EIMI 2010

Conference

EDUCATIONAL INTERFACES between MATHEMATICS and INDUSTRY

Proceedings

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Preface

This book gathers a collection of refereed articles containing contributions presented at the EIMI (Educational Interfaces between Mathematics and Industry) Conference held in Lisbon, Portugal, from 19 to 23 of April 2010. They reflect the contribution and the response of the international scientific and educational communities to the challenge proposed by the International Commission on Mathematical Instruction (ICMI) and the International Council for Industrial and Applied Mathematics (ICIAM), when they have jointly launched the EIMI Study in July 2008, the 20th of the ICMI Studies Series.

This conference which was held in Portugal as a tribute to an initial suggestion of its National Committee of Mathematicians, corresponds to the first part of the Study, as described in its Discussion Document, and its Proceedings intend to be a contribution to better understand the connections between innovation, industry, education and mathematics and to offer ideas and suggestions on how education and training in these areas can contribute to enhancing both individual and societal developments. The second part will correspond to the Study volume which is expected to appear in 2011, as a result of the further works of the Conference, of the invited speakers and the Working Groups.

The EIMI Conference, in addition to the contributions talks corresponding to these papers, consisted of six plenary lectures by Thomas A. Grandine, (Applied Mathematics, The Boeing Company, Seattle, USA), Celia Hoyles (Institute of Education, University of London, UK), Arvind Gupta (MITACS, University of British Columbia, Canada), Masato Wakayama (Faculty of Mathematics, Kyushu University, Japan), Helmut Neunzert (Fraunhofer ITWM, University of Kaiserslautern, Germany) and Henk van der Kooij (The Freudenthal Institute, Utrecht University, The Netherlands) and six Working Groups, covering the following topics: The mathematics-industry interface; Technology issues in mathematics and in education; Mathematics-Industry communication; Education in schools; University & academic technical/vocational education and Education/training with industry participation.

The Scientific Committee of the conference coincided with the International Programme Committee of the EIMI Study, that was composed by Alain Damlamian (France, co-chair), Rudolf Strässer (Germany, co-chair), José Francisco Rodrigues (Portugal, host country), Helmer Aslaksen (Singapore), Gail Fitzsimons (Australia), José Gambi (Spain), Solomon Garfunkel (USA), Alejandro Jofré (Chile), Gabriele Kaiser (Germany), Henk van der Kooij (Netherlands), Li Ta-tsien (China), Brigitte Lutz-Westphal (Germany), Taketomo Mitsui (Japan), Nilima Nigam (Canada), Fadil Santosa (USA), Bernard Hodgson (*ex-officio*, ICMI), Rolf Jeltsch (*ex-officio*, ICIAM).

The Editors, as members of the Local Organizing Committee of the EIMI Conference, wish to acknowledge the ICMI and the ICIAM for the promotion of this important and timely Study and the thank the financial support of the sponsors of the Conference, namely the Fundação Calouste Gulbenkain, the Fundação para a Ciência e a Tecnolologia, the Universidade de Lisboa and the Centro Internacional de Matemática, that hosted its organization. A special acknowledgement is due to COMAP (the Consortium for Mathematics and Its Applications) for taking in charge the publication of these Proceedings.

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Mathematical modelling in making linkages or mechanics: Using LEGO located on elementary mechatronics tools

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Abstract In old authorized textbooks "Second Categories in Mathematics" of Japan, tools or machines that were familiar to students in those days were taken up as materials, and the problems about those motions or structures were intended to be solved by mathematically and scientifically approaches. To make linkages or mechanism with using LEGO can be one of the modern approaches of mathematical modelling. I will introduce example of teaching practices of mathematical modelling for upper secondary students, and that aims are to reproduce or actualize various mechanisms. Students could learn mathematical modelling through finding properties of mathematics from each model or making models reflected properties of mathematics. We can confirm those from LEGO models that are reflected students' ideas.

Introduction

I have developed mathematical modelling materials and instructed with using LEGO located on elementary mechatronics tools (Isoda & Matsuzaki, 1999, 2003; Matsuzaki & Isoda, 1999a, 1999b; Matsuzaki, 2006, 2007). We can see properties of mathematics from motions of tools or machines (Ito, 1983; Mankiewicz, 2000). As one of the studies on modern handling of tools or machines in mathematics education (*e.g.* Isoda & Mariolina, 2009), there is a web site "History Museum of Mathematics" produced by Isoda. Contents about studies of mechanism with using LEGO in this web site are updated as web pages "Various Mechanism and Mathematics Experiments" from viewpoint of integration between mathematics and the other subject. In this report I will introduce teaching practices of mathematical modelling in making of linkages or mechanism with using LEGO for upper secondary students.

Mathematics and linkages or mechanics shown in old authorized textbooks of Japan

During World War II, old authorized textbooks "*First and Second Categories in Mathematics*" of Japan take up integrated contents between mathematics and science. Tools or machines which one of concrete contents includes familiar to students in those days are taken up as a problem. Especially there are same called section "motions of machines" in both textbooks; at "Chapter 1. Trace" in "*Second Categories in Mathematics 3"* and at "Chapter 2. Power and Motions" in "*Second Categories in Mathematics 5"*. In these textbook, same materials are taken up depending on learning contents by spiral, and tasks of making models or considering with using models are included in some of problems. For example, problems that took up four sections mechanism are the following figures 1 and 2.

Mathematical modelling materials and teaching practices examples with using LEGO

Components constructed mechanism are called link that provide necessary motions for machines. In the case of LEGO, various parts of block are equivalent to link, as well as block parts that are basic components, there are also gears and sticks, joints etc...(Fig.3), and these are able to reproduce or actualize various mechanism. Figure 4 is reproduction of four sections mechanism by LEGO that took up in a previous page (see Fig.1 & Fig.2). As an advantage of reproducing mechanism with using LEGO, we can confirm motions of each point by moving mechanism. In addition, by exchanging or changing blocks or joints, we can understand formation conditions of linkages or mechanism easily. The point where we should pay attention to when we use LEGO as mathematical modelling materials is works themselves with using LEGO become some models in mathematical modelling (Matsuzaki,

問 2. 下ノ圖ノ装置デ，D ヲ中心ニシテ C ヲ廻スト，B ハドンナ運動ヲスルカ。模型ヲ作ツテコレヲシラベヨ。マタ，圖ヲ書イテシラベヨ。

Problem 2. What kind of motions does B carry out when turn C around D in the device of the lower figure? Research its motions with making a model. In addition, draw figures and inquire its motions.

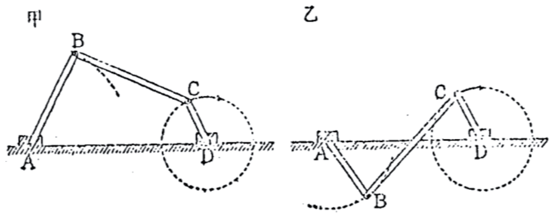
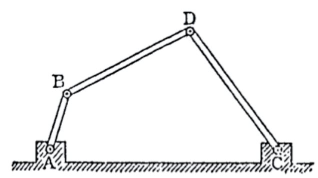


Figure 1—"Chapter 2. Observation of figures §7: Rotaly motion" in "*Second Categories in Mathematics 1*" (p.57)

問 2. 右ノ圖デ，B ガ A ノ周リヲ回轉スルト，D ハドンナ運動ヲスルカ。B ガ一様ナ速サデ回轉スルトキ，D ノ速サハドノ邊デ速ク，ドノ邊デ遅イカ。マタ逆ニ，D ヲ動カシテ B ニ回轉運動ヲサセルコトガデキルカ。



Problem2. When B turns around A, what kind of motion does D carry out in a right figure? When B turns with a same speed, which side is the speed of D fast or late? In addition, can you let move D adversely, and B do rotary motion?

Figure 2—"Chapter 1. Trace §1: Motions of Machines" in "*Second Categories in Mathematics 3*"(p.1)

2006). In other words, we apply characteristics of LEGO that can make or modify models easily, and could propose a new mathematical modelling instruction.

At Next, I will introduce examples of teaching practices of mathematical modelling for upper secondary students, and that aims are to reproduce or actualize various mechanism. Students could learn mathematical modelling through finding properties of mathematics from each model or making models reflected properties of mathematics. Special lessons for students are planned, and these lessons are applied characteristic of each subject or specialty of each teacher. The period of these lessons is two years from the 11th grade to the 12th grade, and the lessons for the 11th grade students are called 'seminar' and the lessons for 12th grade students are called 'theme study'. These lessons for two grades connect as a rule, and in 'theme study' students have to settle the matter learned in 'seminar'. In addition, the 12th grade students have to study in details as a graduation paper.

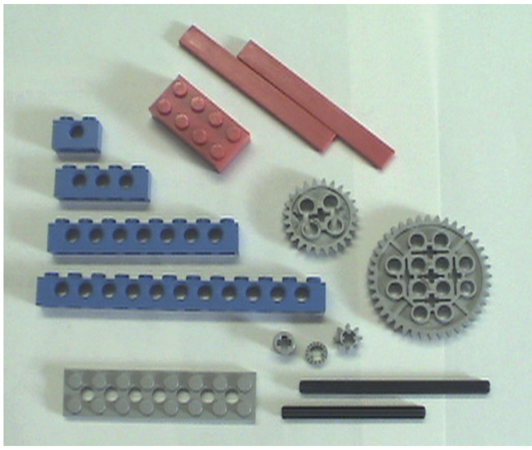


Figure 3—Parts of LEGO

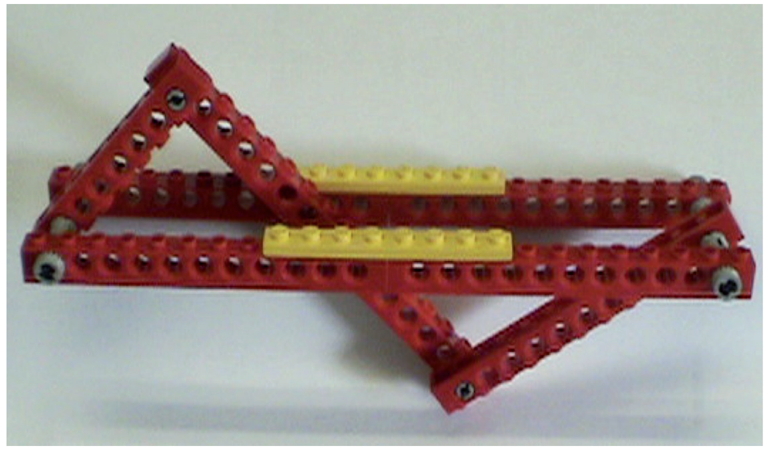


Figure 4—Four sections mechanism by LEGO

"Seminar" for the 11th Grade Students

There were seven times lessons (total 18hours) from June 2006 to February 2007. I planned the theme "the world come true by LEGO and the world of mathematics" as mathematics "seminar" and 26 students selected. In these lessons two mathematics teachers instructed by team teaching.

At the first period, teachers explained the fill-in of "seminar" and machines or mechanics. So teachers proposed reproducing "the vehicle in an amusement park" as one of tools or machines familiar to us with using LEGO. At that time, teachers introduced the following web sites as references; "Pages of LEGO" within "Mathematics History Museum" that is one of projects of Center for Research on International Cooperation Education Development (CRICED) by Isoda in University of Tsukuba (Fig.5) and Kinematic Models for Design Digital Library (KMODDL) in Cornel University that display various collections of mechanics (Fig.6).

At the second period, students really tackled reproduction or actualization of tools or machines which each student chose with using parts of LEGO. At first, each student decided a tool or a machine to reproduce or actualize, and considered mechanism to hide behind in a tool and a machine with referring to books and web sites. And next teachers let students illustrate linkage of mechanism while exemplifying lazy tongs which a certain student produced.

At the third period, students continuously tackled reproduction or actualization of tools or machines, and teachers suggested students to focus on mechanism. So teachers instructed two directions for inquiring; first one is to research about mechanism to hide behind in a tool or a machine, and second one is to apply mechanism and to create a new tool or ma-

Figure 5—"Pages of LEGO"

Figure 6—"KMODDL"

Figure 7—Products by students on web page

chine. Students presented progress report about the past activity and future development to another students or groups.

At the fourth period, students looked back on past activities, and each student or each group confirmed problem setting and policies of activities to be jointed to ''theme study'' in the next grade.

"Theme Study" for the 12th Grade Students

24 students selected "theme study" from April 2007 continuously. "Theme study" is different from "seminar", and teachers have to indicate each student, because each student set the theme according to their own interests. Method to indicate is teachers read students' papers and tell the results, and students revise their papers. This process is repeated several times like referee process of academic journal. Now completed students' papers make up into a booklet and most of them are shown on the web within CRICED (Fig.7).

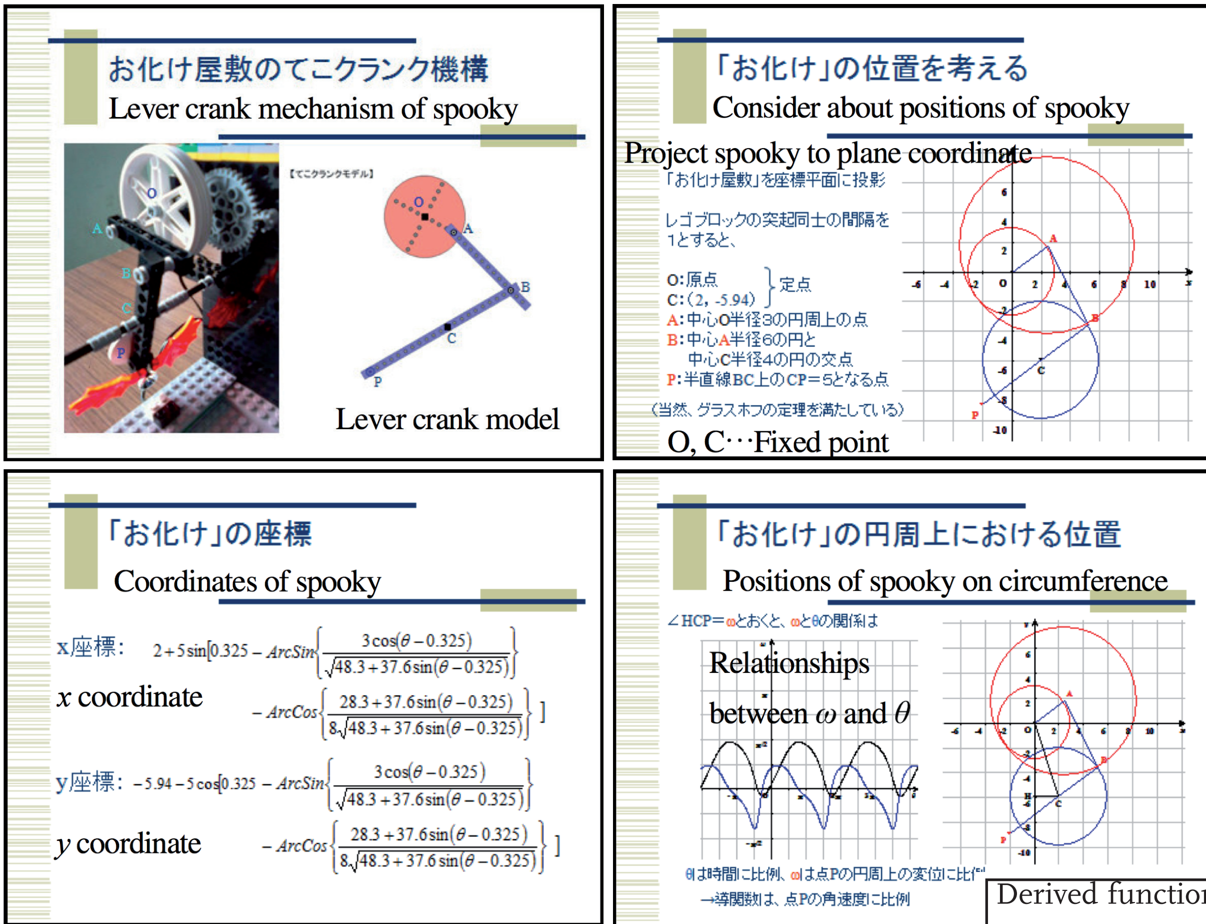


Figure 8—“Linkage and its functions of spooky house: lever crank mechanism” by KK

On September 2007, some of students presented about their study findings. The following presentation slides in Fig.8 & Fig.10 are parts of presentations.

KK reproduced the situation of spooky that hide in a well appear with LEGO, and analysis the motion mathematically (see also Fig.9). He imaged motion of spooky as “fly out with drawing an arc” and tried to actualize motion by converting rotary motion of motor into reciprocating motion of a sector. So he focused on lever crank mechanism as a mechanism which actualized such motion. And he reflected LEGO model on plane coordinates to analyze the motion mathematically. At that time he set standard interval between the holes of parts of LEGO to decide the unit of axis of coordinates. As a result, the motion of spooky was not drawn circle and was equal to his image that “fly out with drawing an arc”. He expressed coordinates of spooky that is a motion point expressed in angle of rotation and drew graphs of trace of motion point and its derived function, and he pointed out these relationships.

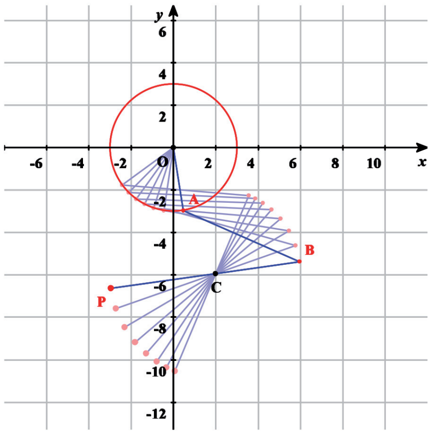


Figure 9—Motion of spooky

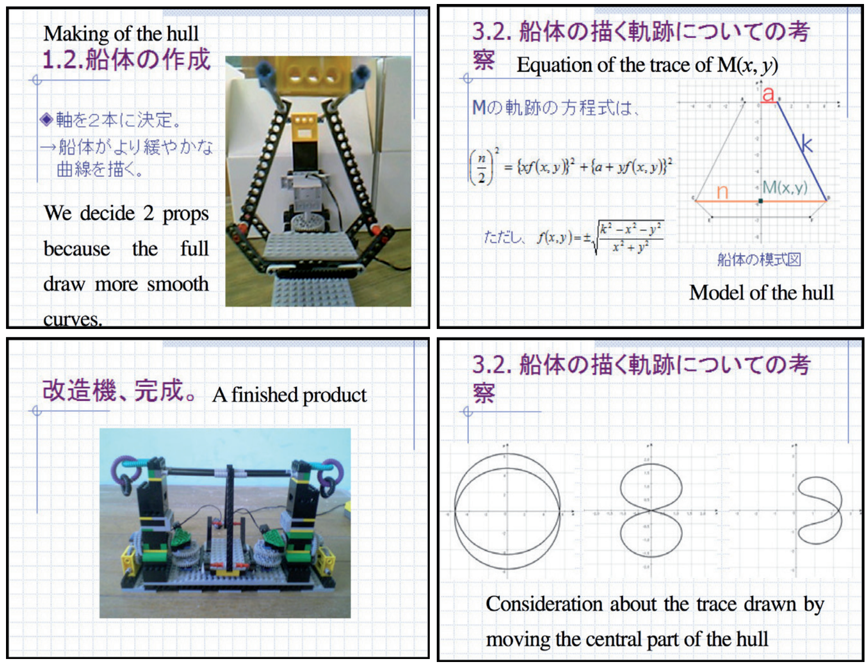


Figure 10—“Mathematical analysis of Viking: from manufacturing to mathematics” by KT & SH

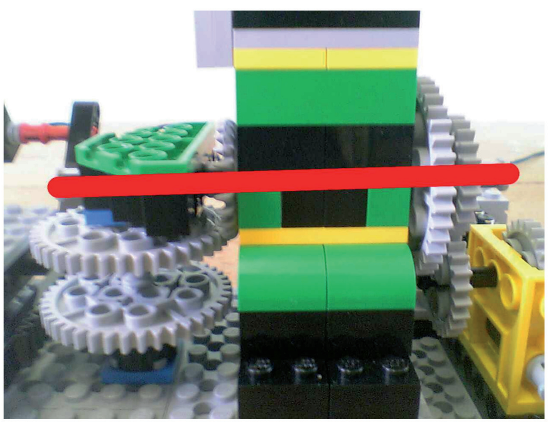


Figure 11:—Setting a shaft with shade gears

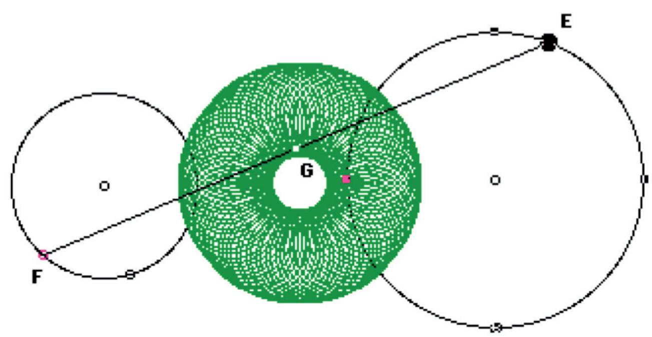


Figure 12—A trace of the middle point of freedom point

KT & SH worked collaborate to reproduce of viking. They improved models several times to actualize the smooth motion of the hull with maintaining the strength of the viking. So they used shade gear or worm gear (Fig.11).

