/tazevedo

June 27-29, 2007: EPSA 2007: Workshop and Advanced School on Eigenvalue Problems, Software and Applications

Organizers

Paulo B. Vasconcelos and Maria J. Rodrigues (Univ. of Porto), Osni Marques (Lawrence Berkeley National Lab., USA), José Roman (Technical Univ. of Valencia, Spain).

Aims

The goal of the Workshop and Advanced School is to bring together leading researchers in the numerical solution of eigenvalue problems to survey the state-of-the-art methods and computational tools to solve large eigenvalue problems.

It aims to encourage the interchange of new ideas, to create a suitable environment for the participants to get acquainted and involved in today's computational mathematics, in particular research and applications that involve eigenproblems and spectral analysis.

Specific objectives are

(i) to survey and present recent developments in both theoretical and computational aspects of matrix eigenvalue problems,

(ii) to report on important practical applications and on challenging problems using high performance computing and

(iii) to foster new collaborations between the participants.

This event comes as a follow up of successful events, such as the Advanced Summer School on Recent Development on Large Scale Scientific Computing (a CIM event organized in 2001) and six Workshops on ACTS - Advanced CompuTational Software Collection (organized by DOE/LBNL). It will include a range of tutorials on methods and tools for the solution of eigenvalue problems and hands-on practices using the high performing clusters from the new Grid Computing infrastructure available at the University of Porto.

The target attendees are researchers and post-graduate students on Mathematics, Biomathematics, Engineering, Computer Science, Computational Economics and Finance, and other branches of Social Sciences. The course in the Advanced School will interest also graduate students and computational scientists whose research require the use of robust numerical algorithms, novel techniques, large amounts of eigenvalue calculations, or combinations of these.

The event is a Satellite Conference of ICIAM 07 and will take place at the Faculty of Science of the University of Porto.

Invited speakers

James Demmel (Univ. of California at Berkeley, USA)

Peter Arbenz (ETH, Zürich, Switzerland)

Filomena Dias d'Almeida (Univ. of Porto)

Tony Drummond (Lawrence Berkeley National Lab., USA)

Rui Ralha (Univ. of Minho)

Other speakers

Osni Marques (Lawrence Berkeley National Lab., USA)

José Roman (Technical Univ. of Valencia, Spain)

Paulo Vasconcelos (Univ. of Porto)

For more information about the event, see

www.fep.up.pt/epsa2007

July 18-20, 2007: LQCIL'07 Workshop on Quantum Cryptography

Organizers

Pedro Adão, Paulo Mateus (Chair), Cláudia Nunes and Yasser Omar (all Technical University of Lisbon).

Aims

This workshop will inaugurate the biannual Lisbon Quantum Computation, Information and Logic Meetings Series. It will be devoted to quantum cryptography and security, bringing together researchers from both classical and quantum information security to exchange ideas and discuss the latest results and future directions of the field. The workshop will be constituted by 7 invited lectures and 15 contributed talks. It is organized within the scope of the QuantLog project of SQIG - Security and Quantum Information Group, IT (formerly CLC - Center for Logic and Computation).

The event will take place at the Instituto Superior Técnico, Lisbon.

Invited speakers

Claude Crépeau (McGill Univ., Canada)

Artur Ekert (Cambridge Univ., U.K.)

Virgil Gligor (Univ. of Maryland, U.S.A.)

Hoi-Kwong Lo (Univ. of Toronto, Canada)

Mike Mosca (Univ. of Waterloo, Canada)

Andre Scedrov (Univ. of Pennsylvania, U.S.A.)

Umesh Vazirani (Univ. of California, Berkeley, U.S.A.)

For more information about the event, see

wslc.math.ist.utl.pt/lqcil07

July 22-27, 2007: CIM/UC Summer School "Topics in Nonlinear PDEs"

Scientific Coordinators

José Francisco Rodrigues (CMUC and Univ. Lisbon), José Miguel Urbano (CMUC and Univ. Coimbra).

Aims

Nonlinear Partial Differential Equations (PDEs) are central in modern Applied Mathematics, both in view of the importance of the concrete problems they model and the novel techniques that their analysis generates. The subject has developed immensely in recent years, in many unexpected and challenging directions, and a new range of applications emerged with the advent of Biomathematics.

The Summer School is a joint venture of the CIM and the University of Coimbra (UC), and is sponsored by the Gulbenkian Foundation. It will gather a group of leading specialists working on Partial Differential Equations and its main applications to Biology, Engineering and Physics, and will highlight emerging trends and issues of this fascinating research topic.

The School will consist of four short courses of six hours each, and of short communications by PhD students and Post-Docs. It will be held at University of Coimbra.

Short courses

Luis Caffarelli (Univ. of Texas at Austin, USA)
Problems and methods involving free boundaries

Charlie Elliott (Univ. of Sussex, UK)
Critical state models in superconductivity

Felix Otto (Univ. of Bonn, Germany)
Analysis of pattern formation in physical models

Benoit Perthame (École Normale Supérieure, France)
Nonlinear PDEs in Biology

For more information about the event, see

www.cim.pt/pdes07

September 17-19, 2007: ROBOMAT 2007 Workshop on Robotics and Mathematics

Organizers (Chairpersons)

Hélder Araújo (University of Coimbra), Maria Isabel Ribeiro (Technical University of Lisbon).

Aims

This workshop will aim at discussing several problems from Robotics from the perspective of the mathematical problems that they raise. It will be a forum where specialists with backgrounds both in Engineering and Mathematics will have an opportunity to discuss relevant research issues not from the point of view of the application but essentially from the point of view of the mathematical models and principles required to solve them.

This workshop will be relevant for:

PhD and MSc students working in Robotics and in relevant mathematical aspects;

Established researchers in Robotics and in Mathematics interested in strengthening the domains of their research work that are relevant for both Robotics and Mathematics;

The following mathematical disciplines are likely to have strong relevance for robotics: Algebraic and differential topology, Dynamic systems theory, Optimization algorithms, Combinatorics, Differential algebraic inequalities, Statistical learning theory.

The workshop will be held at Hotel D. Luís, Santa Clara, Coimbra.

Invited speakers

Henrik Christensen

(Director of the Centre for Autonomous Systems at the Royal Institute of Technology, Stockholm, Sweden, and a chaired professor of computer science specialising in autonomous systems, in the Department of Computer Science and Numerical Analysis.)

David Mumford

(Professor at the Division of Applied Mathematics of the Brown University, USA; in 1974 he was awarded the Fields Medal at the International Congress of Mathematics; his main topic of current research is Pattern Theory.)

Raja Chatila

(*Directeur de Recherche CNRS*, he is the Head of the Robotics and Artificial Intelligence Group, of the LAAS, Toulouse, France; he has numerous contributions in Mobile Robotics, Intervention Robots and

Planetary Rovers, Service Robots, Personal Robots, Cognitive Robots; he has an extensive number of publications in all these fields and he has been responsible for several important European projects.)

For more information about the event, see

http://labvis.isr.uc.pt/robomat

October 26-27, 2007: Follow-up Workshop on Optimization in Finance

ORGANIZER

Luís Nunes Vicente (Univ. of Coimbra).

OTHER CIM EVENTS IN 2007:

WORKING AFTERNOONS SPM/CIM

CIM, Coimbra

A joint initiative of the Portuguese Mathematical Society (SPM) and CIM. Programme for 2007:

January 13, 2007 - Computational Mechanics
Organizer: Isabel Figueiredo (Univ. Coimbra)

March 3, 2007 - Probability and Stochastic Analysis
Organizer: Ana Bela Cruzeiro (Tech. Univ. Lisbon)

May 5, 2007 - Computation
Organizer: Cristina Sernadas (Tech. Univ. Lisbon)

September 29, 2007 - Calculus of Variations
Organizer: Luísa Mascarenhas (New Univ. Lisbon)

November 24, 2007 - Algebraic Topology
Organizer: Margarida Mendes Lopes (Tech. Univ. Lisbon)

For more information, see

www.spm.pt/investigacao/spmcim/spmcim.phtml

CIM SHORT COURSES

Hotel Quinta das Lágrimas, Coimbra

February 9, 2007
Title: *Mathematical Models for Trading Volatility*
Lecturer: Marco Avellaneda (Courant Institute, USA)

February 12, 2007
Title: *Flat Surfaces*
Lecturer: Marcelo Viana (IMPA, Brazil)

May 25, 2007
Title: *Trends in Theoretical Epidemiology*
Lecturer: Gabriela Gomes (Inst. Gulbenkian Ciência)

October 13, 2007
Title: *Mathematics and Games*
Lecturer: Jorge Nuno Silva (Univ. Lisboa)

MEETINGS SPE/CIM

Hotel Quinta das Lágrimas, Coimbra

A joint initiative of CIM and the Portuguese Statistical Society (SPE) with the support of the National Institute for Statistics (INE).

May 12, 2007
Probability and Statistics in Telecommunications
Coordinator: António Pacheco (Tech. Univ. Lisbon)

November 17, 2007
Methodological Issues in Official Statistics
Coordinator: Pedro Corte-Real (INE)

For updated information on these events, see

www.cim.pt/?q=events

CIM News

Annual Scientific Council Meeting 2007

Hotel Quinta das Lágrimas, Coimbra

The CIM Scientific Council will meet in Coimbra on February 10, to discuss the CIM scientific programme for 2008.

Timetable:

10:30-16:30 Scientific council working session.

17:00 Mario Ahues (Univ. of Saint-Etienne, France), *Superconvergence of projection methods for weakly singular integral operators.*

18:30 Ludwig Streit (Univ. of Bielefeld, Germany and Univ. of Madeira, Portugal), *Frontiers and Applications of Infinite Dimensional Analysis.*

20:00 Dinner.

For the detailed programme and registration, see

www.cim.pt/?q=cscam07

Meeting of the General Assembly of CIM

May 26, 2007, Coimbra

The General Assembly of CIM will meet on May 26, 2007 in the CIM premises at the Astronomical Observatory of the University of Coimbra. The programme includes a Seminar given by Gabriela Gomes (Inst. Gulbenkian de Ciência, Portugal).

Research in Pairs at CIM

CIM has facilities for research work in pairs and welcomes applications for their use for limited periods.

These facilities are located at Complexo do Observatório Astronómico in Coimbra and include:

- office space, computing facilities, and some secretarial support;
- access to the library of the Department of Mathematics of the Univ. of Coimbra (30 minutes away by bus);
- lodging: a two room flat.

At least one of the researchers should be affiliated with an associate of CIM, or a participant in a CIM event.

Applicants should fill in the electronic application form in

www.cim.pt/?q=research

CIM on the Web

For updated information about CIM and its activities, see

www.cim.pt

NEWS FROM OUR ASSOCIATES

- **WORKSHOP ON STATISTICAL EXTREMES AND ENVIRONMENTAL RISK**

  February 15-17, 2007, Univ. of Lisbon, Portugal

  The workshop will bring together some leading experts on extreme value theory and other experts working on environmental sciences. The objective of the workshop is to look at recent advances on all aspects of extreme value theory with specific emphasis on its application in environmental sciences.

