BULLETIN

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EDITORIAL

The International Center for Mathematics (CIM) in 2009 has continued to promote and to organize several meetings and interdisciplinary conferences in the Mathematical Sciences. This Bulletin, as well as CIM's web page, announces the continuation of initiatives for 2010 that is able to sponsor and/or promote. In spite of the difficulties of its current financial situation, CIM aims to pursue the development of research in mathematics and the promotion of international cooperation between researchers in the Mathematical Sciences. In particular, its Direction intends to establish protocols with institutional associates with a view to a greater participation by these institutions in interdisciplinary activities. In particular, to stimulate and to facilitate programme activities among its more than forty associates, CIM calls for individual or joint proposals of mathematical research activities to be done in Portugal for 2011 and 2012, namely for the organisation of short or longer thematic programmes, conferences, workshops, specialized summer courses, or courses in partnership with associates, specially for interdisciplinary subjects within mathematics and other sciences, technology and society, from climate and energy to mathematics education, from nanotechnology to bio-sciences, from computation to complex systems. Those interested in proposing initiatives to CIM are invited to contact the Direction members.

CONTENTS

Digital Mathematics Libraries: *The Challenges of the Pt-DML*
by Assis Azevedo, José Borbinha, Pedro J. Freitas, Eugénio Rocha 2

Report: *On the Madeira Klein Conference* by Bill Barton 6

Article: *Trends in Mathematics: How they could Change Education?* by László Lovász 10

A Conversation with Mats Gyllenberg by José Francisco Rodrigues 16

Feature Article: *All you should know about your "Companion"* by Robert E Hartwig 22

The International Conference: *History of Astronomy in Portugal: Institutions, Theories, Practices, an International Conference* by Luis Saraiva 32

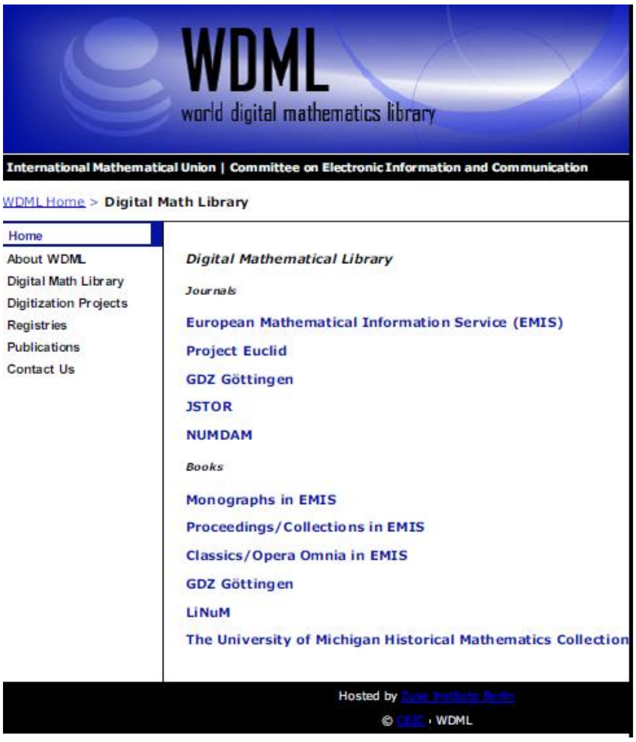
Coming Events 34

Pedro Nunes Lectures by *Sir* Michael Atiyah 36

The Challenges of the Pt-DML

Assis Azevedo [1], José Borbinha [2], Pedro J. Freitas [3], Eugénio Rocha [4]

Everyone agrees that the world is now more digital than ever. Information technology is profoundly transforming the ways in which scholars consume research and disseminate their outputs. Content may still be used in different formats (e.g., people find articles online and print them locally) and on different devices (e.g., iPods, Kindles, handheld readers), but increasingly it must be at least discoverable online to reach readers. For that reason, literature in non-digital format is losing its audience. In many disciplines, people are aware of such problems and are developing initiatives to transform the relevant non-digital resources (e.g., books, journals, maps) to digital formats, organizing them in coherent collections (e.g., see the World Digital Library by UNESCO at [1]).



The mathematicians have contemplated an effort to digitize the past mathematical literature (estimated in 75 million pages) in order to make it available online, this effort is commonly designated by the World Digital Mathematics Library (WDML) [2]. This IMU (International Mathematical Union) virtual project has stimulated several other concrete projects and workshops, such as the European Science Foundation Workshop on the European Virtual Library in Mathematics held at the University of Santiago de Compostela, Spain, $13 - 14^{\text{th}}$ March 2009 [3].

The aim is to make as much of the past literature available as possible, linked to the present literature in suitable ways, pushed by the principle that mathematics is an accumulative science (so the past is very relevant) with a remarkable importance to other disciplines. Common examples of such importance are: (a) *Medical Imaging (tomography)* is possible because of early 20^{th} century measure theory; (b) *Secure Transactions* between banks or over the internet are possible because of cryptography results in number theory from the last two centuries; and (c) *Modern String Theory* depends on algebraic geometry from 19^{th} century, besides many others. Hence it is expectable that present and future scientific development will use decisively past mathematical literature.

In the end, the key point of WDML is to preserve knowledge and to make it worldwide and effectively available to whoever need it.

The present situation

The WDML effort is still under active development and, so far, it has been materialized in a set of projects and initiatives, for which we list a few (including comments from their own homepages):

- mini-DML [4]: *a French project with the goal to collate in one place basic bibliographical data for any kind of mathematical digital article and make them accessible to the users through simple search or metadata retrieval. It is based at NUMDAM and provides a simple search interface for the collected data;*
- European Mathematical Information Service (EMIS) [5]: *The first electronic mathematical library;*

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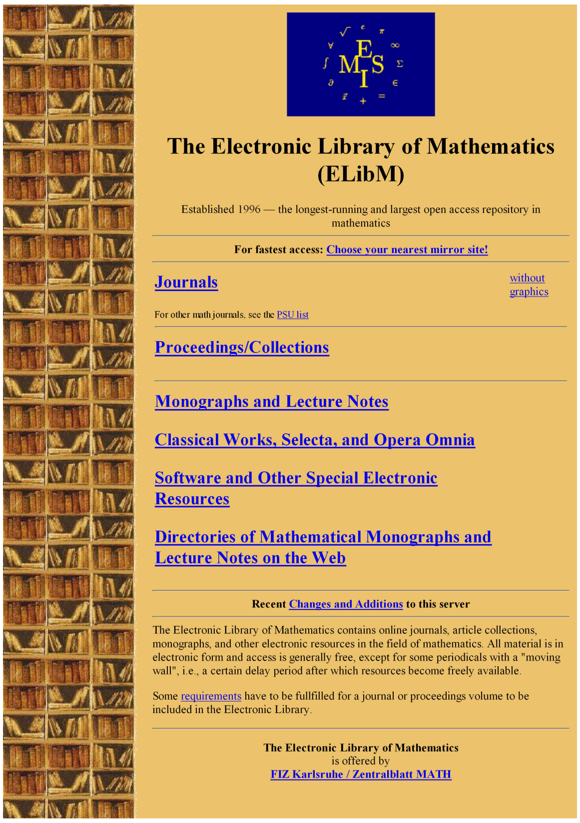

- JSTOR [6]: *is a general digitization project for the cultural heritage of the USA. It includes a large collection of mathematical journals, but has some interface limitations as single page view only;*
- "Göttinger Digitalisierungs-Zentrum" (GDZ) [7]: *the digitization center of the State and University Library Göttingen, with more than 4 Million pages of digitized content available (including the digitized Gutenberg Bible). This encompasses more than 1.2 Million pages of mathematics, accessible through the Mathematica collection;*
- "Jahrbuch Project" or Electronic Research Archive for Mathematics (ERAM) [8]: *provides an electronic database containing the reviews from the "Jahrbuch ber die Fortschritte der Mathematik" as well as digitized versions of the most relevant publication of the time;*
- "Numérisation de documents anciens mathématiques" (NUMDAM) [9]: *a French digitization project, providing access to the digitized versions of the French major mathematical journals, extending the collection continuously. The articles are available in PDF and DjVu format. Descriptions of the articles are provided, including the identifying numbers of the reference journals Mathematical Reviews and Zentralblatt der Mathematik (not the MSC classification, though), an abstract and the complete bibliography, with each item linked to a digitized version if available, MR and Zbl;*
- Project Euclid [10]: *a user-centered initiative to create an environment for the effective and affordable distribution of serial literature in mathematics and statistics. Project Euclid is designed to address the unique needs of independent and society journals through a collaborative partnership with scholarly publishers, professional societies, and academic libraries. In particular, Project Euclid has digitized back-files of the given journals and provides free access for many issues. Metadata come with abstract, keywords and classifications (no links to review journals);*
- "Biblioteka Wirtualna Matematyki" [11]: *a Polish project that is based at the ICM of Warsaw University, with the goal of digitizing Polish mathematical journals;*
- Czech DML [12]: *one of the European pilot projects having both a repository and a search engine;*
- "Biblioteca Digital Española de Matemáticas" [13]: *a project for the Spanish journals promoted by the Spanish Committee of Mathematics;*
- Ulf Rehmann's Collection [14]: *sited at the Bielefeld University, gives links and information regarding digitization projects and a fairly comprehensive list of digitized mathematics available.*

The above projects and initiatives are generally funded by national foundations or academic resources (with some exceptions as JSTOR which is a private enterprise). These (retro-)digitalized materials are (and more will be made) available online, at a reasonable cost or free of charges, in the form of an authoritative and enduring digital collection, developed and curated by a network of institutions. For the WDML goal to happen, many issues and challenges need to be addressed, and this is being done through a number of committees, sub-committees and projects.

In Portugal, in spite of the organization of the 2006 Internatinal Workshop held at the University of Aveiro that produced a collective book [15], we are somehow delayed, e.g. when compared to France or USA, but it seems that now things are starting to roll on. The inertia not only is due to financial issues but it is also related with the general unawareness of scholars to this topic. There are two main initiatives that are OAI aggregators of some of the Portuguese digital resources available (mainly preprints and thesis; without digitalization concerns), namely, the *Repositório Científico de Acesso Aberto de Portugal* (RCAAP) at [16] and the *Portuguese Archive of Mathematics* (PAM) at [17].

Another aspect which is crucial to build the Portuguese DML (PtDML) is the identification and catalog of portuguese relevant resources to be digitized or transformed in order to be available in an aggregator web portal. In particular, the journal *Portugaliae Mathematica*, up to 1993, is retrodigitized and available at the site of the National Digital Library [18]. The journal is now published by the European Mathematical Society Publishing House. For instance, a search at mini-DML for "von Neumann" will yield now eight resuts, five from Project Euclid, one from the Czech DML, one from Numdam and one from *Portugaliae Mathematica*:

00238014. von Neumann, John
A Further Remark Concerning the Distribution of the Ratio of the Mean Square Successive Difference to the Variance
Ann. Math. Statist. 13, no. 1 (1942), 86-88
Article (Euclid)

00120986. von Neumann, John
Approximative properties of matrices of high finite order
Portugaliae mathematica 3(1), 1-62 (1942)
Article (Portugaliae Mathematica)

00238024. von Neumann, John
Distribution of the Ratio of the Mean Square Successive Difference to the Variance
Ann. Math. Statist. 12, no. 4 (1941), 367-395
Article (Euclid)

00310345. von Neumann, John
Miscellaneous. On the position of mathematics. The mathematician
Applications of Mathematics, Volume 10, number 5 (1965), 444-451
Article (Czech-DML)

00223444. von Neumann, John; Goldstine, H. H.
Numerical inverting of matrices of high order
Bull. Amer. Math. Soc. (N.S.) 53, no. 11 (1947), 1021-1099
Article (Euclid)

00104938. von Neumann, J.
On infinite direct products
Compos. Math. 6, 1-77 (1939)
Article (Numdam)

00238001. Hart, B. I.; von Neumann, John
Tabulation of the Probabilities for the Ratio of the Mean Square Successive Difference to the Variance
Ann. Math. Statist. 13, no. 2 (1942), 207-214
Article (Euclid)

00238053. von Neumann, J.; Kent, R. H.; Bellinson, H. R.; Hart, B. I.
The Mean Square Successive Difference
Ann. Math. Statist. 12, no. 2 (1941), 153-162
Article (Euclid)

But the journal *Portugaliae Mathematica* is not the only mathematical portuguese heritage. In fact, other periodical publications from several institutions exist and need to be addressed in order to be in the PtDML, e.g. from the Portuguese Mathematical Society, there are the *Bulletin* at [19] and *Gazeta* at [20]; among other publications from other institutions which are currently being surveyed [16, 17].

The EuDML project

EuDML, European Digital Mathematical Library, is a project that will build a new service for search and browsing to serve as a proxy to existing European portals of mathematical content. That will be achieved by implementing, on top of a rich metadata repository, a single access portal for heterogeneous and multilingual collections of digitized and born digital contents (papers, books, manuscripts, etc.). The service will be constructed by merging and augmenting the information available about each document from each collection, and matching documents and references across the entire combined library. Relevant elements such as authors, bibliographic references and mathematical concepts will be singled out and linked to matching items in the collections; similar mechanisms will be provided as public web-services so that end-users or other external mathematical resources will be able to discover and link to EuDML items. This way, EuDML will be a new major player in the European (and, in general, international) emerging landscape of scientific information discovery services, enabled for reuse in new added-value chains (such as in mashups). EuDML also will be aligned with the purposes of Europeana, The European Digital Library at [21], an initiative willing to reach the same objectives but considering the overall scientific and cultural European heritage. In that sense, it is expected EuDML will interoperate with Europeana for the area of mathematics! The EuDML consortium will comprise 14 international partners (including the Zentralblatt Math, based in Germany and the actual European larger database in mathematics, plus also partners from Portugal, Spain, France, UK, Poland, Czech Republic, Hungary, Bulgaria and Greece) and will be coordinated by a Computer Science research team from the IST/Tecnical University of Lisbon with a large scientific and technical experience on design and development of digital libraries. The EuDML project will be co-funded by the ICT Policy Support Programme (ICT PSP) of the European Commission. It will start the 1st February 2010, and will last for 3 years.

It is expected that the participation of the IST in this initiative will result in a strong incentive to the Portuguese mathematics community, to boost new activities towards the development of the PtDML - Portuguese Digital Mathematical Library, in an effort to be lead by the CIM in close collaboration with its associates including the mathematical societies SPM and SPE and the mathematical libraries of Portuguese universities.

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[4] The mini-DML: http://www.numdam.org/minidml/

[5] The European Mathematical Information Service (EMIS): http://www.emis.de/

[6] JSTOR: http://www.jstor.org/

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[8] Jahrbuch Project: http://www.emis.de/projects/JFM/

[9] NUMDAM: http:\www.numdam.org

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[18] Portugaliae Mathematica at the Portuguese National Library: http://purl.pt/index/pmath/vol/PT/index.html

[19] Bulletim of the Portuguese Mathematical Society: http://nautilus.fis.uc.pt/bspm/

[20] Gazeta of the Portuguese Mathematical Society: http://www.mat.uc.pt/~gazeta/GazetaOnline/online.php

[21] Europeana - The European Digital Library (EuDML): http://www.europeana.eu

On the Madeira Klein Conference

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The programme of the conference "Didactics of Mathematics as a Mathematical Discipline", that was held in Funchal last October 1-4, 2009, can be found in http://glocos.org/index.php/dm-md/. *This conference was the first one associated with the IMU/ICMI Klein project,* http://www.mathunion.org/index.php?id=805*, The Klein Project, inspired by the Felix Klein's famous book* Elementarmathematik vom höheren Standpunkte aus*, published one century ago, is intended as a stimulus for mathematical teachers, so to help them to make connections between the mathematics they teach, or can be asked to teach, and the field of mathematics, while taking into account the evolution of this field over the last century. The project will have three outputs: a book simultaneously published in several languages, a resource DVD to assist teachers wishing to bring some of the ideas to realisation in their classes, and a wiki-based web-site seen as a vehicle for the many people who will wish to contribute to the project in an on-going way. This report to the Design Team of the Klein project will focus on themes and contributions that arose in discussion at the Madeira meeting, that were strongly debated, or that received some consensus.*

After an introductory session from the Rector of the University of Madeira, José Manuel Castanheira, the Presidents of ICMI and CIM, respectively Michèle Artigue and José Francisco Rodrigues, who also presented a brief overview on Felix Klein, and Bill Barton, as convenor of the Klein Project, the discussion raised two important points that need further consideration. The first was an organisational issue, the idea of local or regional "Writing Workshops". It was suggested, that when the project was progressed a little more, it would be possible to have writing workshops involving a group of people coming together to draft material for the Klein Project (either for the book or for web-pages or resources). It would be understood that the product of these workshops would not necessarily be included, but would be submitted to the Design Team for consideration (and maybe further development). However, the Workshops would provide a wide opportunity for involvement, and, if they included both mathematicians and mathematics educators, would become part of the process of the project. I had not thought before about the process of the Klein Project having some developmental aims separate from the project itself. The second point raised was that of "problems sets" as either on organising idea for the book and/or a technique for writing. It was not clear exactly what constitutes a "problem set" (and more than one idea appeared to be present in the discussion), but there was general agreement about the usefulness of this idea.

Figure 1: *The opening session chaired by the Rector of the University of Madeira.*

Thomas Banchoff's talk on Midpoint Polygons using a geometric environment raised the general issue of the way technology has changed geometry itself as well as its pedagogy. As an aside, he reminded us of the power of counterexampleswith a nice example of a conjecture that appears to be true and is then (moving one corner of a pentagon left the area of the midpoint polygon constant, but moving another changed it). Gert

[1] Bill Barton (University of Auckland, New Zealand) is the Chair of the Design Team of the IMU/ICMI Klein project and will succeed Michèle Artigue as President of International Commission on Mathematical Instruction (ICMI) in January 2010.