And they expressed a trace of the central part of the hull as an equation and chased the motions of the hull as a trace. On this situation they can get various traces by changing parameters. They also found that a trace of the middle point of the freedom point on the circumference of two circles became in the shape of a doughnut (Fig.12). The equations of the trace are added a condition that 'central part of the hull always took fixed number *n*' as this. Finally they generalized a equation of the trace.

Conclusion

In old authorized textbooks "*First and Second Categories in Mathematics*" of Japan, tools or machines that were familiar to students in those days were taken up as materials, and problems about those motions or structures were intended to be solved by mathematically and scientifically approaches. The characteristics of those problems include tasks to make models of tools or machines, and consider with using those models that students own made. To make linkages or mechanism with using LEGO located on elementary mechatronics tools can be one of the modern approaches of mathematical modelling.

In this paper, I took up the theme "the world come true in LEGO and the world of mathematics" by team teaching of mathematics teachers and introduced teaching practices and students' activities in "seminar" for the 11th grade students and "theme study" for the 12th grade students. The problem is to reproduce or actualize "the vehicle in an amusement park" that were one of tools or machines that were familiar to students. It is difficult to actualize them precisely with using LEGO because limit of parts or strength problems, and it is possible to reproduce or actualize parts of structures or models of them. At that time, characteristics of models are reflected students' ideas and we can confirm those from LEGO models. For example, KK applied lever crank mechanism and actualized motion of spooky based on his image, and KT & SH tried to modify devices of the hull at several times. Then students referred various web sites about linkages or mechanisms. Furthermore students analyzed LEGO model mathematically. For example, KK set standard interval between holes of parts of LEGO and reflected LEGO model on plane coordinates. KT & SH considered patterns of the trace of the hull based on the combination of parts of LEGO in process to lead an equation of the trace.

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PAGES OF LEGO' RETRIEVED 15 OCTOBER 2009 FROM THE WORLD WIDE WEB:

http://math-info.criced.tsukuba.ac.jp/museum/lego/lego.html

'THE WORLD COME TRUE BY LEGO AND THE WORLD OF MATHEMATICS' RETRIEVED 6 MAY 2008 FROM THE WORLD WIDE WEB: http://math-info.criced.tsukuba.ac.jp/museum/kikou/lego-komaba07

Mathematics for Engineering and Engineering for Mathematics

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Presentation

The Spanish Committee for Mathematics (CEMat) and its Committee for Education celebrates the launching of the International ICMI Study 20 devoted to Educational Interfaces between Mathematics and Industry as a joint collaboration of the International Commission for Mathematical Instruction (ICMI) http://www.mathunion.org/ICMI/ together with the International Council for Industrial an Applied Mathematics (ICIAM) http://www.iciam.org/.

The Committee for Education of the CEMat, that plays the role of ICMI-Spain, has studied and valued positively the convenience of contributing to the 2010 EIMI Conference (which will be held in Portugal) by a report about the current situation and perspectives of the educational relations and connections between mathematics and industry in Spain.

ICMI-Spain has realized that there are several activities that are being developed in this field in Spain, both at university and secondary education levels. Some Spanish universities and research centres are active members of the European Consortium for Mathematics and Industry (ECMI) http://www.ecmi-indmath.org/ and some of their directors are members of CEMat and its Committee for Education. There are instructors and innovation teams in secondary education involved in the development of studies about the relations between industrial activities and mathematics in secondary education. The "Dirección General de Formación Profesional" of the "Secretaria de Estado de Educación y Formación" of the Ministry of Education (MEC) http://www.mepsyd.es/mecd/jsp/plantilla.jsp?id=144&area=organigrama has expressed its will to deepen into this field from the point of view of a professional education reform.

According to this approach, the Committee for Education of the CEMat has promoted the presentation of a joint work addressing some of the issues raised in the Discussion Document of the EIMI Study.

1. Introduction

The present contribution brings together three works that consider, at secondary and university levels, the educational relationships between engineering, in a very broad sense, and mathematics. Such works take as its starting reference some experiences developed in Spain.

The common framework of the contribution is exploring (from the formative viewpoint) the two-way road between mathematics and engineering. On one way, it is interesting to determine and include in the secondary education and university syllabuses the math that play a role in the real world when it comes to problem solving and applications. The other way is related to value the contribution to mathematical education of the methods, techniques, procedures and experiences from the real world and engineering.

The works we mentioned above are arranged as follows. The first work is devoted to explore certain questions related to the previous issues in the context of secondary education. This work has been developed by instructor Miquel Albertí. The second work (due to Profs. Amat and Busquier) analyzes, in a particular case, what type of math is more suitable to be taught to engineers. The third and last one deals with the development in several Spanish formative proposals of the concept of Mathematical Engineer, work developed by Profs. Romero and Tejada (coordinator).

2. Mathematics and "Engineering" in secondary education

During the last year a research concerning mathematical activity at work contexts has been developed in the town of Sabadell (Catalonia, Spain). As a result of this study we have reached several conclusions related to certain statements presented in the ICMI-ICIAM's *Discussion Document* of the *International Study on Mathematics and Industry*. From our viewpoint, these conclusions are basic for a didactic approach of the problem in the secondary school. The main questions are the following.

First, it has been always said that Mathematics are used almost everywhere, but the contexts of its use and the way it is used may have changed through the last years. What several decades ago had to be solved with paper and pencil was solved later with calculator and it can be solved now with computer. Software often appears incorporated to machinery. Which kind of Mathematics is used today at work?

Second, the presence of Mathematics in real life contexts has usually been seen, by professional mathematicians as well as by mathematics educators, as applications of Mathematics. However, it's not often like this. History of Mathematics shows that before becoming academic, many elements of Mathematics were grounded on practical problems. Axioms, theorems and proofs are not the beginning, but the final stage of a process through which academic mathematics are settled. The use of Mathematics at work, is limited to applications of academic Mathematics? Do workers solve a problem in the same way than it does a mathematics educator?

Third, almost all mathematics educators come from an academic context and never have worked in another job different from teaching. There are three reasons for which practice work mathematical aspects should be considered in a truly competence mathematics education. First, educate citizens to be mathematically competent, and not only at work, but culturally. Second, to prepare students for the work world anticipating at class some of the activities they will have to face once finished the secondary school. And third, make clear the relationship between Mathematics and social and cultural environments. Then a question arises: How can a teacher promote mathematical competency coherent with a non-academic work or industrial environments if his or her mathematical education was developed outside these environments?

We suggest solutions to these three points of interest considering both teacher and student interests from an educational point of view. Teacher need to educate people mathematically competent. This means real life competency. To achieve this goal they need to put their students in real life situations of which work situations are a part of utmost importance. Students must learn mathematics and realize that the mathematical knowledge they have acquired is really useful to answer or to create answers to the problems they will face in their lives, also as workers, after secondary education.

Concerning to secondary education, the approach could be determined by three main questions related to the role to be developed by the different educational stages towards Industry:

(a) Industry and secondary education curriculum

Which are nowadays the general and actual frameworks between Education and Industry? What is studied concerning to Industry at the secondary school? Which relation or presence has Industry in the mathematics curriculum? Which mathematics is involved in those topics related to Industry that are studied in another matters?

(b) Mathematics for Industry

To know directly after interviews to workers and industrial business men which mathematics is used and/or needed at industrial field. Such a field is so huge that research should be limited to the most outstanding industrial sectors in the aim to precise a fundamental mathematical kernel.

(c) Industry and competency education

Which mathematical aspects and skills should be studied deeply at secondary school in the benefit of people's competency? It's desirable that someone who has finished secondary education can face his or her incorporation to the industrial work field, vocational training or university technical studies.

The following are our considerations concerning to the six focus of attention of the EIMI-ICMI project, in the context of secondary school:

Starting point is the identification, development and valorisation of a curriculum including innovative applications of mathematics and highlighting industrial mathematical problems. But then the point is: Which are the 'industrial mathematical problems'?

To specify an adequate level for workers means an agreement between educators and professionals. It can be achieved through a mathematical education by competences. After passing secondary school stage one should be mathematically competent to face a professional training industrially oriented. But, do they the teacher and the professional share the same idea of competency? Maybe they complement each other? Anyhow, it's essential to know differences and intersections to know which ones must enter into mathematical secondary education.

Student activities at secondary school should fit three aspects. First, they should not be isolated from reality. Second, extra academic activities for students should be addressed towards the knowledge of industrial world, visiting *in situ* some industrial enterprises according to their relevance, availability or spatial proximity. Third, education authorities should promote, as it's the case of Catalonia (Spain) the realization of work practices out of school in industrial enterprises.

But not only students must be the focus. Teachers must know what happens outside of their academic context. Teacher training by non academic professionals can lead to mathematical developing, both in its knowledge and educative aspects.

This work is concerned with secondary school perspective. Industrial view should be obtained interviewing industrial professional and business men.

Industrial Sectors for this EIMI-Secondary School Study

Let us start with the identification of the Industrial Sectors for this EIMI-Secondary School Study to know which Mathematics is used. An Spanish ICMI study should be focused on Catalonia, Madrid, Valencia and Basque Country, Main Spanish industrial branches are found in Catalonia, Madrid and Valencia, and then at the Basque Country and Andalusia. Catalonia-Madrid becomes the fundamental axis of Spanish industry ahead of the transversal axis Valencia-País Vasco, but given the actual accessibility of the author, and concerning to secondary school, these work had to be developed around Catalonia. Mercè Sala study in 2000 was intended to know which the most significant branches of Catalonian Industry. She took three indicators: occupancy, gross added value y number of establishments. Her work shows that importance of Industry is due to its 28% of the Catalonia's PIBpm as well as of its occupancy. Also, 'Catalonia creates more than the 25% of the VABpm and almost a 24% of the Spanish industrial occupancy' (Sala, 2000: 4).

Considering as more productive branches those leading the classification according to the three former indicators and that altogether give more than the 70% of employment, added value and number of establishments, we get a list of the main industrial sectors:

i) Textile industry
ii) Engineering industry
iii) Food, drink and tobacco industries
iv) Chemical industry
v) Paper, edition, graphic arts and reprography industries
vi) Machinery and mechanical equipment
vii) Car industry
viii) Electric machinery and implements

According to its technologic complexity, Sala (op. Cit.: 10) classifies these eight industrial sectors in the following way (table 1):

Technologic complexity	Branch
Very high	Electric machinery and implements
High	Chemical industry
	Machinery and mechanical equipment
	Car industry
Meddle	Engineering industry
Low	Textile industry
	Food, drink and tobacco industries
	Paper, edition, graphic arts and reprography industries

Table 1—Outstanding industrial sectors and technology.

Taken into account that the study concerns secondary school, it should be appropriated only to consider industrial sectors for which no very specific instruction is needed. This means to reduce the eight former sectors with high or very high technologic complexity. Doing it that way, only four are left: Textile industry; Engineering industry; Food, drink and tobacco industries; and Paper, edition, graphic arts and reprography industries

Unfortunately, contemporary crisis made difficult to contact and to get response from enterprises belonging to these sectors. Due to this reason the scope of the study was limited to different kind of jobs, some outside industrial field. But anyway, and given the elementary education level as it's the secondary school, the reader will see that the results and conclusions we show here are easily applicable to the industrial field.

Industrial Culture, Education and Mathematics

Several characteristics of the industrial field are absent in secondary education and in mathematics education as well: competitiveness, production and valorisation method of results.

Lack of competitiveness, often existing between pupils, is usually vanished by teachers and educators, probably as a consequence of the traditional lack of it between public education professionals. Nevertheless, some kind of competitiveness between educators is fomented now by education authorities.

In secondary education very rarely a product is elaborated, and when it's the case, it usually happens outside the mathematical context. It's so in academic activities belonging to the technologic field. But although mathematics can be used in the elaboration process, they do not receive the proper attention. Industrial productions are almost inexistent as there is not available technology to produce them.

Another critical aspect to be considered at work is that, be it or not industrial, it's not enough to get a 5 to pass the exam, you must get a 10. Otherwise, the product is not correct. If it is not perfect, then it's wrong. Moreover, the result is not only valued by the chief, but by lots of customers. Hence, there is an external and social valorisation totally absent in secondary school that transcends teacher's evaluation. In academic education contexts, pupils work rarely transcends teacher or professor evaluations.

Here we find a reason for introducing and intensify the academic weight of realistic and productive activities to be developed individually or in group, in collaboration and cooperation, the aim of which transcends teacher numerical qualification. Education in search of the 10, and not only the teacher's one, but also the student's and social environment's one. This means to introduce a bit of *industrial culture* into the secondary school.

Mathematics at Work

To know which Mathematics is used and how it's used at work it has been elaborated a *Guide of Work Mathematical Activity* (Albertí, 2009, pp. 100-102). Given the wide extension of the context, the study was limited to actually accessible jobs for a people after having finished secondary school in the town of Sabadell (Catalonia, Spain). This guide relates mathematical activities developed at work with the five content groups of the Spanish secondary school curriculum. All these work mathematical activities were identified after interpellation to professional workers of every job. It has been confirmed, according to the ICMI document, that workers still think that only something truly hard deserves to be considered as mathematical. For instance, workers do not see as mathematics the calculation of the area of a room or calculated estimations of the measure of a magnitude.

Jobs studied came from different labour sectors. But all of them were closely related to Industry in one of two senses. The first, because the job belonged to an Industrial sector, as it was the case car industry. The second, because nowadays most of the tasks are delegated to special technology and such a technology usually comes from Industry.

As mentioned above, the main mathematical activities at each of these works were related to the five contents of secondary education: (i) Numeration and calculation; (ii) Change and relations; (iii) Space and shape; (iv) Measure; and (v) Chance and statistics.

After all we got a series of mathematical activities rooted on work contexts. Five activities shared by almost every job were identified as UWA (universal work activities), where often one find mathematical activity:

UWA1: To follow hygienic and security rules at work.
UWA2: To follow chief work orders and instructions.
UWA3: Budget estimations, stocktaking and cashing up.
UWA4: Label, map and operating manual interpretations (including software).
UWA5: Creation and interpretation of graphics and maps.

Teaching-learning capabilities were checked to bring them to academic contexts in the aim that students experiment at classroom several of the real world mathematical activities they will need to overcome once finished their secondary school education.

Not only applications of Mathematics

Are Mathematics actually applied at practice? If so, do its applications agree with those applications referred by mathematics educators in the classroom? In this sense, we have found several discrepancies.

In *Building* right angles are traced on the basis of the so called by workers as 'Egyptian triangle', i.e., a triangle of sides 3m, 4m and 5m or a smaller proportional one of sides 30cm, 40cm and 50cm. Workers know that such a figure produce a right angle, although they don't know a proof of it. In fact, they don't need to know such a proof. Nevertheless, Mathematics educators not only should know this proof, but the proof should belong to the mathematical education they offer to their students. Pythagorean theorem is made of two implications. The first one is the most popular and usually presented in secondary education classrooms in Spain. But people working in building make right angles thanks to the other implication, which never is proved, studied or showed in the classroom.

In *Multimedia Design Production* is interesting to save money inserting the maximum number of visiting cards (just to put an example) into one sheet of paper. Experts do the task experimentally, but it could be done easily preparing an EXCEL page to determine their maximum possible number and the way they should be placed into the sheet of paper. So we did it. It was a spatial optimization problem solved with EXCEL which solution was truly welcomed by the professional.

An important geometric problem in *Agriculture* as well as for *Gardening* is to design the shape of a trees plantation. This means to create a grid which intersection points determine the places were trees will be planted. A type of these is the so-called 'tresbolillo' plantation in Spain. When the interviewed gardener explained his system he referred to his triangular grid as equilateral although it was actually made of isosceles triangles.

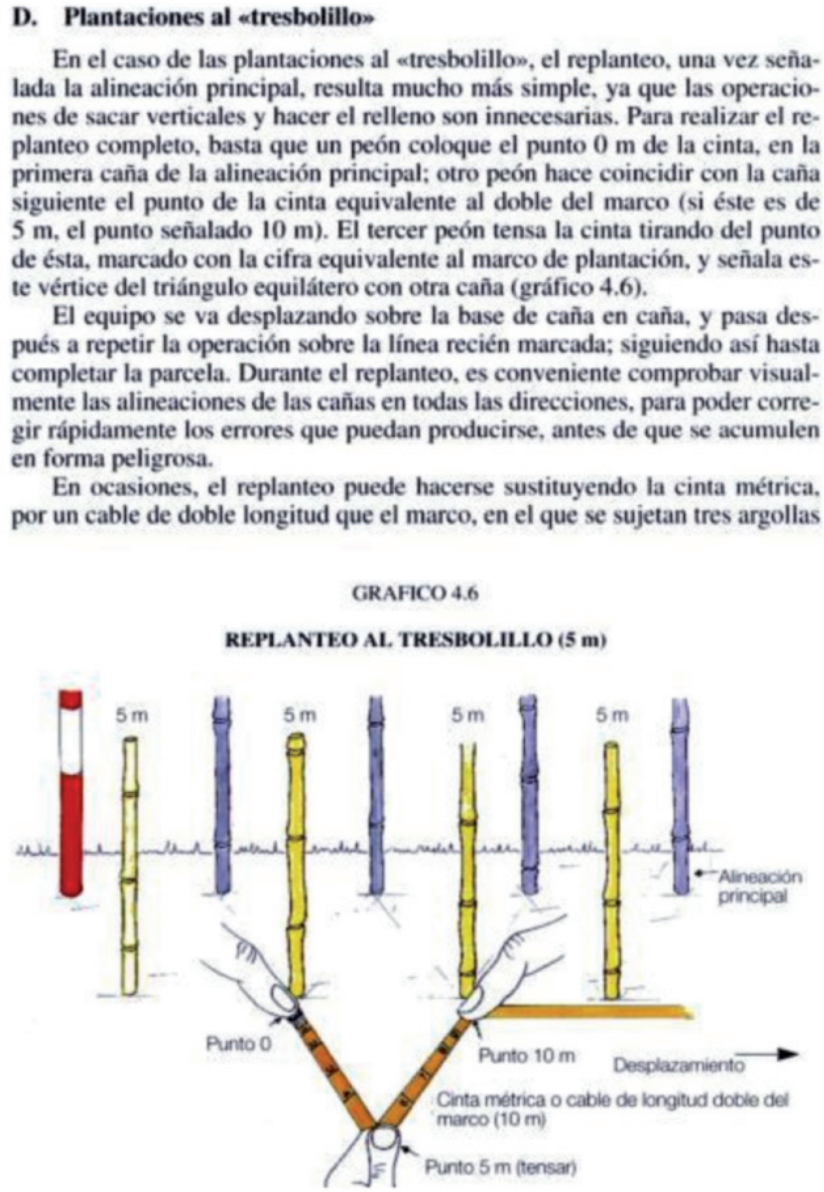
The researcher, as mathematician and mathematics educator, asked himself how would he do or explain to his students the construction of an equilateral triangular grid. He thought that such a construction should be practical because it had to be done on the land, not on a sheet of paper. He considered the Euclidean solution for the construction of an equilateral triangle could fit the requirements if put into practice with long enough strings. However, reality didn't go that way. At practice, the Euclidean solution is not used to build such grids, but another one describe by Gil-Albert (1999, p. 84) where the equilateral triangle is obtained as a limit case of an isosceles one (fig. 1).