  Webpage: seer2007.fc.ul.pt

- **CT2007 - INTERNATIONAL CATEGORY THEORY CONFERENCE**

  June 17-23, 2007, Carvoeiro, Portugal

  The Annual Conference on Category Theory will bring together leading experts on category theory and its applications. It is organized by the Category Theory Group of the Centre for Mathematics of the University of Coimbra and will celebrate the 70th birthday of F. William Lawvere.

  Webpage: www.mat.uc.pt/~categ/ct2007

- **NONUNIFORMLY HYPERBOLIC DYNAMICS AND SMOOTH ERGODIC THEORY**

  June 25-29, 2007, Instituto Superior Técnico, Lisbon, Portugal

  Conference dedicated to Yakov Pesin on the occasion of his 60th birthday. Topics will include the subjects of his landmark works and those on which he exerted the strongest influence, including: nonuniform hyperbolicity, smooth ergodic theory, partial hyperbolicity, thermodynamic formalism, dimension theory in dynamics, and related subjects.

  Webpage: www.math.ist.utl.pt/camgsd/pesin

- **ICDEA2007 - INTERNATIONAL CONFERENCE ON DIFFERENCE EQUATIONS AND APPLICATIONS**

  July 22-28, 2007, Lisbon, Portugal

  The purpose of the conference is to bring together both experts and novices in the theory and application of difference equations and discrete dynamical systems. The main theme of the meeting will be "Discrete Dynamical Systems and Nonlinear Science".

  Webpage: www.math.ist.utl.pt/icdea2007

- **56TH SESSION OF THE ISI - INTERNATIONAL STATISTICAL INSTITUTE**

  August 22-29, 2007, Lisbon, Portugal

  This is the most important world meeting in Statistics, gathering usually more than 2000 participants. So as to allow for a large participation in this event, the Portuguese Statistical Society has scheduled its XV Annual Conference to August the 19th-21st in Lisbon (a combined registration is available at www.spestatistica.pt).

  Webpage: www.isi2007.com.pt

- **ORP^3 - OPERATIONS RESEARCH PERIPATETIC POSTGRADUATE PROGRAMME**

  September 12-15, 2007, Guimarães, Portugal

  ORP^3 is a new instrument of EURO designed for young OR researchers and practitioners. ORP^3 aims at being a forum promoting scientific and social exchanges between the members of the future generation of OR in academic research and industry. ORP^3 is a European peripatetic conference each edition of which is hosted by a renowned European centre in OR.

  Webpage: www.norg.uminho.pt/orp3

- **ANNOUNCEMENT AND FIRST CALL FOR PAPERS FOR THE INTERNATIONAL JOURNAL FOR COMPUTATION VISION AND BIOMECHANICS**

  The main goal of the International Journal for Computational Vision and Biomechanics consists in the provision of a comprehensive forum for discussion on the current state-of-the-art in these fields.

  Webpage: www.fe.up.pt/~ijcvb

For updated news, see www.cim.pt/?q=newsassoc

Chaotic Dynamics: physical measures and statistical features

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Abstract

We present some results on the statistical features of certain chaotic dynamical systems. We shall focus on the existence of physical measures, decay of correlations and statistical stability.

1. Introduction

Take a mathematical space M and think of its points as representing physical, biological or some other variables. Endow this space with a function (rule) $f\colon M \circlearrowleft$ which, given any point in M, comes up with another point in M. The combination is a *discrete-time dynamical system* for which M is the *phase space*, and the function gives the *evolution law*. The *orbit* (or *trajectory*) of a given point $x \in M$ is the sequence of successive iterates $(f^n(x))_n$, where $f^n = f \circ \cdots \circ f$ (n times).

In broad terms, one may refer two main goals of Dynamical Systems theory: *i)* to describe the typical behavior of trajectories, specially as time goes to infinity, and *ii)* to understand how this behavior changes when the law that governs the system is sightly modified. Even in cases of simple evolution laws, orbits may have a rather complicated behavior, which makes its description a very difficult task, specially when the system has *sensitivity to initial conditions*: a small change in the initial state produces large variations in the long term behavior of the trajectory. A well succeeded strategy for studying this kind of systems is through a probabilistic viewpoint: if one is not able to predict the future configuration of the system, let us try at least to find out the probability of certain configurations. In this approach we are particularly interested in *physical measures*, which characterize asymptotically, in time average, a large set of orbits.

Starting with classical results, in this work we present recent developments on the probabilistic theory of chaotic dynamical systems, specially about the existence of physical measures and some of their statistical features.

2. Physical measures

Let $(M, \mathscr{A}, \mu)$ be a probability space and $f : M \to M$ be such that $f^{-1}(A) \in \mathscr{A}$ for each $A \in \mathscr{A}$. We say that f *preserves the measure* μ, or μ is an f*-invariant measure*, if $\mu(f^{-1}(A)) = \mu(A)$ for all $A \in \mathscr{A}$. A direct consequence of this definition is that $\{x \in M : x \in A\}$ and $\{x \in M : f^n(x) \in A\}$ have the same μ measure for every $n \in \mathbb{N}$. This means that the probability of finding a point in a measurable set does not depend on the moment we are considering. One of the first results on the probabilistic features of dynamical systems was obtained by Poincaré for conservative systems, and can be translated to our context in the following way:

[1] Work carried out while visiting UFBA, Brazil. Partially supported by CNPq, FCT through CMUP and POCI/MAT/61237/2004.

Poincaré Recurrence Theorem. *Assume that f preserves a probability measure μ. If A is a measurable set, then for almost every $x \in A$, there are infinitely many $n \in \mathbb{N}$ for which $f^n(x) \in A$.*

The previous result says nothing about the *frequency* on which typical orbits visit A, i.e. it gives no information on

$$\lim_{n\to\infty} \frac{\#\{0 \le j < n \colon f^j(x) \in A\}}{n}. \tag{2.1}$$

Does this limit exist? Where does it converge to? Birkhoff Ergodic Theorem gives answers to these questions and, in fact, much more general conclusions. Before we state it, let us introduce some important concept on this subject. Assume that f preserves a measure μ. We say that f (or μ) is *ergodic* if $\mu(A) = 0$ or $\mu(M \setminus A) = 0$ for any $A \in \mathscr{A}$ with $f^{-1}(A) = A$. Observing that $f^{-1}(A) = A$ implies that $f(A) \subset A$ and $f(M \setminus A) \subset M \setminus A$, this means that the space cannot be decomposed into two significant parts that do not interact.

Birkhoff Ergodic Theorem. *Assume that f preserves a probability measure μ. If φ is integrable, then there is an integrable function φ^* such that for μ almost every $x \in M$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \varphi^*(x).$$

Moreover, $\varphi^*(x) = \int \varphi d\mu$ for μ almost every $x \in M$, provided μ is ergodic.

Taking φ as the characteristic function of a measurable set A, we easily deduce that the limit in (2.1) exists for μ almost every $x \in M$. Furthermore, if μ is ergodic, then it is precisely $\mu(A)$. This means that the frequency of visits to A coincides with the proportion that A occupies in the phase space.

The results we have presented so far concern dynamics over a probability measure space with no additional structure on the underlying phase space M. Frequently M has a Riemannian manifold structure and a volume form on it which gives rise to a Lebesgue measure m on the Borel sets of M. Birkhoff Ergodic Theorem gives that asymptotic time averages exist for almost every point, with respect to an invariant measure μ, and they coincide with the spatial average, provided μ is ergodic. However, an invariant measure can lack of physical meaning, in the sense that sets with full μ measure may have zero Lebesgue measure. This problem can be overcome by the notion that we present below, which has been introduced by Sinai, Ruelle and Bowen in the context of hyperbolic dynamical systems.

An invariant probability measure μ is said to be a *physical measure* for $f \colon M \to M$ if for a positive Lebesgue measure set of points $x \in M$

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi \, d\mu, \tag{2.2}$$

for all continuous $\varphi : M \to \mathbb{R}$. This means that the averages of Dirac measures over the orbit of x converge in the weak* topology to the measure μ. We define the *basin* of μ as the set of points $x \in M$ for which (2.2) holds for all continuous φ.

It easily follows from Birkhoff Ergodic Theorem that if μ is an ergodic probability measure which is *absolutely continuous with respect to the Lebesgue measure*, i.e. it does not give positive weight to sets with zero Lebesgue measure, then μ is a physical measure. Indeed, if μ is ergodic, then by Birkhoff Ergodic Theorem its basin has full μ measure. By absolute continuity, the basin of μ cannot have zero Lebesgue measure.

3. Low dimensional dynamics

There is no need of great complexity in evolution laws for which intricate dynamical behavior occurs. To illustrate this, the basic model is the family of quadratic maps $q_a(x) = 1 - ax^2$, where $x \in [-1, 1]$ and $a \in [0, 2]$ is a parameter[2]. In spite of its simple appearance, the dynamics of these maps presents many remarkable phenomena. From the topological point of view, the situation is quite well understood in most situations.

Theorem 3.1 ([Ly1], [GS])**.** *There is an open and dense set of parameters $a \in [0, 2]$ for which q_a has a periodic orbit*[3] *attracting Lebesgue almost every point.*

In spite of its simple formulation, this remained as a long term conjecture in one dimensional dynamics. From a probabilistic point of view, the situation is completely different. Its richness first became apparent with the work of Jakobson, where it was shown that a positive measure set of parameters corresponds to quadratic maps with chaotic behavior.

Theorem 3.2 ([Ja])**.** *There is a positive Lebesgue measure set of parameters $a \in [0, 2]$ for which q_a has an absolutely continuous ergodic measure μ_a.*

By the considerations at the end of Section 2 we have that μ_a is a physical measure. Some extra knowledge on the properties of μ_a allows us to show that $\log |q_a'|$ is μ_a

[2] Here is where the rich part of the dynamics lies. For parameters out of this range or points out of this domain the dynamics is well understood.

[3] The orbit of a given point x is called *periodic* if some positive iterate of it coincides with x.

integrable and $\int \log |q_a'| d\mu_a > 0$. By Birkhoff Ergodic Theorem

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} \log |q_a'((q_a^j(x))| = \int \log |q_a'| d\mu,$$

and so, using the chain rule, we have a positive *Lyapunov exponent* at almost every x:

$$\lim_{n\to\infty} \log |(q_a^n)'(x)| > 0.$$

The existence of this positive Lyapunov exponent gives one pervasive feature of chaos: *sensitivity to the initial conditions.*

As we have seen, at least two types of distinct behavior are present on the quadratic family, and they alternate in a complicate way. Besides these two types, different behaviors were shown to exist, including examples with bad statistics, like absence of a physical measure or a physical measure concentrated on a hyperbolic repeller. Finally Lyubich depicted a nice picture of the global situation.