Schubring's talk on Klein and his vision included the key ideas of historical shifting (that is the gradual elementarisation of mathematical topics over time) and the consequential hysteresis, a gap of more or less 30 years between the origin of a mathematical idea in its original complexity and the integration of the concept as an organic part of mathematics. Nevertheless, it was noted, new unprocessed (but suitably presented) mathematical ideas can motivate and inspire teachers. In discussion, it was noted that we must solve contemporary problems, which are different from those facing Klein although similarities exist, (but what are they exactly?). In particular, the Wiki-site frees the book from the tyranny of having to choose. Nevertheless, we should not shy away from the fact that the Klein project will be a filter of the essence of mathematics. The choice of examples/topics/problems which can convey this essence is not unique. So focus on the vision we wish to present for teachers. One suggested vision was: mathematics as a human construction in order to resolve classes of problems in a certain domain, where mathematicians pursue both the techniques for solution but also the underlying structure that makes those techniques work. Another point raised was that we know that doing mathematics supports its understanding. This applies for teachers as well as students. How can we incorporate this idea into the Klein Project?

Sebastian Xambó introduced us to Clifford's conception of geometric algebra, the link with physics and the writing of David Hestenes http://modelingnts.la.asu.edu/. Hestenes paper on the occasion of the Oersted Medal and Xambó's slide 18 of his presentation in particular. The session prompted discussion about the idea of "Chapters" in the book, the linking of ideas, and raised the idea of Case Studies (e.g. of elliptic curves, codes, complex numbers, algebraic topology, FLT, etc) rather than (as well as?) Chapters. It also highlighted the way mathematics is part of the frontier to human knowledge geometric algebra could be an example of a living research area exemplifying the culture of modern mathematics. Will we have a summary, somewhere, of the achievements of mathematics in the 20th century? Mário Dias Carneiro showed us more interconnections, topological ideas in differential equations (tent maps), and illustrated the importance of normal forms. He spoke more about the way research has changed with new technology. He introduced the idea of "The Better Book", that is a book that continues to evolve with new contributions as they mature being contained in new editions.

Ulrich Kortenkamp demonstrated the power of computing in many ways, both as a mathematical tool, and as a presenter of mathematical ideas, e.g. a lovely illustration of the many notions of angle, and another of the midpoint theorem showing how theorems arise from definitions but are not true in a universal or mystical sense. We need to utilise technology. See ¡madepedia.de¿. He spoke of criteria for the Wiki-site: citable, authors visible, interactive, and an editorial board. What else? Manuel Silva spoke of algorithmic thinking, giving examples of algorithmic proofs, including an induction example. He argued for Erdos' Probabilistic Method to be included. Discussion questioned how profoundly we can (or should) study algorithms, and then asked metaquestions about algorithms: how do we choose them, how do we critique them, how do we choose between them, how do we know if they are correct or not, etc? It was argued that programming is part of mathematics, or, rather, that formulating mathematics in a programming language is mathematics. Is a programming language a new language of mathematics? The SAGE Project (William Stein at Washington State) was mentioned.

Figure 2: *An aspect of the audience with B. Barton, B. Hogdson and T. Banchoff in the first row.*

Jaime Carvalho e Silva reminded us that Klein's was not the only vision of his era, alerting us to wonder what people will say about the Klein Project book 100 years from now. João Caramalho Domingues spoke about a proof of Cunha, which raised the issue of results no longer used showing us mathematical development. Discussion included the following formulation of the Klein project as saying to teachers "You are teaching elementary mathematics, but this is why what you are teaching is important". That is, it is neither exposition, curriculum, text, nor popularisation. National schools of thought were suggested as needing inclusion. Similarly for different approaches: the genetic (historic) approach, the intuitive approach, the experimental approach, the axiomatic (logico-deductive) approach, and the pedagogical approach. (Any others?). It was noted that "the work of logical analysis is to distinguish the acts of intuition and help successive abstractions and so proceed in the development of mathematical intuition into higher spaces".

Abraham Arcavi refocused our attention on school mathematics, and the students whom our target audience will be teachingnot all are going to be mathematicians. Joao Pedro Ponte pursued this theme by focussing on the need for students to experience mathematical discovery and investigationthereby raising the issue of how those themes will be represented in the Klein Project book. James King brought us back to mathematical development with a discussion of affine geometry. In discussion the inverse phrase "advanced mathematics from an elementary viewpoint" was raised again is this what the Klein Project is about? (Hans Rademacher, Higher Mathematics from an Elementary Point of View, Birkhäuser, 1983). How far can you go mathematically without getting into formal mathematics or doing a transposition or didactically engineer an advanced topic? Another question asked was how the book can capture both the present state and what is still to be done. Part of the answer is to ensure mathematics is presented openly, open problems (in the sense of showing that solved problems lead to other problems, and in the sense of unsolved problems, or that problems can be expanded).

Figure 3: *The workshop was held at the University of Madeira sixteenth century building.*

Margarida Oliveira presented dynamical modelling and simulation using a geometry environment, and Elsa Fernandes presented the use of robots in the classroom. Together the presentations inspired reflection on the way that teachers are able to take new ideas and transform them in the classroomfreeing the Klein Project to the task of presenting interesting ideas, not directing classroom practice. Frank Quinn ranged over some historical developments in the methodology of mathematics, and thereby raised the questions of who defines "significant" change, when does change in mathematics imply change in classrooms, and the deep nature of the discontinuity between schools and research mathematics. It also raised again the issue of the diversity of mathematicians philosophies, ways of working, and approaches to mathematics.Yuriko Baldin emphasised the importance of the concept of manifold and transformation groups, and linear algebra as a basic tool. She suggested that topology is directed towards global results in geometry, analysis towards local properties. She referred us to a television documentary on the Poincaré conjecture. Emanuel Martinho, Maria Margarida Pinto and Virgínia Amaral reminded us of the difficulties of writing a text, thereby pointing out some problems we can avoid. Arsélio Martins spoke of some negative influences of technology, giving an example where dynamic geometry can lead to important mathematical thinking being avoided. He noted the importance of examples that "look right, but are not". He urged the Klein Project to present problems to teachers so that they are not problems to be solved, but are rather situations by which to develop further mathematics. Discussion on counterexamples mentioned Falsehoods in Mathematics by Maxwell; Ed Barbeau, Mathematical Fallacies, Flaws and Flimflam, MAA, 2000; and Counterexamples by Dudley. Another reference is Proofs from the Book by Erdös, where we will find proofs that are not key results as much as paradigmatic of proofs. There needs to be something about proof and how they help us understand. The idea of showing proofs where the "obvious" way was not the right one was put forward. Another related idea mentioned later was that it is also important to prove that some things cannot be true. Another phrase that caught attention was "we can take students and teachers to the beach to see the openness of the seabut only the brave can sail to the edges of the horizon".

Luís Esteves used trigonometry to model a fun park, and Adelaide Carreira, Leila Ângelo, and Ana Valdez discussed topics in analysis and calculus. This led us to consider the way that software generates problemsdoes this happen in research? What are examples? Another possibly guiding idea to arise was that of a digest of books: think of the set of available books, and ask what is missing or what genre is missing. The Klein Project might also provide a guide to these books. The teachers who had presented were then asked what they would like. After some comments referring to curriculum/text issues, the following emerged: "I hope that the book might close the gap between secondary school and the intentions of university"; "I want it to broaden my horizons in an accessible way". I also need some simple examples to be able to answer my students when they ask me "what is this for? Both applications, careers and mathematics"; "Recent mathematicians results can be applied in schoolsbut we don't know how to do it. We know GPS has mathematics insidebut how?"; "I want to share the beauty of advances in mathematics. I teach linear algebra very well, but what is it good for? Where is this going?"; "I would like to see the main topics that can be foundations and then the links with the development of mathematics." The website/book For All Practical Purposes Comat, was mentioned as a resource that is updated, but note it is for students who are ending their mathematical study. The book needs to explain WHY: why do we need to study algebraic fractions, factoring polynomials, rational functions up to the graphis it because we have been trained to do so, or is it fundamental?

José Carlos Santos argued that Group theory needs to be in the Klein Project introduced through group actions, in particular on geometrical objects. João Fernandes, speaking about mathematics in astronomy, gave criteria for examples to be used in the Project. Pedro Patrício reiterated the complexity (and importance) of crypto-coding. Discussion mentioned extending groups to crystals and noted that this is a nice example because it was not done by mathematician. Another astonishing application of group theory is the work of a Polish mathematician who almost broke ENIGMA code, but it got changed. He sent his discoveries to UK when he knew Poland was being invaded and that helped the English break the code later in the war. It was noted that a contemporary feature of mathematics is that it is digital, so key theorems include compressing information, and capacity limits for transmit information. Further discussion on cryptography mentioned the headaches presented to mathematicians by security; and asked how it was possible to make elliptic curves "elementary". It was suggested that the Klein Project will affect the curriculum whether it is intended or not. But that nudging the curriculum (in no particular direction, just asking questions of it) is a good outcome.

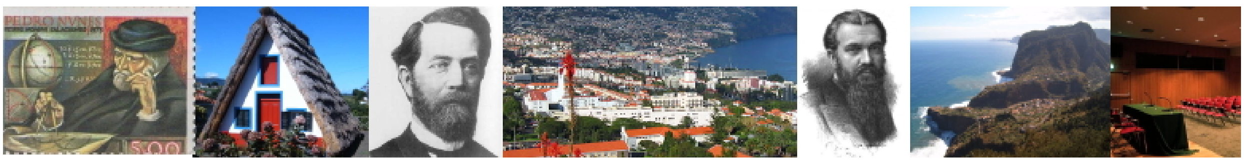
In the final sessions, Luis Sanchez and Michèle Artigue both discussed analysis, its foundations and how it might be presented. Dinis Pestana discussed statistics and the Central Limit Theorem, and John Mason presented five possibilities for the Klein Project:

- presenting contemporary mathematics to teachers (and others)
- presenting mathematics as the solving of problems
- as the explaining of phenomena
- ways to bridge school-university divide
- a unification of mathematics and its didactics.

Bernard Hodgson emphasised: the importance of integrating and using explicitly an historical vision; the role of visual proofs; and a chapter on logic, presenting some topics with a strong mathematical logic connection. Discussion noted that each of Mason's possibilities implies a different genre of the book (or resource). Also that we must represent a 20th century vision of how mathematics CAN be presented. Is there an opportunity for a fresh voice. We need to ask for whom examples are illustrative or inspiring, and what we expect people to do with them.

Figure 4: *Bill Barton in the summarizing session chaired by M. Artigue.*

In my summing up, and thanking the organisers, I emphasised the value of the conference, especially as the first conference and as the model it represented of productive discussions between mathematicians and mathematics educators. The Project is indebted to Centro Internacional de Matemática (CIM) and its director for organising this first Klein conference. Appreciation also to the Centres of Mathematics at the University of Coimbra, and at the Universidade do Minho, Centro de Matemática e Aplicações Fundamentais at the University of Lisbon, and the University of Madeira for their support, particularly the latter who provided a magnificent venue. We give thanks to the Programme Committee of José Francisco Rodrigues (Pres. CIM), Elfrida Ralha (Univ. Minho), Jaime Carvalho e Silva (Univ. Coimbra), Suzana Nápoles (Univ. Lisboa), Pedro Patrício (Univ. Minho) and the Local Organising Committee of José Manuel Castanheira (Univ. Madeira), Elsa Fernandes (Univ. Madeira), Sandra Mendonça (Univ. Madeira). There is no doubt that this was an extremely successful conference thanks to their efforts.



Article

Trends in Mathematics: How they could Change Education?

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1 Introduction

Mathematical activity (research, applications, education, exposition) has changed a lot in the last 50 years. Some of these changes, like the use of computers, are very visible and are being implemented in mathematical education quite extensively. There are other, more subtle trends that may not be so obvious. Should these influence the way we teach mathematics? The answer may, of course, be different at the primary, secondary, undergraduate and graduate level. Here are some of the general trends in mathematics, which we should take into account.

1. *The size of the community and of mathematical research activity* is increasing exponentially; it doubles every 25 years or so. This fact has a number of consequences: the impossibility of keeping up with new results; the need of more efficient cooperation between researchers; the difficulty of identifying "core" mathematics (to be mastered at various levels); the need for better dissemination of new ideas. How can mathematical education prepare future researchers and appliers of mathematics, future decision makers and the informed public for these changes?
2. *New areas of application, and their increasing significance.* Information technology, sciences, the economy, and almost all areas of human activity make more and more use of mathematics, and, perhaps more significantly, they use all branches of mathematics, not just traditional applied mathematics. How can we train our students to recognize problems where mathematics can help in the solution?
3. *New tools: computers and information technology.* This is perhaps the most visible new feature, and accordingly a lot has been done to introduce computers in education. But the influence of computers on our everyday life and research is also changing fast: besides the design of algorithms, experimentation, and possibilities in illustration and visualization, we use email, discussion groups, on-line encyclopedias and other internet resources. Can education utilize these possibilities, keep up with the changes, and also teach students to use them in productive ways?
4. *New forms of mathematical activity.* In part as an answer to the issues raised above, many new forms of mathematical activity are gaining significance: algorithms and programming, modeling, conjecturing, expository writing and lecturing. Which of these non-traditional mathematical activities could and should be taught to students?

I will say some more about these trends, and discuss the question of their influence on mathematical education. I will make use of some observations from my earlier articles [6, 7].

2 The size of the community and of mathematical research activity

The number of mathematical publications (along with publications in other sciences) has increased exponentially in the last 50 years. Mathematics has outgrown the small and close-knit community of nerds that it used to be; with increasing size, the profession is becoming more diverse, more structured and more complex.

Mathematicians sometimes pretend that mathematical research is as it used to be: that we ¯nd all the information that might be relevant by browsing through the new periodicals in the library, and that if we publish a paper in an established journal, then it will reach all the people whose research might utilize our results. But of course 3=4 of the relevant periodicals are not on the library table, and even if one had access to all these journals, and had the time to read all of them, one would only be familiar with the results of a small corner of mathematics.

A larger structure is never just a scaled-up version of the smaller. In larger and more complex animals an increasingly large fraction of the body is devoted to "overhead": the transportation of material and the coordination of the function of various parts. In larger and more complex societies an increasingly large fraction of the resources is devoted to non-productive activities like transportation information processing, education or recreation. We have to realize and accept that a larger and larger part of our mathematical activity will be devoted to communication.

This is easy to observe: the number of professional visits, conferences, workshops, research institutes is increasing fast, e-mail is used more and more. The percentage of papers with multiple authors has jumped. But probably we will reach the point soon where mutual personal contact does not provide sufficient information flow.

There is another consequence of the increase in mass: the inevitable formation of smaller communities, one might say subcultures. One response to this problem is the creation of an activity that deals with the secondary processing of research results. For lack of a better word, I'll call this expository writing, although I'd like to consider it more as a form of mathematical research than as a form of writing: finding the ramifications of a result, its connections with results in other fields, explaining, perhaps translating it for people coming from a different subculture.

Are there corresponding changes in mathematical curricula and, more generally, in the way we teach mathematics? The first, and most pressing, problem is the sheer size of material that would be nice (or absolutely necessary) to teach. In addition, as we will see, we should put more emphasis on (which also means giving more teaching time to) some non-traditional mathematical activities like algorithm design, modeling, experimentation and exposition. I also have to emphasize the necessity of preserving problem solving as a major feature of teaching mathematics.

How to find time to learning concepts, theorems, proofs, especially with the rapid expansion of material, and at a time when class time devoted to mathematics is being reduced in many countries? Which of the new areas should make its way to education (on the secondary or college level), and which of the traditional material should be left out? This is not a one-time crisis: mathematical research is not showing any signs of slowing down.

One possible answer to this question is to leave the teaching of any recently developed area of mathematics to later in the education, to Masters and PhD programs. The trouble with this approach is that many educated people will never meet the mathematics of the last 200 years, which will contribute to the unfortunate but persistent misconception that mathematics is a closed subject. Many of the new areas of mathematics are important for understanding developments in technology and science, and by not teaching them we give up illuminating the increasing role of mathematics in modern life.

The other possible answer is to remove from the curriculum traditional material that is deemed less important. This approach has the negative effect of eroding well-established methods for teaching mathematical thinking. For example, elementary geometry has been purged from the curriculum in many countries. While this kind of geometry is indeed peripheral in modern mathematical *research*, it is of course still important in applications, and, perhaps even more important, its study is very instrumental in the development of spatial conception, and, perhaps even more significantly, in understanding the real nature of mathematical proofs, the "Aha" event when an incomprehensible connection becomes clear through looking at it the right way.

I have no easy answer to this question. Probably one must concentrate on mathematical competencies like problem solving, abstraction, generalization and specialization, logical reasoning and use of mathematical formalism, along with the non-traditional skills mentioned above (see e.g. [10]). One could select a mixture of classical and more modern mathematical topics that are best suited to develop these competencies and (of course) basic skills, and at the same time give some sort of picture of the historical roots as well as contemporary applications.

Another question raised by the increasing complexity of the world of mathematics is whether exposition style mathematics has any place in education. One aspect of this is teaching students to explain mathematics to "outsiders", teaching them how to summarize results without getting lost in the details. This is not easy to do, but to teach such skills would be very useful indeed.

A more heretical thought is to do some expository style teaching. In most sciences like chemistry or astronomy, it is natural to teach in high school or even college the facts without explaining all the technical details of their discovery (or even of their exact meaning). Some of this is done in mathematics too: many students learn that the regular pentagon can be constructed with ruler and compass but the regular heptagon cannot, or that equations of degree 5 or more cannot in general be solved by radicals. But these examples are almost 200 years old! Can we solve the problem of exposing students to modern mathematics by working out appropriate non-exact but still mathematical blocks of material? I hesitate to answer "YES", but the question is valid.

3 New areas of application, and their increasing significance

The traditional areas of application of mathematics are physics and engineering. The branch of mathematics used in these applications is analysis, primarily differential equations. But in the boom of scientific research in the last 50 years, many other sciences have come to the point where they need serious mathematical tools, and quite often the traditional tools of analysis are not adequate.