Car industry workers have to know how to interpret graphics. But often they need to do in a way not usually studied in academic contexts. Academic graphic interpretations too often look at the function involve too closely, the teacher asking for images and anti images, growing and decaying intervals. But in reality, sometimes, these are not the most relevant factors. This is the case for the following one, where colours play an important role to show that it's possible to make a new car more respectful with the environment along improving the former power response (fig. 2).

Hence, the mathematician and the mathematical educator must be careful and work together with professionals when developing mathematical models to fit real practice. Let's not forget that theoretic solutions to practical problems often are not the best practical solutions of the problem.

Work and Industry: a source of real mathematical activities

Math activities from work contexts are real, practical and contextualized problems where teachers can situate students. Sometimes, work math activity consists in an application of



D. Plantaciones al «tresbolillo»

En el caso de las plantaciones al «tresbolillo», el replanteo, una vez señalada la alineación principal, resulta mucho más simple, ya que las operaciones de sacar verticales y hacer el relleno son innecesarias. Para realizar el replanteo completo, basta que un peón coloque el punto 0 m de la cinta, en la primera caña de la alineación principal; otro peón hace coincidir con la caña siguiente el punto de la cinta equivalente al doble del marco (si éste es de 5 m, el punto señalado 10 m). El tercer peón tensa la cinta tirando del punto de ésta, marcado con la cifra equivalente al marco de plantación, y señala este vértice del triángulo equilátero con otra caña (gráfico 4.6).

El equipo se va desplazando sobre la base de caña en caña, y pasa después a repetir la operación sobre la línea recién marcada; siguiendo así hasta completar la parcela. Durante el replanteo, es conveniente comprobar visualmente las alineaciones de las cañas en todas las direcciones, para poder corregir rápidamente los errores que puedan producirse, antes de que se acumulen en forma peligrosa.

En ocasiones, el replanteo puede hacerse sustituyendo la cinta métrica, por un cable de doble longitud que el marco, en el que se sujetan tres argollas

GRAFICO 4.6

REPLANTEO AL TRESBOLILLO (5 m)

Figure 1—Equilateral triangle as an isosceles one.

academic math knowledge; while others, becomes a melting pot to math creation. But anyway, both are authentic situations not invented nor supposedly real according to teacher's imagination. Demidovich (1976: 91) contextualizes in Building a popular problem of constructing an optimal fence from a wall. *Pisa 2003 Study* (2005) declares as work context its problem 22, consisting in calculating how many different pizzas can be prepared combining the available ingredients. During our research we had the opportunity to ask about these questions to workers. No one answered affirmatively, pizza makers looked at us astonishingly.

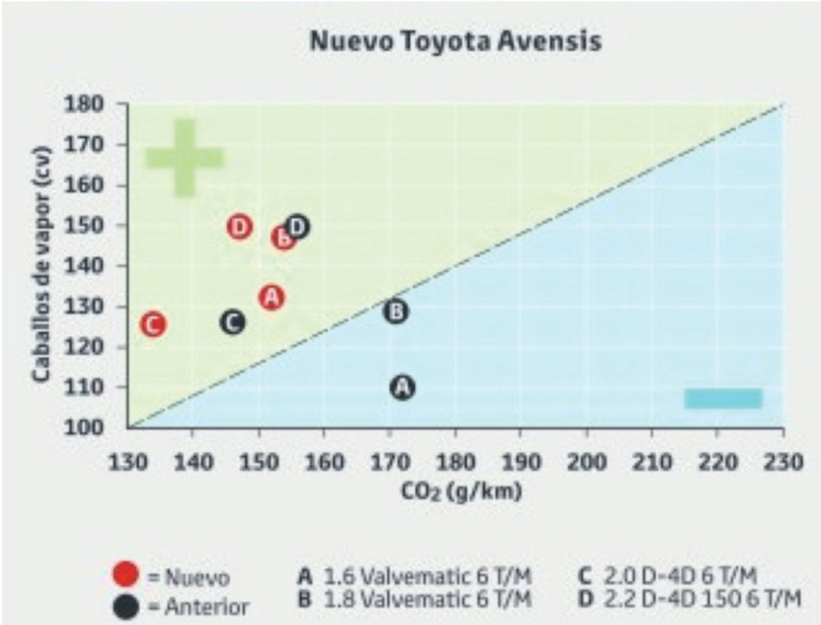


Figure 2—A new car good for us and good for the environment.

Software as EXCEL is so much used today that its academic treatment should be reinforced and expanded. Pupils should not learn to use it only to pass exams or to get teacher's approval, but to realize that it will be a good help to situate them in life after school.

Mathematical knowledge learned after work mathematical activities has the meaning that often student demand or cannot find in other academic mathematical practices. Real contexts should play a fundamental role in mathematical education. If we see Industry not as part of a taken for granted larger label as Work, but as its generalization, we'll see how great role can play in secondary education.

Work Mathematical competence versus Cultural Math Competency

If Mathematics educators pretend to offer an authentic competency education coherent with contemporary social and cultural environments, they should know which Mathematics are used in the world they live in to prevent realizing mathematical projections over it. Introducing at classroom some mathematical activities from the work world no only favours interaction between Work or Industry and Mathematics, but reveals the presence of Mathematics in real daily life. It also helps students' motivation, as pupils will find in such activities a sense and a meaning rarely found in other classroom mathematical tasks.

We want to mention a decisive event in our research planning. It happened with two students with a high lack of motivation towards Mathematics and towards their general learning process. They were addressed to follow special sessions in a specialized professional training centre one day of the week. They were asked to write down and show to their Mathematics teacher whatever they did related to Mathematics when they learned to be bar waiters, cookers or carpenters. Astonishingly they actually did it. They worked out these outside school' mathematical tasks, while they did not work school ones. Later on we knew they were very good job workers.

Industry for Mathematics Education

Mathematics coming from work contexts cannot and should not determine the scope of Mathematics secondary school curriculum, but part of it should be incorporated at it. If not, on which aspects can we base the relation of our matter with reality? Such a relation should not be left to teacher's imagination and creativity. This would not be honest and it could transmit the perception that such a relationship is arbitrary, depending on the teacher hobbies. To prevent it it's necessary that teachers of Mathematics know what happens outside their academic context. So they will see Industry as a source for mathematical problem solving and creation from where to get authentic and real mathematical activities to be developed at classroom.

Continuum changes of our society (technological, social and cultural) demand a continuum up to date of knowledge of our environments (technologic, social and cultural). Mathematicians and mathematical educators will find a task to do with any kind of workers, especially in Industry. They can promote adult workers' mathematics education through the use of advanced technology (software, if it can make work easier), and helping workers to solve mathematical problems.

3. Mathematical Competences in the Engineering Curricula versus Industrial demands

Nowadays there is a considerable attention directed towards increasing the achievement, participation and involvement of mathematical tools in engineering. Mathematics, including statistics and computing, are accepted as an important part of technological applications. The relation between university, research institutes and industry is now more active. Mathematics provides the language for communication and discovery in many scientific disciplines and modern industry. For this reason is very important to incorporate the use of mathematics in the new engineering curricula in a satisfactory way. However, this task is not trivial and many problems have to be solved.

Mathematics has to be one of the most important parts in almost any science or engineering education. Mathematicians provide a logical thinking and a language that explains many complicated processes. It is crucial in the technological development.

It is important to remark that different historical periods required different types of mathematics. Nowadays, mathematicians and engineers have to work together in real applications. The technological progress also demands, every day, a progress in mathematics. The link between pure mathematics and its applications in the industrial progress is a crucial task. In particular, the engineer has to obtain the necessary competences to develop this collaboration.

In the last decade the collaboration between the university and the industry has increased considerably in the Region of Murcia (Spain). Many research groups in the Polytechnic University of Cartagena (UPCT) have industrial projects. This fact has pointed out the need to review the mathematical competences in the new curricula of the engineers. The different departments are working in these new curricula and in the next year the new programs will start.

The aim of this part of the paper is to analyze the competences in mathematics in the new curricula of the engineers. It is a preliminary work and special attention is paid to the particular case of UPCT. We would like to compare the competences obtained by the student in this university with the ones that the industry demands.

This study is organized in four parts. The first will present the wished competences. The new curricula elaborated (to start in 2010) are analyzed in the next part. As a particular example, that is very important in our region, we detail the mathematical competences in the Agricultural Engineering. The following part explains the type of collaboration with the Industry of the Region of Murcia in Spain. Finally, we present some discussions and conclusions by analyzing the acquired mathematical competences in the university in comparison with the demanded in the agricultural industry.

The new curricula encourage the acquisition of both transversal and specific competences. The transversal competences can be divided in three subgroups.

A) Instrumental transversal competences

T1.1	Ability to analyse and synthesise
T1.2	Ability to organise and plan
T1.3	To communicate suitably both orally and written in Spanish
T1.4T	o communicate suitably both orally and written in English
T1.5	Basic computational abilities

T1.6	Ability to use the information
T1.7	Resolution of problems
T1.8	Decision making

B) Personal transversal competences

T2.1	Critic and auto critic ability
T2.2	Team work
T2.3	People relations
T2.4	Ability to work in a team
T2.5	Ability to communicate with other experts
T2.6	To show knowledge and understanding of the diversity
T2.7	Ability to work in an international context
T2.8	Ethical approach in the different fields of the profession

C) Systematic transversal competences

T3.1	Ability to apply the knowledge in practice
T3.2	Ability to knowledge
T3.3	Adaptation in new situations
T3.4	Creativity
T3.5	Ability to become a leader
T3.6	To show knowledge and understanding of others cultures
T3.7	Ability to work on your own
T3.8	Initiative
T3.9	Qualité
T3.10	Motivation

The specific competences depend of each degree. The mathematics has a transversal nature and the above competences can be incentivized in our courses. Moreover it can be used to obtain specific competences in many different situations. The specific competences are related with the type of work that the student can be done. The UPCT is a polytechnic. There are several engineering degrees with different specific competences, however only one department of Applied Mathematics and Statistics. The material prepared by our department has several particularities depending on the specific characteristics of each degree. Basically, in order to obtain its specific competences, the real applications are always related with the degree.

The specific mathematical competences are: Ability for the resolution of mathematical problems in engineering. Applications of mathematical tools related with: linear algebra; geometry; differential geometry; differential and integral calculus; ordinary differential

Competences			
Specific	**Transversal**		
Ability for the resolution of mathematical problems in engineering. Applications of mathematical tools related with: linear algebra; geometry; differential geometry; differential and integral calculus; ordinary differential equations and partial differential equations; numerical analysis, statistics and optimization. The use of MATLAB in engineering problems.	✓ T1.1	✓ T2.1	✓ T3.1
	✓ T1.2	✓ T2.2	✓ T3.2
	✗ T1.3	✓ T2.3	✓ T3.3
	✓ *T1.4*	✓ T2.4	✓ T3.4
	✓ T1.5	✓ T2.5	✓ T3.5
	✓ T1.6	✗ T2.6	✗ T3.6
	✓ T1.7	✗ T2.7	✓ T3.7
	✓ T1.8	✗ T2.8	✓ T3.8
			✓ T3.9
			✓ T3.10

Table 2

equations and partial differential equations; numerical analysis, statistics and optimization. The use of MATLAB program in engineers problems.

Traditionally mathematics and engineering have been learning in different separate courses. Many students of engineering have not been especially motivated to study mathematics. Mathematicians do not see how to apply the theory in engineering. In particular, there is usually too big a gap between the different fields during the learning process.

One goal in the new curricula is to decrease the distance between the different areas in order to obtain students with the necessary competences. In the last two years we have worked in this direction in our university. The collaboration between researchers, teachers and industry is very important.

A particular situation can be found in the Agricultural sector. Many groups in our university work with the industry of this sector and many deficiencies in the actual curricula have been detected. With the new curricula we try to respond to new challenges by taking into account mathematical working life requirements in a better and a more appropriate way than earlier.

Now, we describe the specific and transversal competences acquired in the Agricultural Engineering at UPCT with the mathematical courses on Algebra and Calculus, Differentials Equations, Applied Statistics, Numerical Analysis and Informatics (see table 2).

As we can see, in these courses the student seems to learn the necessary mathematical tools to obtain a good background in the topic. Moreover, it is motivated both the own work and the team work. The learning process is active and the use of computer tools has been increased. Finally, the applications and the relation of the mathematics with real problems emerge as the key point.

But, is it enough for the industrial applications? To answer this question, we will analyze the acquired mathematical competences in comparison with the demanded in the agricultural industry

The agro food sectors in the region include: Horticulture, Fruit culture, Canned, Citric, Almonds and Cereals, Olive and Olive Oil, Vineyard and Wine, Fishers and Aquaculture, Farming, Meat Industry. The contribution of the agro food sector at the region economy is the most important. Our principal companies as: Hero, Juver alimentación, Cofrusa, García Carrión-Don Simón, EL Pozo alimentación, Ricardo Fuentes e Hijos S.A, Cafés Celdrán y Bernal, El Barranquillo, Harimsa, Macopan, Durán or Bonduelle are very important in our economy.

The implication of the government is also very important. For example, the public expenditure in Euros for I+D+I in the region is increasing every year:

2007	2008	2009	2010	2011
8.210.520	8.8867.362	9.576.751	10.342.891	11.170.322

On the other hand, in our university, the Higher Technical School of Agricultural Engineering has a very nice international reputation. This School has some leader research groups that have active collaboration with these companies. In most of these projects high mathematical competences are necessary. Actually, our department works with these groups in order to improve these competences. The necessary mathematical tools include: basic statistic, nonlinear least-square, ANOVA, mathematical models using statistic or differential equations and computer applications.

It is clear that Agricultural industry needs professionals with high mathematical competences. As we have described, the different sectors use all the mathematical tools that compose the new curricula. Thus, the goal will be to obtain an engineer that can work fluently with a mathematician (or any scientific) and give clever solutions to the industrial problems. Of course, we need also a compromise from the other side: the mathematicians. In fact, the need for a strong collaboration between the different parts: schools, universities, businesses and governments, concerning the promotion of mathematics, science and technology, should be unanimously recognised. The main reason for such an initiative is the dissatisfaction of industry with the quality of specialists emerging from universities. In our opinion the new curricula and the new pedagogical techniques can improve the mathematical competences of our engineers. The pedagogical proposal in our department could be classified halfway between the classical teaching techniques and the new pedagogical proposals introduced for adapting teaching methods to the new European Framework for Higher Education. The innovation is focused on the use of teaching videos developed by the teachers themselves as a complement for classroom lessons. The proposal was divided into three stages. The first stage was developed during the months of September and October (the first two months of the first semester in Spain). At this stage, the contents of the subject were introduced with lectures. Then workgroups of 2-3 people were formed, and personalized meetings were held with all the groups. Next, during the second stage in November, the videos made by the teachers were given to the students. The videos explained both the theory and the problems derived from that theory. The exams from previous years were provided. During this month, we had only computer-practice lessons. In the last stage, during the months of December and January, questions raised by the students were resolved, and the students were evaluated. The results were strongly positive. Students and teachers alike were highly satisfied with the experience.

In conclusion, we would like to point out that mathematical technologies have the potential to bring real world applications to life in the mathematics classroom. Collaboration between researchers, teachers and industry seems crucial. Moreover, this task should be iterative introducing the necessary changes during the process. Many people are working in order to improve the curricula of the engineers and a ccollaboration between schools, universities, businesses and governments seems also very important.

The new curricula seem to go the correct direction but we would appreciate the real change in the near future. But, it is a very difficult task. The gap between the different areas and between the university and the industry has to decrease. Moreover, most research takes place at departments of universities. Professors and are involved teaching of basic courses and at many universities their research is considered a personal hobby. Moreover, the majority of first year students we receive at our universities are poorly prepared. In particular, the industry is unable to find enough engineers who are sufficiently educated by our universities. We hope that with the new curricula and the new methodologies this situation will change. In our opinion, we have performed the first step in the good direction.

4. Mathematicians as engineers

Finally, in the third work, we shall consider the main characteristics of some master programmes developed in Spain under the common title of Mathematical Engineering. These kinds of programs have been developed in Spain in the last years although they have a long standing tradition in, for instance, United Kingdom, France or Italy.

In the previous section, an analysis of mathematics contents and skills has been made, that, in a particular case, must be a part of the engineers training, i.e. it is about mathematics for the engineering, about mathematics for industry. From other point of view, in this section we examine some proposals that could develop skills or contents that may contribute to the formation of a specific type of mathematician who has the skills and attitudes that could be considered typical of an engineer training.

Taking into account the training proposals that are being offered in this sense, two typical denominations can be found in Europe: Industrial Mathematics and Mathematical Engineering.

The first one is a usual denomination within the titles offered by universities and colleges belonging to the ECMI, which are consistent with the name of the Consortium itself. There are also training proposals (degrees, masters and doctorates) named Mathematical Engineering in countries like France, United Kingdom, Italy or Spain, which in some cases, are offered by universities that also belong to the ECMI.

The professional performance of a mathematician in the industry, business or management can vary from one country to another depending on the economical structure of each of them. That is, the development opportunities as an industrial mathematician, in a country with a rich and autonomous industry such as France, may be very different from those one could find in a country such as Spain, whose economy is focused on services and highly dependent on the foreign know-how. This situation may partly explain why the denomination of Mathematics Engineering has been adopted in Spain against Industrial Mathematics so far.

The first question that we want to address is the following one: Does such training proposals agree with the same professional profile? As we will see the answer is that, even though they have many common features overall (the usage of mathematics in solving real problems), they differ in essential aspects with regard to the professional performance that may be expected from the students of the different training proposals. In such case, the second question we propose is if it is worthy to define a common professional profile of the mathematical engineer.

One previous issue that can not be ignored is that of denomination, that is the extent to which the denominations agree with the contents and how they communicate their objectives. According to our opinion, the name of Industrial Mathematics is an evolution of that of Mathematics for Engineering original concept. Such designation would be appropriate at some times and in some countries where the industrial weight was very important. However, the ECMI itself was forced at one point to explicitly extend the field of its activity beyond the pure industrial environment, in the classical sense, to the more general field of business and management as it is showed with the introduction of the Techno-Mathematics and Econo-Mathematics terms. In the Discussion Document of this study it is written: "In order to better understand these phenomena, the Study starts from a broad definition of Industry (from the Organization for Economic Co-operation and Development) "... *broadly interpreted as any activity of economic or social value, including the service industry, regardless of whether it is in the public or private sector*" (OECD 2008, p.4) The term "industry" obviously refers to a diverse range of activities, producing goods and services. Under constraints such as time and money, these activities generally attempt to optimize limited-sometimes scarce-resources, both material and intellectual. The overarching goal is to maximize benefits for certain groups of people while, ideally, minimizing harm to other groups and the natural environment."

In this sense it can be said that although the denomination of Mathematical Engineering fits better the proposed professional competences, it represents a strong anomaly as it is the only Engineering discipline that references the instruments used, instead of the specific area of developed activity.

However, it can be good reasons to keep, in some countries at least, the designation of Mathematical Engineering. In the economic activity sectors, the applied mathematician competes with disadvantage with classical engineers limiting them to a role of subordination and support to those engineers. This unfavourable competition occurs just from the entrance of the students in degree studies. At least in Spain, there is a very strong social pressure for qualified students in mathematics to choose classical engineering degrees. There are indications that Mathematical Engineering can change somewhat this situation as it is observed with the experience of the Mathematical Engineering degree offered, for instance, by the Polytechnic of Milano and the more recent one of the Complutense University of Madrid.

Leaving aside, for the moment, the issue of the denomination, the question is if these proposals are directly focused to the development of a professional profile with features clearly differentiated from other previously existing. As it has been already said, the answer is not clearly affirmative, even under the same denomination. In order to illustrate this issue, and without wanting to be comprehensive, we expose the content and main characteristics of three Spanish Master Degree Programmes in Mathematical Engineering offered at present by the Carlos III University of Madrid (UCIIIM), the Santiago de Compostela University in cooperation with the Coruña and Vigo Universities (USC-UDC-UVIGO) and the Complutense University of Madrid (UCM), respectively.