Theorem 3.3 ([Ly2])**.** *For Lebesgue almost every $a \in [0,2]$ the map q_a has either a periodic attracting orbit or an absolutely continuous ergodic measure.*

Though we have used the absolutely continuous ergodic measure to obtain a positive Lyapunov exponent, the existence of this exponent can be deduced directly for a positive Lebesgue measure subset of parameters. The big difficulty in carrying this out is that quadratic maps combine regions of the phase space where the dynamics expands, together with a critical region where the derivative becomes arbitrarily small. In [BC1], Benedicks and Carleson implemented a strategy which enabled them to prove the existence of a positive Lyapunov exponent not only for quadratic maps, but also for the Hénon maps

$$\begin{array}{rccc} f_{a,b}: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x,y) & \longmapsto & (1-ax^2+y, bx). \end{array}$$

In [He], Hénon proposed this two parameter family as a model for non-linear two dimensional dynamics. This can be thought as a simplified discrete-time version of the Lorenz flow and interpreted as an unfolding of the quadratic family[4]. Based on numerical experiments for $a = 1.4$ and $b = 0.3$, Hénon conjectured that this system should have a *strange attractor*. It was not at all *a priori* clear that the attractor detected experimentally by Hénon was not a long stable periodic orbit. Benedicks and Carleson managed to prove that Hénon's conjecture was true for small $b > 0$[5].

Theorem 3.4 ([BC2])**.** *There is a positive Lebesgue measure set $\mathscr{BC}$ of parameters such that for each $(a,b) \in \mathscr{BC}$ the map $f = f_{a,b}$ has the following properties:*

(1) *there is an open set $U \subset \mathbb{R}^2$ such that $\overline{f(U)} \subset U$ and $\Lambda = \bigcap_{n=0}^{\infty} f^n(U)$ attracts the orbit of every $x \in U$;*

(2) *there is $z_0 \in \Lambda$ whose orbit is dense on Λ, and there is $c > 0$ such that $\|Df^n(z_0)(0,1)\| \geq e^{cn}$ for all $n \geq 1$;*

(3) *f has a unique physical measure supported on Λ.*

The physical measure was obtained by Benedicks and Young in [BY1]. The second item of the theorem gives the existence of a positive Lyapunov exponent in a dense orbit, thus showing that the attractor displays sensitive dependence to the initial conditions for the parameters in $\mathscr{BC}$.

4. Non-uniformly expanding maps

As seen in the previous section, for one-dimensional maps the existence of absolutely continuous invariant measures is intimately connected with the existence of a positive Lyapunov exponent. Inspired by the remarkable progress for the one dimensional case, one presently aims at obtaining similar conclusions in higher dimensions. The first result we present in this direction is for uniformly expanding maps. A map $f\colon M \to M$ is called *uniformly expanding* if there is $\sigma < 1$ such that $\|Df(x)^{-1}\| < \sigma$ for every $x \in M$.

Theorem 4.1 ([KS])**.** *Let $f : M \to M$ be a C^2 uniformly expanding map. Then f has a unique ergodic absolutely continuous invariant probability measure whose basin has full Lebesgue measure.*

We are also interested in maps admitting (critical) sets where the derivative is not an isomorphism or simply does not exist. We say that $\mathscr{C} \subset M$ is a *non-degenerate critical set* if the derivative of f behaves as a power of the distance close to $\mathscr{C}$. Staying away from technicalities, we refer [ABV] for a precise definition of this concept. Let us just mention that it captures the flavor of non-flat critical points in dimension one.

Let $f\colon M \to M$ be a local diffeomorphism in $M \setminus \mathscr{C}$, where $\mathscr{C}$ is a non-degenerate critical set with zero Lebesgue measure. We say that f is *non-uniformly expanding* if the following conditions hold:

[4]For $b = 0$ orbits eventually lie on $\{y = 0\}$ and dynamics can be thought as that of quadratic maps.

[5] It remains an interesting open question to know if the chaotic attractor exists for Hénon's choice of parameters $a = 1.4$ and $b = 0.3$.

(c_1) there is $\lambda > 0$ such that for Lebesgue almost every $x \in M$

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| < -\lambda;$$

(c_2) for all $\epsilon > 0$ there is $\delta > 0$ such that for Lebesgue almost every $x \in M$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(f^j(x), \mathscr{C}) < \epsilon.$$

Condition (c_1) allows points where the derivative does not expand. Expansion is only attained asymptotically in average for most orbits. We shall refer to (c_2) as *slow recurrence* to $\mathscr{C}$. It essentially says that generic orbits do not hit small neighborhoods of the critical set too frequently.

Theorem 4.2 ([ABV]). *Let $f\colon M \to M$ be a C^2 non-uniformly expanding map. There are absolutely continuous ergodic probability measures $\mu_1, \dots, \mu_p$ whose basins cover a full Lebesgue measure subset of M.*

Uniqueness can be obtained if f is *transitive*, i.e. with a dense orbit in M. Uniformly expanding maps are always transitive.

Condition (c_1) assures that the *expansion time* function $\mathscr{E}(x)$, defined as the minimum $N \geq 1$ such that for all $n \geq N$

$$\frac{1}{n} \sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| < -\lambda,$$

is well defined and finite Lebesgue almost everywhere.

We observe that slow recurrence condition is not needed in all its strength. Actually, it is enough that it holds for some sufficiently small $\epsilon > 0$ and $\delta > 0$ conveniently chosen; see [Al, Remark 3.8]. We fix once and for all $\epsilon > 0$ and $\delta > 0$ in those conditions. This allows us to define the *recurrence time* $\mathscr{R}(x)$, as the minimum $N \geq 1$ such that for all $n \geq N$

$$\frac{1}{n} \sum_{i=0}^{n-1} -\log \operatorname{dist}_\delta(f^j(x), \mathscr{C}) < \epsilon,$$

which is finite Lebesgue almost everywhere. We define the *tail set* (at time n) as

$$\Gamma_n = \{x \in M\colon \mathscr{E}(x) > n \text{ or } \mathscr{R}(x) > n\}.$$

This is the set of points that at time n have not reached the exponential growth or slow recurrence assured by (c_1) and (c_2). Non-uniform expansion guarantees that the Lebesgue measure of this set converges to zero when $n \to \infty$. The speed of this convergence plays an important role in the statistical features of non-uniformly expanding dynamical system, as we shall see later on.

Next we present a family of maps, introduced by Viana in [Vi], that has served as a model for many general results on non-uniformly expanding maps.

Example 4.3 (Viana maps). Let a_0 be a parameter conveniently chosen and take $b : S^1 \to \mathbb{R}$ a Morse function. Consider the cylinder transformation $\hat{f} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ given by

$$\hat{f}(s,x) = (\hat{g}(s), \hat{q}(s,x)),$$

where $\hat{g}$ is an expanding map of the circle $\hat{g}(s) = ds \pmod{\mathbb{Z}}$, for some $d \geq 2$, and $\hat{q}(s,x) = a(s) - x^2$ with $a(s) = a_0 + \alpha b(s)$, for small $\alpha > 0$.

Theorem 4.4 ([Vi]). *If f is close to $\hat{f}$ in the C^3 topology, then f is non-uniformly expanding. Moreover, there is $c > 0$ such that $m(\Gamma_n) \lesssim e^{-c\sqrt{n}}$.*

Viana maps reveal some new phenomenon if comparing to the family one dimensional quadratic maps: the non-uniformly expanding behavior holds for an open set of transformations. Recall that by Theorem 3.1 we have density of parameters for which the corresponding quadratic map has a periodic attractor.

5. Mixing rates

There are several possible ways of measuring the chaoticity of a given dynamical system. One of them is analyzing its mixing rates. An invariant probability measure μ is said to be *mixing* if

$$\mu(f^{-n}(A) \cap B) \to \mu(A)\mu(B), \tag{5.1}$$

when $n \to \infty$, for any measurable sets A, B. We leave it as an easy exercise to the reader to show that mixing implies ergodicity[6].

Roughly speaking, mixing indicates that, as long as sufficiently large iterates are taken, the proportion of points in B arising from A tends to the proportion that A occupies in the whole space. In general there is no specific rate at which the convergence in (5.1) occurs. However, defining the *correlation function* of *observables* $\varphi, \psi\colon M \to \mathbb{R}$,

$$C_n(\varphi,\psi) = \left| \int (\varphi \circ f^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu \right|,$$

it is sometimes possible to obtain specific rates at which $C_n(\varphi,\psi)$ decays to zero, provided φ and ψ have sufficient regularity. Observe that choosing the observables

[6] The converse is not true: irrational rotations of the circle are ergodic and not mixing with respect to the length measure, which is obviously invariant.

as characteristic functions we get the definition of mixing.

Given $\varphi\colon M \to \mathbb{R}$, consider the random variables φ, $\varphi\circ f$, $\varphi\circ f^2$,.... The exponential decay of correlations tells in particular that $\varphi\circ f^n$ and φ become uncorrelated exponentially fast as n tends to infinity.

Theorem 5.1 ([BY2]). *Hénon maps have exponential decay of correlations (with respect to the unique physical measure) for parameters in $\mathscr{BC}$.*

A key ingredient in the proof of this result is the existence of a direction of non-uniform expansion. However, there is a well localized set of "critical" points where orbits suffer setbacks in expansion when they pass near this set. The decay of correlations takes into account the set of points approaching in a counterproductive way the source of non-expansion. The measure of this set decays exponentially fast to zero.

For non-uniformly expanding maps, *a priori* we have no knowledge on the source of "critical" behavior. The decay of correlations ultimately depends on the speed that the Lebesgue measure of the tail set converges to zero, at least for some specific rates.

Theorem 5.2 ([ALP], [Go]). *Assume that $f: M \to M$ is a C^2 transitive non-uniformly expanding map. If $m(\Gamma_n)$ is summable, then some power of f is mixing with respect to the (unique) physical measure μ. Moreover, for Hölder continuous φ,ψ one has:*

(1) *if there is $\gamma > 1$ for which $m(\Gamma_n) \lesssim n^{-\gamma}$, then $C_n(\varphi,\psi) \lesssim n^{-\gamma+1}$;*

(2) *if there are $\gamma > 0$ and $0 < \eta \le 1$ for which $m(\Gamma_n) \lesssim e^{-\gamma n^\eta}$, then there is $\gamma' > 0$ such that $C_n(\varphi,\psi) \lesssim e^{-\gamma' n^\eta}$.*

Using Theorem 4.4 we easily deduce that the decay of correlations for Viana maps has order $e^{-c\sqrt{n}}$ at least[7].