For example, biology studies the genetic code, which is discrete: simple basic questions like finding matching patterns, or tracing consequences of flipping over substrings, sound more familiar to the combinatorialist than to the researcher of differential equations. A question about the information content, redundancy, or stability of the code may sound too vague to a classical mathematician but a theoretical computer scientist will immediately see at least some tools to formalize it (even if to find the answer may be too difficult at the moment).

Even physics has its encounters with unusual discrete mathematical structures: elementary particles, quarks and the like are very combinatorial; understanding basic models in statistical mechanics requires graph theory and probability.

Economics is a heavy user of mathematics—and much of its need is not part of the traditional applied mathematics toolbox. The success of linear programming in economics and operations research depends on conditions of convexity and unlimited divisibility; taking indivisibilities into account (for example, logical decisions, or individuals) leads to integer programming and other combinatorial optimization models, which are much more difficult to handle.

Finally, there is a completely new area of applied mathematics: computer science. The development of electronic computation provides a vast array of well-formulated, difficult, and important mathematical problems, raised by the study of algorithms, data bases, formal languages, cryptography and computer security, VLSI layout, and much more. Most of these have to do with discrete mathematics, formal logic, and probability.

One must add that which branches of mathematics will be applicable in the near future is utterly unpredictable. Just 30 years ago questions in number theory seemed to belong to the purest, most classical and completely inapplicable mathematics; now many areas in number theory belong to the core of mathematical cryptography and computer security.

A very positive development in recent decades is the decreasing separation between pure and applied mathematics. I feel that the mutual respect of pure and applied mathematicians is increasing, along with the number of people contributing to both sides. The diversity of applications should also strengthen the flow of information across all of mathematics. No field can retreat into its ivory tower and close its doors to applications; nor can any field claim to be "the" applied mathematics any more.

How to give a glimpse of the power of these new applications to our students? Perhaps some nonstandard mathematical activities like programming and modeling (to be discussed later) can be used here.

4 New tools: computers and information technology

Computers, of course, are not only sources of interesting and novel mathematical problems. They also provide new tools for doing and organizing our research. We use them for e-mail and word processing, for experimentation, and for getting information through the web, from the MathSciNet database, Wikipedia, the Arxives, electronic journals and from home pages of fellow mathematicians.

Are these uses of computers just toys or at best matters of convenience? I think not, and that each of these is going to have a profound impact on our science.

It is easiest to see this about experimentation with Maple, Mathematica, Matlab, or your own programs. These programs open for us a range of observations and experiments which had been inaccessible before the computer age, and which provide new data and reveal new phenomena.

Electronic journals and databases, home pages of people, companies and institutions, Wikipedia, and e-mail provide new ways of dissemination of results and ideas. In a sense, they reinforce the increase in the volume of research: not only are there increasingly more people doing research, but an increasingly large fraction of this information is available at our fingertips (and often increasingly loudly and aggressively: the etiquette of e-mail is far from solid). But we can also use them as ways of coping with the information explosion.

Electronic publication is gradually transforming the way we write papers. At first sight, word processing looks like just a convenient way of writing; but slowly many features of electronic versions become available that are superior to the usual printed papers: hyperlinks, colored figures and illustrations, animations and the like.

The use of computers is an area where often we learn from our students, not the other way around. The question here is: how to use the interest and knowledge in computing, present in most students today, for the purposes of mathematical education? Most suitable for this seem to be some nonstandard mathematical activities, which I discuss next.

5 New forms of mathematical activity

5.1 Algorithms and programming

The traditional 2500 year old paradigm of mathematical research is defining notions, stating theorems and proving them. Perhaps less recognized, but almost this old, is algorithm design (think of the Euclidean Algorithm or Newton's Method). While different, these two ways of doing mathematics are strongly interconnected (see [6]). It is also obvious that computers have increased the visibility and respectability of algorithm design substantially.

Algorithmic mathematics (put into focus by computers, but existent and important way before their development!) is not the antithesis of the "theorem-proof" type classical mathematics, which we call here *structural*. Rather, it enriches several classical branches of mathematics with new insight, new kinds of problems, and new approaches to solve these. So: not algorithmic *or* structural mathematics, but algorithmic *and* structural mathematics!

What does this imply in math education? As we discussed above, mathematical education must follow, at least to some degree, what happens in mathematical research; this is especially so in those (rare) cases when research results fundamentally change the whole framework of the subject. So set theory had to enter mathematical education (one would wish with more moderation and less controversy than happened with "new math"). Algorithmic mathematics is another one of these.

However, the range of the penetration of an algorithmic perspective in classical mathematics is not yet clear at all, and varies very much from subject to subject (as well as from lecturer to lecturer). Graph theory and optimization, for example, have been thoroughly reworked from a computational complexity point of view; number theory and parts of algebra are studied from such an aspect, but many basic questions are unresolved; in analysis and differential equations, such an approach may or may not be a great success; set theory does not appear to have much to do with algorithms at all.

Our experience with "New Math" warns us that drastic changes may be disastrous even if the new framework is well established in research and college mathematics. Some algorithms and their analysis could be taught about the same time when theorems and their proofs first occur, perhaps around the age of 14. Of course, certain algorithms (for multiplication and division etc.) occur quite early in the curriculum. But these are more recipes than algorithms; no correctness proofs are given (naturally), and the efficiency is not analyzed.

The beginning of learning "algorithmics" is to learn to *design*, rather than *execute*, algorithms [8]. The euclidean algorithm, for example, is one that can be "discovered" by students in class. In time, a collection of "algorithm design problems" will arise (just as there are large collections of problems and exercises in algebraic identities, geometric constructions or elementary proofs in geometry). Along with these concrete algorithms, the students should get familiar with basic notions of the theory of algorithms: input- output, correctness and its proof, analysis of running time and space, etc.

In college, the shift to a more algorithmic presentation of the material should, and will, be easier and faster. Already now, some subjects like graph theory are taught in many colleges quite algorithmically: shortest spanning tree, maximum flow and maximum matching algorithms are standard topics in most graph theory courses. This is quite natural since, as I have remarked, computational complexity theory provides a unifying framework for many of the basic graph-theoretic results. In other fields this is not quite so at the moment; but some topics like primality testing or cryptographic protocols provide nice applications for a large part of classical number theory.

One should distinguish between an algorithm and its implementation as a computer program. The algorithm itself is a mathematical object; the program depends on the machine and/or on the programming language. It is of course necessary that the students see how an algorithm leads to a program that runs on a computer; but it is not necessary that every algorithm they learn about or they design be implemented. The situation is analogous to that of geometric constructions with ruler and compass: some constructions have to be carried out on paper, but for some more, it may be enough to give the mathematical solution (since the point is not to learn to draw but to provide a field of applications for a variety of geometric notions and results).

Let me insert a warning about the shortcomings of algorithmic language. There is no generally accepted form of presenting an algorithm, even in the research literature (and as far as I see, computer science text books for secondary schools are even less standardized and often even more extravagant in handling this problem.) The practice ranges from an entirely informal description to programs in specific programming languages. There are good arguments in favor of both solutions; I am leaning towards informality, since I feel that implementation details often cover up the mathematical essence. For example, an algorithm may contain a step "Select any element of set S". In an implementation, we have to specify which element to choose, so this step necessarily becomes something like "Select the first element of set S". But there may be another algorithm, where it is important the we select the first element; turning both algorithms into programs hides this important detail. Or it may turn out that there is some advantage in selecting the *last* element of S. Giving an informal description leaves this option open, while turning the algorithm into a program forbids it.

On the other hand, the main problem with the informal presentation of algorithms is that the "running time" or "number of steps" are difficult to define; this depends on the details of implementation, down to a level below the programming language; it depends on the data representation and data structures used.

The route from the mathematical idea of an algorithm to a computer program is long. It takes the careful design of the algorithm; analysis and improvements of running time and space requirements; selection of (sometimes mathematically very involved) data structures; and programming. In college, to follow this route is very instructive for the students. But even in secondary school mathematics, at least the mathematics and implementation of an algorithm should be distinguished.

An important task for mathematics educators of the near future (both in college and high school) is to develop a smooth and unified style of describing and analyzing algorithms. A style that shows the mathematical ideas behind the design; that facilitates analysis; that is concise and elegant would also be of great help in overcoming the contempt against algorithms that is still often felt both on the side of the teacher and of the student.

5.2 Problems and conjectures

In a small community, everybody knows what the main problems are. But in a community of 100 000 people, problems have to be identified and stated in a precise way. Poorly stated problems lead to boring, irrelevant results. This elevates the formulation of *conjectures* to the rank of research results. Conjecturing became an art in the hands of the late Paul Erdöos, who formulated more conjectures than perhaps all mathematicians before him put together. He considered his conjectures as part of his mathematical *œuvre* as much as his theorems.

Of course, it is difficult to formulate what makes a good conjecture. (There is even a lot of controversy around Erdös's conjectures.) It is easy to agree that if a conjecture is good, one expects that its resolution should advance our knowledge substantially. Many mathematicians feel that this is the case when we can clearly see the place of the conjecture, and its probable solution, in the building of mathematics; but there are conjectures so surprising, so utterly inaccessible by current methods, that their resolution *must* bring something new – we just don't know where.

In the teaching style of mathematics which emphasizes discovery (which I personally find the best), good teachers always challenged their students to formulate conjectures leading up to a theorem or to the steps of a proof. This is time-consuming, and there is a danger that this activity too is eroding under the time pressure discussed above. I feel that it must be preserved and encouraged.

5.3 Mathematical experiments

In some respects, computers allow us to turn mathematics into an experimental subject. Ideally, mathematics is a deductive science, but in quite a few situations, experimentation is warranted:

(a) Testing an algorithm for efficiency, when the resource requirements (time, space) depend on the input in a too complicated way to make good predictions [1].

(b) Cryptographic and other computer security issues often depend on classical questions about the distribution of primes and similar problems in number theory, and the answers to these questions often depend on notoriously difficult problems in number theory, like the Riemann Hypothesis and its extensions. Needless to say that in such practically crucial questions, experiments must be made even if deductive answers would be ideal.

(c) Experimental mathematics is a good source of conjectures; a classical example is Gauss' discovery (not proof) of the Prime Number Theorem. Among the contemporary examples of this, let me mention the most systematic one: the graph-theoretic conjecture- generating program GRAFFITI by Fajtlowicz [2, 3].

There are several excellent books about experimental mathematics (see e.g. [1]). Programs like Derive, Maple or Mathematica offer us, and the students, many ways of experimentation with mathematics. A simple example: a student can develop a real feeling for the notion of convergence and convergence rate by comparing the computation of the convergent sums $\sum 1/k^2$ and $\sum 1/2^k$.

[1] I do not include here verification of the correctness of a program, which is not a mathematical issue, but rather software engineering.

Mathematical experimentation has indeed been used quite extensively in the teaching of analysis, number theory, geometry, and many other topics. The success seems to be controversial; my feeling is that, similarly as in the teaching of algorithms, the development of large well-tested sets of experimental tasks takes time, and is the most crucial element of the success of these teaching methods.

5.4 Modeling

To construct good models is the most important first step in almost every successful application of mathematics. The role of modeling in education is well recognized [9], but its weight relative to other material, and the ways of teaching it, are quite controversial.

Modeling is a typical interactive process, where the mathematician must work together with engineers, biologist, economists, and many other professionals seeking help from mathematics. A possible approach here is to combine teaching of mathematical modeling with education in team work and professional interaction.

A good example is the course "Discrete Mathematical Modeling" at the University of Washington [4] (similar courses are taught at several other universities, e.g. at the Eötvös University in Budapest). The main feature of this course is that the students, in groups of 2 or 3, must find a real-life problem in their environment. They have to develop a model, gather data, find and code the algorithms that answer the original question, and give a presentation of the results. The real-life problems raised are quite broad in scope, from problems on favorite games to attempts to help family or friends in their business, and some of the answers obtained turn out quite useful.

5.5 Exposition and popularization

The role of this activity is growing very fast in the mathematical research community. Besides the traditional way of writing a good monograph (which is of course still highly regarded), there is more and more demand for expositions, surveys, minicourses, handbooks and encyclopedias. Many conferences (and often the most successful ones) are mostly or exclusively devoted to expository and survey-type talks; publishers much prefer volumes of survey articles to volumes of research papers. While full recognition of expository work is still lacking, the importance of it is more and more accepted.

On the other hand, mathematics education does little to prepare students for this. Mathematics is a notoriously difficult subject to talk about to outsiders (including even scientists). I feel that much more effort is needed to teach students at all levels how to give presentations, or write about mathematics they learned. (One difficulty may be that we know little about the criteria for a good mathematical survey.)

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A Conversation with Mats Gyllenberg

About biomathematics and other contemporary mathematical issues

José Francisco Rodrigues

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On the occasion of the 150th anniversary of the publication of "On the Origins of Species by Means of Natural Selection", a landmark in Biology, CIM has organised an international conference on "The Mathematics of Darwin's Legacy", in collaboration with the European Society for Mathematical and Theoretical Biology. This conference brought to Lisbon Mats Gyllenberg, professor of applied mathematics (biomathematics) and chairman of the Department of Mathematics of the University of Helsinki. This article is an edited conversation held during the conference with one of the conference organisers and currently director of CIM.

Gyllenberg is currently the president of the Finish Mathematical Society, one of the two editors-in-chief of the Journal of Mathematical Biology and has been appointed Chairman of the permanent PESC (Physical and Engineering Sciences) standing committee of the European Science Foundation (ESF) for the three-year period 2009-2011. This ESF Committee is one of the five science units of the ESF, and its fields of interest include physics, chemistry, mathematics, technical sciences, computer sciences and material sciences and introduces new programmes and networks, and through its opinions, it also has a more extensive influence in the ESF's research policies, being the current Forward Look on "Mathematics in Industry", proposed by the applied mathematics committee of the European Mathematical Society, a recent example.

Working in Biomathematics

You made your PhD already in mathematical biology. What was your training as a mathematician?

Indeed I took my undergraduate studies at the University of Technology in Helsinki. So my main topic was mathematics, abstract mathematics. Functional analysis was my favorite topic and I wrote a master thesis on von Neumann algebras. But already during my university studies I took microbiology and biochemistry as a minor. So I have really done the some laboratory work. I have grown bacteria and I have done a lot of real experiments. From the very beginning I knew that I would become a mathematician but I was very interested in biology and then I realized that I can combine these two. My PhD thesis was already on "Dynamics of Structured Populations"

In the "Mathematics Genealogy" I found that you are a "descendent" of the Finnish mathematician Lindelöf, because Lehti was your adviser and he was advised by Järnefelt that was a student of Lindelöf, which by the way had a Swedish name. At the time he became professor at the University of Helsinki, Finland was still a Grand-Duchy of the Russian empire.

Yes! That is true. It is a pity that Lindelöf's adviser is not known. He probably didn't have one. Most Finnish mathematicians have a common ancestor and that is Lindelöf. In fact, still today we have a Swedish speaking minority. I belong myself to this minority, as well as Lars Ahlfors, probably the great Finnish mathematician of all time.

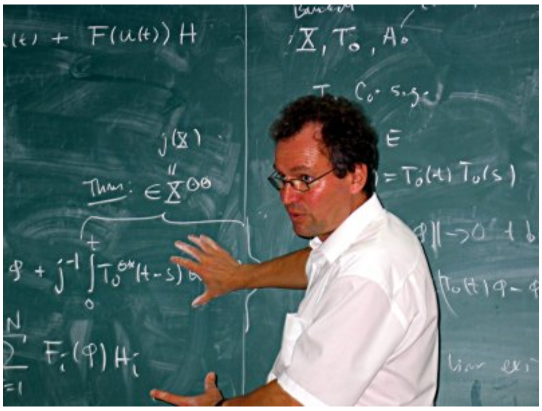

Figure 1: *Matts Gyllenberg.*

In fact, there exits a strong mathematical tradition in Finland, that started before Finland became an independent country. But even if Ahlfors started in Europe he spent most of his career in the United States.

Indeed, he moved to the United States, quite early. He was the first Fields medalist in 1936, if I remember correctly. Ahlfors was also a student of Lindelöf. I would say that it was really Lindelöf who started the Finnish school of analysis and complex function theory, which after Nevanlinna and Ahlfors have become quite famous.

After your PhD, that was already towards biomathematics you stayed in Finland or you did your career *elsewhere?*

In fact, during my PhD work I spent a year in Amsterdam working together with Odo Diekmann who had influence on my work. I mean, I wouldn't be the mathematician I am today without the collaboration with Odo, which started then and is still going. We still write papers together and so on. Then I was a visiting professor at Vanderbilt University, Nashville, Tennessee and in 1989 I got my first chair of mathematics. That was in the Luleå University, Sweden. I stayed in Sweden for three and a half years and then I came back to Finland, working in the University of Turku. I spent also some time as a Visiting Professor in Santa Barbara, Gothenburg and Utrecht and now I have been for five and a half years in the University of Helsinki.

You have a large number of publications within mathematical biology and applied mathematics but you have also many publications in dynamical systems. In fact, we can see your interest by bio-mathematics, in particular structured population dynamics and in the mathematical theory of evolution. But you have also works in differential equations. How do you relate this to analysis and differential equations?

It is a little different. Of course it is analysis, for sure, but actually my interest in differential equations or integral equations came from the population dynamics because the basic thing in population is reproduction. So the birth rate is something which you should determine in order to know how the population evolves and those who are born today are born from parents that were born in the past. So it's very easy to understand that we get some sort of functional delay equation or Volterra integral equation. So it was really from biological problems that I got my interest in integral and differential equations and, of course, I have also done work in this area without direct applications to biology.

Biomathematics, Euler and Darwin

I remember our conversation last February in Barcelona when I met you during a CRM meeting in biomathematics. You mentioned then the 1760 Euler model of human population with an integral operator and, later, you sent me your article on "Mathematical aspects of physiologically structured populations", where you referred to Euler's description of an exponentially growing population with a steady age distribution (balanced exponential growth) done more in the spirit of an actuary than a biologist. That means that you also have some interest in the history of mathematics or it is just a side interest?