Although, the general aim of these three Masters is to develop mathematical, statistical and computational skills to solve technologic o scientific problems in business and industry and other related task with focus on competencies as modelling, programming and simulation as well as ability to handle huge amounts of data by integrating mathematical, numerical and statistical methods, we have found different results when trying to reach the declared goals.

First, the durations of the Master Programmes are different; ranging from the UCIIIM Master, which is spread over two years and covers a total of 120 ECTS (European Credit Transfer System) to the UCM Master with 60 ETCS in total (and till 60 ECTS as previous complementary formation depending of the background of the entering students), corresponding to a year study. On the other hand, in the USC Master are required courses up to 90 ETCS.

A more detailed analysis of the course content will offer us deeper information about the differences in the knowledge of the mathematical contents and practical competences that these Masters provide to students, and make us to easily identify the core mathematics curriculum needed to achieve the goal of providing qualified students with extensive, detailed knowledge of the theory and methods combined with a high level of practical competence in Mathematical Engineering.

It must be said that all the used information about the Masters on Mathematical Engineering has been obtained from the web pages. It is assumed that all the essential characteristics, in order to interest to potential students, are reflected on them.

Master in Mathematical Engineering UCIIIM

The information about this master is extracted from its web page:

http://www.uc3m.es/portal/page/portal/postgrado_mast_doct/masters/math_engineering

Presentation

The evolution of technology and applied sciences conveys the need for a flexible and multidisciplinary education.

The Master in Mathematical Engineering offered by Universidad Carlos III de Madrid aims to provide students with the necessary tools to face technological or scientific problems by means of:

- Sound mathematical, physical and statistical foundations
- Mathematical modelling

Analysis and numerical solution of models

Validation of models and results

Aims

The Programme aims to gain the adequate tools in a Graduate degree referring to acquisition, integration and communication of the knowledge obtained, development of skills in solving problems and introduction to research.

Obtaining the Master degree endows the student to know the most advanced computer science tools (software) in the field of Applied Mathematics and Statistics. These tools are useful to deal with problems and simulation of sceneries to evaluate the different methods and techniques acquired.

The Master qualifies for admission to the PhD Programme and is divided in two academic years with 60 ECTS each. Each branch has a different structure of compulsory and elective subjects. The Research project/Master Thesis can have also a different number of ECTS ranging from 18 to 30 ECTS.

The UCIIIM Master appears split from the beginning into fourth separated branches:

- Statistical Sciences and Techniques,
- Mathematical Foundations in Engineering,
- Fluid Mechanics, and
- Modelling and Numerical Simulation.

The corresponding courses offered in each branch are:

Statistical Sciences and Techniques

1st semester—Preparatory courses on Mathematics, Statistics, Matlab and R language and Compulsory courses on Numerical Methods, Advances Methods in Matrix Analysis and Statistics.

2nd semester—Compulsory courses on Regression Analysis, Operational Research, Numerical Linear Algebra, Stochastic Processes and Time Series.

3rd semester—Compulsory course on Optimization and Elective courses on Multivariate Analysis, Mathematical Statistics and Advanced Techniques in Multivariate Statistics (Neural Networks).

4th semester—Elective courses on Topics in Advanced Statistics or Bayesian Inference.

Mathematical Foundations in Engineering

1st semester—Compulsory course on Numerical Methods and Eligible courses on Partial Differential Equations, Dynamic Systems and Bifurcations, Advances Methods in Matrix Analysis, Methods in Real Analysis with applications, Introduction to wavelets with applications, Statistical Mechanics and Introduction to Complexity Sciences.

2nd semester—Compulsory course on Numerical Methods in Linear Algebra and Eligible courses on Applied Harmonic Analysis, Advanced Complex Analysis and Transformed Theory, Mechanical Advanced Elements, Special Functions with Applications, Optimal and Adaptive Filtering, Interdisciplinary Applications of Statistical Mechanics and Advanced Complex Systems.

3rd semester—Elective courses on Approximation Theory with Applications, Asymptotic for Parabolic Equations, Mathematical Information Theory and Econophysics.

4th semester—Elective courses on Geometric Theory of Functions, Orthogonal Polynomials and Special Functions for Engineering, Complex Analysis and Operator Theory for Engineering, Iterative Advanced Methods for Signal Processing, Mathematical Aspects in Sampling Theory and Signal Processing, Hamiltonian Systems, Numerical Methods for Geometry and Topology, Mathematical Methods for Control Theory and Robotics and Dynamics of Evolution.

Fluid Mechanics

1st semester—Compulsory courses on Numerical Methods for Ordinary Differential Equations, Singular Perturbations, and Fluid Mechanics.

2nd semester—Compulsory courses on Numerical methods for partial differential equations, Hydrodynamic Stability and Introduction to Turbulence, Singular Perturbations, Fluid Mechanics, Heat and Mass Transfer and Eligible course on Experimental Methods in Fluid Mechanics or Nonlinear Dynamical Systems and Chaos.

3rd semester—Elective courses on Computational Fluid Dynamics, Numerical Methods for Conservation Laws, Finite and boundary elements Spectral methods.

4th semester—Elective courses on Modelling in Science and Industry, Advanced Topics in Fluids Mechanics and Combustion.

1st semester—urbations, Modelling with Partial differential equations and Fluid Mechanics.

2nd semester—Compulsory course on Numerical methods for partial differential equations and Elective courses on Quantum physics and solid state physics, Nonlinear dynamics and bifurcations in extended systems, Nonlinear dynamical systems and chaos and Inverse Problems and Imaging.

3rd semester—Elective courses on Numerical Methods for conservation Laws, Stochastic Numerical Methods, Finite and boundary elements, Spectral methods, Boltzmann Equation with Applications, Electronic Transport in Micro and Nanostructures and Astrodynamics and Space Geodesy.

4th semester—Elective courses on Modelling in Science and Industry, Advanced Statistical Physics and Advanced Numerical Seminar.

Master in Mathematical Engineering USC-UDC-UVIGO

The information about this master is extracted from its webpage:

http://www.dma.uvigo.es/MASTER/curso0910/index.php

The Master of the USC-UDC-UVIGO on Mathematical Engineering is divided in three semesters with 6 blocks to cover a total of 90 ECTS. Five blocks (Mathematical Models, Equations, Numerical Methods, Computing and Numerical Simulation) in the first year with 5 compulsory courses and 26 elective subjects. The sixth block corresponds to the third semester and it includes a Workshop on Industrial Problems and a Master Thesis of 18 ECTS.

The aim of this Master is to form technical specialists in modelling and numerical simulations of business and industry process with focus in the use of Professional Software, without forgetting the educational background in numerical methods and equations.

Specifically, the structure of the programme tries to cover the following aims:

- To know and to understand the problems that arise in the Engineering and in the Applied Sciences areas as a beginning point for a suitable mathematical modelling.
- To be able to determine if the model of a process is well set out and formulate it mathematically in a functional suitable frame.
- To be able to select the best numerical technologies to solve a mathematical model.

- To know and use the professional software tools most used in the industry and in the company for the simulation of processes.
- To acquire skills to integrate the knowledge of the previous points appointed numerical simulation of processes or devices arisen in the industry or in the company in general, and to be capable of developing new software applications of numerical simulation.
- To acquire skills of learning those allow them to join teams of I+D+i of the business world.

The courses offered in the USC-UDC-UVIGO Master are:

Block I: Mathematical Models—Compulsory course on Mathematical Models on Mechanics of Continuous Media, and Electives courses on Mathematical Models in the following topics: Finances, Fluid Mechanics, Solid Mechanics, Electromagnetism and Optics, Acoustic or Environmental Sciences.

Block II: Equations—Compulsory course on Partial Differential Equations and Eligible courses on System Control and Optimization and Advanced Topics in Partial Differential Equations

Block III: Numerical Methods—Compulsory courses on Numerical Methods and Finite Elements and Eligible courses on Numerical Methods for Optimization, Finite Differences, Finite Volume Methods, Advanced Topics in Numerical Methods and Advanced Topics in Finite Elements

Block IV: Computing—Compulsory course on Programming Languages and Systems and Eligible courses on Parallel Computing, Computer Architecture and Operative Systems, Computer Networks, Distributed Computing and Advanced Topics in Programming Languages and Systems

Block V: Numerical Simulation—Eligible courses on Computer Aided Design (CAD) and on Professional Software in the following topics: Fluid Mechanics, Solid Mechanics, Electromagnetism and Optics or Acoustic

Block VI: Project—Including courses on Software Engineering and Project Methodology, modelling activities in a Workshop on Industrial Problems and a Master Thesis.

Master in Mathematical Engineering UCM

The information about this master is extracted from its webpage:

http://www.mat.ucm.es/mambo/estatico/titul/pop/ingemat/

The objective of the UCM Master in Mathematical Engineering is to produce professionals with broad based problem-solving skills that can be used in the industrial, services or public sectors, using the language and tools of mathematics and other related areas. Graduates will be proficient in the main techniques and tools for analyzing, modelling, solving and optimizing a great variety of problems and systems with the required skill, knowledge and experience. The methodological approach is the degree's most important feature: it seeks to develop the concept of engineering in the field of mathematics, focusing mainly on problem modelling and solving, and case studies.

The Master Structure is flexible with a minimum of 60 ECTS to a maximum of 120 ECTS according to the applicant's previous mathematical learning and it is divided in a First and a Second level. The Programme Coordination Committee will decide if student direct entry into Level 2 is possible, or if the applicant has to register (fully or partially) for one of the two following options of a First Level: option A up to 60 ECTS of second cycle modules from the degrees of Mathematics or Statistical Science and Technology or option B which includes a Refresher/Adaptation course, including up to 12 ECTS in: Numerical linear algebra (Matlab), Mathematical analysis and ODEs, Probabilities and statistics and Programming languages (FORTRAN and C). The Second Level consists of 60 ECTS (Compulsory modules: 42 ECTS, Optional modules: 9 ECTS and Placement or Research Project: 6 ECTS and Modelling Week: 3 ECTS).

Compulsory courses are: Databases, Applied Statistics and Data Mining, Numerical Methods, Modelling and Dynamic Systems, Optimization and Simulation and Introduction to Financial Mathematics, Placement or Project and Modelling Week.

The optative courses offered are Signal Theory, Artificial Vision, Cryptography, Java and Web Services, Advanced topics on Java and Web Services, Mathematical Models for Food Technology, Modelling of Deformable Mediums for Industrial Processes, Stochastic Calculus and its Applications, Applications of Computational Algebra in Artificial Intelligence, Interest Rates, Introduction to Financial Risk Management, Numerical Methods for Finance, Numerical simulation in paralel computing, Optimization in Orbital Control, Geodetic Networks and GPS and Analysis and Inversion of Gravimetric Data. This offer of optative courses can change from year to year.

Comparative analysis

After exploring the details of these Programmes, it is obvious that curricular issues are different in some aspects described in what follows:

- First at all, a general distinction can be made into Master Programmes orientated to one specific field, let us say, for example to Econo-Mathematics or Techno-Mathematics, as UCIIIM, and, on the other hand, with a more global orientation as the UCM Master Programme.

- Although it is not excluded an academic or professional goal, the UCIIIM and USC-UC-UCVIGO Masters are considered as Research Masters and, therefore, linked to a Ph.D. programme. On the other hand, the UCM Master programme is seen, as a whole, as a professional Master programme. This distinction is crucial since these goals could mark differences between the contents and the skills required in order to obtain them.

- The structure and contents of the Masters also stablish importat differences between them. On one hand, the MUSC-UC-UVIGO Master presents a classical offer in Industrial Mathematics that is supported by an excellent research group in the field. On the other hand, the UCIIIM Master, since it lacks a common part, is actually a group of four Masters. Finally, the UCM Master contains an important common part, which helps to ensure almost identical competences between the students. It deepens less in a particular area, but it tries to get the students adapted easily to different professional challenges.

- The contents do not completely determine the competences attained in a Master's degree, since the methodology is especially relevant when we try professional training. Only the UCM Master refers explicitly to a learning methodology based on case studies and on experience modelling and solving real problems but some activities developed in the other masters could be based in the same methodology.

- The duration of the bachelor degrees in Spain affects in an important way to the Master degree offers. After a four-year degree, to complete a training period equally in ECTS to those offered in other European countries, it is only necessary a year Master's degree of 60 ECTS. However, this approach makes difficult the harmonization between Spanish Masters and European Masters without penalising the Spanish students with an extra year or without declining the possibility to join the Spanish masters to other European students. The disparity of proposals regarding the duration of the three studied masters faithfully reflect this situation. Particularly complicated could be adapting them to the general guidelines that are being discussed within the ECMI orientated to a proposal of European Master on Industrial Mathematics. For instance, the duration of the Master Thesis of the UCM Master is very far away from the mentioned guidelines.

- About the employment factor, there are also differences between them. Due to the current situation in the Spain industrial activities, it is hard to obtain permanent positions for the students with the mentioned profiles. Actually, for certain singular problems a consultant is hired, whose survival will depend on the continuous realization of projects. In this type of consulting highly specialized professionals could be needed, professional with an excellent mathematical background to tackle highly complex problems. On the other hand, if what we want is to train people to make them have a strong quantitative profile in several different areas and with experience in adaptation to others, it might be needed a more transversal orientation towards problem solving. It is predictable that in the near future we will be having a higher number of companies interested in this last type of professional, a kind of quantitative engineer since they are used to work on different areas.

Finally, and in view of dispersion and variety of training proposals that have illustrated, would be worthy to define at European level a versatile professional profile in the field of mathematics with a broad orientation to employment in industry, business and management having engineer's own competences and requirements? I.e. would be worthy to formally create the figure of Mathematical Engineer? We think that any proposal on this figure must take into account the main characteristics of an engineer training. Next section transcribes a proposal of those characteristics.

The Skills, Attributes and Qualities of an Engineer

Among a great variety of proposal addressing this issue we think that the work of Maddocks et al., 2002, represents very well most of them. In that work, the Subject Benchmark statement for engineering details skills, attributes and qualities that are thought necessary to enable the engineer to practice effectively in a professional manner. It is expected that an engineering degree programme will foster, develop and inculcate such attributes, skills and qualities. These attributes, skills and qualities are listed in the Subject Benchmark Statement under five headings. The demonstration of Skills / Attributes / Qualities in each one of them is the following:

Knowledge and Understanding

A Graduating Engineer should be able to demonstrate:

- Specialist (Discipline) knowledge
- Understanding of external constraints
- Business and Management techniques

- Understanding of professional and ethical responsibilities
- Understanding of the impact of engineering solutions on society
- Awareness of relevant contemporary issues

Intellectual Abilities

A Graduating Engineer should be able to demonstrate:

- The ability to solve engineering problems, design systems etc. through creative and innovative thinking
- The ability to apply mathematical, scientific and technological tools
- The ability to analyse and interpret data and, when necessary, design experiments to gain new data
- The ability to maintain a sound theoretical approach in enabling the introduction of new technology
- The ability to apply professional judgement, balancing issues of costs, benefits, safety, quality, etc.
- The ability to assess and manage risks

Practical Skills

A Graduating Engineer should be able to:

- Use a wide range of tools, techniques, and equipment (including software) appropriate to their specific discipline
- Use laboratory and workshop equipment to generate valuable data
- Develop, promote and apply safe systems of work

General Transferable Skills

A Graduating Engineer should be able to:

- Communicate effectively, using both written and oral methods
- Use Information Technology effectively
- Manage resources and time
- Work in a multi-disciplinary team
- Undertake lifelong learning for continuing professional development

Qualities.

A Graduating Engineer should be:

- Creative, particularly in the design process
- Analytical in the formulation and solutions of problems
- Innovative, in the solution of engineering problems
- Self-motivated,
- Independent of mind, with intellectual integrity, particularly in respect of ethical issues
- Enthusiastic, in the application of their knowledge, understanding and skills in pursuit of the practice of engineering

The full Benchmark Statement is available at

http://www.qaa.ac.uk/crntwork/benchmark/engineering.pdf

As a conclusion, we think that is worth to propose the professional profile of a Mathematical Engineer. The training process of such a professional must take into account an appropriate balance between Mathematics, modelling activities coupled with another discipline such as Statistics or Physics with the aim to develop knowledge, understanding and experience of the theory, practice and application of selected areas of Mathematics but also including Statistics and Operations Research and learning mathematics not only in mathematics courses but including context-based mathematics courses in other disciplines that employ significant mathematical methods so that students will be able to use this skills and techniques of these areas to solve problems both of a routine and of a less obvious nature arising in industry, commerce and the public sector and they could function comfortably in work situations that require quantitative and analytical skills to solve more general mathematical problem. But we think that in order to define the professional profile of the Mathematical Engineer, these specific requirements must be developed under a learning methodology that guaranties the acquisition of the Skills, Attributes and Qualities of an Engineer that we have considered above.

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UCM Master:

http://www.mat.ucm.es/mambo/estatico/titul/pop/ingemat/

Methodological reflections on capturing the mathematical expertise of engineers

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Abstract For providing a mathematical education of engineering students which is relevant for their later work as engineers, one needs studies that try to capture the mathematical expertise of engineers. There are only a few studies of this kind because they are not easy to conduct. In this contribution we describe and discuss the methods applied in the available studies. We present general problems and methods from qualitative research which were used. We distinguish between approaches observing engineering students and those that deal with real workplaces, and discuss risks and advantages. It is the intention of this contribution to provide the basis for a discussion on how one can appropriately investigate the mathematical expertise of engineers.

Introduction and general remarks

In order to provide a mathematical education of engineers which is not restricted to the academic realm but also useful in later practical work, it is necessary to investigate the mathematical expertise an engineer still needs in his/her daily life. Although there are quite a few studies on the use of mathematical thinking at workplaces, most of these are concerned with non-academic workplaces. One major obstacle for conducting such studies might be that it is not easy for didactical researchers to become familiar with engineering work within a reasonable timeframe. The studies performed so far (as to the knowledge of the author) can be found in (Alpers, 2006-2009), (Cardella/Atman, 2005, 2007), (Cardella, 2009), (Gainsburg, 2006, 2007) and (Kent/Noss, 2002). In this contribution, we intend to present and discuss the advantages and risks of the methods applied in these studies in order to provide a basis for further studies which are urgently needed to get a "broader picture". We grouped the studies according to whether investigations were made with students (next section) or with real engineers (third section). Finally, we discuss what might be used in future studies and how the results could inform the mathematical education of engineers. We do not present the results of the respective studies (a summary can be found in Alpers, 2009b).

We start with some general remarks and problem statements which are important for investigating the mathematical expertise of engineers. The first problem is concerned with the fact that there is nothing like "the" engineer. There are different branches of engineering (e.g. civil, mechanical, electronical engineering) and within these branches there are again various job profiles (e.g. computational engineering, design, sales). In order to avoid inadequate generalizations, any study has to make clear which branch of engineering and which job profile is considered. In (Kent/Noss, 2002) e.g., the authors identified different job profiles in civil engineering by discussing these with managers and then decided to concentrate on structural engineers. In (Alpers, 2006–2009) the study is restricted to practice-oriented mechanical engineers who are graduates of universities of applied studies and who usually work in design or test department but not in computational departments or at the forefront of R&D. An identification of different job profiles is also helpful for investigating the interfaces between engineers. This is important for investigating the necessary understanding "beyond" interfaces, e.g. the understanding of what kind of results computational engineers can deliver.

The methods applied come from qualitative research, mostly based on the "grounded theory" by Glaser, Strauss and Corbin (see Flick, 2005):

- Participating observation: Zevenbergen (2000) distinguishes between a "hard-core" version where the researcher participates in the working process for a longer period of time, and a “soft-core” version which is restricted to several shorter observations. The soft-core one is the more realistic one. When it comes to engineering it is even questionable in how far a researcher is able to participate. The observations are recorded electronically or by notes taken by the researcher. This approach is also known as “ethnographical” research.

- Interviews: By interviewing people working on tasks one can get a better understanding of what they do and why they do it in the way observed. Impressions and hypotheses stemming from former observations can be checked.
- Investigation of tools and artefacts: Work is often mediated by tools like computational programs, and many computations are hidden within tools. An investigation of tool usage is necessary to obtain information on what mathematical knowledge is still required for reasonable use. Artefacts also include guidelines or conventions the work is based upon.