Let us now give some consequence of the decay of correlations. Starting with the Lebesgue measure m, one may consider the sequence of push-forwards $f^n_* m$, for $n \ge 1$, where these measures are defined for each $n \ge 1$ as $f^n_* m(A) = m(f^{-n}(A))$. In many situations (e. g. uniformly expanding maps) the absolutely continuous invariant measure is actually equivalent to the Lebesgue measure m, in such a way that we may take $\psi = dm/d\mu$ in $C_n(\varphi,\psi)$ and, assuming m normalized, we obtain

$$C_n(\varphi,\psi) = \left|\int(\varphi\circ f^n)dm - \int \varphi d\mu\right|,$$

Supposing $C_n(\varphi,\psi) \to 0$ as $n\to\infty$, one has

$$\int(\varphi\circ f^n)dm \longrightarrow \int\varphi d\mu.$$

This means that $f^n_* m$ converges in the weak* topology to μ. Hence, the faster correlations decay, the better physical measure are approximated by the push-forwards of Lebesgue measure.

6. Statistical stability

One is interested in studying the variation of physical measures in certain classes of dynamical systems. Its continuous variation points in the direction of stability of the dynamical system, at least in terms of the statistical distribution of orbits for nearby dynamics.

Let $\mathscr{F}$ be a family of C^k maps, for some $k \ge 2$, from a manifold M into itself, and consider $\mathscr{F}$ endowed with the C^k topology. Assume that each $f \in \mathscr{F}$ admits a unique physical measure μ_f. We say that $\mathscr{F}$ is *statistically stable* if

$$\mathscr{F} \ni f \longmapsto \mu_f$$

is continuous with respect to the weak* topology on the space of probability measures.

As shown in Theorem 3.4, though highly unstable in terms of the evolution of its individual orbits, Hénon attractors at $\mathscr{BC}$ parameters are fairly regular in statistical terms. The next result shows that the statistics of the these maps does not change dramatically when one perturbs parameters in $\mathscr{BC}$.

Theorem 6.1 ([ACF]). *The family $\mathscr{BC}$ is statistically stable.*

The physical measures of Hénon maps at $\mathscr{BC}$ parameters are supported on attractors with zero bidimensional Lebesgue measure. Consequently, those physical measures are necessarily singular with respect to the Lebesgue measure. In cases where the physical measure is absolutely continuous with respect to the Lebesgue measure m on the phase space, we may even aim at *strong statistical stability*: the map

$$\mathscr{F} \ni f \longmapsto \frac{d\mu_f}{dm}$$

is continuous with respect to the $L^1(m)$ norm in the space of densities.

The following result holds for families $\mathscr{F}$ of non-uniformly expanding maps. We denote by Γ^f_n the tail set associated to $f \in \mathscr{F}$.

Theorem 6.2 ([AV],[Al]). *Assume that there are $C > 0$ and $\gamma > 1$ such that $m(\Gamma^f_n) \le Cn^{-\gamma}$ for all $f \in \mathscr{F}$ and $n \ge 1$. Then $\mathscr{F}$ is strongly statistically stable.*

Using Theorem 4.4 we easily deduce that the family of Viana maps is strongly statistically stable[8].

[7] It remains an interesting open question to know if the estimate for the measure of the tail set given by Theorem 4.4 is optimal.

[8] Though not explicitly stated in Theorem 4.4, the rate at which $m(\Gamma_n)$ decays to 0 is uniform on the set of Viana maps.

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Math in the Media

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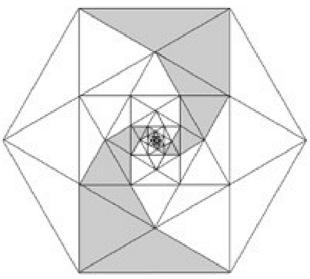
Spidrons.

This spidroball, a rhombic triacontahedron assembled from 30 spidron pairs, appeared on the cover of the October 21 *Science News*. Image courtesy Marc Pelletier, Walt van Ballegooijen, Dániel Erdély and Amina Buhler Allen.

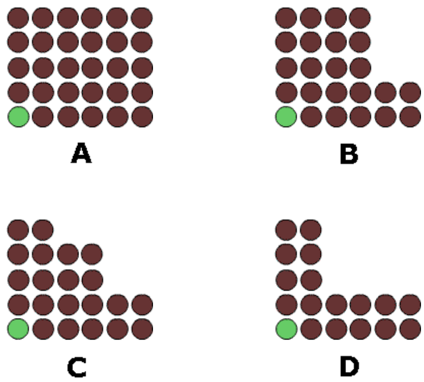
The easiest way to draw spidrons is to start with a hexagon, inscribe a six-pointed star, and repeat with the star's interior hexagon ad infinitum. Then attaching to each flat isosceles triangle the equilateral triangle to its right, and to each equilateral triangle the isosceles triangle below it, you get six spiralling polygonal chains: each one is half a spidron. (Alternatively, always attach to the left; this construction makes it clear that the area of a spidron is one third of the area of the hexagon you started with).



Two semi-spidrons assemble into one full spidron.

Spidrons were recently featured in *Science News Online* (October 21, 2006), where Ivars Peterson tells us that they were invented and named in the early 1970s by Dániel Erdély, a Hungarian industrial designer and a student of Rubik, and that Erdély soon discovered that when creased properly a spidron array takes on interesting 3-dimensional behavior, with potential practical applications. Erdély has recently been collaborating with other graphic designers and with sculptors; some of their work was presented at last summer's Bridges conference (www.lkl.ac.uk/bridges). One sample: the spidroball shown above. Additional images are available on Erdély's Spidron website (www.spidron.hu).

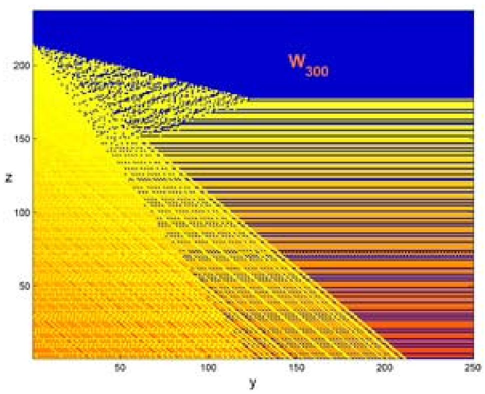
Next year in Marienbad: chaos. Chomp is a 2-dimensional version of Nim (www.csm.astate.edu/Nim.html), the game popularized in *L'année dernière à Marienbad.*



The first 3 moves in a 5 x 6 game of Chomp. A: the initial configuration; the object is not to be forced to select the green cookie. B: after Player 1's first bite. C: after Player 2's first bite. D: after Player 1's second bite. Each bite takes a cookie and all the cookies north and east of it.

But while a simple strategy exists for Nim, Chomp is much harder. It is known that there is always a winning strategy for Player 1 but the strategy itself is unknown in general, except for a few special cases like $n \times n$, $2 \times n$, and $n \times 2$. In "Chaotic Chomp" (*Science News Online*, July 22, 2006, www.sciencenews.org/articles/20060722/bob10.asp) Ivars Peterson reports on developments in the analysis of the $3 \times n$ case. Chomp dates back to 1974 (in fact, it is equivalent to a game discovered in 1950) but was taken up a few years ago by Doron Zeilberger, a mathematician at Rutgers, who decided it would be "an ideal problem for illustrating the role that computers can play in mathematical research."

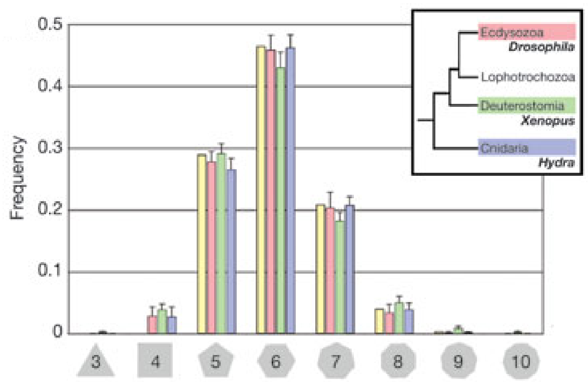
Zeilberger introduced the notation (x, y, z) to describe the position in $3 \times n$ Chomp which has x columns of 3 cookies, y columns of 2, and z columns of 1, and published in 2000 an algorithm generating for each x an algorithm for playing the game with an arbitrary y and z. He returned to the problem in 2003 with faster algorithms and on the basis of the results speculated "It seems that we have 'chaotic' behavior, but in a vague, yet-to-be-made-precise sense." Peterson focuses on the recent work of Eric Friedman (Computer Science, Cornell) and Adam Landsberg (Physics, Claremont colleges), who have fleshed out this intuition: "By using mathematical tools originally developed for calculating properties of physical systems, Friedman and Landsberg show that the exact location of winning and losing cookies in Chomp varies unpredictably with small changes in the size of the initial array." The figure below uses Zeilberger's notation and shows in yellow/red, for $x = 300$, the "instant winner" positions (positions from which you can leave your opponent in a losing position with smaller x). The chaotic region is clearly visible.



Winning positions (yellow/red) for a 3-row Chomp game with 300 height-3 columns. The y and z coordinates refer to the number of height-2 and height-1 columns, respectively. Image courtesy Adam Landsberg.

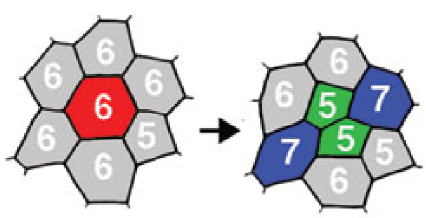
Furthermore, they made the remarkable discovery that Chomp is renormalizable. As Peterson explains it, "the geometry of winning positions for small values of x and winning positions for large values of x is roughly the same, after a suitable change in scale." Specifically, the W600 figure, scaled down by a factor of 2 in each direction, is essentially indistinguishable from the W300 shown here. Zeilberger's papers (excellent reading) are available at `www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/chomp.html` and `www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/byrnes.html`. Friedman and Landsberg's paper is also available online, as a PDF file (`people.cornell.edu/pages/ejf27/pfiles/chomptr.pdf`). For a history of the problem, see Andries Brouwer's page (`www.win.tue.nl/~aeb/games/chomp.html`) on the game.

Epithelial topology. Epithelial tissue is typically a 2-dimensional array of cells. Topologically the average cell shape must be a hexagon. Remarkably, an identical, asymmetric distribution of polygonal shapes shows up over an enormous range of organisms. Drosophila is a fly, Xenopus is a frog and Hydra is a tiny fresh-water relative of jellyfish.



"Drosophila wing disc (pink), Xenopus tail epidermis (green) and Hydra epidermis (blue) all exhibit a similar non-gaussian distribution of epithelial polygons with less than 50% hexagonal cells and high (and asymmetric) percentages of pentagonal and heptagonal cells. The inset indicates relative phylogenetic positions for Drosophila, Xenopus and Hydra." Yellow bars represent the theoretical distribution derived in this article (see below). Image from *Nature* 442 1038-1041, used with permission.

A theoretical explanation for this phenomenon is given in "The emergence of geometric order in proliferating metazoan epithelia," by Matthew Gibson (Harvard) and collaborators, in *Nature* for August 31, 2006. It relies on the observation that when a cell in a 2-dimensional array divides, each of its daughters typically has one fewer neighbor, while two of its neighbors pick up an extra side.