It is a side interest. But I really think that most mathematicians are too lazy. We know that many results are rediscovered over and over again because it is easier to prove a lemma then go to the library and try to find the original or previous results. But Euler was absolutely fantastic! He knew a lot of things, from an intuitive point of view, and only later on his results have been made rigorous mathematics. But it is often the case, when we go back to the old masters, we learn a lot by reading Euler or Riemann.

Figure 2: *Poster of the* International Conference "The Mathematics of Darwin's Legacy"*, November 23-24, 2009.*

I should mention that the exponential growth in population is basically due to Euler. In fact we can find already in his book "Introductio in analysin infinitorum", from 1748 where he lays the foundations of infinitesimal analysis, four interesting examples about geometrical growth of populations in the chapter about logarithms and the exponential. And that was not the first time that mathematicians had interest in quantifying population. It is also quite well-known the pioneer 1760 memoire of Daniel Bernoulli on "Essai d'une nouvelle analyse de la mortalité causée par la petite vérole et des avantages de l'inoculation pour la prévenir", and its role in understanding the benefit of vaccination in the diminishing smallpox mortality. So, there was already biomathematics in some special examples before Darwin. What do you think about the conference on "The Mathematics of Darwin's Legacy", that brought you this time to Portugal?

As a matter of fact, I think this was a wonderful conference. I like this small workshop type of conference more than the big ones. It was also good that there were only a small number of speakers, and we had one full hour to our lectures. I think that this is much better then to have 20 minutes communications. I think that we really heard during these two days very good lectures and I learnt a lot from them. On the other hand, the subject of the conference is absolutely central in biomathematics today. Now, my main interest in mathematical biology, aside from structured population is the mathematical theory of evolution, in particular, adapted dynamics, the interaction between ecology and evolution by natural selection. As we have heard during this conference also evolution by natural selection is something inherently mathematical, although Darwin did not formulate it at all the mathematical relations.

He could not because he had no genetics in his theory. But soon afterwards, the English biometric tradition started with Dalton and Pearson with the development of the statistical theory, in particular, for the scientific treatment of biological data...

Figure 3: *An aspect of the International Conference on "The Mathematcis of Darwin legacy" at the University of Lisbon.*

In the very first lecture of this conference it was pointed out by Warren Ewens that there is an obvious problem, a contradiction, because at Darwin's time it was thought that the inheritance of the mother and the father were blended at conception. It is so evident that the future generations would all look alike and there would be no variation upon which natural selection could operate. And this somehow shows the great genius of Darwin, that although he was not mathematically trained, he saw that there was some sort of contradiction that he could not really resolve it, but he just stepped aside to continue to develop his theory. That was great. Darwin took from Malthus the idea of exponential growth, although Malthus has called it geometrical growth, but that's the same thing, and he also had this wealth of examples from breeding domestic animals. So Darwin knew artificial selection, and, combining the exponential growth with the selection principles known from animal breeding, he could arrive at his theory of evolution by natural selection. Usually, when big leaps are made in science, they are usually done by somebody combining ideas from two completely different theories and then make a synthesis. And the success of Darwin is also one of those examples, and, of course, there was a strong mathematical element in Darwin's syntheses.

Which confirmation came afterward with the help of mathematics, because the "synthesis" of Mendelism and Darwinism was done basically using mathematical ideas and models, such as the Hardy-Weinberg law in population genetics. On one hand, the evolution theory was very important in the development of mathematical statistics. For instance, the new methods and concepts that Fisher invented in statistics were done in the beginning of the 20th century and were strongly motivated by biology and by the evolution theory. Their application in his 1930 book on "The Genetical Theory of Natural Selection" lead to conclusions that were confirmed by the works of biologists such as Haldane, Wright and Dobzhansky, among others. On the other hand, those decades were also crucial for the mathematisation of biology using differential equations. So, Lotka-Volterra models, for instance, that appeared in 1925-1926 are very well-known and still have a tremendous influence in population dynamics. It's very interesting the comparison between these two independent contributions, the one by Lotka, with a statistical-physics approach to biology, and the other by Volterra with its mechanical approach. The evolution theory in the second half of the 20th century raised other aspects that Darwin did not really foresee. Besides the reproduction, the mutation and the selection there is the cooperation between live entities and, of course, its mathematical models, like evolutionary game theory for instance, that are current research topics. What do you think about this?

It's extremely important. I would say about Lotka-Volterra system that these models have been extremely productive because they are sufficiently simple as equations. Of course, a lot of idealization is made, but that's always the case in the mathematical models of biology. They are simple, but at the same time they have sufficiently rich behaviour that you can really get biological insight and, on other hand, they have been a wonderful inspiration for mathematicians. Of course, for two dimensional systems almost everything is known, but already in three dimensions there are a lot of very interesting mathematical questions. For instance, the whole theory of monotone dynamical systems really has got its source in the Lotka-Volterra system. When we have cooperation or competition then you can have this ordering depending whether is cooperation or competition, you have to change the direction. They have some sort of monotonicity. That has been a wonderful source of inspiration for mathematicians. I like very much this two way interaction. Mathematics is needed to get biological insight; biological questions are useful to give inspiration to create new mathematics. In my work I collaborate quite a lot with biologists and their intuition often helps me to find the right way of proving my terms. One should always bear in mind that this dialogue between mathematicians and biologists is extremely important.

Editing Mathematical Biology

You are now one of the Editors-in-Chief of the Journal of Mathematical Biology. Is there a difference between mathematical biology, biology or mathematics papers? How do you cope with this interaction?

Originally, the Journal of Mathematical Biology was founded about 30 years ago. The name of the journal was, I think, analog with mathematical physics, which is mathematics. Mathematical biology was also viewed as being really mathematics and not biology. But now this has changed a lot and we really require for a paper to be published in the Journal of Mathematical Biology that there is real biology in it. So, there must be some new biological insight that you get from mathematics. Of course, mathematics should not be elementary or trivial. So, the ideal papers are when new mathematical methods are developed, some new mathematics created in order to get insight into some biological questions.

You mean Interdisciplinary? But this is a very difficult issue. I'm also an editor myself of a European Mathematical Society journal, Interfaces and Free Boundaries, aiming the combination of Mathematical Modelling, Analysis and Computation. Those ideal interdisciplinary papers are very rare.

Yes! Of course, we also publish some quite mathematical papers, which get the inspiration from biology, and are potentially, at some later time, applicable to some real biological system. So, I think that my predecessor as Editor-in-Chief, Odo Diekmann, made a wonderful work making this journal the best one in mathematical biology or biomathematics, whatever you want to call it, and really in quality is of course the number one. I think we have been able to keep this high quality.

I think it is essential in a mathematical journal, in any scientific journal, to have a good referee system. So, in a journal, like Journal of Mathematical Biology, do you have a referee for mathematics and another for biology? When you have a conflict with the two points of view, how do you solve it? Do you act as an Editor-in-Chief or ask a third opinion?

This is quite often the case. It depends on the submitted manuscript, but, as I said, we require that there must contain real biology. There must be interpretation and insight in biology. This actually often requires that we have referees from both biology and mathematics. The other question concerning conflict is quite difficult. It is impossible to give a general answer, because every paper is different. Of course there are two completely opposing views. I have to make up my mind myself. As an Editor-in-Chief I have the last word. There are basically two options. Either I reject the paper or I ask the author to make major revisions in order to be able to publish it within the scope of the journal. I mean, it depends from case to case. It's very difficult to give a general opinion.

I have to ask one more question to you as an editor. Nowadays it is quite hard to find good referees who wish to give some constructive answer in time...

It is extremely difficult, terribly difficult. This has to do with the general hectic speed of life. Everybody is so busy. Many people don't answer at all, others say yes, and then promise, and then they forget, and you don't hear anything about it... I would say that the Journal of Mathematical Biology has such a good reputation, most referees who really take the job to review a paper, they do it quite well. They write detailed reports and produce rather long and constructive reviews. As an editor I'm quite happy with the referees we have.

Figure 4: *Mats Gyllenberg, on the right, with José Francisco Rodrigues at University of Lisbon in November 24, 2009.*

Mathematicians and scientific organisations

You are also the President of the Finnish Mathematical Society. How large is the Society? Does that function take much time from you?

It doesn't take so much time, no. The society is quite small. We have about 350 members, professional mathematicians, teachers, graduate students, some working in the industry... We have to accept every membership application in the Board of the Society, and the criteria being that they must be somehow connected to mathematics. For instance, they must have a master in mathematics, that's good, and if they work with mathematics in industry or, of course, if they are in the Academia and if mathematics is the main subject. Sometimes we reject applications when we think that they do not have anything to do with mathematics. The majority of members is working in the Academia or are PhD students. I don't know exactly the fraction, but it's not relevant the number of high-school teachers in our Society.

Is there a relation between the Finnish Mathematical Society and the National Committee of Mathematicians that represents Finland, now in Group III, at the International Mathematical Union?

No. Formally there is none. The National Committee is something separated from the Society and depends on the Finish Academy of Science and Letters. Pertti Mattila is currently the Chairman of the Finish National Committee of mathematicians.

You are now also the Head of the Department of Mathematics and Statistics of the largest university in Finland. In your department, what is the role of applied mathematics or other applied subjects as biomathematics, industrial mathematics ... ?

A few decades ago, it was really pure mathematics, classical functional theory, functional analysis... But re cently we have been able to recruit new professors in applied mathematics, and very good ones. We have, besides the mathematical biology, which we have already discussed, we have a big group of mathematical biology, but there is also a very strong group in inverse problems. And this year we have appointed a professor in industrial mathematics. So, I think we have a very strong applied mathematics. We have a good bond between pure and applied. We have actually also two Centres of excellence at our department. One is on Inverse Problems Research, the other is in Analysis and Dynamics Research. I myself and the biomathematics group belong to this second one, because we deal with dynamical systems, modeling, biological phenomena. This is actually quite exceptional, because there are not so many of these national Centres of excellence in our country. Two are in mathematics and they are at our department. I'm quite proud of this.

That is the research component of the department. But there is also a teaching component, I presume. So, how are in Finland the mathematical courses with the Bologna degrees: 1st cycle, 2nd cycle and 3rd cycle? What is required to be a mathematical teacher in high-school?

To be honest we have not changed much in Finland. We had to introduce the bachelor degree after three years of university studies, and have two for the master. Mathematics teacher in high-school have to have a master in mathematics. And that is what is very good at the Finnish school system. We have highly educated teachers. They have also to take the pedagogical education, during the master. So, it means that they have slightly less courses in mathematics. But, still, they know the subject very well since they follow mathematical courses corresponding to at least four years.

And are you still accepting enough students in your mathematical courses at the University of Helsinki, or are you having problem as in many other European universities?

We get a lot of students, but we also have a drop out, which is rather high. I suppose that there are many young people who want to study medicine or law. But it's very difficult to get to the Medical and in the Law Schools. And then they think: "OK! Let's go and study mathematics!", and then, they shift.

Interesting! So, these students to get into medicine or law school, some of them go through mathematics. That is not a bad idea...

Of course not! It's a very interesting idea. They gain them from us. So, we get about two hundred new students in mathematics, every year, and about 110 bachelors, for the master courses.

Do you also have technology or engineering courses in Helsinki University? And how is that going along with the others universities in the Finnish system?

We have everything at the University of Helsinki, except for engineering, which is at the Technical University and is a separate university. In fact we have quite a number of universities in Finland, which is a big drawback in the Finnish system. We still have twenty universities and the population is five millions. Now, we are trying to decrease the universities by merging some of them, which I think is a good idea. For instance, the Helsinki University of Technology, the Business School, and then the School for Industrial Design, they have merged from the beginning of this year, for one bigger university.

In Finland, there is Nokia which is a world known leader in the telecommunications industry, that requires a lot of mathematics. Is there any particular role or cooperation between Nokia and the mathematical sciences in Finland?

In a sense, yes... I mean, the leaders of Nokia always emphasize the role of mathematics. It's extremely important. In this case we get moral support but we don't get any money from it. They hire mathematicians, they hire computer scientists...

You are also the first mathematician which is chairing of the Standing Committee for Physical and Engineering Sciences (PESC) of the ESF (European Science Foundation) since 1st January 2009. ESF has now a very good cooperation with the European Mathematical Society and had lunch the Forward Look in Mathematics and Industry. PESC has last June in Berlin the 12th Round Table Meeting of ESF Member Organisations exactly about Mathematics. If you allow me will I finish our conversation quoting some of your statements when you started your functions at the PESC/ESF enhancing the focus on interdisciplinary collaboration.

"During the past decade we have witnessed an unprecedented technological revolution. Progress in telecommunication, the world wide web and Google are only a few examples of technological advancements that have profoundly changed everyday life. These achievements are all based on fundamental research in disciplines like mathematics, physics and computer science - fields covered by PESC. This interaction between the pure and the applied makes the PESC environment important and intriguing. (...) The traditional division of the natural sciences into "hard" sciences like mathematics, physics, chemistry and computer science and the "soft" life sciences is old fashioned and in fact obsolete. Modern (molecular) biology could not exist without collaboration with chemistry and physics. (...) The life sciences increasingly use mathematical and statistical modelling and are often dependent on heavy computing."

Figure 5: *Groupe Picture,* International Conference "The Mathematics of Darwin's Legacy".

The Jornada Matemática SPM/CIM on Mathematical Biology

The *Jornadas SPM/CIM* are a joint initiative of the Sociedade Portuguesa de Matemática and the Centro Internacional de Matemática, with the purpose of enhancing collaboration between Portuguese mathematicians in all areas of research. This session toke place on the 25th of November 2009, in the Complexo Interdisciplinar da Universidade de Lisboa, and was organised by Nico Stollenwerk, (CMAF/FCUL, University of Lisbon) and gathered both senior and junior researchers in Mathematical Biology, in a relaxed and lively atmosphere for discussions.

The opening lecture was given by Peter Jagers, from Chalmers University in Sweden, titled *Extinction versus persistence*.

The list of the other presentations given is as follows: *The hidden potential of recombination inhibitors: sidestepping the Darwinian inevitability of resistance?*, Philip Gerrish (Univ. de Lisboa/ Univ. of New Mexico), *Stability for Lotka-Volterra models with delays*, Teresa Faria (Univ. de Lisboa), *Evolutionary branching of a magic trait*, Tadeas Priklopil (Univ. of Helsinki), *Multi-scale models in tuberculosis: the case of drug resistance*, Paula Rodrigues (Univ. Nova de Lisboa), *Animal growth in random environments*, Carlos Braumann (Univ. de Évora), *Structured populations in the N-person snowdrift game*, Marta Santos, (Univ. de Lisboa), *Steady-state topologies of SIS dynamics on adaptive networks*, Stefan Wieland (Univ. de Lisboa), *Trimorphic generalist-specialist coexistence on two special resources*, Ilmari Karonen (Univ. of Helsinky), *Prediction of protein-protein interactions based on amino-acid sequences*, Valeria Manna (ICAR/CNR), *Hereditary maximum parsimony trees and not so hereditary ones*, by Mareike Fischer (Univ. of Vienna), *Stationary in moment closure and quasi-stationarity of the SIS model*, Alberto Pinto (Univ. do Minho).

The day closed with a round table discussion, chaired by José Francisco Rodrigues (CMAF/ Univ. de Lisboa) with the theme "Open questions and future prospects of mathematical biology". Further details, including abstracts of the talks, can be seen at http://www.cim.pt/?q=spm_cim_jornada_mathematical_biology_2009

All you should know about your "Companion"

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Every mathematician will meet a good dose of linear algebra in his/her battle to reach nirvanna, be it as stepping stone to infinite dimensions, or as an entry into more abstract algebra, or as a computational tool in numerical analysis.

In this article we examine the story of the "companion matrix"

$$L[f(x)] \quad = \begin{bmatrix} 0 & \dots & & -f_0 \\ 1 & & & -f_1 \\ & \ddots & & \\ 0 & & 1 & -f_{n-1} \end{bmatrix},$$

which is associated with the monic polynomial $f(x) = f_0 + f_1x + \cdots + f_nx^n$ with $f_n = 1$, over a field $\mathbb{F}$.

These matrices are the "molecules" that lie at the heart of the whole field of linear algebra and its many applications, ranging from Canonical Forms to Systems Theory, Digital Image Processing and Numerical Analysis.

Their membership includes the "least periodic" matrix in the form of a nilpotent Jordan block, as well as the "most periodic" matrix, which undoubtedly is the circulant matrix.

Companion matrices really represent polynomials and appear whenever polynomials are involved. Add to this that polynomials are one of the most important concept in applicable mathematics, and so there is some ground for examining them.

They are in fact a realization of "finiteness". Indeed, finiteness implies periodicity, periodicity implies the existence of annihilating polynomials and these in turn imply the existence of companion matrices.

Companion matrices and annihilating polynomials are two concepts which like "foot soldiers" are always "there" in the background! In fact, as in real life, it is the behavior of these "molecules" that dictates the behavior of matrices in general.

There are numerous reasons why these matrices play such a dominant role, and we shall not attempt to give a complete "all you should know about your companion" presentation.

In this note we shall demonstrate that many of the useful results involving companion matrices are a consequence of the *companion shift* property. We shall go through several of these and develop the notation as we go along. Since there are so many areas of application, some secrets of our companion will be left to the literature.

Before we present the shift condition, we mention some of its well known properties.

- Its transpose L^T is similar to L, which translates into the fact that any matrix is similar to its transpose.
- $f(x)$ is a (left) annihilating polynomial for $L(f)$. Indeed $f_\ell(L_f) = L^n + L^{n-1}f_{n-1} + \cdots + If_0 = 0$, which ensures that any matrix has an annihilating polynomial.
- Companion matrices are irreducible Hessenberg matrices that represent polynomials.
- They are the simplest matrices that one can write down with a prescribed characteristic polynomial. As such they are crucial building blocks in the construction of Canonical Forms.
- They are *non-derogatory*, i.e. the characteristic and minimal polynomials are equal. (blame Sylvester!)
- They are sparse (= most of their entries are zero).
- They are closely associated with **cyclic** subspaces, which are the most important subspaces of linear algebra, module theory, etc. Indeed, their use ranges from Canonical Forms to chain computations, as found in coding and numerical analysis (GMRES).
- The link to the cyclic subspaces is provided by the cyclic *chains*, which play a key role in the question of minimal polynomials and basis changes.