Based on observations, interviews and tool usage investigations, researchers try to develop interpretation categories and hypotheses. Zevenbergen (2000) distinguishes between research work that mainly tries to detect school mathematics in work places, and real ethnographical studies which try to capture the understanding of workers and their construction of mathematical meaning and understanding. The latter also search for hidden or embedded mathematics and for mathematisable activities. She criticizes the former approach as too restricted. On the other hand, Gainsburg (2005) states that for informing mathematical education it is quite worthwhile to gain information on the usage of mathematical concepts taught in schools. The same could be said for the use of university mathematics in engineering environments. We discuss this question in the next section when describing the investigation frameworks of studies observing engineering students.

Observing students working on tasks

The studies performed by Cardella/Atman (2005, 2007), Cardella (2009) and Alpers (2006-2009) are not based on observations of the work of real engineers but are concerned with how engineering students work on engineering tasks. In the sequel, we will describe their methods in more detail and discuss the benefits and pitfalls.

Cardella and Atman observed industrial management students working on their capstone project (large application-oriented final project). Having an education in industrial engineering themselves certainly facilitated the investigation of such a group. Furthermore, they conducted interviews with four engineering students from other areas. They segmented, grouped and coded their observation and interview notes using qualitative methods, and then mapped their results onto the categories of mathematical thinking developed by Schoenfeld (1992). These categories are the following: knowledge base; problem solving strategies; effective use of resources; beliefs and affects; mathematical practices. Some of the categories like problem solving strategies are quite general and might also be considered as categories belonging to other fields of engineering education like physics or engineering mechanics. In another study (Cardella/Atman 2007), Cardella also observed 50 engineering student when working on a project task for three hours, and four mechanical engineering master students working on a project that was concerned with designing a mobile compressor unit for dentists. The methods of investigation were the same as in previous studies. She also looked specifically for modeling activities.

The author identified together with a colleague who worked in the car industry for several years, "typical" tasks for a practice-oriented mechanical engineer (a graduate of a university of applied sciences). Since the author is a lecturer for mathematics in the department of mechanical engineering and is also involved in developing engineering guidelines for mechanisms with the German Association of Engineers (VDI), a certain understanding of the work of mechanical engineers helped in understanding the tasks. Two students in their last (8.) semester worked on each task for 100 hours (being paid). The colleague involved acted as mentor (playing the role of a group leader in industry). The students had access to industry strength programs (like CAD, FEM, mechanism analysis tools, machine element dimensioning tools) which are also used in the university education. The students were asked to make notes on their work, problems and thinking processes. They delivered these notes and the files they produced when using the programs to the author who investigated the notes in order to understand the mathematical thinking involved and also to get a deeper understanding of the task. Based on these investigations the students were interviewed to clarify questions and to demonstrate the tool usage. These interviews were audio-taped and the demonstrations were screen recorded. This allowed for later investigation and own trials with the software which was also available to the author. The author often used the opportunity to ask the students and the colleague to clarify questions that came up later in the investigation process.

When interpreting the material and the tool usage the investigation of the author was guided by the following questions formulated in advance (these were partially based on the work of Kent/Noss (2002) described in the next section):

- Which visible mathematical concepts and procedures were used? Is there a difference between their usage in practical tasks and in mathematical education?

- What is the relationship between intuition, work in coarse models and work in more detailed models? Where in the spectrum from qualitative to quantitative thinking can the work be located?
- Are there mathematical concepts hidden or embedded in application contexts? Which kind of mathematical understanding is necessary to work effectively and efficiently in the contexts, i.e. to solve the problems? Would an explicit mathematisation improve efficiency, i.e. by reduction of trials?
- Are there any guidelines or design rules that are applied instead of performing own calculations?
- Which cognitive models are required for making reasonable use of the software? Which objects are visible at the interface ("boundary objects" after Kent/Noss, 2002)? Which mathematical competencies are required or at least helpful when operating at the interface? What is the role of goal oriented experimentation when solving problems and for understanding the role of model parameters?
- What is the role of mathematical concepts when interpreting the program output and checking whether it makes sense?
- Which knowledge is important beyond using the software programs (knowledge on material properties, production processes, availability, costs etc.)?
- Are there any problem situations where the "usual" procedures (guidelines, procedures from books etc.) do not work any longer for fulfilling the requirements ("breakdown situations" after Kent, Noss 2002)? Would a more mathematical approach be helpful in overcoming the problem situation?
- Which further computational tasks would be transferred to a computational department and what would the interface to such a department look like?

In the sequel we state potential problems and benefits of the above approaches:

- Unrealistic circumstances: The investigation of student work has principle restrictions since there are no real customers and there is no time pressure which is normal for engineers working in industry. Moreover, the students were not part of an organizational structure. In the study of Cardella, the students worked in project teams which might resemble teams in industry whereas in the study by the author the students mainly worked as individuals. The colleague involved acted as sort of a group leader in industry.

- Engagement of students: It might be questioned whether students are as engaged as real engineers who earn their money by their work. The author chose very engaged students for working on the task and Cardella/Atman observed students working on their capstone projects which are quite important to the students as final major projects.
- Choice of task: It is quite important to identify tasks which resemble those occurring in real industry and which can be treated without having an embedding into larger projects. In the study by the author, the colleague involved played a very important role because it was up to him to judge on the authenticity of the task. He also could give the students background information when necessary. In the work of Cardella the task concerning the capstone project for industrial engineering students came from an industrial partner, such that it was also based on a real problem.
- Representativeness of tasks: It is clear that one task cannot be representative for the whole spectrum of tasks in a certain engineering branch or for a certain job profile. In the study by the author, four different tasks were taken from various areas (design, mechanisms, machine element dimensioning, test and data processing) but it cannot be claimed that this covers the whole range of tasks for a practice-oriented engineer.
- Representativeness of students: It is clear that the work of a student cannot be representative for the work of senior engineers but students in their final semester can be considered as being similar to junior engineers. In the study by the author, the colleague involved was questioned whether or not particularly interesting parts of the students' work were due to the specific project situation or could also be found in real industry workplaces.

The main advantage of the approach where students are observed consists of the possibility of deep probing. The time of the students is not subject to strong restrictions as is often the case with real engineers. The author had several discussions with the students and was able to clarify questions several weeks after they had finished their work. The students spent quite some time demonstrating their tool usage and explaining their thinking. Since the author is not an engineer, it was often necessary to pose questions in order to get a better understanding. The colleague involved could also be questioned, particularly concerning the question when and under which circumstances a more mathematical approach including deeper quantitative modeling might have been indicated in order to improve results.

As stated above, Zevenbergen (2000) criticised the approaches where researchers looked mainly for the usage of mathematical concepts treated in schools. One could argue that the same might hold with respect to the work by the author and Cardella. Since the catalogue of questions set up by the author and the categories of mathematical thinking used by Cardella are very broad including questions on potential additional mathematisations, the risk of having too restricted a perspective seems to be small.

Observing real work places

In the studies by Kent/Noss (2002) and Gainsburg (2005–07), real workplaces of civil engineers were observed. Kent and Noss started with a document analysis (literature on engineering mathematics and on design activities) and a review of the software used by civil engineers in order to obtain a "first" understanding. One certainly cannot expect to understand the real work of civil engineers based on a short period of document analysis but it helped to formulate questions and interpret the answers in subsequent interviews with managers and engineers. In these interviews engineers brought with them working documents and sometimes also demonstrated software usage. The interviews were recorded and transcribed. Furthermore, Kent and Noss studied records of project meetings, documents produced by engineers, and e-mail sequences of cooperating engineers.

Kent and Noss used the information obtained in first interviews to get an overview of the job profiles which led them to concentrate themselves on structural engineers. They tried to discover mathematical concepts used either overtly or hidden in application contexts. Moreover, they identified a spectrum from qualitative-intuitive thinking to detailed quantitative work and used this to order and interpret their observations. They also recognized that many computational aspects of the work were delegated to software, simplified by coarse rules or guidelines or performed by specialized analysts (either internal or external). An example for a interpretative category they formulated in their analysis is "understanding through use" by which they mean that engineers gained an understanding not by having a formal computational model but by developing rules of usage by observing the results of variations.

The investigations by Gainsburg which are also concerned with civil engineers interpret the engineering work from a modeling perspective. She distinguishes between three kinds of corresponding activities: creation of new mathematical models for solving a problem; selection and adaptation of existing and accepted models that are applied to real problems; usage of situation-specific procedures and routines which avoid the usage of mathematical models. Gainsburg conducted ethnographical studies in two civil engineering companies where she observed structural engineers. She concentrated on four tasks comprising two to eight days, a wide spectrum of quantitative problems, use of technology, and participation of a range of junior and senior engineers. Collected data consisted of observation notes, audio recordings of discussions among engineers and between engineers and the investigator, copies of artefacts (drawings, documents, computation sheets), written and oral answers of engineers on questions of the investigator and 24 hours of interviews. Gainsburg worked through the quantitative problems, tried to understand the thinking processes, the usage of tools, and the actual usage of mathematics. The four cases were coded and themes and patterns of mathematical practices were identified. Gainsburg found two specific challenges in the engineering work she described as "understanding the phenomenon" and "keeping track" (of the different level of models involved). She also investigated how engineers build up expertise. Here, the categories of the modeling perspective were helpful. She identified the development of a thorough understanding of models and their applicability as those components of knowledge which can be reused in other situations and hence make up the expertise of engineers. She also coined the notion of "engineering judgment" in order to describe the components of engineering expertise which comprise mathematical and extra-mathematical capabilities. Within this framework she identified a suitable attitude of engineers towards mathematics which she describes as "skeptical reverence".

The main advantage of both investigations is that authenticity is guaranteed by observing real workplaces and real engineers working on real tasks. All the potential problems stated above with respect to approaches observing just students are not present. Both investigations chose civil engineers. This branch of engineering is probably the one that can be best understood within a shorter timeframe by non-engineers because everyone has a basic understanding of buildings and can imagine potential problems like stability. This way, the researchers were able to overcome the principle problem of such a direct observation of unfamiliar workplaces.

In the sequel we discuss some potential problems with investigating workplaces directly:

- Understanding the work processes: Very often engineers have very little time for their projects, and even more for an external researcher who has a lot of questions to get a better understanding of the technical problems, challenges and potential alternatives by applying a more mathematical approach. Gainsburg states that she had 24 hours of interview material, so in her study the engineers seemed to be quite willing to spend their time with the researcher but in other companies this could constitute a serious problem. When performing the study with students, the author made the experience that his understanding grew bit by bit, so he had to question the students and the colleague and students over and over again which is also more problematic in a real industrial setting. There is also a danger that a researcher restricts his investigation to those parts which he is able to understand.

- Detecting opportunities for further mathematisation: This issue is strongly related to the former one. If one does not want to restrict oneself to mere observations of mathematical thinking but also wants to investigate where a more mathematical approach might have made work more efficient and effective, one needs a very good understanding of the problems discussed in a company. For non-academic workplaces Noss and others looked for the potential benefits of using more mathematics in what they called "breakdown situations" where the normal routine did not work any longer. Gainsburg encountered such a situation in her studies when there were two different approaches to compute the propagation of forces in a building which were both unsatisfactory, and the discussion between engineers ended up with the decision to adopt the one that was well established in civil engineering. For the author who is not a civil engineer it is not clear whether a better understanding of the mathematics of both approaches would have led to a quicker and better solution but this would be interesting for detecting the potential of better mathematical understanding.
- Representativeness of companies and tasks: When visiting workplaces for some time, one can just choose a few companies and departments within companies. Hence, it is questionable in how far the findings can be generalized for a certain branch of engineering. Therefore, it is necessary to discuss the findings with other engineers, and – in the end – to conduct a variety of studies.

Further considerations

Although the above approaches have been discussed separately, this does not mean that just one of them is adequate. As stated above, both have their benefits and pitfalls and both also depend on the background of the researcher for being successful. Moreover, both can inform each other. The author found the results of Gainsburg and Kent/Noss very helpful for interpreting the work of the students. Looking for interesting breakdown situations helps to focus the attention when studying the material of the students. The results were also used when setting up the research questions formulated in section 2. The other way around, a study with students can also be a useful first stage before visiting companies. Hypotheses generated in such pre-studies can be checked with real workplaces.

Studies on the mathematical expertise of engineers are usually performed with the intention to improve university education. Therefore, as Gainsburg (2005) already observed, it is useful to relate the goals of mathematical education to the observations made in the study. Cardella used Schoenfeld's categories of mathematical thinking for her analysis which are very general but it might also be worthwhile to use Niss' (2003) mathematical competencies (and in particular those relating to modeling), adapted for the university level, for analyzing the mathematical parts of engineering work. This would be a refinement of Gainsburg's approach who already made her observations from a coarse modeling perspective.

In any case more studies of all kinds are indicated for corroborating hypotheses generated so far and for gaining further insights.

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Mathematical Modelling in the Singapore Curriculum: Opportunities and Challenges

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Abstract In an educational system that is entrenched in traditional modes of lesson delivery, it is likely that its school mathematics curriculum would also be traditional. Within such an environment, it would not be easy to make mathematical modelling a part of the mathematics curriculum. Using the Singapore school mathematics curriculum as a backdrop, we discuss the challenges faced by teachers, as well as opportunities made available to teachers who wish to engage in mathematical modelling activities in the classroom. Notwithstanding the fact that the Singapore mathematics curriculum has undergone several major changes in recent years, it remains largely traditional in form and nature. This paper also discusses the various attempts and efforts to promote the idea of mathematical modelling in schools and among school teachers.

Introduction

Mathematical modelling as a topic or a discipline provides an excellent interface between the mathematics that can be taught in a classroom and the mathematics that can be applied in the real world. Traditionally, school mathematics tends to emphasize acquisition of skills and understanding of concepts in various topics or fields of mathematics, with occasional mention of their utilitarian value in industries or connections to other disciplines. In fact, all too often, mathematics has been perceived by many as a subject consisting of formulae, theorems and proofs.

In the late 1980s, there was a shift in mathematics education towards making problem solving as a goal of learning mathematics. In Singapore, a new mathematics curriculum based on a framework in which mathematical problem solving is the central theme was designed and implemented in 1990 (Lim, 2002). The intention is to provide a suitable platform to engage students in learning mathematics for the purpose of solving mathematical problems. Since then, the curriculum has undergone several changes; some of these are minor refinements while others are major overhaul. The central theme, however, has remained unchanged. The latest framework for the Singapore mathematics curriculum is shown in Figure 1 (Ministry of Education, 2006a). Surrounding the central objective are five areas that the curriculum strives to develop in the mathematics learner: Skills, Concepts, Processes, Attitudes and Metacognition.

This so-called "pentagonal framework" applies to Singapore's school mathematics for all levels (that is, from Year 1 to Year 12), and serves as the basis for designs of mathematics curricula at these levels. It should be noted that Singapore's curricula for all subjects are controlled by the Curriculum Planning and Development Division (CPDD), which is a division of the Ministry of Education (MOE) of Singapore.

With mathematical problem solving as the central theme, it would be reasonable to expect mathematical modelling to fit in naturally in the school mathematics curriculum. In practice, however, this is not so. It was only a few years ago that "Applications and Modelling" was explicitly mentioned as one of the areas of development in the revised framework. Even then, not much has changed in the syllabuses of all the mathematics subjects offered in Singapore schools to promote or include mathematical modelling in the learning of mathematics.

It seems somewhat ironical that at the same time when an emphasis on mathematical modelling was made, for some reason, the topic "Particle Mechanics" was removed from the GCE "A" Level Mathematics, a subject offered by Pre-University students. One possible reason could be the shift towards Statistics and the use of graphic calculators means that

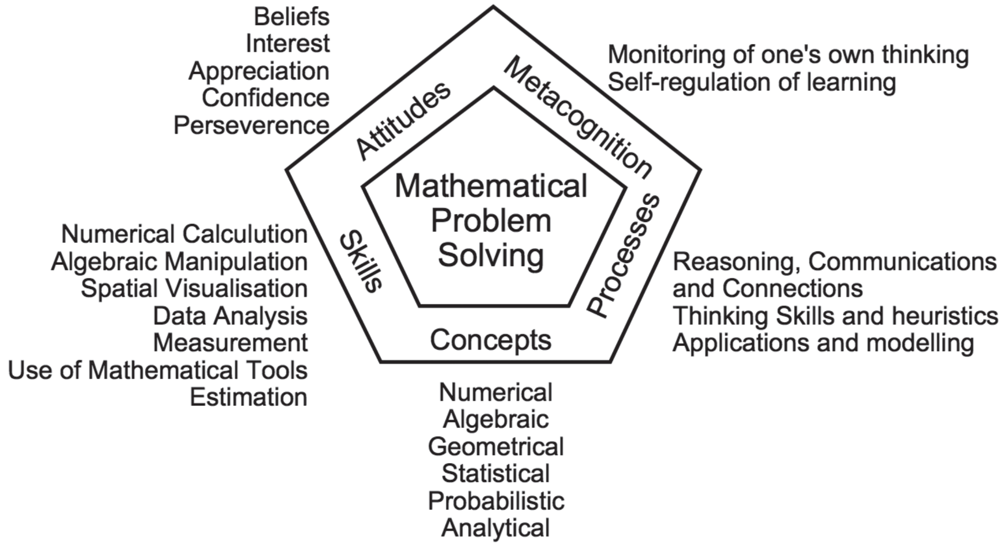


Figure 1—The Singapore Mathematics Framework.

something has to be deleted from the syllabus. Nonetheless, there appears to be a mismatch between the intended curriculum and the implemented syllabus.

Mathematical Modelling in Singapore schools -Challenges

There are many factors that make it difficult or even unattractive for Singapore schools and teachers to adopt mathematical modelling as an approach or a strategy in teaching mathematics in a typical classroom. In this section, we examine some of these challenges.

The Teacher

In general, it is fair to say that a typical mathematics teacher in Singapore is competent and hardworking. For a mathematics teacher, a typical lesson cycle would involve some lesson planning, designing of worksheets, lesson delivery, giving in-class work and homework problems, and so on. A typical teacher would have been trained to manage all these tasks with some confidence. However, facilitating a mathematical modelling activity may be quite different and may require a different set of skills and knowledge. In Singapore, teachers are neither trained nor sufficiently exposed to such an approach of learning mathematics. Moreover, mathematical modelling is not listed as part of the scheme of work at any level. There is, therefore, very little motivation for the teacher to engage in modelling activities.

Many Singapore mathematics teachers are highly dependent on textbooks and resource books in planning and conducting their lessons. These resources, together with additional worksheets, are often more than enough for the teacher to deliver the required syllabus and fulfil the curriculum. Without appropriate material and ideas, the teacher may not have the capacity or knowledge to carry out good mathematical modelling activities.

While many school teachers in Singapore are university graduates, not many who have been assigned to teach mathematics have had any experience in or exposure to mathematical modelling. Even teachers who have a mathematics degree may not have heard of mathematical modelling, much less acquired the thinking skills needed. It is therefore understandable if Singapore mathematics teachers feel unprepared and inadequate to teach mathematical modelling in the classroom (Ang, 2001).

The nature of mathematical modelling requires one to be open and ready to accept multiple solutions to a problem at times (Yanagimoto, 2005). Sometimes, a solution may not even exist. It may require a significant change in the teachers' mindsets for them to accept this fact. Such a paradigm shift takes time and is not easy to achieve in an environment where the teacher is often viewed as an authority of the subject knowledge. Moreover, the teacher may not have the confidence to undertake or carry out tasks which are so open.

The Learner

In an examination obsessed environment like Singapore, many students are driven by assessment. Assessment appears to be the main reason why students spend time on their school work. There are on-going debates and arguments over whether the assessment system in Singapore should be changed, or at least modified or refined. Nonetheless, it seems that in the minds of many students, something is worth studying only if it is included in the tests or examinations. In other words, if mathematical modelling does not form an assessable component in the final test or examination, students may not be as motivated to be engaged in it. Mathematical modelling is more than just problem solving and very often, students must be prepared to share their experiences or communicate their products (English and Watters, 2004). This is quite a challenge for students in Singapore schools who are known to be lacking in communication skills. Very often, modelling activities are also social experiences requiring students to develop conjectures or manage conflicts, among other things. These are areas which are new to Singapore mathematics students and will take time for them to get used to.