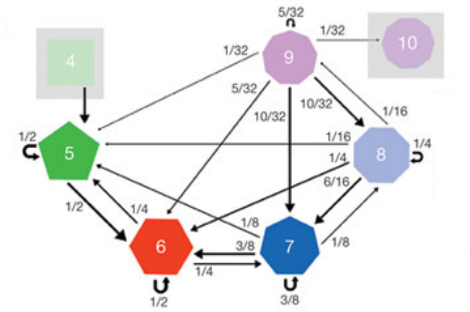


Typically the daughters of a hexagonal cell are pentagons, while two hexagonal neighbors become heptagons. Image from *Nature* 442 1038-1041, used with permission.

To study the way the distribution of polygonal types changes under repeated subdivisions, Gibson and his colleagues axiomatize the situation (each of these statements is given an experimental justification):

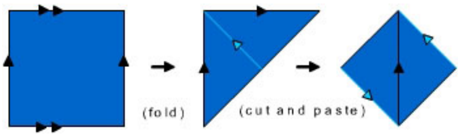
- cells are polygons with a minimum of four sides.
- cells do not re-sort.
- mitotic siblings retain a common junctional interface.
- cells have asynchronous but roughly uniform cell cycle times.
- cleavage planes always cut a side rather than a vertex of the mother polygon.
- mitotic cleavage orientation randomly distributes existing tricellular junctions to both daughter cells.

They use these axioms to construct a Markov-chain model for the distribution of polygonal types. This model predicts the yellow bars in our first image, and also predicts a rapid evolution to this distribution regardless of the initial set of polygonal types.



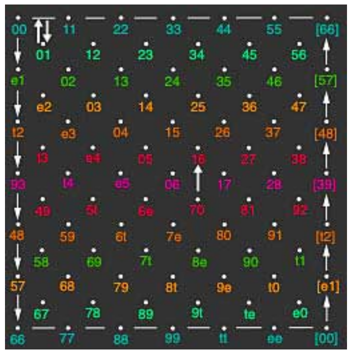
Markov-chain model for the change in polygonal type from one generation to the next. Image from *Nature* 442 1038-1041, used with permission.

The Geometry of Musical Chords. This is the title of a report (music.princeton.edu/~dmitri/voiceleading.pdf) in the July 7 2006 *Science*, written by Dmitri Tymoczko, Professor of Music at Princeton. The abstract begins: "A musical chord can be represented as a point in a geometrical space called an orbifold. Line segments represent mappings from the notes of one chord to those of another." The simplest example of this representation is for the case of intervals, or two-note chords. As Tymoczko says, "Human pitch perception is both logarithmic and periodic." We judge the distance between tones in terms of the ratio of their pitches, and identify tones when that ratio is 2. So the psychological space of tones is a circle, where we can mark off 12 equidistant points corresponding to the pitch classes C, $C\#$, $D, \dots, A\#, B$. It is convenient to identify this circle with $T^1 = \mathbb{R}/12\mathbb{Z}$ and to place the equal-tempered pitches at the integral points $0(C), \dots, 11(B)$. Then the space of pairs of tones is the torus $T^1 \times T^1$ and the space of intervals (unordered pairs) is the quotient of this torus by the relation $(x, y) \sim (y, x)$. The result is a Möbius strip, a manifold with boundary and thus an *orbifold*.



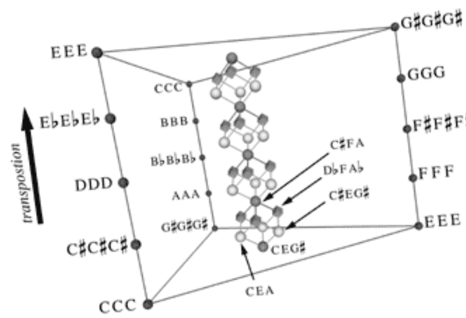
The identification $(x, y) \sim (y, x)$ makes the torus into a Möbius strip, a manifold with boundary.

Here is how the intervals appear on the Möbius strip:



The 2-note chords, or intervals, as they appear on the Möbius strip of unordered tone pairs. "t" is 10 and "e" is 11. 70 = 07 corresponds to the fifth chord $C - G$. Transposition corresponds to sideways motion. "Voice leading," i.e. motion through chords, is represented by paths on the surface: e.g. $C - G \to D - F\#$ is represented by the arrow $70 \to 16$. Note that the voice leading $C - C\# \to C\# - C$ reflects off the upper boundary. Image courtesy Dmitri Tymoczko.

For three or four-note chords the topology becomes more complicated. For example three-note chords live on the 3-dimensional orbifold constructed by taking a 3-dimensional prism with base a triangle, twisting the base so as to cyclically permute the vertices, and identifying it with the opposite face. But it makes musical sense: "Chords that divide the octave evenly lie at the center of the orbifold and are surrounded by the familiar sonorities of Western tonality."



The 3-dimensional orbifold representing the space of 3-note chords. "Chords that divide the octave evenly lie at the center of the orbifold and are surrounded by the familiar sonorities of Western tonality." Image courtesy Dmitri Tymoczko.

It is clearly Tymoczko's intent for these representations to serve not only as a tool in musical analysis, but also as a stimulus for new directions in composition. More information, including free ChordGeometries software, on his website (music.princeton.edu/~dmitri).

Mathematical error control. Barry Mazur has a "News and Views" piece in the September 7 2006 *Nature* about recent steps towards the proof of the Sato-Tate conjecture, which predicts the distribution of the error terms in good approximations for solutions of combinatorial number-theoretic problems. These are not the errors that plague natural scientists measuring things out in the field, but the report is a work of art. Mazur illustrates the conjecture with a nice, and elementary, example. The problem here is to count the number $N(p)$ of ways a prime number p can be written as a sum of 24 squares of integers. Note that zero and negative numbers are allowed to participate, and that all permutations of the terms in a sum must be counted as different "ways". So $N(2)$ is already 1,104. Now there exists a good approximation $A(p)$ for $N(p)$:

$$A(p) = \frac{16}{691}(p^{11} + 1),$$

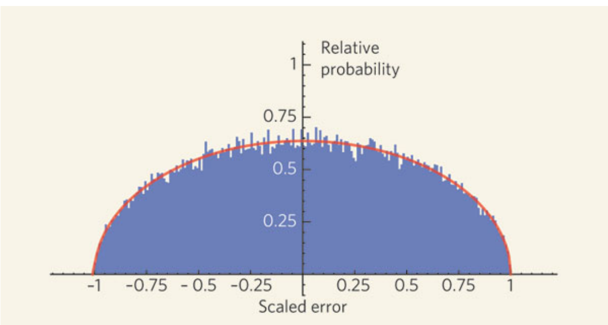
good in the sense that the error scales like the square root of N; in fact there is an explicit least upper bound for the error as a function of p:

$$\mid N(p) - A(p) \mid \leq \frac{66304}{691}\sqrt{p^{11}}.$$

For problems like this, the Sato-Tate conjecture predicts that the distribution of the scaled error

$$\frac{N(p) - A(p)}{\frac{66304}{691}\sqrt{p^{11}}}$$

should be governed by the distribution $\frac{2}{\pi}\sqrt{1-x^2}$, whose graph is a semi-circle normalized to have area 1.

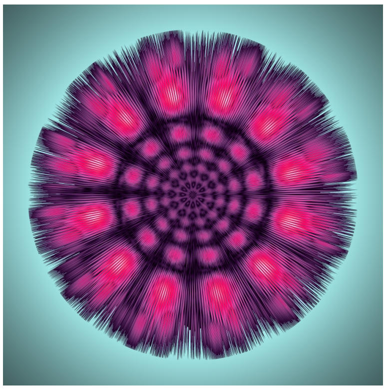


The scaled error distribution for $N(p)$ predicted by the Sato-Tate conjecture (red curve) and the actual distribution for primes less than one million. Image from *Nature*, 443 38-40, used with permission.

For this case of the conjecture the evidence is excellent but there is as yet no proof. Mazur mentions a class of problems, related to elliptic curves, where the conjecture has in fact been proved (through the efforts of his Harvard colleague Richard Taylor and Taylor's collaborators). As Mazur explains it, "The proof came by combining some wonderful pieces of mathematics, and the key to it all is the so-called representation theory. This branch of mathematics, in its various guises, studies abstract groups by representing them as groups of linear transformations of vector spaces. By understanding the profound number-theoretic structure behind enough of the symmetric tensor powers of a certain representation of a certain group, one can compute the probability distribution of the corresponding scaled error terms, and so confirm the Sato-Tate conjecture." Mazur concludes: "This is a magnificent achievement for at least two reasons. First, the method brings synthetic unity to deep results in quite distinct mathematical fields. ...Second, the work answers a question of delicate nature. Number theorists have long held the opinion that the 'error terms', despite the pejorative name, have a mesmerizingly rich structure ... and that the keys to some of the deepest issues in their subject lie hidden in that structure."

Math at the World Cup. According to a news report in the June 15, 2006 *Nature*, it has been established mathematically that soccer goals are contagious, statistically speaking: scoring one goal increases the probability that your team will score more. Michael Hopkin, who write the piece, calls this "one of soccer's classic clichés," and attributes the result to Martin Weigel (Herriot-Watt University, Edinburgh) and his colleagues Elmar Bittner, Andreas Nussbaumer and Wolfhard Janke, all at Leipzig University. The four have posted a preprint on arXiv.org (arxiv.org/abs/physics/0606016) with the title "Football fever: goal distributions and non-Gaussian statistics." As they put it: "modifying the Bernoulli random process underlying the Poissonian model to include a simple component of self-affirmation seems to describe the data surprisingly well and allows to understand the observed deviation from Gaussian statistics." They analyzed "historical football score data from many leagues in Europe as well as from international tournaments, including data from all past tournaments of the 'FIFA World Cup' series" and concluded: "The best fits are found for models where each extra goal encourages a team even more than the previous one: a true sign of *football fever*." The group paid special attention to three German soccer leagues: the East German *Oberliga*, the West German *Bundesliga* and the women's league, the *Frauen-Bundesliga*. They found that their self-affirmation factor κ was higher for the East German league and highest of all for the women.

Math: Whale Songs → Kaleidoscopic Images. "Subtle Math Turns Songs of Whales Into Kaleidoscopic Images" was the headline for a piece in the August 1 2006 *New York Times*, accompanied by four images like this one.



A periodic segment of the song of the Minke Whale *Balenoptera acutorostrata*; graphic generated using wavelet analysis; plotted in polar coordinates with time $=\theta$. Aguasonic image (www.neoimages.net/artistportfolio.aspx?pid=1421) by Mark Fischer, used with permission.