- Companion matrices induce a *companion shift*, on a chain of matrices, which can be manipulated in numerous ways. These shifts may be compensated for when the terms in the string satisfy suitable recurrence relations. It is this cancelation phenomena that is at the bottom of the some of the deeper theorems.

- Cyclic subspaces parallel cyclic groups, which in turn are fundamental building blocks in all of group theory.

- The companion matrix of $L(x^n - 1)$ is special. It presents itself when dealing with permutation matrices and is the linch pin in the whole field of Discrete Fourier Transform. Indeed, the roots of its polynomial generate the "mother of all groups", i.e. the group of n-th roots of unity.

Before we enter the realm of the companion matrix let us first clear up some more of the needed definitions and notations. Throughout this article all matrices will be over a field $\mathbb{F}$, but many results extend to the non-commutative (block) case.

Let $A \in \mathbb{F}_{n\times n}$ and $\mathbf{x} \in V = \mathbb{F}^n$. The characteristic and minimal polynomials of A are denoted by $\Delta_A(x)$ and $\psi_A(x) = \psi_V(x)$, respectively. The minimal annihilating polynomial (m.a.p.) of $\mathbf{x}$ relative to A is the monic polynomial $\psi_x(\lambda)$, of least degree, such that $\psi_x(A)\mathbf{x} = \mathbf{0}$. A Polynomial will be denoted by $p(x)$ or by $p(\lambda)$, when there is no ambiguity, and we use $\partial(.)$ for degree. The reciprocal polynomial of $f(x)$ is given by $\tilde{f}(x) = x^n f(1/x)$. We shall freely interchange L_f and $L(f)$ and will suppress the subscript where convenient. We use rk(.) and $\nu(.)$ for r rank and nullity and denote the Kronecker product by $\otimes$. A is regular if $AA^-A = A$ for some A^- and col$[\mathbf{x_1}, .., \mathbf{x_n}]$ stands for $[\mathbf{x_1^T}, .., \mathbf{x_n^T}]^\mathbf{T}$.

Besides $L(f)$, there are several other matrices that are also determined by $f(x)$. In particular we need its $n \times n$ symmetric Hankel matrix $G = G[f(x)]$, and the $(n+t) \times t$ basic *shift* matrix $S_t(f)$, generated by $f(x)$, which are given by

$$G_f = \begin{bmatrix} f_1 & f_2 & \cdots & & f_n \\ f_2 & f_3 & \cdots & f_n & 0 \\ \vdots & & & & \\ f_{n-1} & f_n & 0 & \cdots & 0 \\ f_n & 0 & & \cdots & 0 \end{bmatrix}, \text{ and}$$

$$S_t(f) = \begin{bmatrix} f_0 & & & 0 \\ f_1 & f_0 & & \\ \vdots & & \ddots & \\ f_n & f_{n-1} & \cdots & f_0 \\ 0 & f_n & & \vdots \\ \vdots & & \ddots & \\ 0 & & & f_n \end{bmatrix}.$$

For later use, we denote the "flip matrix" $G(x^n)$ by F. In many settings it is essential to use the transposed companion matrix L^T, rather than L – as for example with eigenvectors – because left and right multiplication are different. The fundamental relation between L and L^T is given by

$$L_f G_f = (L_f G_f)^T = G_f L_f^T \text{ or } G^{-1}LG = L^T,$$

and even holds in the non-commutative case. This identity is the reason why G is often referred to as the "intertwining" matrix or symmetrizer.

Consider L_f, where $f(x) = f_0 + f_1 x + \cdots + x^n$. We associate the coefficient vectors $\mathbf{f} = [f_0, \ldots, f_{n-1}]^T$ and $\hat{f} = [f_0, \ldots, f_n]^T$ and the chain matrices $X'_n = [1, x, ., x^{n-1}]$ and $Y_n = \begin{bmatrix} 1 \\ y \\ \vdots \\ y^{n-1} \end{bmatrix}$. The companion shift takes the form

$$(i)\ xX'_n - X'_n L_f = f(x)\mathbf{e}_n^T \ \ or\ \ (ii)\ L_f^T X_n - xX_n = -f(x)\mathbf{e}_n \tag{0.1}$$

which is trivial to verify and yet is the most important property of the companion matrix!

The basic shift matrix $S_m(f)$ has been introduced to take care of polynomial multiplication, i.e. of convolution. Again, let $f(x) = f_0 + f_1 x + \cdots + f_n x^n, g(x) = g_0 + g_1 x + \cdots + g_m x^m$, and $h(x) = f(x)g(x) = h_0 + h_1 x + \cdots + h_{m+n} x^{m+n}$, with coordinate columns $\bar{\mathbf{f}}, \bar{\mathbf{g}}$ and $\bar{\mathbf{h}}$. The two key results that we need are

$$\begin{aligned} f(x)X'_n &= X'_{2n} S_n(f) \\ S_{m+1}(f)\bar{\mathbf{g}} &= \bar{\mathbf{h}}. \end{aligned} \tag{0.2}$$

The latter reflects the convolution product $h_k = f_k g_0 + f_{k-1} g_1 + \cdots + f_0 g_k$.

There are numerous types of operations that we can now perform on this shift equation.

(i) We can *multiply* through by a suitable matrix.

(ii) We can *embed* this shift into a unimodular polynomial matrix – which will enable us show the equivalence of $xL - L$ to its Smith Normal Form diag$(f(x), I)$.

(iii) We can consider it as a *matrix equation* of the form $AX - XB =$C, and use the *telescoping trick*, – i.e. repeatedly pre-multiplying by A and post multiplying by B, and add – to arrive at

$$A^k X - XB^k = \Gamma_k = A^{k-1}C + A^{k-2}CB + \cdots + CB^{k-1} \tag{0.3}$$

(iv) We may (formally) *differentiate* to give

$$X'^{(k)}_n (L_f - xI) = kX'^{(k-1)}_n - f^{(k)}(x)\mathbf{e}_n^T$$

(v) We may *evaluate* the identity at $x = a$ or replace x by a matrix B, giving us useful block identities.

(vi) We may *combine* any of the above such as the companion shift with the basic shift $S_n(g)$, or differentiation followed by multiplication.

Let us now present each of the above and see how this application can be used.

1 Multiplication

If we multiply (0.1) by G, we meet the family of *adjoint* polynomials $f_i(x)$ given by

$$F_n' = [f_0(x), f_1(x), \dots, f_{n-1}(x))] = [1, x, x^2, \dots, x^{n-1}]G_f,$$

or in detail

$$f_k(x) = f_{k+1} + f_{k+2}x + \dots + f_n x^{n-k-1}, \; k = 0, 1, \dots, n-1.$$

It should be noted that $f_{-1}(\lambda) = f(\lambda)$ and that $f_{n-1}(\lambda) = f_n = 1$.

If we multiply (0.1)-(i) on the right by G we obtain the *adjoint shift condition*

$$xF_n' - F_n' L_f^T = f(x)\mathbf{e}_1^T, \tag{1.4}$$

which is equivalent to the recurrence relation $f_{k-1}(x) = f_k + xf_k(x)$, $k = 0, 1, \dots, n-1$.

If we right multiply (0.1) by $\mathrm{adj}(xI - L)$ then we generate $f(x)\mathbf{e}_n^T \mathrm{adj}(xI - L) = X_n^T(xI - L)\mathrm{adj}(xI - L) = X_n^T f(x)I$, from which we may cancel $f(x)$ to give

$$X_n' = [1, x, ., x^{n-1}] = \mathbf{e}_n^T[\mathrm{adj}(xI - L_f)].$$

2 Substitution

Given a matrix $A \in \mathbb{F}_{n\times n}$, if we replace x by A in (0.1) and (1.4) we see that

$$\begin{aligned} &A[I, A, \dots, A^{m-1}] = \\ &[I, A, \dots, A^{m-1}][L_A(f) \otimes I] + [0, \dots, 0, f(A)]. \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &A[A_0, A_1, \dots, A_{m-1}] = \\ &[A_0, A_1, \dots, A_{m-1}][L_A^T(f) \otimes I] + [f(A), 0, \dots, 0]. \end{aligned}$$

If in addition $f(x) = \Delta_A(x)$ and $f(A) = 0$, then we obtain the coefficients $A_i = f_i(A)$ in the expansion $\mathrm{adj}(xI - A) = \sum_{i=0}^{n-1} A_i x^i$. Not surprisingly we can use the companion shift to actually characterization a companion matrix. Indeed,

Proposition 2.1. *Given a matrix B with minimal polynomial $p(x)$ of degree n. An $n \times n$ matrix X equals $L(p)$ iff*

$$B[I, B, \dots, B^{n-1}] = [I, B, \dots, B^{n-1}](X \otimes I). \tag{2.6}$$

Proof. If $X = L(p)$ then take $f = p$ in (2.5). Conversely, if (2.6) holds we select $f = p$ in (2.5). Subtracting shows that $[I, B, \dots, B^{n-1}][(L(p) - X) \otimes I] = 0$. Since the powers in the chain are independent it follows that $X = L(p)$. □

We may now introduce a second matrix B, and multiply (2.5) through by $(I \otimes B)$ to give for any monic polynomial $f(x)$

$$\begin{aligned} &A[B, AB, \dots, A^{r-1}B] = \\ &[B, AB, \dots, A^{r-1}B](L_f \otimes I) + [0, \dots, 0, f(A)B]. \end{aligned}$$

Chains of the form $[B, AB, \dots, A^{r-1}B]$ are of considerable importance in linear control and systems theory. We shall mainly focus on the case where B is a column $\mathbf{x}$, in which case the chain matrix $K_r(\mathbf{x}, A) = [\mathbf{x}, A\mathbf{x}, A^2\mathbf{x}, \dots, A^{r-1}\mathbf{x}]$ is referred to as a Krylov matrix. There are now two cases of interest.

(i) Suppose that the m.a.p of ψ_x has degree $\partial(\psi_x) = r$. Then $A^r\mathbf{x}$ is the smallest (= first) power that is a linear combination of the previous powers, and as such the r links in the chain matrix $K_r(\mathbf{x}\ , A) = [\mathbf{x}, A\mathbf{x}, A^2\mathbf{x}, \dots, A^{r-1}\mathbf{x}]$ are linearly independent and

$$AK_r(\mathbf{x}, A) = K_r(\mathbf{x}, A)L[\psi_x(\lambda)].$$

Completing the matrix $K_r(\mathbf{x}, A)$ to an invertible matrix $Q = [K_r, B]$, we then arrive at

$$AQ = Q\left[\begin{array}{c|c} L(\psi_x) & E \\ \hline 0 & D \end{array}\right] \text{ i.e. } Q^{-1}AQ = \left[\begin{array}{c|c} L(\psi_x) & E \\ \hline 0 & D \end{array}\right].$$

This will shortly be used as the first step in the derivation of the Cyclic-Decomposition Theorem.

(ii) If, on the other hand, we take the first n links in the chain we obtain $K_n(\mathbf{x}, A) = [\mathbf{x}, A\mathbf{x}, A^2\mathbf{x}, \dots, A^{n-1}\mathbf{x}]$. Now $A^n\mathbf{x}$ must be a linear combination of lower powers, say $A^n\mathbf{x} = -[(f_0\mathbf{x} + f_1A\mathbf{x} + \dots + f_{n-1}A^{n-1}\mathbf{x}]$. We see that

$$AK_n = K_nL(f),$$

where $f(x) = f_0 + f_1x + \dots + x^n$. It is clear that $K_n(\mathbf{x}, A)$ will be non-singular iff $\partial(\psi_x) = n$, in which case A is non-derogatory and $A = K_nL(f)K_n^{-1}$.

Associated with the above chain is the cyclic subspace $Z_{\mathbf{x}}(A) =< \mathbf{x}, A\mathbf{x}, A^2\mathbf{x}, \dots, >$ generated by $\mathbf{x}$. It is also referred to as the Krylov subspace generated by x, and is used, for example, in the GMRES method of numerical analysis.

A vectorspace V is called cyclic if $V = Z_{\mathbf{u}}(A)$ for some vector $\mathbf{u}$ in V. For the case where $V = \mathbb{F}^n$ this means that $V =< \mathbf{u}, A\mathbf{u}, \dots, A^{n-1}\mathbf{u} >= R(K_n(\mathbf{u}, A))$.

Cyclic subspaces parallel the concept of cyclic groups, which are by far the most important type of group. The following is, for example, the analog to the fundamental theorem of finite cyclic subgroups.

Proposition 2.2. *If $V = Z_u(A)$ and W is an A-invariant subspace then*

(i) W is also a cyclic (sub)space.

ii) $W = Z_{g(A)\mathbf{u}}$ for some polynomial $g(\lambda)$.

It is easily verified that the vector $\mathbf{y} = g(A)\mathbf{u}$ indeed has a m.a.p equal to $\psi_y = \psi_A/g$.

As such, it should come as no surprise that cyclic subspaces also play an important role in several areas of applied linear algebra such as in coding and in linear control and *pole placement.* Indeed, cyclic codes contain the BCH codes, which are one of the most important families of error-correcting codes. The key question: When is a vectorspace V cyclic? is really a question about companion matrices

3 Embedding

Next, we embed X_n' into a unimodular matrix

$$K(x) = \left[\begin{array}{c|cccc} 1 & x & x^2 & \dots & x^{n-1} \\ \hline 0 & 1 & x & & x^{n-2} \\ & & 1 & & \vdots \\ & & & \ddots & x \\ 0 & & & & 1 \end{array}\right],$$

and then compute $K(x)[xI - L(f)] = \begin{bmatrix} \mathbf{0}^T & f(x) \\ -I & \mathbf{u} \end{bmatrix}$, where $\mathbf{u}^T = [f_0(x), f_1(x), \dots, f_{n-2}(x)]^T$. Selecting $R(x) = \begin{bmatrix} \mathbf{u} & -I \\ 1 & 0 \end{bmatrix}$, we see that

$$K(x)[xI - L(f)]R(x) = \begin{bmatrix} f(x) & 0 \\ 0 & I \end{bmatrix}.$$

Since $K(x)$ and $R(x)$ are unimodular with $K(x)^{-1} = I - xJ_n(0)$, we have obtained the Smith Normal Form $\mathrm{diag}(f(x), I)$ of $L(f)$.

Taking determinants shows further that $\Delta_L(x) = f(x)$ so that L_f is indeed non-drogatory.

4 Some basic Identities

Before we progress, we shall need several basic identities that illustrate how companion matrices deal with polynomial properties. The key shift property of L comes from the following.

Proposition 4.1. *If $L = L_f$ and $\mathbf{f} = [f_0, \dots, f_{n-1}]^T$, then*

(i) $L^i\mathbf{e}_1 = \mathbf{e}_{i+1}$ for $i = 0, 1, \dots, n-1$ and $L^n\mathbf{e}_1 = \mathbf{f}$.

(ii) $I = [\mathbf{e}_1, L\mathbf{e}_1, \dots, L\mathbf{e}_{n-1}] = [\mathbf{e}_1, L\mathbf{e}_1, \dots, L^{n-1}\mathbf{e}_1]$.

(iii) $L^k = [\mathbf{e}_{k+1}, L\mathbf{e}_{k+1}, \dots, L^{n-1}\mathbf{e}_{k+1}]$, for $k = 1, 2, \dots$

From these we obtain the curious by-product that

$$col([I, L, L^2, \dots, L^{n-1}]) = col(\begin{bmatrix} I \\ L \\ \vdots \\ L^{n-1} \end{bmatrix}).$$

We next show that polynomials in L and L^T generate Krylov chains and that they are completely determined by their first and last columns respectively.

Lemma 4.2. *Let $L = L(f)$, $g(x) = \sum_{i=0}^n g_i x^i$ and $h(x) = \sum_{i=0}^k h_i x^i$, with $k < n$, $h_k \neq 0$ and associated vectors $\mathbf{g} = [g_0, g_1, \dots, g_{n-1}]^T$, $\mathbf{h} = [h_0, h_1, \dots, h_{n-1}]^T$ and $\boldsymbol{\gamma} = \mathbf{g} - g_n\mathbf{f}$. Then*

(i) $g[L(f)] = [\boldsymbol{\gamma}, L\boldsymbol{\gamma}, \dots, L^{n-1}\boldsymbol{\gamma}]$.

(ii) $\mathrm{rk}[h(L)] \geq n - k$ with equality if $h|f$.

(iii) $g(L)G_f = [f_0(L_f)\boldsymbol{\gamma}, \dots, f_{n-1}(L_f)\boldsymbol{\gamma}]$.

(iv) $g[L(f)]\mathbf{h} = h[L(f)]\boldsymbol{\gamma}$.

(v) $g(L^T)\mathbf{e}_j = f_{j-1}(L^T)g(L^T)\mathbf{e}_n$.