Higher 3 (H3) Mathematics is a subject for Pre-University students. It has been offered as a GCE 'A' Level subject in Singapore since 2006, and is meant for students with a strong aptitude for mathematics. In the current H3 Mathematics syllabus, the application of differential equations as mathematical models is explicitly mentioned. More specifically, the topic includes mathematical models in population dynamics. The syllabus is intended to provide students with the “opportunity to further develop their mathematical modelling and reasoning skills” (Ministry of Education, 2006b). However, from the students’ perspective, these “experiences” in mathematical modelling are neither authentic nor relevant. Very often, these experiences are reduced to typical assessment or textbook exercise items which local students may not be able to relate to.

Promoting mathematical modelling

Despite all the challenges discussed in the preceding section, there is a sense of determination amongst the local mathematics education community to step up efforts in promoting the idea of teaching mathematical modelling within the confines of the current mathematics curriculum. We discuss some of these efforts.

Resources

One of the challenges mathematics teachers face is the lack of good resources on mathematical modelling activities or problems which they can use for their classroom. These problems or activities should be manageable and something that students can understand and undertake. In an effort to help fill the gap, local mathematics educators have made attempts to produce relevant resources in the form of booklets (Dindyal 2009; Ang, 2009b). These booklets contain many examples of mathematical modelling activities suitable for students at different levels, and using a range of topics from the existing school mathematics curriculum.

In addition, teachers may also study exemplars of mathematical modelling problems which are relevant to the local students. As an example, consider the case of an outbreak of the disease, Severe Acute Respiratory Syndrome, or SARS in short. In 2003, Singapore was inflicted with this deadly disease which spread quickly in a short period of time. Over the 70 days of outbreak, 31 lives were lost. A simple model for a SARS epidemic can be constructed using the mathematics taught in H3 Mathematics. In particular, a model using the logistic equation can be constructed (Figure 2). This may be further refined using a “double logistic” curve (Ang, 2004). This example provides a relevant context for students to discuss the model and experience the process of mathematical modelling.

When this idea was shared with a group of H3 Mathematics teachers at a recent workshop, they all agreed that they need more exemplars of this nature. More examples of mathematical modelling activities specifically targeted at local teachers have also been published (see, for example, Ang and Neo, 2005; Ang, 2006; Ang 2009a).

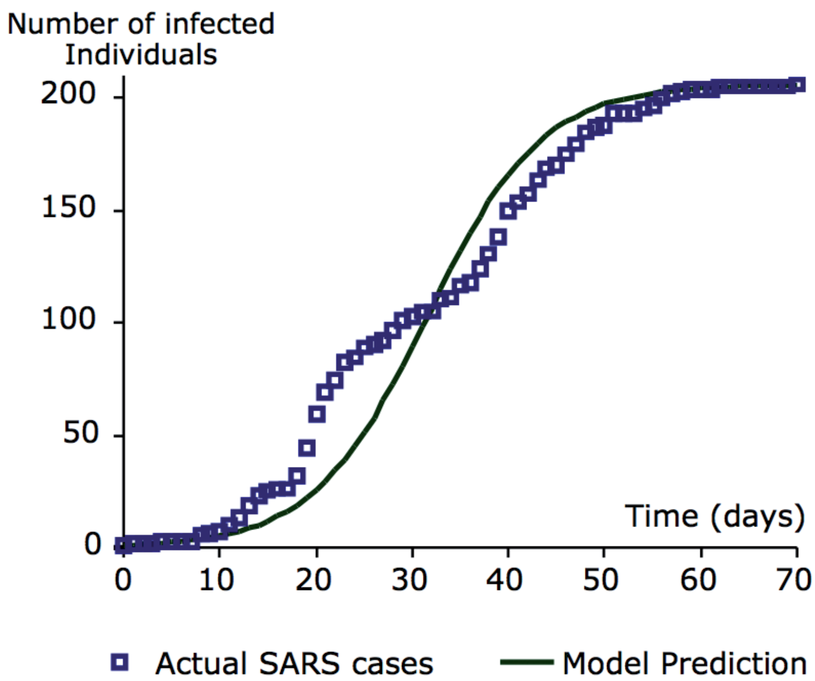


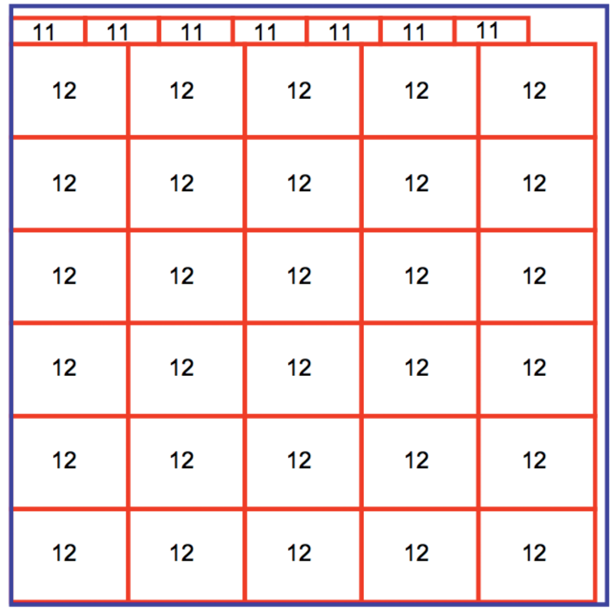
Figure 2—A simple logistic model for a SARS outbreak (Ang, 2004).

Singapore International Mathematics Challenge

The inaugural Singapore International Modelling Challenge (SIMC) was held in May 2008. A total of 208 mathematics students and mathematics educators from 26 institutions in twelve countries participated in this event. This biennial event distinguished itself from the well known Mathematics Olympiad as its focus is on mathematical modelling and applications of mathematics to solve real world problems. In this event, students were given two challenges, known as a "pre-site challenge" and an "on-site challenge". The problem for the "pre-site challenge" for SIMC 2008 is shown below.

> Each week, a glass manufacturing company constructs small rectangular windows by cutting one large 5 metre by 5 metre sheet of glass. The company gets orders for several rectangular window pieces of varying widths and heights. When the glass cutting gets done, there is typically some waste incurred in the large sheet which the company is forced to throw away. The company wants to minimize the wasted left-over glass in the large sheet while fulfilling the demands as much as possible. Develop an algorithm to minimize the wasted left-over glass or alternatively maximize the used glass in the large sheet. Your algorithm should provide a positioning of the smaller rectangular windows that will be cut on the larger sheet while taking care that the problem should be solvable in a reasonable amount of time. Provide a detailed evaluation of the performance of your algorithm on several data instances. A sample data instance

Piece	Width (mm)	Height (mm)	Orders
1	160	875	20
2	208	134	30
3	213	684	50
4	268	745	60
5	305	416	100
6	412	727	100
7	499	439	100
8	576	371	50
9	629	840	50
10	630	608	50
11	644	214	50
12	789	991	100
13	934	440	30
14	961	321	50
15	971	768	50



Possible cutting pattern with 2.3133% wastage

Figure 3.

with a possible cutting pattern (not necessarily the best) is indicated below.

[For graphical covenience the image following the preceeding text is in figure 3.]

Programmes for Teachers

To further expose teachers to the idea of mathematical modelling and raise their awareness of this approach of teaching mathematics, the local community of mathematics educators has planned special programmes for teachers in recent months, and will be planning more activities in the coming year. A few of these are listed below.

Mathematics Teachers Conference

The theme of the Mathematics Teachers Conference 2009 (held at the National Institute of Education) was “Mathematical Applications and Modelling”. Invited speakers for this conference include Professor Gabrille Kaiser (University of Hamburg, Germany) and Dr Gloria Stillman (University of Melbourne, Australia). Close to one thousand local teachers participated in this one-day conference, which feature talks and workshops.

Feedback from participants was collected. While talks and workshops in this conference managed to serve their purpose of raising teachers’ awareness on mathematical modelling, it will probably take some time before their level of confidence and competence can be raised.

Workshops

In view of the fact that many teachers still do not have a clear idea of what mathematical modelling is all about, the CPDD plans to address this problem by running workshops on mathematical modelling in the near future. These workshops will acquaint teachers with the basic processes involved in mathematical modelling and provide ideas for teachers to try some activities with their students.

Mathematical Modelling Outreach

The Mathematics and Mathematics Education (MME) academic group of the National Institute of Education (NIE) will be organising an event called "Mathematical Modelling Outreach" (or MM Outreach) in June 2010. This event is focussed on mathematical modelling and is not a competition. Schools will be invited to send a few students each to participate in this 4-day event, which will culminate in oral presentations of the students' products.

Teachers from participating schools will be trained to be facilitators by faculty members of MME before the actual event. During the 4-day event, MME faculty members will assist these facilitators as they carry out the modelling activities with student participants. As this is the first time this event is being organised, the scale is intentionally kept small. Only Year 5 and Year 8 students will be invited to participate.

Conclusion

A mathematics curriculum does not need drastic changes for mathematical modelling to be a part of a learner's learning experience. Indeed, it is more crucial to provide the necessary support that teachers need for them to embrace this mode of teaching mathematics. Even in a traditional mathematics curriculum, with sufficient exposure to and experimentation in the bolts and nuts of mathematical modelling, it is possible for a teacher to exploit these ideas and use them in a mathematics classroom. Lingefjard argues that mathematical modelling can be used as "a way to summarize and assess the mathematical competencies the students possess" (Lingefjard, 2007, p. 336).

While it is difficult for a teacher to attain the same level of skills in mathematical modelling that an applied mathematician would have acquired over time, it is possible for the teacher to learn alongside his students. The bigger challenge is for the teacher is to change his mindset and become a partner in learning; perhaps the teacher can continue to be an active learner and show the way for the students to learn mathematics in a fun and practical way through mathematical modelling.

It seems certain that Singapore's future will be transformed by technology, global trends and the changing demographics. The profile of our students continues to evolve and change. All these will impact the overall education curriculum of the nation. The latest initiative in curriculum planning recommends four important student outcomes, of which "self-directed learner" is one. Because of its very nature, mathematical modelling could play an important role in achieving this outcome. There is hope that mathematical modelling could feature prominently in the Singapore mathematics curriculum in the near future.

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First results from a study investigating Swedish upper secondary students' mathematical modelling competencies*

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Abstract This paper reports on the first results from a study investigating Swedish upper secondary students' mathematical modelling competency with the aim to inform and enlighten relations and discrepancies between upper secondary mathematics and mathematics as applied in industry. Using non-parametric statistical methods the data from 381 students where analysed and the students' modelling competency described in terms of seven sub-competencies. Possible factors affecting the students' mathematical competency such as attitudes toward modelling, previous experiences, last taken mathematics course, grade, class and gender were also investigated.

* A slightly different version of this paper is submitted for publication in the proceedings of the ICTMA 14 conference held in Hamburg 27-31 July, 2009.

Introduction and purpose

Pollak (1969) in discussing the relationship between mathematics as taught in schools and applications of mathematics as used outside the school setting notes that genuine applications of mathematics steaming from real life and other scientific disciplines, different industrial context included, as presented to the students in the textbooks are rare. However, as pointed out by De Lange (1996) the use of applied problems in mathematics education can function as a motivation factor and that the recent and continuing development in computers makes more real problems available for the mathematics classrooms. In terms of ICMI 20 study the aim of this paper is "to broaden the awareness of industry with respect to what school and university graduates can and cannot do realistically in terms of mathematics" (Damlamian & Sträßer, 2009, p. 526).

What do Swedish upper secondary students know about mathematical modelling and how capable are they of solving modelling problems? These questions are asked in the background of the internationally growing interest in the field of educational research in mathematics focused on applications and modelling and part of the aim of the ICMI Study 20. This paper discusses the first results of part of an empirical study conducted to enlighten the present situation at the upper secondary level in Sweden with respect to these issues.

Since 1965 there has been an increasing explicit emphasis on mathematical modelling in the written curricula document governing the Swedish upper secondary mathematics education (Ärlebäck, submitted). In the present mathematics curriculum it is stressed that "[a]n important part of solving problems is designing and using mathematical models" and that one of the goals to aim for is to "develop their [the students'] ability to design, fine-tune and use mathematical models, as well as critically assess the conditions, opportunities and limitations of different models" (Skolverket, 2000). Indeed, *using and working with mathematical models and modelling*, *problem solving*, *communication* and *the history of mathematical ideas* are emphasized as four important aspects of the subject that should permeate *all* mathematics teaching (ibid.). However, a more explicit definition is not given and thus this description of mathematical modelling opens up for interpretations.

The aim of this paper is to get an initial indication of the level of the mathematical modelling competency of Swedish upper secondary students. In addition, it also investigates if factors such as grade, gender, last taken mathematics course, and different attitudes might affect the level of success of students solving modelling problems.

The research questions we addressed in this research in general terms were:

1. What modelling competency do Swedish upper secondary students in 12th grade display?
2. Are there any connections between the students' modelling competency relation to mathematical achievement in general (grade), gender, students' interest, last taken mathematical courses, or to previous experiences?

Methodology, theoretical considerations and method

Rather than to device a research instrument of our own we decided to try to use an already existing and tried out tool, and after having scanned the research literature we decided to use the research instrument developed and constructed by Haines, Crouch and Davis (2000). The instrument, also reported in Haines, Crouch and Davis (2001), originally consisted of 12 multiple-choice questions (five alternative choices) together with a partial credit assessment model assigning a score of 2 to one preferred of the five alternatives in each question; 1 to one or more other choices since "an alternative response could indicate knowledge and understanding in mathematical modelling" (Haines et al., 2000, p. 5); and 0 to the remaining alternatives. Using the words by Houston and Neill (2003a, pp. 156-159) the 12 questions, grouped in six pairs used in a pre-post test setting, focused on the following aspects of the modelling process (see also section 2.1 below): *making simplifying assumptions*; *clarifying the goal*; *formulating the problem*; *assigning variables, parameters, and constants*; *formulating mathematical statements*; and *selecting a model*. The instrument was "devised both to address the need for a base level assessment of modelling skills and for application during or on completion of an experience in mathematical modelling" (Haines et al., 2000, p. 2) and the authors argue that using the instrument, "it is possible to obtain a snapshot of student' [modelling] skills at key developmental stages without the student carrying out a complete modelling exercise" (ibid., p. 10).

The number of test items was extended to 18 by Houston and Neill (2003b), adding one new question to each of the six aspects above. In addition, Haines, Crouch and Fitzharris (2003) extended the numbers of items adding two questions involving *graphical representations* and two questions *exploring real and mathematical world connections*, making a total set of 22 test items covering eight aspects of the modelling process.

Besides being used in the research referred to above, the research instrument has also been drawn on and used in different settings with a variety of objectives in Haines and Crouch (2001); Izard et al. (2003); Ikeda, Stephens and Matsuzaki (2007); Lingefjärd and Holmquist (2005); and Kaiser (2007), to among other things, investigate the levels of students' modelling competencies.

The view of mathematical modelling underlying the construction of the research instrument mentioned in previous section is represented by the left diagram in figure 1. Note that the 'content' of the boxes in this diagram are of different types and on different levels; *real world problem* represent the real word situation or phenomena under consideration; *formulating model*, *solving mathematics*, *interpreting outcomes*, *evaluating solution*, and *reporting* are all processes, or to use the terminology of Haines et al. (2000) — *skills*, involved in mathematical modelling; *refining model* is also a process but on another level in the sense that it is often compound by the other processes just mentioned, meaning that the modeller(s) goes back to the real world problem and possibly *re*-formulate, *re*-solve, *re*-reinterpret and *re*-valuate her/his/their work. However, the right diagram in figure 1 makes a more clear distinction between the corresponding processes and *refining model* in this representation of mathematical modelling means engaging in another cycle. It also situates the processes in relation to the intra- and extra-mathematical worlds. In spite of the differences between the two diagrams in figure 1 we believe that the respective authors as a matter of fact share more or less the same overall view on mathematical modelling, but it must be stressed that both these views of mathematical modelling are highly idealized and schematic representations of the complex processes involved. A similar view is presented by Palm et al. (2004) in their interpretation of the written curriculum documents governing the Swedish upper secondary mathematics education, and this suggests that the research instrument described above could adequately be applied also to the Swedish context.

Mathematical modelling as presented in figure 1 is often described using the notion of *modelling competencies*. However, the meaning and content of this concept varies among its users; Blomhøj and Højgaard Jensen's (2003) definition is "[b]y mathematical modelling competence we mean being able to autonomously and insightfully carry through all aspects of a mathematical modelling process in a certain context." (p. 126), whereas Maaß (2006) definition is "[m]odelling competencies include skills and abilities to perform modelling processes appropriately and goal-oriented as well as the willingness to put these into action." (p. 117). The latter definition we find ambiguous and problematic for two reasons; first it is not clear what *skills* or *abilities* are, as to the relation between these two concepts; and secondly, the emphasis on *willingness* seems to lack reasonable motivation and has weak grounding, and in addition makes *modelling competencies* a concept hard to operationalise. We believe that the definition suggested by Maaß is incompatible with the research instrument initiated by Haines et al. (2000) and that the solely use of this can not productively be used to analyse students' mathematical modelling competencies defined in this manner as suggested by Kaiser (2007). Hence, in this paper we chose to define modelling competence in

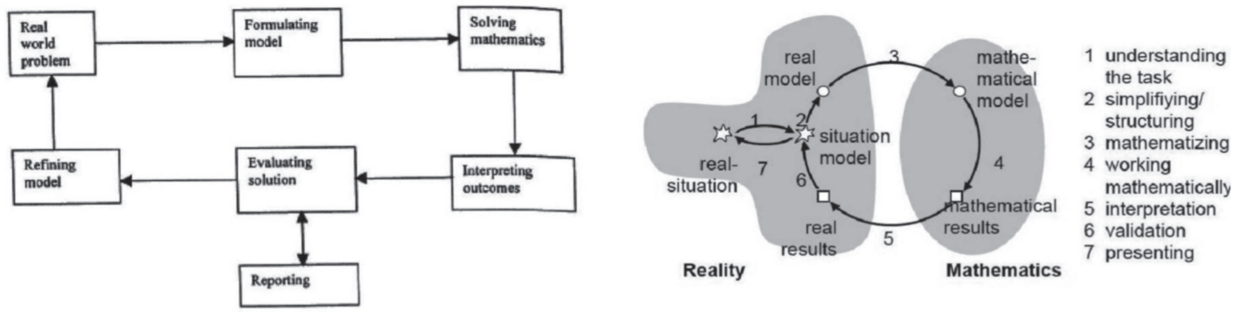

Figure 1—To the left are the "[s]tages in the mathematical modelling process" such as presented by Haines et al. (Haines et al., 2000, p. 3) and to the right is the "modelling cycle" of Blum and Leiß (2007) as presented by Borromeo Ferri (2006, p. 92).

line with Blomhøj and Højgaard Jensen (2003) quoted above, and refer to the processes involved in mathematical modelling, previously described in terms of the eight aspects of the modelling process, as *modelling sub-competencies*.

Developing an instrument

All the 22 items where translated into Swedish and effort was made to make the translations as true to the original formulations as possible only adjusting some of the details to the Swedish context where this was appropriate. The translations were checked by a third independent researcher before the items where piloted in a group of 16 students, each individually assigned eight items distributed so that we roughly got the same number of responses on all 22 items. In addition, for each item the students were asked to answer the following three questions: (1) *Do you think that the problem you just solved is relevant for a mathematics class?*; (2) *Do you think the problem is interesting?*; and (3) *Do you think the problem is connected to reality?* by choosing 'Yes', 'I don't know' or 'No' and to give a short motivation for their choices. The pilot study served to check how the translated items worked in practice, how much time the students needed to complete the eight items, and to give us a first impression of the students' feelings for, and attitudes towards, working on the items. During the time the students worked on the pilot test, which varied between 20 and 30 minutes, the first author surveyed the class and recorded comments.

Taking some of the recorded students' comments from the piloting into account together with the wish to push the test time down to approximately 20 minutes we decided to cut out one of the items. In doing this, we also decided to incorporate the aspects of modelling listed as *graphical representations* with *selecting a model* since these two both focus on the selecting of a mathematical model; in terms of a graph in the first case and a formula in the second. Hence there are totally four items focusing on the aspect of *selecting a model*. So, the view taken on *modelling competency* in this paper is that it as least is constituted of the following sub-competencies: (sC1) *to make simplifying assumptions concerning the real world problem*; (sC2) *to clarify the goal of the real model*; (sC3) *to formulate a precise problem*; (sC4) *to assign variables, parameters, and constants in a model on the basis of sound understanding of model and situation*; (sC5) *to formulate relevant mathematical statements describing the problem addressed*; (sC6) *to select a model*; and (sC7) *to interpret and relate the mathematical solution to the real world context* (cf. Kaiser (2007, p. 115–116)). In addition, the three follow-up questions subsequent to each item were also cut out and replaced with the following seven four-alternative-Likert attitude questions ending the test: *I consider the problems on the test to be* (Q1) *fun*, (Q2) *easy*, (Q3) *interesting*; (Q4) *I think the problems on the test invite you to use mathematics to answer the questions*; (Q5) *I think that problems of this type are well suited for the upper secondary mathematics courses*; (Q6) *In the upper secondary mathematics courses we often work(ed) on similar problems*; and (Q7) *I would like (would have liked) to work more often on similar problems in the upper secondary mathematics courses*.

