Finsler Geometry, Rejuvenation of a Classical Domain

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Pedro Nunes Lectures 2016,
Aveiro,
October 20, 2016

Summary

- Some Definitions and Examples.
- A Brief Historic Perspective
- Relevance in Modelling
- Basics of (Modern) Finsler Geometry
- Using the Formalism
- Some Recent Results

Section 1

Some Definitions and Examples.

A Brief Historic Perspective

Definitions

Let M be a C^∞ manifold. A *Finsler metric* on M is a function $F : TM \longrightarrow \mathbb{R}^+$ that is C^∞ outside the zero section such that:

1. F is positively homogeneous, i.e.

$$\forall v \in TM, \forall \lambda \in \mathbb{R}^+, F(\lambda\, v) = \lambda\, F(v)\ ;$$

2. F is positive on $TM - \{0\}$;
3. F is convex.

The Finsler metric F is said to be *reversible* if, for all $v \in TM$, $F(-v) = F(v)$.

Examples

- **Calculus of variations**. A typical problem considers a functional $\mathcal{F}$ on the space of curves on M

$$c : [a, b] \longrightarrow M, \quad \mathcal{F}(c) = \int_a^b F(\dot{c}(t))\, dt \ ;$$

- if one insists that the problem be invariant under *reparametrization* of the curve, then F must be positively homogeneous. This leads to a *degeneracy* in the Euler-Lagrange equations for extremal curves, but the convexity of F ensures that it is the only one.
- **Riemannian Geometry**. A Riemannian metric g gives rise to an associated (reversible) Finsler metric F_g defined, for $v \in TM$, by $F_g(v) = \sqrt{g(v, v)}$.

Examples (continued)

- **Randers metrics.** Let g be a Riemannian metric and β be a differential 1-form; the associated *Randers metric* is $F_g + \beta$ (typical of the problem of navigation in a flow).
- **Hilbert metrics.** Let C be a bounded convex domain in $\mathbb{R}^n$. For p, $q \in C$, we set

$$d_H(p, q) = \frac{1}{2} \log(\mathrm{crossratio}[a, p, q, z])$$

where a and z are the intersections of the line pq with the boundary of C.
This comes from the Finsler metric F_C on C defined, for $v \in T_p C$, by

$$F_C(v) = F_e(v) \left(\frac{1}{d_e(p, p^+)} + \frac{1}{d_e(p, p^-)} \right) .$$

Section 2

A Brief Historic Perspective

A Brief Historic Perspective

- 1788 Joseph-Louis de LAGRANGE, Méchanique Analytique
- 1854 Bernhard RIEMANN's oral defense and essay
- 1869 Edwin Bruno CHRISTOFFEL's equivalence problem
- 1894 David HILBERT
- 1900 Hilbert's Problems 4 and 23 at the Paris International Congress of Mathematicians
- 1922 Paul FINSLER's habilitation in Cologne
- 1926-1929 Paul FUNK, Ludwig BERWALD
- 1931 A seminar at Chekiang University
- 1934 Élie CARTAN

A Brief Historic Perspective (continued)

- 1934 Élie CARTAN's book *"Les espaces de Finsler"*

Les Espaces de Finsler. By E. Cartan. (Actualités Scientifiques et Industrielles, No. 79.) Paris, Hermann, 1934. 40 pp.

In this pamphlet the author presents the recent developments in the geometry of Finsler spaces. He proposes five postulates for measurements in the space and by means of them obtains his tensor calculus. As he points out, his affine connection is essentially different from that of Berwald and it is more nearly analogous to that of a Riemann space; its main advantage is in the fact that lengths are preserved in parallel displacement. Of course here, as is usual, only spaces leading to a regular problem in the calculus of variations are considered. Of the scalar differential forms the author considers only the most important one—the angular metric of Landsberg, and he shows that it has curvature +1. There are three sections on the geometry of curves and surfaces in a three-dimensional Finsler space which show to what extent or with what modifications classical differential geometry can be carried over to apply to Finsler spaces.