Proof. (i) $g(L)\mathbf{e}_1 = \sum_{i=0}^{n-1}(g_i - g_n f_i)L^i\mathbf{e}_1 = \sum_{i=0}^{n-1}(g_i - g_n f_i)\mathbf{e}_{i+1} = \boldsymbol{\gamma}$.
(ii) Matrix $[\mathbf{h}, \dots, L^{n-k-1}\mathbf{h}] = S_{n-k}(h)$ has rank $n-k$. If $f = hq$, then $\partial(q) = n-k$ and $\mathrm{rk}[q(L)] \geq n-(n-k)$. Also $0 = h(L)q(L)$, which shows that $\nu[h(L)] \geq k$.
(iii) $[f_0(L_f)\boldsymbol{\gamma}, \dots, f_{n-1}(L_f)\boldsymbol{\gamma}] = [I, L, \dots, L^{n-1}](G_f \otimes I)(I \otimes \boldsymbol{\gamma}) = [I, L, \dots, L^{n-1}](I \otimes \boldsymbol{\gamma})G_f = [\boldsymbol{\gamma}, L\boldsymbol{\gamma}, \dots, L^{n-1}\boldsymbol{\gamma}]G = g(L)G$.
(iv)$[\boldsymbol{\gamma}, L\boldsymbol{\gamma}, .., L^{n-1}\boldsymbol{\gamma}]\mathbf{h} = \mathbf{h}(\mathbf{L})\boldsymbol{\gamma}$.
(v) $g(L^T)\mathbf{e}_j = G^{-1}[g(L)G]\mathbf{e}_j = G^{-1}f_{j-1}(L)\boldsymbol{\gamma} = [G^{-1}f_{j-1}(L)]g(L)\mathbf{e}_1 = [G^{-1}f_{j-1}(L)]g(L)G\mathbf{e}_n = f_{j-1}(L^T)g(L^T)\mathbf{e}_n$. □

Part one shows that the map must have degree n. Part two shows that if $d = (f, g)$ then $\mathrm{rk}[g(L_f)] = \mathrm{rk}[d(L_f)] = n - \partial(d)$, which illustrates the close connection between gcds and companion matrices, and is crucial in systems theory.
Next we observe that G does symmetrize all powers of L, i.e. $L_f^k G_f$ is symmetric. In fact it follows by induction that, for $k = 1, 2, \dots$,

$$L_f^k G_f = \mathrm{diag}(-\begin{bmatrix} 0 & & f_0 \\ & \cdot^{\cdot^{\cdot}} & \vdots \\ f_0 & \cdots & f_{k-1} \end{bmatrix}, \begin{bmatrix} f_{k+1} & \cdots & f_n \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ f_n & & 0 \end{bmatrix}).$$

We now come to a couple of useful inverses.

The inverse of $L(f)$ exists exactly when f_0 is invertible and has the form of flipped companion matrix, i.e.

$$L(f)^{-1} = -[I(f_1 f_0^{-1}) + L(f_2 f_0^{-1}) + \cdots + L^n(f_0^{-1}].$$

The inverse of G on the other hand, is again a Hankel matrix. Indeed,

$$\begin{aligned} G[\mathbf{e}_n, L^T\mathbf{e}_n, \ldots, (L^T)^{n-1}\mathbf{e}_n] &= [G\mathbf{e}_n, GL^T\mathbf{e}_n, \ldots, G(L^T)^{n-1}\mathbf{e}_n] \\ &= [\mathbf{e}_1, LG\mathbf{e}_n, \ldots, L^{n-1}G\mathbf{e}_n] \\ &= [\mathbf{e}_1, L\mathbf{e}_1, \ldots, L^{n-1}\mathbf{e}_1] = I. \end{aligned}$$

Transposing now gives

$$G^{-1} = [\mathbf{e}_n, L^T\mathbf{e}_n, \ldots, (L^T)^{n-1}\mathbf{e}_n] = \begin{bmatrix} \mathbf{e}_n^T \\ \mathbf{e}_n^T L \\ \mathbf{e}_n^T L^2 \\ \vdots \\ \mathbf{e}_n^T L^{n-1} \end{bmatrix}.$$

Using this in turn yields

$$I = GG^{-1} = G \begin{bmatrix} \mathbf{e}_n^T \\ \mathbf{e}_n^T L \\ \mathbf{e}_n^T L^2 \\ \vdots \\ \mathbf{e}_n^T L^{n-1} \end{bmatrix} = \begin{bmatrix} \mathbf{e}_n^T f_0(L) \\ \vdots \\ \mathbf{e}_n^T f_{n-1}(L) \end{bmatrix},$$

establishing that $\mathbf{e}_n^T f_i(L) = \mathbf{e}_{i+1}^T$. We conclude this section with some of the interaction between $L(f)$ and the matrix $N = E_{1,n}$ and assume that $\partial(g) < n$.

Proposition 4.3. *The following hold:*

(i) $L[f(x) + g(x)] = L(f) - \mathbf{e}_n \mathbf{g}^T$.

(ii) $L[f(x) - 1] = L + N$ and $f(L + N) = I$.

(iii) $L[f(x) - x^k] = L + E_{k+1,n}$ and $f(L + E_{k+1,n}) = (L + E_{k+1,n})^k$.

(iv) $NL_f^k N = 0$ for $k = 0, 1, \ldots, n-1$ and $NL^{n-1}N = N$.

(v) $g(L_f + N)N = g(L_f)N + g_{n-1}N$.

(vi) $f(L_f + N)N = N = Nf(L_f + N)$.

(vii) $(L + N)^r - L^r = \Gamma_r(L, N, L) = L^{r-1}N + L^{n-2}NL + \cdots + NL^{r-1}$, $r = 1, \ldots, n-2$.

Proof. (ii)-(iii) $f(x) - x^k$ is an ap for $L + E_{k+1,n}$. (iv) Follows from (4.1)-(i). (v) $g(L+N)\mathbf{e}_1 = g[L(f-1)]\mathbf{e}_1 = L(f-1)\mathbf{g} = (L+N)\mathbf{g} = L\mathbf{g} + N\mathbf{g} = g(L)\mathbf{e}_1 + g_{n-1}\mathbf{e}_1$. (vi) $N\Gamma_r = 0$ for $r = 0, \ldots, n-1$. □

5 Corner matrices

Besides multiplication or evaluation there are two other operations that we can apply to the companion shift, and these are telescoping or differentiation. Actually the corner matrix acts very much like "differentiation". For example, for any polynomial $g(x)$,

$$\begin{aligned} g(\begin{bmatrix} x & 1 \\ 0 & x \end{bmatrix}) &= \begin{bmatrix} g(x) & g'(x) \\ 0 & g(x) \end{bmatrix} \\ &= g(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}) + g'(x) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

More generally, the corner matrices Γ_k appear in the powers of $\begin{bmatrix} A & C \\ 0 & D \end{bmatrix}$. The difference form now becomes $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}^k - \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^k = \begin{bmatrix} 0 & \Gamma_k \\ 0 & 0 \end{bmatrix}$, where Γ_k satisfies the down-shift recurrence relation

$$\Gamma_{k+1}(A, C, D) = A\Gamma_k + CD^k = A^k C + \Gamma_k D.$$

If we now have a second polynomial $g(x) = g_0 + g_1 x + \cdots + x^N$ then $g(M) = \begin{bmatrix} g(A) & \Gamma_g A, C, B) \\ 0 & g(D) \end{bmatrix}$, where

$$\begin{aligned} \Gamma_g &= \sum_{i=1}^{N} g_i \Gamma_i = \sum_{i=0}^{n-1} g_i(A) C B^i \\ &= [I, A, \ldots, A^{n-1}][G_g \otimes C] \begin{bmatrix} I \\ B \\ \vdots \\ B^{n-1} \end{bmatrix} \end{aligned} \tag{5.7}$$

and the $g_i(x)$ are the adjoint polynomials affiliated with $g(x)$. The difference now becomes $g(\begin{bmatrix} A & C \\ 0 & D \end{bmatrix}) - g(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}) = \sum_{i=1}^{N} \begin{bmatrix} f_i(A) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & B^i \end{bmatrix}$.

6 Companion matrices, Resultants and Bezoutians

We now present an example in which we telescope the companion shift equation resulting in a simple relation between companion matrices, resultants and Bezoutians. The latter enters the realm of root location, stability analysis, and gcd-degree computation.

Given two monic polynomials $f(x)$ and $g(x)$ of degree n. The bilinear form associated with $f(x)$ is the difference quotient

$$\frac{f(x) - f(y)}{x - y} = X_n' G_f Y_n = \sum_{i=0}^{n-1} f_i \Gamma_i(x, y).$$

With the polynomial pair $f(x)$, $g(x)$ we may associate, the form

$$\mathbb{B}(f, g) = \frac{f(x)g(y) - f(y)g(x)}{x - y} = \sum_{i,j=0}^{n-1} b_{ij} x^i y^j = X_n' B Y_n.$$

It is bilinear and anti-symmetric, i.e. $\mathbb{B}(g, f) = -\mathbb{B}(f, g)$. The $n \times n$ matrix $B = \mathbb{B}(f, g)$ is called the Bezoutian (of Hankel type), and is symmetric. It is clear that

$$\begin{aligned}\mathbb{B}(f, g) &= g(x)\frac{f(x) - f(y)}{x - y} - f(x)\frac{g(x) - g(y)}{x - y} \\ &= g(x)[X_n' G_f Y_n] - f(x)[X_n' G_g Y_n]\end{aligned}$$

In order to tackle the bilinear form $[g(x)X_n']G_f Y_n$ we use the basic shift on $g(x)X_n'$ to simplify the result. Since $t = n$, it is convenient to split $S_n(f) = \begin{bmatrix} S_n^-(f) \\ S_n^+(f) \end{bmatrix}$, where

$$S_n^-(f) = \begin{bmatrix} f_0 & & 0 \\ \vdots & \ddots & \\ f_{n-1} & \cdots & f_0 \end{bmatrix},\ S_n^+(f) = \begin{bmatrix} f_n & \cdots & f_1 \\ & \ddots & \\ 0 & & f_n \end{bmatrix}.$$

Because of their Toeplitz structure, representing polynomial multiplication, we know that $S_n^-(f)$ and $S_n^-(g)$ commute, as do their "plus" counter parts. We can now split the basic shift (0.2) into

$$f(x)X_n' = X_{2n}' S_n(f) = X_n' S_n(f)^- + x^n X_n' S_n(f)^+. \quad (6.8)$$

Next we recall the companion shift (0.1), which has the form $AX - XB = C$, where $A = x$, $B = L(f)$, $X = X_n'$ and $C = f(x)\mathbf{e}_n^T$. Telescoping as in (0.3) we obtain

$$x^k X_n' - X_n' L^k = \Gamma_k(xI, f(x)\mathbf{e}_n^T, L) = f(x)\mathbf{e}_n^T \Gamma_k(xI, L) \quad (6.9)$$

If the second polynomial is given by $g(x) = g_0 + g_1 x + \cdots + x^n$, then we pre-multiply (6.9) by g_k and sum, giving

$$g(x)X_n' - X_n' g(L_f) = f(x)\mathbf{e}_n^T \Gamma_g(xI, L) = f(x)\mathbf{q}^T(x), \quad (6.10)$$

We next use the adjoint polynomials in

$$\begin{aligned}\mathbf{q}^T &= \mathbf{e}_n^T \Gamma_g = \sum_{i=0}^{n-1} g_i(x)\mathbf{e}_n^T L_f^i \\ &= [g_0(x), \ldots, g_{n-1}(x)] \begin{bmatrix} \mathbf{e}_n^T \\ \mathbf{e}_n^T L_f \\ \vdots \\ \mathbf{e}_n^T L_f^{n-1} \end{bmatrix} \\ &= X_n' G_g G_f^{-1} = X_n' Q, \quad \text{where } \mathrm{Q} = (G_g G_f^{-1}).\end{aligned}$$

Applying the basic shift in (6.10) now gives

$$X_{2n}' S_n(g) - X_{2n}' \begin{bmatrix} g(L_f) \\ 0 \end{bmatrix} = X_{2n}' S_n(f) Q,$$

from which we obtain the identity

$$S_n(g) G_f - \begin{bmatrix} g(L_f) \\ 0 \end{bmatrix} G_f = S_n(f) G_g.$$

We could also have telescoped the adjoint shift equation. Splitting this then produces

$$\begin{bmatrix} S_n^-(g) G_f \\ S_n^+(g) G_f \end{bmatrix} - \begin{bmatrix} S_n^-(f) G_g \\ S_n^+(f) G_g \end{bmatrix} = \begin{bmatrix} g(L_f) G_f \\ 0 \end{bmatrix}$$

in which we equate blocks to yields

$$S_n^-(g)G_f - S_n^-(f)G_g = g(L_f)G_f \text{ and } S_n^+(g)G_f = S_n^+(f)G_g) \quad (6.11)$$

The latter follows from the fact that $S_n^+(f)$ and $S_n^+(g)$ commute and

$$S_n^+(f)F = G_f,\ FG_f = S_n^-(\tilde{f}),\quad S_n^+(f)^T = S_n^-(\tilde{f}), \quad (6.12)$$

where $\tilde{f}$ is the reciprocal polynomial. Using the split basic shift (6.8) we see that

$$[g(x)X_n']G_f Y_n = X_n' S_n^-(g) G_f Y_n + x^n X_n' S_n^+(g) G_f Y_n$$

while

$$[f(x)X_n']G_g Y_n = X_n' S_n^-(f) G_g Y_n + x^n X_n' S_n^+(f) G_g Y_n\ .$$

Subtracting them, we obtain

$$\begin{aligned}\mathbb{B}(f, g) =& X_n'[S_n^-(g)G_f - S_n^-(f)G_g]Y_n \\ &+ x^n X_n'[S_n^+(g)G_f - S_n^+(f)G_g]Y_n,\end{aligned}$$

in which the second term vanished because of (6.11). Extracting the matrix we see that $B(f, g) = S_n^-(g)G_f - S_n^-(f)G_g$ which on account of (6.11) gives Barnett's formula

$$B(f, g) = S_n^-(g)G_f - S_n^-(f)G_g = g(L_f)G_f.$$

Lastly we introduce the resultant matrix $M(g, f) = \begin{bmatrix} S_n^-(g) & S_n^-(f) \\ S_n^+(g) & S_n^+(f) \end{bmatrix}$ and the two row matrices

$$T = \begin{bmatrix} I & -S_n^-(f)[S_n^+(f)]^{-1} \\ 0 & I \end{bmatrix} \text{ and } \mathrm{U} = \begin{bmatrix} I & 0 \\ -Q & I \end{bmatrix}.$$

We subsequently compute the triplet TM(g,f)U in two ways, establishing that

$$\begin{aligned}\begin{bmatrix} \zeta & 0 \\ 0 & S_n^+(f) \end{bmatrix} &= \begin{bmatrix} \zeta & 0 \\ S_n^+(g) & S_n^+(f) \end{bmatrix} \begin{bmatrix} I & 0 \\ -Q & I \end{bmatrix} \\ &= \begin{bmatrix} I & -S_n^-(f)[S_n^+(f)]^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} g(L_f) & S_n^-(f) \\ 0 & S_n^+(f) \end{bmatrix} \\ &= \begin{bmatrix} g(L_f) & 0 \\ 0 & S_n^+(f) \end{bmatrix}.\end{aligned}$$

As such we see that

$$g(L_f) = \zeta = S_n^-(g) - S_n^-(f)[S_n^+(f)]^{-1} S_n^+(g)\ ,$$

which is the (2,2) Schur complement of M(g,f). In conclusion we may use (6.12) to compute

$$M(f, g)M^T(\tilde{g}, -\tilde{f}) = \begin{bmatrix} S_n^-(f) & S_n^-(g) \\ S_n^+(f) & S_n^+(g) \end{bmatrix} \begin{bmatrix} S_n^+(g) & S_n^-(g) \\ -S_n^+(f) & -S_n^-(f) \end{bmatrix},$$

which reduces to diag(K, N), where

$$K = -B(f, g)F \ \ and \ \ N = -FB(\tilde{f}, \tilde{g}).$$

Needless to say, there are numerous generalizations of this concept to multivariate or non-commutative settings.

7 Generalized Adjoint chains

The idea of an adjoint chain may be extended by using a *block* Hankel matrix G. Indeed, suppose we are given the $m \times n$ polynomial matrix $\mathscr{F}(x) = F_0 + F_1 x + \cdots + F_N x^N$. The *adjoint polynomials* associated with this master polynomial are defined by

$$\mathscr{F}_k(x) = F_{k+1} + F_{k+2}x + \cdots + F_N x^{N-k-1}, k = 0, \ldots, N-1,$$

with in addition $\mathscr{F}_N(x) = 0$, $\mathscr{F}_{-1}(x) = \mathscr{F}(x)$ and $\mathscr{F}_{N-1}(x) = F_N$. As in the scalar case they can be expressed in block matrix form as

$$[\mathscr{F}_0(x),\ \mathscr{F}_1(x), \cdots, \mathscr{F}_{N-1}(x)] = [1, x, \cdots, x^{N-1}]G(\mathscr{F}),$$

where $G = G(\mathscr{F})$ is the block Hankel matrix

$$G = \begin{bmatrix} F_1 & F_2 & \ldots & & F_N \\ F_2 & F_3 & & F_N & 0 \\ \vdots & & & & \\ F_{n-1} & F_N & 0 & & \\ F_N & 0 & \ldots & & 0 \end{bmatrix}.$$

The key feature of these polynomials is that they satisfy the down-shift Recurrence Relation

$$x.\mathscr{F}_k(x) = \mathscr{F}_{k-1}(x) - F_k \quad k = 0, 1, \cdots, N-1,$$

in which we may replaced x by a suitable square matrix A on the left, or D, on the right.

Using the block recurrence we may write

$$\begin{bmatrix} \mathscr{F}_1(x) \\ \vdots \\ \mathscr{F}_N(x) \end{bmatrix} .x = \begin{bmatrix} \mathscr{F}_0(x) \\ \vdots \\ \mathscr{F}_{N-1}(x) \end{bmatrix} - \begin{bmatrix} F_1 \\ \vdots \\ F_N \end{bmatrix}$$

and by using the companion structure we also have

$$(L_f \otimes I) \begin{bmatrix} \mathscr{F}_1(x) \\ \vdots \\ \mathscr{F}_N(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \mathscr{F}_1(x) \\ \vdots \\ \mathscr{F}_{N-1}(x) \end{bmatrix} - \begin{bmatrix} f_0 I \\ \vdots \\ f_{N-1} I \end{bmatrix} \mathscr{F}_N(x).$$

Subtracting these we arrive at the generalized adjoint-companion shift identity

$$(L_f \otimes I) \begin{bmatrix} \mathscr{F}_1(x) \\ \vdots \\ \mathscr{F}_N(x) \end{bmatrix} - \begin{bmatrix} \mathscr{F}_1(x) \\ \vdots \\ \mathscr{F}_N(x) \end{bmatrix} x = \begin{bmatrix} F_1 \\ \vdots \\ F_N \end{bmatrix} - \begin{bmatrix} \mathscr{F}_0(x) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{7.13}$$

together with a row analog. It should be noted that the indices differ by one from those in (1.4), and it goes without saying that we may now again replace x by a suitable matrix D.