The pilot study was also used to make a selection of 14 of the original 22 test items; two each representing the seven sub-competencies. The selection was based on the students' results on the test items, and those items which displayed non-extreme answer distribution, which was not considered to be due to interpretational issues of item formulations, was selected out. This means that items in which the students all got full score was discharged in favour of items in which they achieved more moderately. After a discussion, the 14 selected items where grouped into two groups, our first (FHC) and second (SHC) hand choice respectively, from which four tests consisting of seven item each where constructed; test T1 and T2 containing solely items from the two respective groups, and test T3 and T4 consisting of a mixture of items from the two. The total score a student achieves on either of these four tests is what we take as a measure of the students' modelling competency.

For the final version of our research instrument we decided the first instant to be two quotes from the curriculum guidelines for the Swedish upper secondary mathematics courses (the ones we used in the background section) followed-up by two question; *Have you ever encountered the word 'mathematical modelling' during your upper secondary education?* requiring just a 'yes' or 'no' answer; and the open question *Describe in your own words the meaning you ascribe to the concepts mathematical model and modelling*. Next, the seven items followed and then the attitude questions Q1–Q7. In addition, the students where asked to state their gender, latest taken upper secondary mathematics course (in Sweden there are in principle five such courses, one per term, Mathematics A – Mathematics E), and their latest received grade in mathematics (every course is graded as either IG = Fail, G = Pass, VG = Pass with distinction, MVG = Pass with special distinction).

In a national science and mathematics teachers developmental program with participants form all over Sweden, 41 sets of test were distributed in the spring of 2009 and asked to be brought back to their respective school and given to mathematics teachers teaching 12th graders in the science program (normally, at this time of year the students have completed the Mathematics D course). Each set of tests contained 30 of the tests numbered from 1 to 30 with every fourth test a T1, T2, T3 and T4 test respectively starting randomly in that sequence. A letter was also attached to the mathematics teacher informing on the aim of the research and ethical considerations taken, as well as practical requests for how to distribute the tests in their class; to use approximately 20-25 minutes; not to allow the students to use calculators; and for the students to solve the problems individually. In addition, the teachers were asked to fill in a teacher questionnaire, but this will not be reported on in this paper.

In all, we got back 21 sets of test (51%) resulting in test scores from totally 400 students. However, for the statistical analysis, which was made using SPSS, we only analysed the 381 students who answered at least four of the seven items on the tests, since we consider the instrument not to give a reliable measure of the students' modelling competency otherwise. In this paper we will only report on the quantitative data; the results of the analysis of the students' answers on the open questions in the test will be reported on elsewhere.

Statistical analysis

A first statistical analysis using a Kolmogorov-Smirnov test with Lilliefors significance correction showed that the data was not normal distributed, which is not surprising since our sample (N=381) is rather big. However, this lead us to use non-parametric tests for the continued analysis although a rough analysis at hand using a parametric approach reviled the same result at approximate the same level of significance.

The particular tests used were the Mann-Whitney test, the Kruskal-Wallis test, and the Kendall's tau; the Mann-Whitney test is the non-parametric equivalent test to the parametric independent t-test which uses a ranking procedure to compare two independent groups; the Kruskal-Wallis test also uses ranking techniques to compare more that two independent groups; and the Kendell's tau is the non-parametric equivalent to the Pearsons' correlation coefficient.

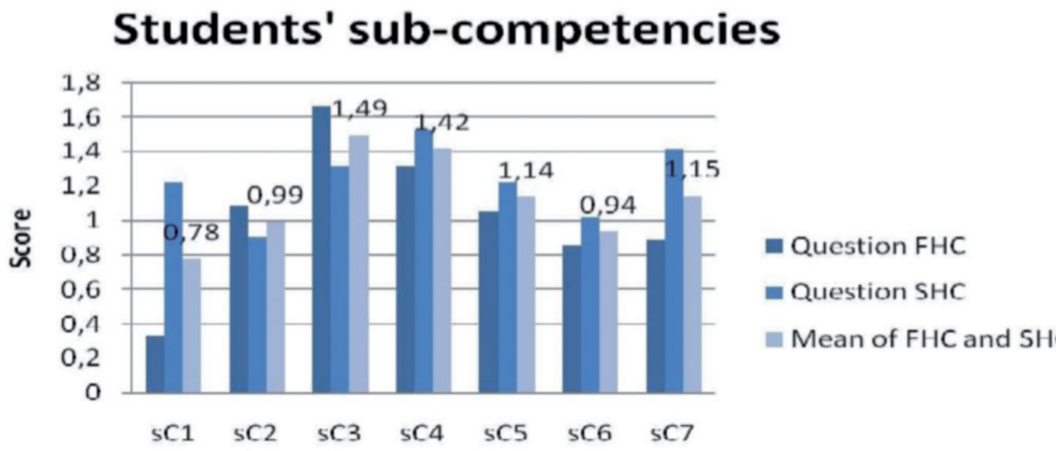

Figure 2—Students' sub-competencies with items relating to FHC and SHC

Results

With a maximum score of 14 the students scored in average 7.78 (Sta.Dev. 2.26). On the 14 different test items, corresponding to the seven sub-competencies, the students' scored in average between 0.33 and 1.67 as illustrated in figure 2.

Comparing pairs of items in respective sub-competence (at a level of $p<.05$) indicate a comparability of the pairs only in sC2 ($H(3)=5.868$, $p=.118$) and sC6 ($H(3)=6.397$, $p=.094$). The pair of items in sC5 are a borderline case ($H(3)=7.809$, $p=.050$).

The mean and standard deviation of the students' total score with respect to gender, classes, grades, last taken course and tests are summarized in the table 1.

When analysing grades and students' modelling competency the four students with grade IG were excluded, because these scored in average 7.5 which are not representative of the IG grade. The grades G, VG and MVG showed a significant affect ($p<.05$) on the students' total scores ($H(2)=27.853$, $p=.000$). However, just considering the grade VG compared with MVG appeared not to have an affect on students' total scores ($U=6720$, $p=.132$).

The students' last taken mathematics course also had a significant affect ($p<.05$) on the students' modelling competency ($H(2)=10.772$, $p=.005$). A further investigation showed that mathematics course D compared with course E had affect on students' total scores ($U=9439$ $p=.005$), but no affect was found between the courses C and D ($U=342.5$ $p=.568$).

Other factors significantly affecting ($p<.05$) the students' total scores were which test they took ($H(3)=27.996$, $p=.000$) and from which class the student belonged to ($H(20)=0.437$, $p=.004$). However, no effect was found with respect to gender ($U=13503.5$, $p=.363$).

Results	Gender		Classes	Grade				Course			Test			
	Female	Male		IG	G	VG	MVG	C	D	E	T1	T2	T3	T4
Mean	7.95	7.76	6.38-9.27	7.50	7.07	7.96	8.43	6.67	7.17	7.99	6.82	8.37	7.63	8.26
Std.Dev	2.22	2.29	1.57-2.84	2.38	2.16	2.34	2.05	1.73	2.12	2.28	1.93	2.24	2.04	2.48
Number	121	237	8-29	4	130	118	128	9	84	281	95	94	93	99

Table 1—Mean value with respect to gender, classes, grades, courses and tests

Results	Attitudes and previous experiences						
	(Q1) Fun	(Q2) Easy	(Q3) Interest	(Q4) Invite math	(Q5) Good mom.	(Q6) Done similar	(Q7) Work more
Mean	2.89	3.25	2.63	2.33	2.41	3.37	2.61
Std.Dev	0.90	0.70	0.94	0.81	0.93	0.86	0.99
Number	376	374	377	372	374	375	375

Table 2—Mean values of the questions about attitudes and previous experiences (1=strongly agree; 2=agree; 3=disagree; 4=strongly disagree; mean 2.5)

Table 2 summaries the students' responses to the questions (Q1)–(Q7) and the only attitude having a significant affect ($p < 0.5$) on the students' modelling competency was if the student considered the problems in the test to be (Q2) *easy* ($H(3)=10.912$, $p=.012$) and to be (Q3) *interesting* ($H(3)=18.292$, $p=.000$).

Investigating par wise correlations among the affecting factors on the students' modelling competency, significant correlations ($p<.05$) were found between grades and respectively course ($\tau=-.342$, $p=.000$), easy ($\tau=-.164$ $p=.000$) and interest ($\tau=-.116$, $p=.010$). In addition, interest also correlated with course ($\tau=-.109$, $p=.021$) and easy ($\tau=.217$, $p=.000$).

It is notable that only 22.5% of the students have heard/used mathematical models or modelling in school. For those how have heard/used mathematical models or modelling in school, it did not show any affect on their total score ($H(1)=.041$, $p=.839$).

Discussion

In comparing the sub-competencies of the Swedish upper secondary students' modelling competency, fig. 2 shows that they were most proficient in questions relating to sC3 and sC4, but exhibited more difficulties in questions relating to sC1, sC2 and sC6. The sub-competence sC2 has also been proved to be difficult for the students in previous research (e.g. Houston & Neill, 2003a; Kaiser, 2007). The notable difference between the two items in sC1 might be an effect due to translation or interpretational problems. Not surprisingly, students' grade and students' last taken mathematical course have positive affect on students' modelling competency.

Looking at the results from the attitude questions Q1–Q7 there are is an overall negative tendency towards working with mathematical modelling as represented in the test items in all answers. In general, the students found the problems very hard (Q2) and did not express any excitement or joy in tackling them (Q1). Neither did the students express that they found the problems especially interesting (Q3) nor that they wanted to (have) work(ed) more on similar problems in their mathematics classes (Q7). However, the students to some extent seemed to recognize the value to use mathematics to solve the problems on the tests (Q4), and in addition regarded the types of questions asked relevant and good to use in mathematics classrooms (Q5). One explanation to these results might be due to the fact that the students expressed that they in principle never worked on similar problems before (Q6). However, such student attitudes may present an obstacle for implementing mathematical modelling at this school level.

In line with Haines et al. (2000) we agree that all the individual stages of mathematical modelling represented in the sub-competencies are part of the modelling process. However, the instrument lack other aspects of the modelling process such as the use of ICT, the fact that not a 'whole modelling problem' is solved, and collaborative work, which means that the research instrument does not provide a complete picture. Nevertheless, in fulfilling the aims of the study, the tests items are adequate in that they allow many students to be tested in a short time and give a first preliminary overview of the present state in the Swedish upper secondary mathematics regarding students' mathematical modelling competency.

In evaluating their research instrument Haines and Crouch (2001) concludes that "the analogue pairs of items are predicted to perform in a comparable manner" (p. 133), except for the two items used in sC3. Due to our big sample we suspected to conclude approximately the same comparability. However, in our study the only comparable pairs of items in respective sub-competencies are sC2 and sC6 (and possibly sC5). Note that the analysis in Haines and Crouch (2001) only investigate 'the originally six stages' in the modelling process and that only the first five are comparable to our sC1–sC5 respectively.

The results on the relations between and among the students' modelling competency and their expressed attitudes (Q2) and (Q3) together with the many correlations found between the attitudes indicate that a more advanced analysis might be fruitful using a more sophisticated statistical model and method. This we plan to do in a forthcoming paper.

Conclusions

The investigation of the modelling competency of Swedish upper secondary 12th grade students revealed that the students' were most proficient in the sub-competencies *to formulate a precise problem* and *to assign variables, parameters, and constants in a model on the basis of sound understanding of model and situation*, and least proficient in *clarify the goal of the real model* and *to select a model* (if *to make simplifying assumptions concerning the real world problem* is disregarded). The study also shows that the students' grade, last taken mathematics course, and if they thought the problems in the tests were easy or interesting, were factors positively affecting the students' modelling competency. In addition, only 23% of the students stated that they had heard about or used mathematical models or modelling in their education before, and the expressed overall attitudes towards working with mathematical modelling as represented in the test items were negative.

In connection to the aim of the ICMI Study 20, the result clearly indicates that Swedish upper secondary students do not have much experience in working with real situations and modelling problems, and that the incorporation of real problems from industry in the secondary mathematics classroom might be problematic. A closer collaboration with representatives from the industry working directly with classroom teachers and didacticians could provide an opportunity to enhance the students' proficiency in this respect. Together such a team could engage in the designing, implementing and evaluating teaching were the connection to industry, and the relevance of mathematics to industry, is made both explicit and accessible.

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Laboratory of Computational Mathematics: an interface between academia and industry

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Abstract During the last decades there has been a growing demand for mathematicians for working in industry (SIAM, 1998). The importance of mathematical knowledge in an industrial context, became more and more relevant. Business, industry and government provide a fertile domain for application of advanced mathematics. To face this reality, universities feel the need to adapt their curricula in mathematics and to create better interfaces with industry.

The Laboratory of Computational Mathematics (LCM) of the Centre for Mathematics of the University of Coimbra (CMUC) was created to identify relevant social and industrial problems that could benefit with the expertise of mathematicians and to give visibility to the research done in CMUC that could be applied by the industry.

The aim of this paper is to analyse the work done in the LCM in two perspectives: the relationship between the university and industry and the relevance of the LCM activities in educational mathematical programmes

Introduction

According to several studies, "if Europe is to achieve its goal of becoming the leading knowledge-based economy in the world, mathematics has a vital role to play" (MACSInet, 2004). In spite of the fact that, in many industrial sectors, the value of mathematics is already proven, there is a need for positive action to promote the use of mathematics by European industry. A dynamic mathematical community interacting actively with industry and commerce, on the one hand, and the science base, on the other, have been pointed out as important ingredients for competing in the global market of the future where innovation will be the key to success.

The OECD report (OEDC, 2008) asserts that "while mathematics presents industry in the 21st century with major opportunities, it faces significant structural challenges in the industrial environment. A strong pressure to organise research and development around well-defined projects, combined with an increasing trend to outsourcing, has led even large companies to significantly reduce their investment in the mathematical sciences.[...] Nonetheless, strongly innovative companies that properly exploit mathematics can rapidly gain a commercial edge over their competitors. This is illustrated dramatically by the success of start-up companies selling custom-designed software."

It is widely perceived that graduate education in mathematics focuses almost exclusively on preparation for traditional academic research careers. Also, because of the interdisciplinary and diversity which non-academic employers typically demand, the knowledge of technical areas outside mathematics is of utmost importance in nonacademic positions.

In Portugal, the number of PhD and Master graduates with a degree in mathematics is small and the quantity of those having non-academic careers is almost imperceptible. Nevertheless, some indicators show the tendency for a slight change. With this scenario, there is the perception that the creation of institutional high level connections between academic mathematics groups and industry could produce a favorable impact (Vicente, 2006).

The activity in LCM

The Laboratory of Computational Mathematics was created in April 2005, integrated in CMUC, a research center that comprises most of the research-active members of the Department of Mathematics of the University of Coimbra (DMUC). Currently, the Centre has 65 members holding PhDs and 17 research students. The Centre makes a significant contribution to fundamental mathematics and includes research in Algebra, Analysis, Numerical Analysis, Optimization, Probability and Statistics, Geometry, History and Methodology of Mathematics.

The foundation of LCM followed the recommendations regarding the CMUC activities of the research unit evaluation panel in 2002 (FCT, 2002): “Computational Mathematics and Numerical Analysis are important subjects on which Portugal is somewhat lagging behind in spite of isolated pockets of expertise. The panel believes that this is well positioned and has the capacity needed to lead a nationwide initiative, and to provide a solid foundation on which to build a major Center of Excellence in Scientific Computing/Computational Mathematics in Portugal”.

During its five years of existence, LCM has been promoting research in computational mathematics and scientific computing as techniques for the solution of challenging problems arising in biological sciences, engineering, finance, and management. The activity of LCM includes interdisciplinary research, high performance computing and the development of numerical software, in collaboration with industry, and also promotes Ph.D.’s and Master’s graduate education.

The relationship with industry

The term “industry” has been used to denote business and commercial firms, research and development laboratories, commercial and not-for-profit research, and production facilities, i.e., activities outside the sphere of education and purely academic research.

As an interface with the University and industry, one of LCM’s major tasks is to identify relevant social and industrial problems that should be tackled. The Portuguese Science Foundation, which supports most of the projects in LCM, has been considering the relevance of the project towards obtaining comparative advantages for Portugal a core criteria for funding, in accordance with the objectives stated in its strategic plan.

To identify such problems, contacts outside the mathematics community are important. To pursue this goal, LCM organized several workshops and an ECMI study group, where industry was invited to present problems that could be worked by mathematicians.

The project portfolio of the LMC includes 12 application-oriented projects, some having industrial partners, namely hospitals, banks and bank holding companies, data and software companies, and industrial firms.

As an example, we highlight the project “Simulation of a Moving Bed Reactor used in the Pulp and Paper Industry”. The pulp and paper business is indeed one of Portugal’s most important industries and an important mill of the major Portuguese firm Portucel, which is one of the world’s biggest producers of bleached eucalyptus Kraft pulp for the packaging industry and one of Europe’s top five producers of uncoated wood-free paper, is located near Coimbra. In order to optimize the quality of the pulp, this industry has a real need for tools that enable the simulation of experiments that cannot be afforded or that might be risky in a real industrial context. The most critical piece of equipment in a Kraft pulp and paper plant is the digester, known as the heart of the mill. It is a very special and complex heterogeneous reactor where a moving bed of wood chips contacts and reacts with sodium hydroxide and sodium sulphide in a liquid phase (Kraft process), in order to dissolve lignin and therefore to release the fibers of cellulose. In this context, the incidence of the work developed at LCM is twofold: from an engineering point of view, the system of equations presented furnishes a description of the transient behaviour of the digester which allows the prediction of the quality of the pulp when some changes in the wood properties occur; from a mathematical point of view, the project gave the possibility to study a new kind of numerical methods, specially tailored to the phenomena that take place in each part of the digester.

"Reaction-diffusion in porous media" is another research project with many relevant real applications, namely the contaminant transport in groundwater which, in Portugal, has a big strategically interest. Nowadays, the pollution of the soils by fertilizers or pesticides is one of the most relevant environmental problems. The diffusion phenomenon can dramatically contribute to the contamination of groundwater and this can have serious social and economical implications. The evolution of contaminants in soils and the subsurface contaminant transport can be analyzed with the simulation of the models considered in this project. In particular, some areas in central Portugal are extremely polluted due to chemical industries. In the past, liquid effluents produced by these industrial units were discharged directly into several nearby streams. These effluents contained many different types of contaminants. Therefore, it is imperative to prevent and control this type of pollution and to have reliable information on mechanisms that track its evolution. The final goal of the project is to predict the evolution of this kind of pollution and to define strategies to reduce its environmental impact. The models studied during this project and the software package that are expected to be developed can also be used to study biological filters.

We must point out that in most of the projects at LCM, the degree of commitment of the companies involved is still less than it would be desirable since, in general, they don't contribute with funding. In addition, many of the on-going projects are engineering real problems but they don't target any particular industrial application.

However, the laboratory has strong interactions with engineers, physicists, physicians and finance specialists from Portuguese universities and also with computational mathematicians from elsewhere in Europe and in the US. Apart from numerous contacts with colleagues all over the world, the cooperation is formalized in a number of organizational frameworks.

There has been a strategical effort to develop a variety of mechanisms to facilitate a constructive relationship between mathematics and industry. To achieve this objective, contacts and collaboration with industrial partners are vital.