The author also considers some special n-dimensional spaces characterized by the vanishing of some tensor invariant; of particular interest is the one for which the determinant of the fundamental tensor is a point-function ($A_{ik}{}^{k}=0$). Another interesting problem considered is the representation of the geometry of line elements (éléments d'appui) as the geometry of surface elements. The whole seems to be very closely related to the study of conservative dynamical systems as treated by contact transformations.

The pamphlet as a whole is extremely well done. The formulas are usually given geometrical content—an attribute often lacking in works on tensor analysis—and the proofs are clear and not too formal. Above all, brief as this pamphlet is, it contains many interesting ideas that seem to be worth elaborating.

M. S. Knebelman

A Brief Historic Perspective (continued)

- 1942 Herbert BUSEMANN

THE GEOMETRY OF FINSLER SPACES

HERBERT BUSEMANN

The term "Finsler space" evokes in most mathematicians the picture of an impenetrable forest whose entire vegetation consists of tensors. The purpose of the present lecture is to show that *the association of tensors* (or differential forms) *with Finsler spaces is due to an historical accident, and that*, at least at the present time, *the fruitful and relevant problems lie in a different direction.*

Finsler spaces were discovered by Riemann in his lecture [1]:[1] *Über die Hypothesen, welche der Geometrie zu Grunde liegen* (1854). The goal which Riemann set for himself was the *definition and discussion of the most general finite-dimensional space in which every curve has a length derived from an infinitesimal length or line element.* In modern terminology Riemann's approach is this: Let a differentiable manifold M of a certain class be given. In any local coordinate system $(x_1, \cdots, x_n) = (x)$ a length $F(x, dx)$ must be assigned to a given line element $(x, dx) = (x_1, \cdots, x_n; dx, \cdots, dx_n)$ with origin x. If $x(t)$ is a (smooth) curve in M then $\int F(x, \dot{x})dt$ is its length.

In order to insure that the length of a curve is positive and independent of the sense in which the curve is traversed, Riemann requires $F(x, dx) > 0$ for $dx \neq 0$ and $F(x, dx) = F(x, -dx)$.

Next Riemann assumes [1, p. 277] that the length of the line element remains unchanged except for terms of second order, if all points undergo the same infinitesimal change. This amounts to the condition $F(x, kdx) = kF(z, dx)$ for $k > 0$. Nowadays we rather justify this condition by requiring that a change of the parametrization of the curve does not change its length.

Riemann then turns immediately to the special case where $F(x, dx) = [\sum g_{ik}(x)dx_i dx_k]^{1/2}$, that is, to those spaces which are now called Riemann spaces. The general case is passed over with the following remarks: the next simplest case would comprise the manifolds, in which the line element can be expressed as the fourth root of a biquadratic differential form. The investigation of these more general types would not require any essentially different principles, but it would be time consuming and contribute comparatively little new to the theory of space (verhältnismässig auf die Lehre vom Raume wenig neues Licht werfen), because the results cannot be interpreted geometrically (see [1, p. 278]).

Sectional Meetings; received by the editors December 8, 1948.

[1] Numbers in brackets refer to the references cited at the end of the paper.

A Brief Historic Perspective (continued)

- 1942 Herbert BUSEMANN
- 1944-1948 Shiing-Shen CHERN's papers on Finsler Geometry
- 1955-1970 André LICHNEROWICZ, Hassan AKBAR-ZADEH, Pierre DAZORD
- 1986-... Patrick FOULON's dynamic approach
- 1992 Shiing-Shen CHERN's Compte-Rendu note at the Paris Science Academy
- 1993-... David Dai-Wai BAO, Zhongmin SHEN, Robert BRYANT, Juan Carlos ALVAREZ-PAIVA, Daniel EGLOFF, ...
- 1993-... Viktor BANGERT, Hans-Bert RADEMACHER, Yiming LONG, Wei WANG,...
- 2009 Shin-Ichi OHTA
- 2012 Thomas BARTHELMÉ
- ...