Gretchen Cuda tells how Mark Fischer, a California-based former engineer, has been using "wavelets — a technique for processing digital signals — to transform the haunting calls of ocean mammals into movies that visually represent the songs and still images that look like electronic mandalas." Cuda checked with Gil Strang of the MIT Math Department, and reports that wavelets, once relatively obscure, "are being used in applications as diverse as JPEG image compression, high definition television and earthquake research." The song and the video, where the pattern shown above can easily be recognized, are available at Minke-Boing on Google.uk (video.google.co.uk/videoplay?docid=-5022402114614151095&q=minke+boing). More from Mark Fischer on his website (www.aguasonic.com).

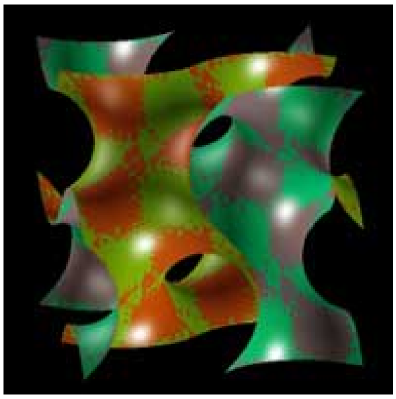
Nanoscale Minimal Surface? "Mesostructured germanium with cubic pore symmetry," by the MSU chemists Gerasimos Armatas and Mercouri Kanatzidis, appeared in the June 29, 2006 *Nature*. The article describes a preparation of germanium resulting in "two three-dimensional labyrinthine tunnels obeying $Ia\bar{3}d$ space group symmetry and separated by a continuous germanium minimal surface." The thickness of the walls of this germanium structure is given as one nanometer. The "minimal surface" separating the labyrinths is identified as the gyroid (www.msri.org/about/sgp/jim/geom/minimal/library/G), a triply periodic surface first described by Alan Schoen in an NASA Technical Note dated May 1970. Schoen gives the (x, y, z) coordinates of a point on the surface in terms of complex integrals:

$$x = \Re \int \frac{e^{i\theta_G}(1-\tau^2)}{\sqrt{1-14\tau^4+\tau^8}}\, d\tau$$

$$y = \Re \int i\, \frac{e^{i\theta_G}(1+\tau^2)}{\sqrt{1-14\tau^4+\tau^8}}\, d\tau$$

$$z = \Re \int 2\, \frac{e^{i\theta_G}\tau}{\sqrt{1-14\tau^4+\tau^8}}\, d\tau$$

where $\theta_G = 38.0147740^o$ approximately is calculated using elliptic integrals. (The 3-integral format goes back to Weierstrass; the specific $\sqrt{1-14\tau^4+\tau^8}$ was used, with 0^o and 90^o instead of θ_G, in H. A. Schwarz's 1865 construction of the first known triply periodic minimal surfaces). Armatas and Kanatzidis, on the other hand, use the much more simply defined level surface $\cos x \sin y + \cos y \sin z + \cos z \sin x = 0$. What is going on? As David Hoffman explained to me, these two surfaces, although extremely close, are not the same. The coincidence is mysterious. As I understand it, chemists start with the symmetry group, which they determine by Fourier analysis of transmission electron micrographs of their sample. From the symmetry group they calculate the equation of a periodic nodal surface as a Fourier series. Our level surface equation comes from setting the sum of the lowest order terms to zero.



The Gyroid (red) and the surface $\cos x \sin y + \cos y \sin z + \cos z \sin x = 0$ (green) plotted together. Image: James T. Hoffman and David Hoffman, Scientific Graphics Project (www.msri.org/about/sgp/jim/geom/level/minimal), used with permission.

Taking more terms gives better approximations to the gyroid, but why this procedure leads to a minimal surface is, as far as I can tell, unknown.

An Interview with James Yorke

This is the mathematical part of a conversation with James Yorke, that took place at the University of Aveiro on the 21st of July 2006, the first day of the Conference "Views on ODEs" — in honor of Arrigo Cellina and James A. Yorke. It ranged from his recent research interests to opinions on how to teach mathematics and how to write a paper. It was fun playing Watson to his Sherlock — I hope you enjoy the conversation as much as I did![1]

I'll start where Cellina was calling you "Maestro". You seem to like to work with other people: I've stopped counting at 140 collaborators. Nowadays it is more common for mathematicians to do collaborative work than when you started.

One time, I found — around 1970 — I was writing a paper that I really liked. But for months it was sitting around and all I had to do was finish off some references, and I couldn't force myself to do it. So I find that it is simply not productive to try and write something by myself. I think it is the interaction that drives it. People add their ideas and I add my ideas and it becomes something better. So I have stopped trying to collaborate with myself.

Well, other people give you pressure and feedback.

That's right, but you get it all, you get different directions, different viewpoints. I got into chaos by switching areas.

You started with Lyapunov functions and control, and ordinary differential equations and things like that.

Differential delay equations... I did a lot of switching even in graduate school. And I feel that one should continue to switch, putting different ideas together.

My most cited paper is with Ed Ott and Celso Grebogi, on controlling chaos. The ideas we describe in the paper are quite simple, but it had a big impact on physicists. There is one reason, namely, that physics ought to be about observing, not disturbing, and control theory is not part of the literature of physics, even elementary linear control theory. So, we got physicists interested in control theory, and they found they liked to control things and there has been way over a thousand papers referring to our paper on controlling chaos.

Well, it also had chaos as an ingredient, right?

Yes, but people had known about chaos for quite a while. One of the things we've tended to do is to try to describe situations by inventing concepts that are appropriate for the physicists. Very often the mathematicians will say there is nothing interesting here, because they do not see what interests the physicists, because they are not physicists. Sometimes the mathematicians think my work is stupid, although they're more happy with the work that is more aimed at mathematicians.

James Yorke

The next question was on the interplay between theory and application which is also something you seem to have worked a lot on.

I'm having problems with the concept — what is an application?

Mathematics and other things?

One of the most applied topics I work on is what I call the billion dollar logic puzzle. This is about genomes: how do you figure out what the DNA is in a chimpanzee or a rat or whatever? I got interested in this 10 years ago just reading the newspaper. And I got some of my collaborators interested, sometimes I can't get them interested and then nothing happens. The idea was, could we find better methods than people already had? It is a question of taking little bits of DNA, and you figure out what the little fragments are, and then you want to figure out which ones overlap which ones, and you have no idea where in the genome it came from. You have these millions of fragments several hundred or a thousand letters long and you try to put them together. And there are errors, and all kinds of problems, but these are not biological problems, you don't have to know anything about the biology.

[1] Isabel S. Labouriau (University of Porto).

That is where the mathematician starts, look at the problem and take out the context.

But this is the way they do it.

I call it a billion dollar problem because that is how much the National Institute of Health (NIH) is spending over a period of about ten years on making these little fragments for lots of different species. I'm not including the human genome, that's a separate project, another billion dollars.

And what sort of maths is involved?

Just trying to be smarter than the next person.

(laughs) Well, that's maths in general.

Well, yes... Little bits of probability theory, and just trying to understand the problem. I don't usually think about the kind of mathematics when I get into a problem, I just try to get into the problem.

I'm not asking what you think beforehand, I'm asking what you have been doing.

OK, I have to say this defensively, you see.

First of all, a lot of the people who have been involved in this problem are extremely smart people. So the problem of trying to outthink these very smart people, is somewhat difficult. That's why I call it a logic puzzle, rather than a math puzzle, you want to find algorithms which work. And there is a little bit of probability theory, and just trying to understand the problem. It is very hard to think about it in simple terms. Sometimes you come up with a simple idea, after working for months.

So it is a problem you have to work with almost no tools.

Right. A lot of computer programming to implement these things, but the ideas are pretty simple.

That sounds very hard.

Yes, it is very hard. Another aspect about this is that there are several centers that create these fragments for the NIH and they are basically paid to create the fragments and by the way put them together into a genome. Nobody checks how good a job they do. If someone else puts it together in a different way, the original center gets to pick the best answer, which by chance, almost always seems to be their own. You see, other people don't want to generate answers, because the answer will be ignored. The data is on the web.

Now, some of these guys are very much afraid of using other people's ideas. This is not true of everybody in the field. One group has a particularly weak set of tools, and so (this is very non scientific) they feel that if they use other people's tools it will make their old stuff look bad.

We are trying to talk to all of the labs, and with some we are having considerable progress. We have more success with the better ones.

Not surprising is it? Maybe that's why they're better.

Right. They don't have to fear looking bad. But this is one project that we've been working on and it's a very big effort.

So that is really switching areas, because that's - no tools.

No tools, right.

Which is really hard to face. A lot harder than, say, a slightly different form of dynamical systems.

One thing that you'll find is that mathematicians tend to emphasize how difficult the problem is, not how big the impact is. And this is counterproductive, it allows people to get deeper and deeper into little problems, which remain difficult but it is hard for them to have much impact.

For me the concept of compactness, for example, at least today it is a very simple idea, and I would love to create an idea like compactness, you see? But for a Fields medals what you need to do is have extremely difficult stuff with very little impact.

Maybe that is a way maths is different from physics.

I disagree. It is the way mathematicians are different from physicists.

And it is not all mathematicians or all physicists. It is not healthy to not worry about impact. Mathematicians will say, "well, maybe it will have impact and maybe it won't, but how can I predict?" Well, life is about predicting the results of your efforts.

Like, I can get food faster.

Or I try to pick up a girl or whatever, you see? One cannot abdicate the responsibility of worrying about the impact of one's efforts. One talks about archival journals, which are journals where you write results up and then they get stuck on shelves, so they'll be there forevermore. This is a terrible concept.

(laughs) On the other hand it is nice to have 2000 year old results that we know about.

Absolutely. Now that's a way mathematics differs from physics, the results that we prove are, hopefully, true forever. I lecture my class in advanced calculus and the Riemann integral about Archimedes, and how he calculated the value of π by upper and lower bounds and moving the lower bounds up and the upper bounds down, and it's converging down to a number, and this is exactly what you do when you define an integral. It's wonderful stuff...

...done more than two thousand years ago.

twenty five hundred...

Let's go back to collaborators. Any one you would like to talk specially about?

I gave a thank you speech for the Japan prize where I listed collaborators who I worked a lot with. These people are absolutely crucial to the whole operation, they contributed a lot of the ideas, I contributed some of the ideas and it all hangs together. The person I have written the most papers with is Edward Ott. He is a physicist and electrical engineer. My view is that if I wanted to talk about chaos, I should talk to non-mathematicians. The reason being that very few mathematicians really have to know about dynamical systems. Everybody else has to understand how things change in time, so there is a huge possible audience out there.

However, I have found that it is very difficult to communicate with an audience who speaks a little bit differently from what you do. You don't realize how different the language is. With Ed Ott who was a very creative guy and a physicist, we're able to come up with ideas that could be expressed in a way that physicists would understand. I collaborated also with Celso Grebogi, but he was more the junior partner in many cases, but he is a very bright guy too.