8 The Cyclic Decomposition Theorem

This theorem is a statement about the periodicity of finite dimensional objects, and is realized in terms of companion matrices.
There are essentially *two* versions of this theorem. A weak version, which is easier to prove, and a strong version, which requires much more firepower. The weak version says that any matrix A in $V = \mathbb{F}_{n \times n}$ is similar to a direct sum of companion matrices. Or equivalently, that any vectorspace over a field $\mathbb{F}$ can be decomposed as a direct sum of "cyclic" subspaces. It may be considered as a special case of the fundamental, theorem of abelian groups.

The best example is that of a permutation, which is a product of distinct cycles. In terms of matrices this says that any permutation matrix is similar to direct sum of matrices of the form $L(x^r - 1)$.

We shall use the adjoint-companion shift (7.13) to derive the "strong"' version of this theorem – which is often called the Rational Canonical Form – in which, in addition, the minimal polynomials of the companion matrices **interlace**. The proof is short and does not use quotient spaces.

When this theorem is combined with the Primary Decomposition Theorem, they will spawn the Jacobson and Jordan Canonical Forms.

Given a matrix A in $V = \mathbb{F}_{n \times n}$, with minimal polynomial ψ_A. Select a maximal vector $\mathbf{x}$, for which $\psi_{\mathbf{x}} = \psi_A = f(\lambda) = f_0 + f_1\lambda + \cdots + \lambda^m$. The existence of such a vector follows as for finite abelian groups G, when we replace the order $O(.)$ of an element by the m.a.p of a vector. Indeed in G, if $O(a) \nmid O(b)$ then there exists z in G such that $O(b)|O(z)$ but $b \neq z$ and if $O(y)$ is maximal then $O(a)|O(y)$ for all a in G.

We then form the chain matrix $K = [\mathbf{x}, A\mathbf{x}, \ldots, A^{m-1}\mathbf{x}]$, which has rank m, and complete it to a basis $Q = [K, B]$ for V. Then $AQ = Q \left[\begin{array}{c|c} L_f & C \\ \hline 0 & D \end{array}\right]$, for some C and D. Since Q is invertible, $Q^{-1}AQ = \left[\begin{array}{c|c} L & C \\ \hline 0 & D \end{array}\right] = M$, and thus $\psi_M = \psi_A = f$. It now follows that $0 = f(M) = \begin{bmatrix} f(L) & \Gamma_f \\ 0 & f(D) \end{bmatrix}$, in which the corner block takes the form

$$\Gamma_f = \sum_{k=0}^{m} f_k \sum_{j=0}^{k-1} L^{k-j-1} C D^j = \sum_{i=0}^{m-1} f_i(L) C D^i = 0 \tag{8.14}$$

and the $f_k(\lambda)$ are the usual adjoint polynomials of $f(x)$. Suppose now that $C = \begin{bmatrix} \gamma_1^T \\ \vdots \\ \gamma_m^T \end{bmatrix}$ and set $\mathscr{F}_k(x) =$

$\sum_{i=k}^{m-1} \gamma_{i+1}^T x^{i-k}$ and $\mathscr{F}_m(x) = 0$. It is easily seen that $\mathscr{F}_k(x)$ satisfies $\mathscr{F}_k(x).x = \mathscr{F}_{k-1}(x) - \gamma_k^T$ so that we can use the adjoint-shift identity (7.13)

$$L(f)\begin{bmatrix} \mathscr{F}_1(D) \\ \vdots \\ \mathscr{F}_m(D) \end{bmatrix} - \begin{bmatrix} \mathscr{F}_1(D) \\ \vdots \\ \mathscr{F}_m(D) \end{bmatrix} D = C - \begin{bmatrix} \mathscr{F}_0(D) \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

in which $\mathscr{F}_0(D) = \sum_{i=0}^{m-1} \gamma_{i+1}^T D^i = \sum_{i=0}^{m-1} \mathbf{e}_{i+1}^T C D^i$. Using the fact that $\mathbf{e}_m^T f_i(L) = \mathbf{e}_{i+1}^T$, this reduces to $\mathscr{F}_0(D) = \sum_{i=0}^{m-1} \mathbf{e}_m^T f_i(L) C D^i = \mathbf{e}_m^T \sum_{i=0}^{m-1} f_i(L) C D^i = \mathbf{e}_m^T \Gamma_f$, and thus, by (8.14), vanishes. In other words we have constructed a solution X to the matrix equation $L(f)X - XD = C$. This means that $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}\left[\begin{array}{c|c} L(f) & C \\ \hline 0 & D \end{array}\right]\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} = \left[\begin{array}{c|c} L(f) & 0 \\ \hline 0 & D \end{array}\right]$, and consequently $A \approx M \approx \left[\begin{array}{c|c} L(f) & 0 \\ \hline 0 & D \end{array}\right]$, in which $\psi_D | \psi_M = f$. It goes without saying that we may repeat the above steps with D to obtain a direct sum decomposition

$$A \approx \text{diag}[L(\psi_1), L(\psi_2), \ldots, L(\psi_t)],$$

where $\psi_t \mid \psi_{t-1} \mid \cdots \mid \psi_2 \mid \psi_1$.

The polynomials $\mathscr{J}_A = (\psi_1, \ldots, \psi_{t-1})$ are unique, and are called the **invariant factors** of A. They completely characterize similarity and do not depend on any possible factorization of polynomials, and were obtained by only using "rational operations". Hence the alternative name of "Rational Canonical Form".

The uniqueness of this Canonical Form follows at once, if we recall that $\psi_1 = \psi_A$ is unique and then apply the following elementary result.

Lemma 8.1. *If* $M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \approx \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} = N$ *then* $\psi_B = \psi_C$.

Proof. $\psi_B(M) = \begin{bmatrix} \psi_B(A) & 0 \\ 0 & 0 \end{bmatrix} \approx \begin{bmatrix} \psi_B(A) & 0 \\ 0 & \psi_B(C) \end{bmatrix}$. Taking ranks shows that $\text{rk}[\psi_B(C)] = 0$ and thus $\psi_B(C) = 0$ and $\psi_C \mid \psi_B$. By symmetry, it also follows that $\psi_B \mid \psi_C$, ensuring equality. □

Now if $A \approx \text{diag}[L(\psi), D] \approx \text{diag}[L(\psi), E]$ then, by applying Lemma (8.1), we see that $\psi_D = \psi_E$, so that we can indeed continue the reduction process with D or with E. The same polynomials will be obtained.

It is of interest to note that the above method can actually also be used to prove that similarity of $\begin{bmatrix} A & C \\ 0 & D \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$, ensures that $AX - XD = C$ has a solution.

9 Differentiation

If the field is closed we may use the companion shift to obtain the Jordan form for $L(f)$, but since we use right eigenvectors, it is more convenient to use L_f^T. The transposed companion shift takes the form

$$L_f^T X_n(x) - X_n(x)x = -f(x)\mathbf{e}_n.$$

Now $f(a) = 0$ iff $X_n(a)$ is an eigenvector for L^T associated with eigenvalue a. The corresponding eigenvector for L_f will be $\mathscr{F}(a) = G_f X_n(a) = \begin{bmatrix} f_0(a) \\ \vdots \\ f_{n-1}(a) \end{bmatrix}$. Now because $\text{rk}[L^T - aI] = n - 1$, it follows that there can only be one independent eigenvector for a, and thus there is exactly one Jordan block per eigenvalue, as expected, for a non-derogatory matrix.

In the simplest case $f(x) = \prod_{i=1}^{n}(x - \lambda_i)$ has n *distinct* roots and $L^T(f)$ has n distinct evalues, and as such is diagonalizable via its evector matrix $V = [X_n(\lambda_1), \ldots, X_n(\lambda_n)]$. Needless to say this is the celebrated Vandermonde matrix

$$V = \begin{bmatrix} 1 & 1 & & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & & & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}.$$

We may conclude that if $f(x)$ has distinct roots then

$$V^{-1} L^T(f) V = \text{diag}(\lambda_1, \ldots, \lambda_n) = \Lambda$$

Likewise $L(f)$ is diagonalized by the matrix basis change GV, i.e. $L(GV) = (GV)D$, where $(GV)_{ij} = f_i(\lambda_j)$, $i, j = 0, 1, \ldots, n-1$.

We next note that, because $\frac{f(x)-f(y)}{x-y} = X_n' G_f Y_n$, if α and β are two distinct roots of $f(x)$ then the quotient vanishes and so $X_n'(\alpha) G_f Y_n(\beta) = 0$. Also letting x approach y, or by summing directly, we see that $X_n'(x) G_f Y_n(x) = f'(x)$. This means that $V^T G_f V = \text{diag}(f'(\lambda_1), ., f')\lambda_n)) = D$ or $V^{-1} = D V^T G$, which can then be used to establish that

$$\begin{aligned} V^T B(f,g) V &= V^T g(L_f) G_f V = V^T G_f g(L_f^T) V \\ &= (V^T G V)\Lambda = D\Lambda. \end{aligned}$$

The companion matrix $\Omega = L(x^n - 1) = [\mathbf{e}_2, \ldots, \mathbf{e}_n, \mathbf{e}_1]$ is called the basic *circulant*, and any polynomial $p(\Omega)$ in Ω is a circulant matrix.

For example if $p(x) = p_0 + p_1 x + \cdots + p_{n-1}x^{n-1}$ then

$$p(\Omega) = \begin{bmatrix} p_0 & p_{n-1} & & p_2 & p_1 \\ p_1 & p_0 & p_{n-1} & \cdots & p_2 \\ p_2 & p_1 & p_0 & & \\ \vdots & & & & p_{n-1} \\ p_{n-1} & & \cdots & p_1 & p_0 \end{bmatrix}.$$

The matrix Ω is one of the most important matrices in all of applied mathematics. Indeed, since $\Delta_\Omega(\lambda) = \lambda^n - 1$, its eigenvalues are the n distinct n-th roots of unity, $\sigma = \{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$, where $\omega = exp(\frac{2\pi i}{n})$.

Consequently it also has n independent eigenvectors $\mathbf{v}_n(\omega^k)$, $k = 0, 1, \ldots, n-1$ (called phasers), and as such Ω can be diagonalized via

$$\Omega^T V = VD, \quad \text{and} \quad \Omega V = VD^{-1},$$

where $D = \text{diag}(1, \omega, \ldots, \omega^{n-1})$ and V is the *Vandermonde* matrix

$$V = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & & \omega^{2(n-1)} \\ \vdots & & & & \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)(n-1)} \end{bmatrix} = V^T.$$

Now V is not only symmetric but its columns $\mathbf{v}(\omega^k)$, are also pairwise *orthogonal*! Indeed,

$$\mathbf{v}(\omega^k)^* \mathbf{v}(\omega^r) \quad = \quad \sum_{s=0}^{n-1} \omega^{s(r-k)} = \begin{cases} n & if \quad r = k \\ 0 & if \quad r \neq k. \end{cases}$$

Consequently we may normalize the eigenvectors and use the *unitary* matrix $W = \frac{1}{\sqrt{n}} V$ for which $W^{-1} = \bar{W}^T = \bar{W}$.

The matrix multiplication

$$\mathbf{y} = W\mathbf{x}$$

is referred to as the **Discrete Fourier Transform**. It is of cardinal importance in the theory of *filtering*. We shall now examine the case of repeated roots of $f(\lambda)$.

Like Janus, differentiation is a "two-faced" personality. On the one hand it is used to compute tangents and tangent planes, and as such is all important in optimization, while on the other hand it also serves as the **ultimate counting machine**. This makes it indispensable in combinatorics and in fact anywhere where polynomials are used. Recall that the term λ^k is after all just a place holder, and that its coefficient can be "counted" by differentiating k times. As such we have two counting tools, matrix multiplication and differentiation and our main trick is is to convert *differential* identities into *matrix* identities.

Suppose If $f(\lambda) = \prod_{i=1}^{s} (\lambda - \lambda_i)^{m_i} = (\lambda - \lambda_i)^{m_i} \phi_i(x)$. We shall now **differentiate** the companion shift in column form to solve this problem. Consider

$$L_f^T X_n(x) = X_n x - f(x)\mathbf{e}_n$$

and differentiate both sides k times. Using the product rule gives

$$L_f^T X_n^{(k)} = x X_n^{(k)} + k X_n^{(k-1)} - f^{(k)}(x)\mathbf{e}_n.$$

Dividing by $k!$ and setting $M_k = X_n/k!$, yields $L^T M_k = x M_k + M_{k-1} - (f^{(k)}/k!)\mathbf{e}_n$, which is a step down recurrence relation. Stacking r of these columns in $W_{r,n}(x) = [M_0, ., M_{r-1}]$ shows that

$$L^T W_{r,n}(x) = W_{r,n}(x) J_r(x) - \mathbf{e}_n F_r(x)^T,$$

where $F_r(x)^T = [f(x), \frac{f'(x)}{1!}, \ldots, \frac{f^{(r-1)}(x)}{(r-1)!}]$. If we substitute λ_i for x and take $r = m_i$, then $F(\lambda_i) = 0$ leaving $L^T W_{m_i,n}(\lambda_i) = W_{m_i,n}(\lambda_i) J_{m_i}(\lambda_i)$, where

$$W_{m,n+1}^T(\lambda) = \begin{bmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^{m-1} & \binom{n}{n}\lambda^n \\ 0 & 1 & 2\lambda & \cdots & (m-1)\lambda^{m-2} & \binom{n}{n-1}\lambda^{n-1} \\ & & & & & \vdots \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & \cdots & 1 & \binom{n}{n-m+1}\lambda^{n-m+1} \end{bmatrix}$$

is an $m \times (n+1)$ confluent Vandermonde block. Stacking these blocks for each of the distinct eigenvalues we arrive at

$$\begin{aligned} & L^T[W_{\lambda_1}, \ldots, W_{\lambda_s}] = \\ & [W_{\lambda_1}, \ldots, W_{\lambda_s}]\text{diag}(J_{m_1}(\lambda_1), \ldots, J_{m_s}(\lambda_s)), \end{aligned}$$

which is the desired Jordan form. Several points should now be noted:

(i) $W_{m,}(\alpha)$ precisely equals the chain matrix $K_n[\mathbf{e}_1, J_m^T(\alpha)]$.

(ii) it relates the derivatives to the coefficients of a polynomial in the stacked form

$$\begin{bmatrix} f(\lambda) \\ f'(\lambda)/1! \\ \vdots \\ \frac{f^{(m-1)}(\lambda)}{(m-1)!} \end{bmatrix} = W_{m,n}(\lambda) \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}.$$

(iii) The matrix $W = [W_{\lambda_1}, \ldots, W_{\lambda_s}]$ is the generalized Vandermonde matrix. It is also known as the Caratheodory matrix, which appears in the study of moment problems.

(iv) W also equals the Wronskian Matrix of the set of functions $\{t^j e^{\lambda_k t}\}$, $k = 1, \ldots, s$ and $j = 0, \ldots, m_k - 1$ and appears in the study of differential equations.

(v) If $[f_0, \ldots, f_n]W = 0$ then $f^{(j)}(\lambda_k) = 0$ for $k = 1, \ldots, s$ and $j = 0, \ldots, m_k$, with $m_1 + \cdots + m_s = n+1$. But then $\pi(x) = \prod_{i=1}^{s}(\lambda - \lambda_i)^{m_i} | f(x)$, in which $\partial(\pi) = n+1$ while $\partial(f) = n$. Thus forcing $f(x) = 0$, ensuring that W is non-singular. As a check we can compute

(vi) $\det(W) = \prod_{1 \le j < i \le s} (\lambda_i - \lambda_j)^{m_i m_j}$.

(vii) The generalized Vandermonde matrix also appears naturally in Hermite interpolation where one aims to find a polynomial $f(x)$ with prescribed derivatives at prescribed points.

Lastly, recall that $B(f, g) = g(L_f)G_f$, and suppose that $W^{-1} L_f^T W = J$ is the Jordan form of L. Then

$$\begin{aligned} W^T B(f,g) W &= W^T g(L_f) G_f W = W^T G_f [g(L_f^T) W] \\ &= (W^T G_f W) g(J). \end{aligned}$$

Now W^TGW is made up of blocks $W_\alpha^T G_f W_\beta$, where α and β are eigenvalues of L. When $\alpha \neq \beta$ we see from the difference quotient that this vanishes. On the other hand, when $\alpha = \beta$, we shall need more care. First recall that the adjoint polynomial satisfy $F_n' = X_n' G$, and hence that ${F_n'}^{(k)} = {X_n'}^{(k)} G$. Now since the rows of W_α^T are the derivatives of X_N at α we obtain

$$W_\alpha^T G_f = \begin{bmatrix} \frac{X_n'(\alpha)}{1!} \\ \vdots \\ \frac{{X_n'}^{(k-1)}(\alpha)}{(k-1)!} \end{bmatrix} G_f = \begin{bmatrix} \frac{F_n'(\alpha)}{1!} \\ \vdots \\ \frac{{F_n'}^{(k-1)}(\alpha)}{(k-1)} \end{bmatrix},$$

which is the weighted Wronskian of the adjoint polynomials at α. It is now clear that $(W_\alpha^T G_f W_\alpha)_{pq} = \frac{{F_n'}^{(p)} {X_n}^{(q)}}{p!q!}$. To compute this we first differentiate the adjoin shift equation (1.4) which gives

$$x{F_n'}^{k)} + kx{F_n'}^{k-1)} - {F_n'}^{k)} L^T = f^{(k)}(x)\mathbf{e}_1^T.$$

Now post mutiply by X_n, and substitute the column companion shift (0.1). This gives

$$k{F_n'}^{(k-1)}(x)X_n(x) = f^{(k)}(x),$$

where we used the fact that ${F_n'}^{(k)}\mathbf{e}_n = 0$. It now follows by induction that

$${F_n'}^{(k)} {X_n}^{(r)} = \frac{f^{(k+r+1)}(x)r!k!}{(r+k+1)!},$$

and thus the matrix $W_\alpha^T G_f W_\alpha$ can now be identified as $F\phi(J_{m_i}(\alpha))$.