Section 3

Relevance in Modelling

Relevance of Finsler Geometry in Modelling

- **Physics.** Crystals in electromagnetic fields; binocular visual space; thermodynamics;...
- **Biology.** Ecology; Marine Biology (evolution of corral colonies); Host/Parasite systems;...
- **Stochastic Models.** Beyond the Brownian motion;...
- **Control Theory.**
- ...

4. Modern Basics of Finsler Geometry

Comparing Riemannian and Finsler Geometry

- **Basic Object.** Riemannian metric g, Finsler metric F.
- **Covariant derivative.** Levi-Civita connection D^g (g-metric and torsion free), in the Finsler setting connections introduced by P. BERWALD and É. CARTAN (metric with torsion), also by S.S. CHERN (torsionfree not metric).
- **Curvature.** $R^g_{X,Y}Z$ measures the deviation from flatness, in the Finsler context there are several notions, e.g., *flag curvature*.
- **Geodesics.** Extremals of length, generalize straight lines, behaviour governed by R^g, in the Finsler context they are extremals of the functional $\mathcal{F}(c) = \int_a^b F(\dot{c}(t))\, dt$.
- **Space forms.** Spaces of constant curvature serve as models. Many examples of constant curvature spaces.

Specific Features of Finsler Geometry

To a coordinate system (x^i) on M, one naturally associates a coordinate system (x^i, X^i) on TM.

- The form $\sum_{i-1}^{n} \partial F/\partial X^i \, dx^i = d^v F$ is well defined.
- Also, at a point $v \in TM - \{0\}$,

$$g_F(v) = \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 F^2}{\partial X^i \partial X^j}(v) \, dx^i \, dx^j$$

  is a well defined *Riemannian metric* on $T_p M - \{0\}$. (Note that, for F_g, $g_{F_g} \equiv g$).
- The *Cartan tensor* A_F is defined on TM as

$$A_F(v) = \frac{1}{4} F(v) \sum_{i,i,k=1}^{n} \frac{\partial^3 F^3}{\partial X^i \partial X^j \partial X^k}(v) \, dx^i \, dx^j \, dx^k \ .$$

  (For F_g, $A_{F_g} \equiv 0$).

Different Approaches to Finsler Geometry

- **Tensorial approach.** This was the classical one.
  There is a real difficulty to identify the *geometric* content as the technicalities are quite formidable.
- **Moving frame approach.** This was the one privileged by both É. CARTAN and S.S. CHERN.
  The key feature is the role of *exterior differentiation*.
  Again one needs a very specific *geometric insight* in order to concentrate on the right notions.
- **Dynamic Approach.** This was the one introduced in the late 1980s by Patrick FOULON.
  The starting point is more subtle, but later on only *geomerically relevant* notions come up.

The Basic Setting of Finsler Geometry

- One starts from the Calculus of Variations. The famous *Euler-Lagrange equations* are actually mathematical subtle

$$\frac{d}{dt}\left(\frac{\partial L}{\partial X^i}\right) - \frac{\partial L}{\partial x^i} = 0, \qquad 1 \leq i \leq n\ .$$

- In intrinsic form, this is a *second order differential equation*, i.e. a map $Z : TM \longrightarrow TTM$ which is a section of both $\pi_{TM} : T(TM) \longrightarrow TM$ and $T\pi_M : T(TM) \longrightarrow T(M)$.
- We shall see later that, from the second order differential equation Z_F determined by a Finsler metric F, a lot of *geometry* follows.