We were the first people to talk about fractal basin boundaries, aside from some classical cases of complex variables which are very non-physical in nature. We're the first people to do so, but the problem was, how do you write a paper about this, that many mathematicians understand. If you could take someone like John Guckenheimer and asked him with five minutes notice to give a twenty minute talk on this, he could do it, but in a language which has no impact, sometimes. And John Guckenheimer is an excellent expositor and a very bright guy, and he's from the Smale school.

The point is, you have to reformulate these ideas in such a way that you can communicate them. Give them something that they could measure. The mathematician will not understand the right importance of measuring anything, but that's what physics is about.

So that became the problem, how do you communicate these ideas?

Having a partner that's a physicist and electrical engineer solves the language problem at the home level.

Right. And he has done tremendous things by himself. He came to the University of Maryland from Cornell, where he was a full professor and by chance his office was down the hall from my office. We started talking, and we found we had interests in common. That's the way things sometimes work.

Another example of the same thing: the University of Maryland hired a new head of meteorology. A woman named Eugenia Kalnay. There was a reception for her at the president's house and we started talking. It turned out that a lot of the ideas that she was using in meteorology were like the ideas we were using in dynamical systems, so this seemed like an excellent opportunity to collaborate. I suggested that we apply for a Keck foundation grant. The Keck foundation does not require that the person have an excellent track record in that particular question. Otherwise we would never have been funded. We started a group and brought in our other collaborators and they all contributed a lot of ideas on weather prediction.

The basic question becomes how do you determine the initial conditions. All prediction is extrapolation from the present. So they have methods for determining what the weather is today at noon all over the earth, but we've found that we could come up with different methods, and we did, and they are being tested and that's an ongoing project. The other guys were contributing all of the ideas, so I dropped out, but nonetheless, this is how it has got started. Eugenia Kalnay, by the way, was for several years head of the National Weather Service's group that comes up with new methods, she wasn't at that time, she was then a faculty member of the University of Maryland. She was a member of the National Academy of Engineering, she is a well established and well known person. We have all had a great time. It turned out that other people had similar ideas, so it wasn't totally new, but it was new enough that it was quite worth while.

I would add that, when you get theorems that are extremely difficult, it is very hard to apply those results. It is much easier to apply simple ideas. So I try to tell people, if they write a paper, which is a complicated paper, figure out what is the key, the simple kernel, and display this. Give it a name, and I don't mean "the Yorke method".

(laughs)

Name it after what it does, so people can focus on it and use it. Sometimes these wonderful ideas get buried, they're more proud of the complicated structure, and this is the wrong idea, you should find this kernel, this idea, their compactness idea, you see...

The hardest part of mathematics is throwing away all the technicality that was so much hard work to do.

Speaking of throwing things away, you see, I have dealt with students who deal with complicated questions and discover complicated mathematical algorithms. We have been discussing fluid flow in a pipe, using other people's equations, and so you do a tremendous amount of mathematical analysis to describe what is happening. I try to tell them that what people are really interested in are the results, not how you got them. And all you want to talk about, if you're a student, is how you got them.

That is what you work hard for.

You want to throw away 90%. If people like your results then they might be interested in how you got them, but not before. I find this is a recurrent problem with the students, that they are unwilling to get rid of all the hard work and say "here is why you should pay attention".

Students — that's another interesting issue. You've had a lot.

A bunch, over thirty. We have created an approach at Maryland, of group advising. A person will work with different people, sometimes on different but related topics. The emphasis tends to be on writing a dissertation. As many applied mathematicians do, we prefer to emphasize writing papers.

Because the coin of the realm is writing papers, and someone who writes a dissertation is learning how to write, generally speaking, unpublishable material. So we say, the student should write something like three papers, and should staple these together with as few changes as possible, and call it dissertation. At Maryland we have finally set up a rule that says that when a student graduates, he or she should have at least one paper submitted. Now that's pretty easy, since you can submit total trash. Nonetheless it is a goal, you see, that the student should know about.

The rule is there to focus the attention, I imagine.

Right, and it is a nonenforceable thing, in the sense that you can just submit the phonebook.

The difficult part is getting it accepted.

But it is a goal. People do not get tenure for writing a good unpublishable exposition.

Writing a dissertation, a student often is not focused. A student is usually unable to write more than ten or twenty decent pages, so the idea of a hundred and fifty page dissertation for someone who cannot write decent text is a somewhat contradictory concept. I feel strongly about this, that a big failing of many advisors is this hundred and fifty page dissertation. The student is released upon the world, with nobody to teach him, or her, how to write papers. And so maybe they get a post-doc, well, the people giving you a post-doc don't want to hire someone so they can write up their dissertation, or if they are willing to do that it is very questionable.

What is the point... since they already have done it.

I'll give you some more opinions. In terms of education, when I went to graduate school I had quite a few good professors, excellent educators who were also researchers and so on.

But what I remember most and learned best, was material I read on my own. That's the way I remember it. Stuff that I read on my own, particularly over summers, on homotopy theory and cohomology theory. If I took courses on these things, I wouldn't understand them. I took one course, on one theorem. At the end of the course, I couldn't tell you what the theorem said. I still can't. The Riemann-Roch theorem. The professor wanted me to publish a paper with a new proof I had come up with. So I was doing something, but, somehow I've just never clicked as to what this theorem said, you see?

(laughs) That's a different level of not understanding.

I was betting that there would be no final exam, and I won the bet. Nonetheless, I did not say I had no understanding but nonetheless I didn't know what the theorem said, I mean, I couldn't state it.

Now it may be harder to learn things from books but the idea is, you go at your own pace, if you don't understand a paragraph you read it a couple of times. In class, it gets read once, and the professor goes on, often without a good book to follow. There are countries, I've been told for example that in Germany in physics, people are not supposed to follow a book because following a book would not show they have the expertise to jump from book to book. Thereby leaving the student stranded, I believe. So, whenever possible, courses should be taught with good books, that are readable. And in high school, I have been told, they try not to teach you material, but try to teach how to learn. Do we tell our students we're teaching them how to learn?

No, we don't, because if you wanted to learn something about a fast Fourier transform, the model would be to find a course that meets three times a week, and is lecturing on this material, which is of course impossible. Well, you're not gonna find this. What you need to do is to go and find a book and sit down and read the book.

And, yes, some students do read a fair amount, but this is much less the education than it should be. They should be told they are going to learn things by sitting down and reading. And reading a book in mathematics is a separate skill from reading books about history, or novels. So I try to force them to learn how to read math books, and with some I'm successful, with some I fail. But we should be teaching them how to learn. As far as I can tell, it means they should be able to go out and find materials on their own and read them and understand them.

I tell them that we run a factory at the University of Maryland and at other universities, and the basic unit of work is professors taking courses and students taking courses, but this is not the natural way to learn things. I particularly dislike professors lecturing without questions, writing stuff on the board and students copying it down, because in this double translation from the book there are many errors, and the professor is speaking and writing and the student can only write down what the professor wrote down, not what the professor said, and so you have a very low quality version of the material. Twice translated, unintelligible, I think... And that's what they are left with. Now when someone writes a book, they spend many hours on the equivalent of one lecture. And when I lecture, I cannot spend many hours for each lecture.

So, the book is better. If I write a book and I give a lecture, my book is much better than my lecture.

The lecture is for a different purpose... the problem is trying to use the lecture for the purpose of the book.

I tell the students to call me coach. I sign my e-mails to them "coach Yorke". What is the difference between a professor and a coach? I say, actually I don't know the answer. I know a coach is supposed to help you excel. I don't know what a professor does. He gets up there and writes down what is in a book that the students haven't found, and the students copy it down. Well, perhaps the professor puts in more material, or something. He puts in what the book left out, the book should have put that material in. The books should have the motivation. And sometimes they do, and sometimes they don't. So I keep telling them to call me coach. Some of them do, some will write "Professor Yorke".

I tell them to bring the books to class, I point out materials in the book, so they're familiar with the books, I try to encourage them to read it, I emphasize the high points in the books, the easy points they can read... often failing at getting them to read the books.

I tend to teach more advanced undergraduates. This is not the problem of the professor who is teaching Calculus. I'm talking of advanced maths students, math majors, rigorous courses, or semi-rigorous courses. I teach from my chaos book, I say, I wrote the book, I'm not gonna read it to you. Of course I wrote it with my collaborators.

I am a person with many opinions, you see...

When I was in the seventh grade, they've split the class into two parts those who could do mathematics and those who couldn't, half and half. So they felt that since I wasn't very good at long division because I would make errors, that I wasn't very good at mathematics. And so I got the subjects with people who took Latin, and I certainly was not good at Latin.

Along the way I had to catch up with the other students, who took Calculus but, over the summer before going to college I had to read, in order to get into an advanced course, I had to read Halmos' "*Finite dimensional vector spaces*", where I learned how to read maths books. Based on this reading before I went to college, I never took an actual linear algebra course, because I thought I had learned this, eventually I learned it better and better.

James Yorke presenting his talk ("Views on ODEs", July 2006)

It doesn't sound like a bad starting point, Halmos' book.

No. But I think it is all about reading books, or papers or whatever. But to get basic education you read books, not papers. And there is a lot of basic education.

I talk to our seniors, who are taking Riemann integration, rigorous proofs, and I ask them, how many know about the theory of Fourier series? Anything, what a Fourier series is? And perhaps one in thirty will know something about this. There are so many topics in mathematics that they have no concepts of. Simply because the day is limited, and they have to learn all kinds of materials.

Our graduate students at the University of Maryland, don't have to take any Analysis or Probability theory and they can get a PhD and while they probably need to know some probability to get into graduate school, they will know less when they graduate than when they came in.

There are two ways of thinking about mathematics, one is the axiomatics, and so the traditional way courses are Algebra, Topology and Analysis. There's another way to look at it, which are the applications of mathematics and these are by and large, Differential Equations, Numerical Methods, Probability and Statistics. This is how mathematics interfaces with the world.

Then you have to do the interface, which is hard work. Really hard work.

But you get to talk to people. About their problems, if you're gonna interface and use these ideas to interface.

I do tell students about wrong ideas in lectures, on purpose. A really fantastic book would mention the wrong answer, but nobody can spend too much space and time in a book for that. So that's something that can be done in a lecture that a book doesn't do.

But I think they can. I think whatever you can say in a class they can put in a book.

I acted as a, shall we say, editor, for a colleague who was revising his advanced calculus book. He felt there were many things that professors should do and therefore should be left out of the book. But my greatest ally was the person he had take advanced calculus from, who did a totally terrible job. And I say, think of students in a class, what do you want them to know. I go into class on the first day and I say, let's talk about teacher evaluation, what have people said in the past about my chaos course. First thing they say is "what about organisation?", and the students say "there is none". And the reason there is none is I go into class and I ask students what questions they have about the book. Because the book is prominent. How do I force them to read the book? So we have a discussion about what they're finding difficult in the book. And the easy parts, they can all read. This is the approach, you see. When they ask a narrow question, you respond with a broad discussion which covers a certain amount of material, in the book.