10 The group Inverse of a Companion Matrix

We have seen that the inverse, if any, of a companion matrix L(f), again has companion structure. When L is singular we require in some settings the group inverse $L^\#$ (over a ring R with 1), which satisfies

$$LXL = L, \; XLX = X \text{ and } LX = XL.$$

It exists iff p_0 is regular and $w = p_0 - (1 - p_0 p_0^-)p_1$ is invertible, in which case it has the form $L^\# = [\mathbf{x}, \mathbf{y}, B]$ where $B = \begin{bmatrix} \mathbf{0}^T \\ I_{n-2} \\ \mathbf{0}^T \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} \mathbf{u} \\ x \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} \mathbf{v} \\ y \end{bmatrix}$, $\mathbf{v} = \mathbf{e}_1 + \hat{\mathbf{f}} y$ and $\mathbf{u} = \begin{bmatrix} v_2 \\ \vdots \\ v_{n-1} \\ y \end{bmatrix} + \hat{\mathbf{f}} x$ and $\hat{\mathbf{f}} = [p_1, \ldots, p_{n-1}]^T$.

The parameters x and y can be expressed in terms of w^{-1}, p_0, p_1 and p_2. Its structure is again sparse, but is closer to that of a perturbed companion matrix. The expression for the Drazin inverse however, is still unknown.

11 Conclusions

We have seen that the companion shift equation is central to many of the applications involving L. It is in combination with other shift conditions that cancellation can occur and the best results materialize. There are numerous generalizations of a companion a matrix, such as the comrade and congenial matrices which use other bases besides the powers of x. Many of the chain relations generalize to the block case and provide a inexhaustible supply of challenging problems.

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History of Astronomy in Portugal: *Institutions, Theories, Practices*

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The United Nations 62nd General Assembly, in order to celebrate the 400th anniversary of Galileo's first telescopic observations, has declared 2009 as the International Year of Astronomy (IYA2009). This celebration was intended to make widely known the importance of Astronomy as a science and as a technique. Among the different strategies proposed to reach this aim, the Portuguese National Committee of the IYA2009, formed by the Portuguese Society of Astronomy, emphasized the need to promote events related to the history of Astronomy.

The Conference on the "History of Astronomy in Portugal: Institutions, Theories, Practices", held at the University of Lisbon Science Museum, from September 24 to September 26, 2009, coinciding with the 22nd meeting of the National Seminar for the History of Mathematics, was an excellent opportunity for scholars and Portuguese researchers on the history of astronomy not only to debate these matters among themselves but also to listen and to talk to some of the best international researchers in this area, contributing to include Portugal in the international net of history of astronomy researchers.

The meeting was organized by researchers of the National Seminar for the History of Mathematics and of the Museum of Science of the University of Lisbon (MCUL), with the support of these two organizations, of the Centro Internacional de Matemática (CIM), of three of the main Portuguese Mathematics centres, CMAF (U Lisbon), CMUC (U Coimbra) and CMUP (U Porto), of the Portuguese Societies of Mathematics and of Astronomy, of the Inter Universities Centre for the History of Science and Technology (CIUHCT) and was sponsored by the Foundation for Science and Technology (FCT).

José Francisco Rodrigues, director of CIM, at the opening ceremony emphasized the reciprocal influence between Mathematics and Astronomy on measuring and understanding space and time throughout human history. This, he said, can be seen as far as the early calendars, with their numerical problems about the counting of days, seasons and years, or the ingenious method of Eratosthenes to measure with remarkable accuracy the circumference of the Earth. Three other significant examples of this historical and scientific interaction were also referred: Kepler's laws of planetary motion, which first two were published also in 1609; Le Verrier's 1846 prediction of the existence of the then unknown planet Neptune, using only mathematics and astronomical observations of the planet Uranus (Galle and d'Arrest later confirmed these predictions within 1° of the foreseen location); and the confirmation in 1919, by a team led by Eddington, of Einstein's prediction of gravitational deflection of starlight by the Sun with the photographs of a solar eclipse taken at dual expeditions in Sobral, northern Brazil, and in Príncipe island, then a Portuguese colony in Africa, showing the distortion of the structure of spacetime by matter, a conclusion from the theory of General Relativity, which was built upon earlier contributions to Differential Geometry by mathematicians like Riemann or Levi-Civita.

Figure 1: *The Scholar Observatory of the Polytechnic School is now integrated in the Science Museum of the University of Lisbon (Photo M. Heller, MCUL)*

In Portugal, throughout its history, Astronomy was developed in the context of Mathematical Sciences. During the times of Portugal's Maritime Discoveries, astronomical navigation was based on spherical trigonometry, and therefore it was the mathematicians who taught astronomy to the pilots. During the 19th century the new centres of science teaching, as the Polytechnic School in Lisbon (Figure 1), or the Polytechnic Academy in Porto, developed astronomy teaching and research in the context of the mathematics subjects.

The inheritors of these 19th century institutions, respectively the Faculties of Sciences of Lisbon and Porto, upheld this tradition during the 20th century and continued to consider astronomy as a subject to be taught in their mathematics departments.

The conference organizers decided to have a program that echoed a wide time span, from the dolmen builders of south-west Europe to the echoes in Portugal of Einstein's theory of relativity. There were 17 talks, nine of them by Portuguese researchers. Eleventh-hour problems prevented three of the speakers, Michael Hoskin, Jim Bennett and José Vaquero, from attending the conference, but their texts were read, and the complementary slides for each talk were shown during their readings.

The opening talk, "The cosmovision of dolmen builders of south-west Europe" by Michael Hoskin (St Edmund's College, Cambridge) analyzed the astronomy factor in the dolmens orientation in south-west Europe. In Portugal, for instance, all dolmens faced within the range of sunrise or moonrise. José Chabas (Pompeu Fabra University, Barcelona), in "Traditions in Computational Astronomy in the Iberian Peninsula in the late Middle Ages" presented a review of the traditions in mathematical astronomy that had a major impact on Portuguese astronomical activity, with a special emphasis on authors associated with Portugal, including Abraham Zacut and Judah Ben Verga towards the end of the 15th century. "Giovanni Lembo's lessons in S. Antão" was the theme of the talk by Ugo Baldini (Padua University). These lessons (1615-1617) are known for documenting the first knowledge in Portugal of Galileo's telescopic observations and for spreading non-ptolemaic models for planetary motions. Baldini centered his talk in other topics in Lembo's lessons which were unusual either in S. Antão's courses or in the mathematics teaching in other Jesuit colleges around Europe (hydraulic engines, hydrography of the Mediterranean sea, etc) showing that Lembo's lessons conveyed information on some aspects of the "inner" mathematical practice of the Society of Jesus specialists in the major colleges which went far beyond the official teaching programs and the contents of Jesuit mathematical handbooks prior to 1630/40. Carlos Ziller Camenietzki (Rio de Janeiro Federal University) in "Restoration astronomers" discussed the work of some of the Portuguese astronomers after the regaining of independence from Spain in 1640, in particular discussing Guilherme Casmach, Manoel Gomes Galhano Lourosa and António Pimenta. Luís Tirapicos (MCUL) in "Instruments and Astronomical Observations at the Jesuit College of Santo Antão o Novo, 1724-1759" presented a preliminary survey of the instruments used in Santo Antão and characterized the observations in which they were used, putting this data in the context of eighteenth century astronomical observatories in Europe. Fernando Figueiredo (Coimbra University) focused on "Astronomy in the Faculty of Mathematics of Coimbra University after Pombal's Reform (1772-1820)" and in particular on the founding of the Astronomical Observatory of Coimbra University (Figure 2), analysing the work and astronomical research of Monteiro da Rocha, its first director and the main force behind the founding and publication of Coimbra's Astronomical Ephemeris. António Costa Canas (Escola Naval) in "The introduction of the Nautical Almanach in Portugal" analysed the problem of computing longitude at sea and the proposed solutions, using chronometers and lunar distances. The Nautical Almanach had pre-computed values of lunar distances. In his talk, Canas contextualized the introduction of this almanach in Portugal and the role of the Portuguese mathematician and astronomer José Monteiro da Rocha (1734-1819), presenting other Monteiro da Rocha contributions to solve the longitude problem.

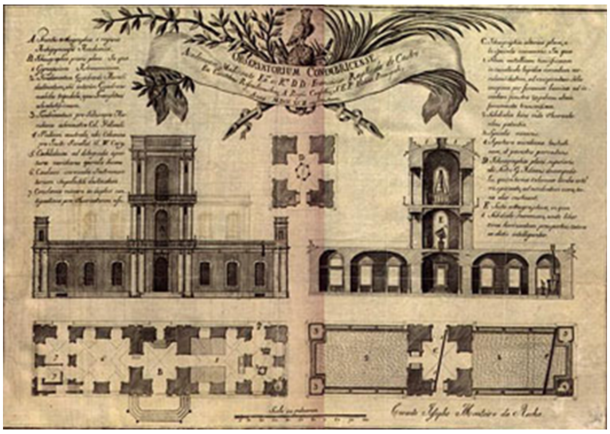

Figure 2: *The Astronomical Observatory of Coimbra (1799-1951) was created by the 1772 Reform of the University and had the mathematician J. Monteiro da Rocha as its first director.*

Pedro Raposo (St. Catherine's College, Oxford) presented "Observatory of Lisbon: the last "big science" undertaking of Classical Astronomy?", an observatory whose cornerstone was laid in 1861. This observatory (Figure 3), which represents and embodies the agenda of stellar astronomy prior to the rise of astrophotography and astrophysics, was strongly modelled on the Pulkovo Observatory in Russia, an observatory which in the first half of the nineteenth represented the foremost astronomical instrumentation and practice. Paulo Crawford (CAAUL) and Ana Simões (CIUHCT) analysed the theme of "Portuguese astronomers and the Principle of Relativity". In their talk they showed that the small network of astronomers of the Astronomical Observatory of Lisbon and those somehow related to them had a positive approach to the theory of relativity, being strongly stimulated by the aspects of that theory which were related to their scientific practice. In particular, by being involved in the founding and development of observatories which were responsible for the time service and legal time they became actively interested in the new concepts of space and time and in the principle of relativity as initially formulated by Einstein.

Figure 3: *The Astronomical Observatory of Lisbon was founded in 1861 at a remarkable site in the capital. (Photo OAL-FCUL)*

Besides these talks, there were interesting presentations by Sérgio Nobre ((UNESP) on "The astronomy presented by Isidore of Seville in his Etymologiae (7th century)"; José Vaquero (Extremadura University) on "Long-term evolution of the sun from Iberian historical documents"; Henrique Leitão (CIUHCT) on "A mathematical and astronomical miracle: the dial of Achaz"; Jim Bennett (Museum of History of Science, Oxford) on "Portugal and the European consensus of eighteenth-century astronomy"; Roberto Martins (UNICAMP) on "The interaction between academic thought and nautical knowledge in Portugal and Spain"; Helmuth Malonek (Aveiro University) and Teresa Costa (Montejunto Secondary School) on "Francisco Miranda da Costa Lobo- a Portuguese astronomer and his attempt to open Portugal to the scientific world"; Isabel Malaquias (Aveiro University) on "Between astronomy and instrumentation: João Jacinto de Magalhães (1722-1790), a remarkable case"; and Vítor Bonifácio (Aveiro University) on "The beginning of Astrophysics in Portugal".

For the record, we state the composition of the Organizing Committee: Luis Saraiva (CMAF/MCUL), Luis Miguel Carolino (MCUL/CIUHCT), António Leal Duarte (CMUC), Marta Lourenço (MCUL/CIUCHT), Samuel Gessner (CIUHCT/MCUL), Vasco Teixeira (MCUL), Paula Gualdrapa (MCUL), Carlos Sá (CMUP). As a final comment, this was a very good meeting, with plenty of stimulating talks and debates. We are looking forward to read the Proceedings of this Meeting, which will be published during the second half of 2010, with the support of CIM.

Coming Events

April, 16-18, 2010: 2nd Porto Meeting on Mathematics for Industry,

Department of Mathematics, University of Porto.

Organizers

Pedro Freitas (UTL/GFM)

Diogo Pinheiro (CEMAPRE/CMUP)

Carla Pinto (ISEP/CMUP)

João Nuno Tavares (CMUP)

José Miguel Urbano (CMUC)

For more information about the event, see

http://cmup.fc.up.pt/cmup/mathindustry/2010/

April, 19-23, 2010: Educational Interfaces between Mathematics and Industry,

Fundação Calouste Gulbenkian and Universidade de Lisboa.

Organizers

José Francisco Rodrigues (Universidade de Lisboa)

Assis Azevedo (Universidade do Minho)

António Fernandes (Instituto Superior Técnico)

Adérito Araújo (Universidade de Coimbra)

For more information about the event, see

http://www.cim.pt/eimi

April, 22-25, 2010: World Congress and School on Universal Logic III,

Estoril, Portugal.

Organizing Committee

Jean-Yves Béziau (Universidade Federal do Ceará)

Carlos Caleiro (Instituto Superior Técnico)

Alexandre Costa-Leite (Universidade de Brasilia)

Katarzyna Gan-Krzywoszynska (Poznan University)

Ricardo Gonçalves (Instituto Superior Técnico)

Paula Gouveia (Instituto Superior Técnico)

Raja Natarajan (Tata Institute, Mumbai)

Jaime Ramos (Instituto Superior Técnico)

João Rasga (Instituto Superior Técnico)

Darko Sarenac (Colorado State University)

For more information about the event, see

http://www.uni-log.org/enter-lisbon.html

June, 15-23, 2010: Summer School and Workshop on Imaging Sciences and Medical Applications,

Universidade de Coimbra.

Summer school: June 15-19, 2010

Workshop: June 21-23, 2010.

Organizers

Isabel Narra Figueiredo (Universidade de Coimbra)

Nuno C. Ferreira (Universidade de Coimbra)

Gil Rito Gonçalves (Universidade de Coimbra)

Pedro C. Martins (Instituto Politécnico de Coimbra)

José Luis E. Santos (Universidade de Coimbra)

For more information about the event, see

http://www.mat.uc.pt/~isma2010/

July, 07-11, 2010: GAeL - Géométrie Algébrique en Liberté,

Universidade de Coimbra.

Organizers

Víctor González Alonso (Univ. Pol. Catalunya)

Nathan Ilten (Freie Universität, Berlin)

Pedro Macias Marques (Universidade de Évora)

Margarida Melo (Universidade de Coimbra)

Kaisa Taipale (University of Minnesota)

Filippo Viviani (Universitá Roma Tre)

For more information about the event, see

http://severian.mit.edu/gael/

July, 9-10, 2010: 8th EUROPT Workshop "Advances in Continuous Optimization",

Aveiro

Organizers

Domingos M Cardoso (Universidade de Aveiro)

Tatiana Tchemisova (Universidade de Aveiro)

Miguel Anjos (University of Waterloo)

Edite Fernandes (Universidade do Minho)

Vicente Novo (Univ. Nac. Educación a Distancia)

Juan Parra (Universidad Miguel Hernández de Elche)

Gerhard-Wilhelm Weber (Middle East Tech. Univ.)

For more information about the event, see

http://www.europt2010.com/

September 26-29, 2010: Raising European Public Awareness in Mathematics,

Óbidos, Portugal

Organizers

Ehrhard Behrends (Freie Universität Berlin)

Nuno Crato (Universidade Técnica de Lisboa)

José Francisco Rodrigues (Universidade de Lisboa)

For updated information on these events, see http://www.cim.pt/?q=events

PEDRO NUNES LECTURES

Pedro Nunes Lectures is a new initiative organized by CIM, in collaboration with SPM (Sociedade Portuguesa de Matemática), with the support of the Fundação Calouste Gulbenkian, for promoting short visits to Portugal of outstanding mathematicians.

Pedro Nunes Lectures are addressed to a wide audience covering broad mathematical interests, particularly PhD students and young researchers. The Pedro Nunes Lectures promote the cooperation between Portuguese universities and are a high level complement of research programs in Mathematics.

Started in July 2009 with the Lectures given by Luis Caffarelli (University of Texas at Austin), the next lectures are scheduled to be given by Michael Atiyah, from the University of Edinburgh, during his visits to the Universities of Minho, Coimbra and Porto, in the period 15 until 24 April 2010.

Professor Sir Michael Atiyah

Sir Michael Atiyah has made fundamental contributions to many areas of mathematics, but especially to topology, geometry and analysis. From his first major contribution – topological K-theory - to his more recent work on quantum field theory, Atiyah has been influential in the development of new theoretical tools and has supplied far-reaching insights. He is a notable collaborator, with his name linked with other outstanding mathematicians through their joint research, such as R. Bott, F. Hirzebruch and I. Singer, or some of his notable students, like G. Singer, N. Hitchin and S. Donaldson.
He was awarded a Fields Medal in 1966, the Copley Medal in 1988 and the Abel Prize in 2004. Atyiah has been President of the Royal Society and Master of Trinity College, Cambridge.
He was the first Director of the Newton Institute at the University of Cambridge and has been the recipient of many honours and awards, including in the UK a knighthood in 1983 and the Order of Merit in 1992.

Titles of the lectures

- *An unsolved problem in elementary Euclidean geometry*, University of Minho, April 16;
- *The index theory of Fredholm operators*, University of Coimbra, April 19;
- *Topology and quantum physics*, University of Porto, April 22.

For more information consult http://www.cim.pt/?q=glocos-pedronunes.

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