The Basic Setting of Finsler Geometry (continued)

- The basic object is the *homogeneous bundle* of M, i.e., the space of *half-lines* of the tangent bundle $\pi_M : TM \longrightarrow M$
$HM = TM/\mathbb{R}^+ \longrightarrow M$.
(It is in some sense an intrinsic, but equivalent, way of dealing with the *sphere bundle* $\{v \in TM \mid F(v) = 1\}$.)
- The *Hilbert form* attached to F, namely $d^v F$, descends to HM where it is called ω_F. The strong convexity of F implies that $\omega_F \wedge (d\omega_F)^{n-1} \neq 0$, i.e., ω_F is a contact form on HM.
- The *dynamics* defined by F is the vector field Z_F which is the *Reeb field* of ω_F satisfying

$$d\omega_F(Z_F, .) = 0 \ , \qquad \omega_F(Z_F) = 1 \ .$$

The Basic Setting of Finsler Geometry (continued)

To any second differential equation Z on HM are associated:

- a splitting of THM

$$THM = \mathbb{R}.Z \oplus VHM \oplus h_Z HM \ ,$$

  where $h_Z HM = \ker \mathcal{L}_Z v_Z$ (for v_Z the *vertical endomorphism* associated to Z defined as $v_Z(Z) = 0$, $v_Z([Z, \tilde{V}]) = -V$, $v_Z(V) = 0$ for any vertical vector $V \in VHM$);
- a *dynamical derivative* $\mathcal{D}^Z$ on vector fields on HM defined by

$$\mathcal{D}^Z(fZ) = (\mathcal{L}_Z f)Z, \quad \mathcal{D}^Z(V) = -\frac{1}{2} v_Z([Z, [Z, \tilde{V}]]) \ ,$$

$$\mathcal{D}^Z(fX) = f\,\mathcal{D}^Z(X) + (\mathcal{L}_Z f)\,X \ ;$$

- a *Cauchy-Riemann structure* J^Z.

The Basic Setting of Finsler Geometry (end)

- The splitting of *THM* and the Cauchy-Riemann structure J^Z associated to Z are parallel for $\mathcal{D}^Z$

$$\mathcal{D}^Z(VHM) \subset VHM \ , \mathcal{D}^Z(h_Z HM) \subset h_Z HM \ , \mathcal{D}^Z Z = 0 \ .$$

- This gives rise to a *Jacobi endomorphism* R^Z on $h_Z HM$ defined as by

$$R^Z(X) = J^Z(proj_{VHM}([Z,\tilde{X}])) \ .$$

- On *HM*, any Finsler metric F gives rise to a natural *Riemannian metric* g_F for which:
  - the splitting $\mathbb{R}.Z_F \oplus VHM \oplus h_{Z_F}HM$ is g_F-orthogonal;
  - $g_F(Z_F,Z_F) = 1$, $g_F(V_1,V_2) = d\omega_F([Z_F,\tilde{V}_1],V_2)$;
  - $g_F(X_1,X_2) = g_F(J^{Z_F}(X_1),J^{Z_F}(X_2))$.

5. Using the Formalism

Second Variation Formula

- The variation of geodesics by geodesics gives rise to an *index form* I_F (as Z_F is naturally attached to F, we use F as index for quantities such as $\mathcal{D}$ and R instead of Z_F)

$$I_F(A_1,A_2) = \int_a^b \Big(-g_F(X_1,\mathcal{D}^F\mathcal{D}^F X_2) - g_F(X_1,R^F(X_2)) \Big)\, dt \ ,$$

where $A_i = \lambda_i\, Z_F + V_i + X_i$.

- This fits completely with the classical situation in Riemannian Geometry if one has identified the Finsler Jacobi endomorphism R^F with the Riemannian one by setting

$$R^{F_g}(X) = R^g(X,\dot{c})\dot{c} \ .$$

- Note that g_F is $\mathcal{D}^F$-parallel and that R^F is g^F-symmetric.