So we ended up talking about teaching. Well, it is a big part of our life.

It is a big part of our life, and there are not many ways in which you can talk about teaching in the mathematics media.

As Cellina was saying, passing it on to the next generation.

In America there's *American Mathematical Monthly*, and they talk about topics which supposedly can be understood more or less by undergraduates, advanced undergraduates. But they tend not to talk about actually teaching. In Portugal the number of students taking classes in math or math majors is decreasing, what can you find that is written about peoples' opinions on this topic? Nothing, probably it is like the United States.

Here it has a very specific reason.

What does it mean?

Here the maths major is the same course as the secondary school teacher training course. So we had a peak some ten years ago because there were lots of positions, now there are no positions, we have a drop. So it is a sort of job related issue.

If you ask your students who have just graduated from studying math, why someone should take mathematics, what is it good for? Clearly one answer is to teach high school. What other answers would they give? This is a very important question. And I think they basically can't give any good answer. Because they have not got into good material. Ask them who has learned any mathematics done in the twentieth century, specially the last half.

In the teacher training course, the only twentieth century part is probability.

I don't know what probability theory is in the twentieth century. You've got Lebesgue integration. Aside from Lebesgue integration.

Well, Kolmogorov axiomatics, that's definitely twentieth century.

Ok. But that's an example which they feel can be used.

Probability in itself, not really, in a student vision. Statistics is used a lot.

We use probability theory all over the place, genomics... Is there one course that discusses a lot of ideas that have come in twentieth century mathematics? No. At least not in the United States.

Everybody went downstairs, I think we should switch this off and leave you to have dinner.

Interview conducted by Isabel S. Labouriau (University of Porto)

James A. Yorke (born August 3, 1941) is a Distinguished University Professor of Mathematics and Physics at the University of Maryland, College Park, and a recipient of the 2003 Japan Prize for his work in chaotic systems.

Born in Plainfield, New Jersey (USA), Professor Yorke earned his bachelor degree from Columbia University in 1963, and came to the University of Maryland for graduate studies, in part because of interdisciplinary opportunities offered by the faculty of IPST (an Institute established in 1950 and committed to interdisciplinary research in the sciences). After receiving his doctoral degree in 1966 in Mathematics, Yorke stayed at the University as a member of IPST. He is perhaps best known to the general public for coining the mathematical term "chaos" with T.Y. Li in a 1975 paper entitled "*Period Three Implies Chaos*".

Professor Yorke has coauthored three books on chaos and a monograph on gonorrhea epidemiology, has supervised 40 Ph.D. dissertations in the Departments of Mathematics and Physics, and has published more than 300 papers. Professor Yorke's current research projects range from chaos theory, weather prediction and genome research to the population dynamics of the HIV/AIDS epidemic.

José Monteiro da Rocha
A Portuguese astronomer and mathematician. The work on comets.

José Monteiro da Rocha was born in a small town in the north of Portugal, named Canavezes (Amarante), in the 25th of June, 1734. We could not find relevant information about his early years. However it is known that the young José went to Brasil as part of the "*Companhia de Jesus*". In 1752 we find him as a teacher in the Jesuit school "*Colégio da Baía*". Coincidentally to the Marquês de Pombal laws against the Jesuits in 1759, José Monteiro da Rocha leaves the religious institution and returns to Portugal. In 1770 he gets the degree of "*bacharel*" from the University of Coimbra.

Monteiro da Rocha had a major impact on the development of the University of Coimbra in the XVIII and XIX centuries. In particular, he gave a fundamental contribution to the creation of the new "*Faculdade de Matemática e Filosofia Natural*" in 1772 by writing the "*Estatutos*" of the faculty. In that same year he becomes a teacher of Physics and Applied Mathematics and later on (1783) we can find him teaching Astronomy. The ability of Monteiro da Rocha in astronomy was particularly appreciated and he was nominated director of the *Astronomical Observatory of the University of Coimbra* in 1785. Until the end of the XVIII century, Monteiro da Rocha was nominated to other important academic positions such as "*Decano e Director Perpétuo da Faculdade*" (1795) and "*Vice-Reitor da Universidade de Coimbra*" (from 1796 to 1799).

Besides his main contribution to the creation of the faculty of mathematics, we can find other improvements in the different university aspects (administration, teaching and science) directly related to Monteiro da Rocha. Among these, we emphasize the creation of the Astronomical Observatory (Fig. 1). The document containing the scientific motivations and administrative rules was written by Monteiro da Rocha. The choice of astronomical instruments was also supervised by him.

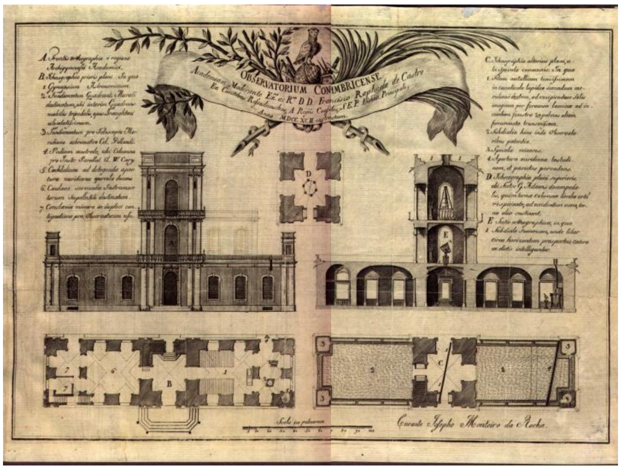


Figure 1. Drawings of the *Astronomical Observatory* of the University of Coimbra, built in 1799 under the supervision of Monteiro da Rocha. In the lower right corner one may read the name of "Monteiro da Rocha".

The scientific work of José Monteiro da Rocha spans between quite different mathematical and astronomical domains. First of all we emphasize the translation to Portuguese of some fundamental textbooks in order to help students and professors from the university. We point out the books of Bézout on arithmetic and trigonometry (both with different editions, respectively from 1773 to 1826 and from 1774 to 1817), the hydrodynamic compendium from Bossut (with two editions in 1775 and 1813) and the text of Marie on mechanics (edited in 1775 and 1812). We can also mention a textbook written by himself, entitled "*Elementa Mathematica*".

Concerning his research work it is also possible to find an interesting sample of publications. Three of them appeared in the first proceedings of the Academy of Sciences of Lisbon (founded in the 24th of December, 1779; Monteiro da Rocha was elected member one month later!) published in 1797 and 1799, namely:

"*Solução geral do problema de Kepler sobre a medição das pipas e tonéis*", "*Aditamentos à regra de Fontaine para resolver por aproximações os problemas que se reduzem às quadraturas*" and "*Determinação das órbitas de Cometas*".

In the sequel, we will focus on the latter publication, taking into account our particular interest in the subject. First of all, let us precise that the determination of a comet orbit means the quantification of the six orbital elements (perihelion distance, eccentricity, orbital inclination, ascending node longitude, perihelion longitude and time at the perihelion) assuming as observables the distances comet-Earth and Earth-Sun in three different epochs. Many famous mathematicians and astronomers were interested in this problem "*hocce longe difficillimum*" (in the own words of Newton, in 1687), even in the case where the orbits of comets are assumed to be parabolas reducing the number of orbital elements from six to five. Among the most famous, we can name Euler, Clairaut, Condorcet, Boscovich, Pingré, Delambre, d'Alembert, Lambert, Lalande and Laplace. The solution appeared only in 1797 by the hand of the Baron of von Zach, presenting the work of Wilhelm Olbers titled "*Abhandlung über die leichteste und bequemst Methode, die Bahn eines Cometen aus einigen Beobachtungen zu berechnem von Wilhelm Olbers*", which gives an easy method to solve the problem. The interesting point concerning Monteiro da Rocha is the following: his work, on the same subject, was effectively published in 1799 (so two years later than the publication of Olbers' work) but the method proposed by Monteiro da Rocha had been, in fact, orally presented to the Academy of Sciences much before, in 1782. This method is formally similar to the one presented by Olbers (both based on the Euler-Lagrange Theorem [3]) and the results are numerically consistent [1]. In order to illustrate this particular point we present, in Table 1, a comparison between three different determinations for the comet *1830 V* (observed by James Watson).

A detailed paper on these comparisons is in preparation [2]. We must add that the work of Monteiro da Rocha was written in Portuguese. This fact certainly was the major cause of the little diffusion of his work at the epoch.

There are other astronomical works of Monteiro da Rocha that deserve to be emphasized: "*Exposição dos métodos particulares no cálculo das Efemérides*" (1797), "*Tábua Náutica para o cálculo das longitudes*" and "*Memórias sobre o uso do retículo romboidal e do instrumento de passagem*" (1806). But it is, probably, "*Efemérides Astronómicas do Observatório de Coimbra*" the contribution of Monteiro da Rocha with major impact in the future of the astronomy in Coimbra and Portugal. In fact, the astronomical ephemeris (including the calculations for the Sun, the Moon, the planets and the brightest stars) have been published during almost 200 years, between 1803 and 2001. The earlier editions were published under the initiative and supervision of Monteiro da Rocha (Fig. 2).

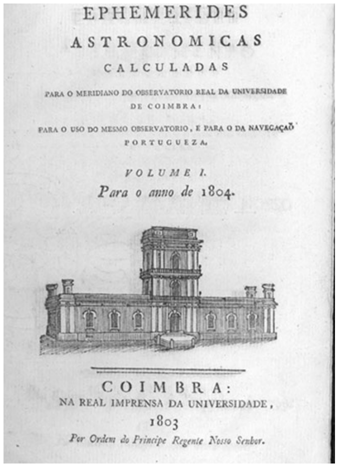


Figure 2. The first edition of the Coimbra astronomical ephemeris, published in 1803 for the year 1804 (copy of the cover).

The global work of José Monteiro da Rocha received the applause and distinction from the Portuguese monarchy at the epoch. In 1800 he was nominated counsellor of the Prince Regent D. João and, four years later, he left the academy and moved to Lisbon, to become a preceptor of the future King D. Pedro IV and his brothers. He also received the "*Comenda da Ordem de Cristo*" in 1802.

José Monteiro da Rocha died in Lisbon in the 11th of December, 1819. According to his will, his manuscripts and scientific works were donated to the Academy of Sciences of Lisbon. His personal library may be found, nowadays, in the Ajuda Palace, in Lisbon.

Orbital Elements	J. Watson	W. Olbers	J. Monteiro da Rocha
perihelion distance (AU)	0.771575	0.7313995	0.777129
orbital inclination	64°31'27.7"	63°43'25"	65°16'29"
ascending node longitude	304°43'11.5"	305°4'55.8"	305°0'16.2"
perihelion longitude	60°23'17.8"	64°24'6"	60°29'23"
time at the perihelion	27-12-1863 (13h33m11s)	28-12-1863 (20h52m)	27-12-1863 (8h13m14s)

Table 1.

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