Finsler versus Riemannian

- Many global theorems in Riemannian Geometry remain true in Finsler Geometry (basically all those who have to do with the calculus of variations of geodesics):
  - Gauß Lemma
  - Myers Theorem
  - Synge Theorem
  - Hopf-Rinow Theorem
  - Hadamard-Cartan Theorem
  - Anosov Theorem
  - ...
- The *fundamental problem* is *to understand well what distinguishes Finsler Geometry from Riemannian Geometry*.

Example of the Hilbert Geometries

- If the boundary of the convex body C is smooth, the Jacobi endomorphism of the associated Finsler metric F is $R^F = -Id$.

Theorem (P. FUNK, L. BERWALD)

Any reversible Finsler metric which is complete and projectively flat with constant negative curvature on a simply connected manifold is a Hilbert Geometry.

Theorem (P. FOULON)

Any projectively flat Finsler metric on S^2 that is reversible with constant positive curvature is Riemannian.

- R. BRYANT showed that projective flatness was actually not necessary.

6. Some Recent Results

Closed Geodesics

- On a compact Riemannian manifold, there are infinitely many closed (distinct) prime geodesics. (The most difficult case is that of the sphere!)
- On S^2 Anatoly KATOK and Wolfgang ZILLER constructed a non reversible Finsler metric with exactly TWO closed geodesics. (The metric is rather simple, even of Randers type.)
- One gets easily ONE closed geodesic on any compact Finsler manifold by standard Calculus of Variations argument.

Theorem (Viktor BANGERT, Yiming LONG)

On S^2, any Finsler metric has at least TWO distinct closed prime geodesics.

Characteristic Forms and Metrics

- The starting point is the Gauß-Bonnet formula following S.S. CHERN's point of view, namely to lift the *characteristic form* to the homogeneous bundle where the form transgresses:
  - in a Riemannian setting $\chi(M^{2m}) = \int_M P(R^g)\, v_g$, where

$$P_1 = \frac{1}{2\pi} K^g \ , \qquad P_2 = \frac{1}{8\pi^2}(|W^g|^2 - |Z^g|^2 + |U^g|^2) \ .$$

  - in a Finsler setting things are not that simple.
- Here is a partial result:

Theorem (D. BAO, S.S. CHERN, Z. SHEN)

A Gauß-Bonnet formula can be established provided the volume of the Finsler sphere at each point is constant, or similar technical assumptions.

Defining a Finsler Laplace-Beltrami Operator

There has been many attempts to define such an operator on M. Here I report on the promising efforts by Thomas BARTHELMÉ.

- One can construct a volume element on M associated to F thanks to the key remark that any volume element Ω on M determines an $(n-1)$-form α^{Ω} on HM so that

$$\alpha^{\Omega} \wedge \pi_M^* \Omega = \omega_F \wedge (d\omega_F)^{n-1} \ .$$

- By normalizing the integral of α^{Ω} to be the standard volume of S^{n-1}, one can determine a unique volume form Ω^F so that

$$\alpha^{\Omega_F} \wedge \pi_M^* \Omega_F = \omega_F \wedge (d\omega_F)^{n-1} \ ;$$

  restricted to VHM, the $(n-1)$-form α_{Ω_F} gives a solid angle.
- Actually, viewing M as a metric space thanks to F, this volume form is also known as the *Holmes-Thomson* measure.

Defining a Finsler Laplace-Beltrami Operator (cont.)

- T. BARTHELMÉ defines the *Finsler Laplace-Beltrami Operator* for a Finsler metric F as follows

$$(\Delta^F f)(x) = \frac{n}{vol_e(S^{n-1})} \int_{H_x M} \mathcal{L}^2_{Z_F}(f \circ \pi_M)\, \alpha^{\Omega_F} \ .$$

- The operator enjoys a number of important properties:

Theorem (T. BARTHELMÉ)

For a Finsler metric F, the Finsler Laplace-Beltrami operator Δ^F is an elliptic linear second order differential operator that is symmetric with respect to Ω_F.
It coincides with the Laplace-Beltrami operator of a Riemannian metric g if the Finsler metric is F_g.

I thank you for your attention.

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