In April 2009, LCM hosted the 69th ECMI European Study Group with Industry (www.mat.uc.pt/esgi69). The purpose of these one-week meetings is to strengthen the links between mathematics and industry by using mathematics to tackle industrial problems which are proposed by industrial partners. The academic participants, who were a diverse group of people with expertise in the mathematical sciences, including PhD students, postdoctoral fellows and professors, allocate themselves to a group, each of which works in one of the proposed problems with the industrial partner. The work was focused on five problems: "Optimizing a complex hydroelectric cascade in electricity market", "Management of stock surplus", "Estimating the price elasticity of water", "Fraud detection in plastic card operations" and "Reliability of a customer relationship management", proposed by the Portuguese firms REN, SONAE, Águas de Portugal, SIBS and Critical Software, respectively.

The experience of the study group was very fruitful, both for the University and the industrial partners. At the end the firms were asked to answer a questionnaire. The first question was "Did the workshop fulfill your expectations?" Next we summarize some of the comments. In the opinion of the representative of one of the companies: "Taking into account the shortness in time of the workshop and the complexity of the proposed problem, we consider that this initiative fulfilled our expectations, considering the given approach in solving the problem and the nice ambience and the contacts made." Other representative answered "It totally corresponded. The work addressed areas and questions which highlight the orientation of the working group for the applied mathematical analysis." Another relevant comment was given: "The initial result of the workshop exceeded our expectations. The work developed during one week was clear. The change of ideas and the results applied to the problem were well developed and presented."

Another question was "How could we improve the links between academia and industry?" One of the companies answered "I suggest the creation of a formal relation between institutions or between a pivot or a representative member of each part, which could often present problems, suggest and analyze possible approaches, debate the possible solutions and measure collaboratively the practical result of the achievements. In summary: partnership to debate ideas, solutions and results." Other idea was: "There are interesting tools like doctoral grants within companies, for relevant topics for the company."

We also asked if they intend to participate in future study groups and if they would recommend these events to other companies and all the answers were "yes". These answers reveal the desire of the industry scientists to stay in contact with current research being carried out at the universities. To enhance connections LCM proposes to promote meetings and study groups on a regular basis and scientific activity spreading actions. Another way to strengthen the links with the outside community is to offer a number of short-courses, with topics of interest to both industry and academia, open to members of the university as well other professionals and industry. These courses could be important to create institutional connections with local industry.

The educational programmes

Traditionally, most of the best undergraduate students in mathematics choose to continue their studies in pure mathematics. In recent reports it has been suggested the need to train more students in applied areas. The current crisis in the academic job market reinforced the attention in the preparation of mathematical students for non-academic employments.

LCM and DMUC think that the transition and integration into the job market of its students is an essential part of its mission. For this reason DMUC runs a Career Service, giving students a first working experience, preparing them for a better integration in the job market. The cooperation between the University and industry from all over the country has made this Career Service a success.

Several of DMUC's former students are now working in Industry. When they were asked about their academic preparation, they all tended to agree, according to what it was also pointed out in the SIAM report (SIAM, 1998), "that they were well educated for several important aspects of non-academic jobs: thinking analytically, dealing with complexity, conceptualizing, developing models, and formulating and solving problems. However, many felt inadequately prepared to attack diverse problems from different subject areas, to use computation effectively, to communicate at a variety of levels, and to work in teams".

Taking into account this scenario, there has been an effort to incorporate modifications in undergraduate mathematics curriculum in order to overcome these drawbacks. We point out that, in the current cycle studies in mathematics of DMUC, some of the courses are really problem-solving oriented and every student must have contact with courses that link mathematics and computing. The students are also encouraged to organize a regular interdisciplinary seminar focusing on a large variety of themes and non-academic mathematicians are invited to meet with students and to talk about their work. Each year DMUC promotes colloquia with former students that work in industry to speak about their experience and to explain the importance of their background in mathematics.

The topics covered by LCM have a prominent place in the educational programmes at DMUC, especially at the Master level. The applied Master programs of DMUC are divided into several areas: Applied Analysis and Computational Mathematics, Computation, Statistics, Optimization and Financial Mathematics. In these Master specialities, which are problem-solving oriented, the students are in contact with real problems. Some companies, like Reuters, Critical Software and Mercer, support these Masters. Students have the possibility to develop their MSc thesis in these companies, being, in this case, supervised by a member of the mathematical faculty and a member of the company staff. Quantitative Methods and Financial Mathematics is another applied Master program of DMUC. This program, shared by DMUC and the Faculty of Economics of the University of Coimbra, receives students in mathematics and economics and involves companies like Bloomberg, Goldman Sachs and the Portuguese banks Millennium BCP and Montepio Geral.

One of the goals of the LCM is to incorporate students of all levels (undergraduate, master and PhD) in its projects. Since its foundation, LCM gives visibility to the work developed in CMUC in applied areas and, as consequence, the number of MSc and PhD thesis made under LCM projects has increased. As an illustration we give some examples. In the thesis "Optimization of a transport network", the work was done in the framework of an European Project named Civitas, a partnership between Critical Software and the local public transport firm, SMTUC. The major goal was to develop a platform to help the users of public transportation in Coimbra to obtain the best path to travel between two points. In the thesis "Variance analysis in the treatment of clinical data", a software to treat the clinical data of the Portuguese Society of Cardiology using both parametric and non-parametric variance analysis was developed. Other dissertations had been done is medical imaging. Two former students developed computational algorithms for the segmentation and registration of medical images and studied their mathematical properties, aiming at applications proposed by the Institute of Biomedical Research in Light and Image (IBILI), a research institution of the Faculty of Medicine of the University of Coimbra.

There are also several students that developed their PhD thesis or have on-going work in the scope of LCM projects. As an example we mention the thesis "Memory in diffusion phenomena", developed within the project "Non-fickian diffusion in polymers and medical applications" which studies mathematical models to simulate diffusion phenomena in materials with memory like polymers. Another PhD student is working on a thesis entitled "Controlled drug release". His goal is to develop a mathematical model and a software package to simulate the drug delivery from contact lens loaded with drug and containing particles, also loaded with drug, dispersed in the polymeric matrix. Both thesis have interdisciplinary character. Chemical engineers from the Chemical Engineer Department and a medical doctor from the Faculty of Medicine are also involved.

Since “applications have been the driving force in the science and mathematics” (Friedman & Littman, 1994), LCM strongly supports the idea that applications are extremely useful to motivate the teaching in mathematics. But, apart from the great effort to introduce real-world applications, we believe, agreeing with OECD report (OECD, 2008), that “curriculum should not become a light version of the accepted curriculum for future researchers”. The students that want to study industrial mathematics should “be familiar with the standard canon of the discipline”.

In spite of the work that has been done, there is an urgent need for more training in the area of industrial mathematics. It is essential to attract bright students to this area and to convey the challenge and the excitement of solving practical problems.

Conclusions and strategy for the future

LCM is a recent structure and for this reason is not yet possible to make a wide quantitative study about the achievements of this project. Nevertheless, the success stories indicate that there has been an increasing interest in strengthening the relation between academia and industry and we feel that mathematics can provide a competitive edge for Portuguese industrial organizations.

Based on our own findings and on the experience of other similar laboratories, we are lead in two directions: building better relations with non-academic organizations promoting the role of mathematics; developing strategies that might be useful to encourage shifts in the curricula in with the objective of promoting closer ties with industry.

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Should Mathematics remain invisible?

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Abstract Mathematical literacy, broadly understood as the ability to reason in terms of abstract models and the effective use of logical arguments and mathematical calculation, became a condition for democratic citizenship. This paper discusses argumentation and proof as two main ingredients in strategies for achieving a higher degree of mathematical fluency in both social and professional life.

Introduction

In the *brave, new world* of Information Society, mathematical literacy became a condition for democratic citizenship. Actually, skills as basic as the ability to think and reason in terms of abstract models and the effective use of logical arguments and mathematical calculation in normal, daily business practice are on demand. Actually this concerns not only highly skilled IT professionals, who are expected to successfully design complex systems at ever-increasing levels of reliability and security, but also specialised workers monitoring, for example, CNC machines.

Even more it concerns, in general, everyone, who, surrounded by ubiquitous and interacting computing devices, has an unprecedent computational power at her fingers' tips to turn on effective power and self-control of her own life and work. Neologism *info-excluded* is often used to denote fundamental difficulties in the use of IT technologies. More fundamentally, from our perspective, it should encompass mathematical illiteracy and lack of precise reasoning skills rooted in formal logic.

Irrespective of its foundational role in all the technology on which modern life depends, Mathematics seems absent, or invisible, from the dominant cultural practices. Regarded as *difficult* or *boring*, its clear and ordered mental discipline seems to conflict with the superposition of images and multiple *rationales* of post-modern way of living. Maybe just a minor symptom of this state of affairs, but *mathphobia*, which seems to be spreading everywhere, has become a hot spot for the media. Our societies, as noticed by E. W. Dijkstra a decade ago, are through an *ongoing process of becoming more and more "amathematical"* [11]. On the surface, at least.

Under it, however, Mathematics is playing the dominant role, and failing to recognize that and training oneself in its discipline, will most probably result in people impoverished in their interaction with the global *polis* and diminished citizenship.

In such a context, this paper aims at contributing to the debate on strategies for achieving a higher degree of *mathematical fluency*. By this we do not have in mind the exclusive development of numerical, operative competences, but the ability to resort to the mathematical language and method to build models of problems, and reason effectively within them. Our claim is that such strategies should be directed towards *unveiling* mathematics contents by rediscovering the relevance of both

- *argumentation* skills, broadly understood as the ability to formulate and structure relationships, justifications and explanations to support an argument;

- and *proof*, as the formal certification of an argument, which encompasses the effective development of proof design and manipulation skills.

Although both aspects are often emphasized separately, the development of educational strategies to bind them together in learning contexts may have an impact in empowering people reasoning skills and, therefore, their ability to survive in a complex world.

The study of *mathematical arguments* is still an issue in Mathematics Education (see, e.g., [1, 16]). On the other hand, the rediscovery of the essential role played by *proofs* (and the associated relevance given to formal logic), has been raised, for the last 3 decades, in a very particular context: that of Computing Science. Actually, the quest for programs whose *correctness* (with respect to a specified intended behaviour) could be established by mathematical reasoning, which has been around for a long time as a research agenda, has recently emerged as a key concern for the Information Society. More and more, our way of living depends on software whose reliability is crucial for our work, security, privacy, and even life (cf, for example wide spread computer-controlled medical instrumentation). Industry is recognising this fact and becoming aware that, at present, *proofs pay V.A.T.*: they are no more an academic activity or an exotic detail, but simply part of the business [9].

The remaining of this paper addresses *argumentation* and *proof*, stressing the need for making them *explicit* in mathematical training at all levels (from middle and high school curricula to profissional education in industrial contexts). Section 2 addresses mathematical argumentation and the development of adequate skills. Part of the discussion is illustrated through the analysis of a class episode registered in the context of a collaborative research project on mathematical communication coordinated by the second author and partially documented in [14]. Section 3, on the other hand, goes from argumentation to proof, building on developments in Computing Science with potential impact for reinvigorating mathematics education. In particular, we focus on the centrality of formal logic and the proposal of a calculational, goal-directed reasoning style which has proven to scale up from the school desk to the engineer's desk tackling complex, real-life problems.

Argumentation

Mathematical learning requires a stepwise construction of a reference framework through which students construct their own personal account of mathematics in a dynamic tension between old and newly acquired knowledge. This is achieved along the countless interaction processes taking place in the classroom. In particular, the nature of the questions posed by the teacher may lead to, or inhibit, the development of argumentation and reasoning skills [3]. A student who is given the opportunity to share what she already knows, her conjectures, and explain the way she though about a problem, will develop higher levels of mathematical literacy in the broad sense proposed in section

1. Team work, which entails the need for each participant to expose his views, argue and try to convince the others, is an excellent strategy to achieve this goal.

Strategies which call students to analyze their arguments and identify its strengths and weaknesses are also instrumental to this aim [13]. Reference [15] singles out a number of basic issues in the development of what is called a *reflexive* mathematical discourse: the ability to go back (either to recover previous arguments in a discussion or to introduce new view points) and the ability to share different sorts of images supporting argumentation (eg, sketches, tables, etc.).

Training argumentation skills is not easy, but certainly an essential task if one cares about mathematical literacy in modern societies. Teacher's role can not be neglected. She/he is responsible for stimulating a friendly, open discussion environment [1], avoiding rejection and helping students to recognize implications and eventual contradictions in their arguments to go ahead [10, 18]. Her role is also to make explicit what is implicit in the students formulation [5], helping them to build up intuitions, asking for generalizations or confronting them with specific particular cases.

Often in school practice conceptual disagreements are avoided (let alone encouraged!), with negative effects in the development of suitable argumentation skills. On the contrary, such skills benefit from exposition to diverse arguments, their attentive consideration and elicitation, as empirically documented in, e.g., [20].

The following episode was recorded in the context mentioned above, in a class of 10 years old students. Although very short, this excerpt illustrates both argumentation in a class and the way a teacher can promote vivid discussions without ruling out any participant. The context was a general discussion in the class on the result of some team work tackling the following problem: *a gardner wants to sow new plants in the flower bed depicted in Figure 1. How much seed should he buy if* 10*g are required for each square meter of the flower bed?*.

The excerpt illustrates what [16] calls emphposition-oriented discussions in which the teacher tries to identify and promote different explicit viewpoints on the problem. She is supposed to analyze and make explicit the logic structure of the arguments in presence and act as a source of both criticism and confidence for all students involved.

T: Let's see this group's solution. Why did you compute the area, instead of, let's say, the perimeter or the volume?

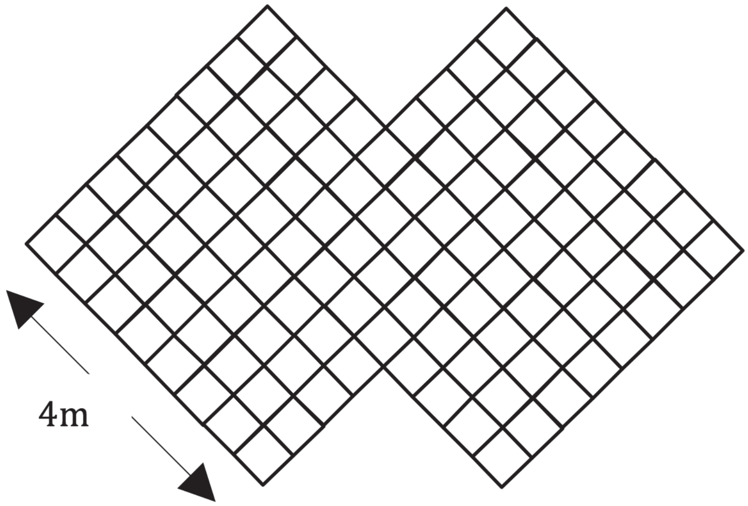

Figure 1—The garden problem.

Magda:	We need to kwon the area because we were told that 10*g* of seed would do for 1 m^2.
T:	Ok. And then?
Magda:	Then we divided the area of the central square by two. We had to get rid of one half.
T:	Why that?
Magda:	We already had the area of one of the big squares ...
T:	Sure. This one for example [pointing to one of them].
Magda:	The problem was they were not complete...
T:	Who was not complete?
Magda:	The squares were not complete.
T:	That's true: this one is not complete, neither is the other.
Magda:	Actually the central square is imaginary. We needed to know the area and divide it by two to get it removed from the area of each of the big squares.

Notice the teacher's effort to make students clarify their reasoning and consider all aspects of the problem. Magda's group strategy was to consider two big squares partially overlapping: the overlap area is refered to as the imaginary central square. Then they took half of latter to subtract to the sum of the areas of the two big squares taken independently. The strategy is correct and shows a clear spatial perception. It is not easy for her, however, to put it into words when other students express doubts about the result. Sue tries to help:

she understood why Magda calls the central square imaginary, but also why other students claim it can't exist:

T: Nice observation. Can you repeat it louder?

Ann: If it were imaginary nobody would have sketched it on the figure.

T: Then we have two theories! One group claims it is an imaginary square, and, being imaginary it does not exist. The other claims the opposite. Any help?

Sue: Ok, it is imaginary because it is there just to signal that one of the big squares is above the other.

T: You mean this square is above the other?

Sue: Yes, it is an incomplete square.

T: What do you think Magda? Would you like to defend your position?

Magda: We claim it is imaginary because otherwise, if it was a real flower bed in a garden, it had to be perceived as such ... and it isn't there, is it?

T: You mean, if you were parallel to the garden you couldn't see the central square, right?

Magda: Exactly.

The debate went on. Just notice, in the following small excerpt, how making an effort to be concrete and come back to the original gardening problem, can help to build up the correct intuitions.

Rachel: Sue claims one of the squares is incomplete, but actually they can be both complete and just one of them be over the other.

T: Ok, you can imagine one square overlapping the other ... your colleagues say this is not possible ...

Rachel: Oh, yes, sure! I forgot the earth in the flower bed. They are right: if the two squares really overlap, the bed will not be flat, but a little higher in the overlapping area.

As a final remark note how classroom interactions can shape the mathematical universe of students. Actually, school mathematics is an iceberg, of which students often only sees what emerges at surface (typically, definitions and procedures). Rendering explicit what is hidden under water is the role of effective mathematical training in argumentation.

Proof

If the development of suitable *argumentation skills* is a first step to a Mathematics-aware citizenship, mastering *proof technology* is essential in a context where, as explained above, *proofs pay V.A.T.* Such is the context of software industry and the increasing demand for quality certified software, namely in safety-critical applications. But what contributions may Computing Science bring to such a discipline? And how could they improve current standards in mathematical education?

As a contribution to a wider debate, we would like to single out in this paper the emphasis on the central role of *formal logic* and the development of a *calculational* style of reasoning. The former is perhaps the main consequence to Mathematics of Computing Science development. An indicator of this move is the almost universal presence of a course on formal logic in every computing undergraduate curriculum.

Proficiency in mathematics, however, would benefit from an earlier introduction and explicit use of logic in middle and high school. Note this is usually not the case in most European countries; the justification for such an omission is that *logic is implicit in Mathematics and therefore does not need to be taught as an independent issue*. Such an argument was used in Portugal to eliminate logic from the high-school curriculum in the nineties. The damage it caused is still to be assessed, but it is certainly not alien to the appalling indicators in what concerns the country overall ranking in mathematics education [17].

High-valued programmers are heavy users of logic. At another scale, this is also true of whoever tries to use and master information in modern IT societies: the explicit use of logic enables critical and secure reasoning and decision making. On the other hand, a heavy use of logic entails the need for more concise ways of expression and notations amenable to formal, systematic manipulation.

The so-called *calculational style* [2, 19] for structuring mathematical reasoning and proof emerged from two decades of research on *correct-by-construction* program design, starting with the pioneering work of Dijkstra and Gries [6, 12], and in particular, through the development of the so-called *algebra of programming* [4]. This style emphasizes the use of systematic mathematical calculation in the design of algorithms. This was not new, but routinely done in algebra and analysis, albeit subconsciously and not always in a systematic fashion. The realization that such a style is equally applicable to logical arguments [6, 12] and that it can greatly improve on traditional verbose proofs in natural language has led to a systematization that can, in return, also improve exposition in the more classical branches of mathematics. In particular, lengthy and verbose proofs (full of *dot-dot* notation, case analyses, and natural language explanations for “obvious” steps) are replaced by easy-to-follow calculations presented in a standard layout which replaces classical implication-first logic by variable-free algebraic reasoning [19, 11].

Let us illustrate with a very simple example what we mean by a *calculational* proof. Suppose we are given the task to find out *whether* $\ln(2) + \ln(7)$ *is greater than, or lesser than* $\ln(3) + \ln(5)$. The ‘classical’ response consists of first formulating the hypothesis $\ln(2) + \ln(7) \leqslant \ln(3) + \ln(5)$ and then verifying it as follows:

(1) function $\ln$ is strictly increasing

(2) $\ln(x \times y) = \ln(x) + \ln(y)$

(3) $14 < 15$

(4) $14 = 2 \times 7$ and $15 = 3 \times 5$

(5) $\ln(14) < \ln(15)$ by (1) and (3)

(6) $\ln(2) + \ln(7) < \ln(3) + \ln(5)$ by (2), (4) and (5)

The proof is easy to follow, but, in the end, the intuition it provides on the problem is quite poor. Moreover, it is hard memorize or reproduce. Most probably it was not made, originally , by the order in which it is presented. This may explain why, in general, this sort of proofs, although dominant in the current mathematical discourse, fail to attract students enthusiasm.

Consider, now, a *calculational* approach to the same problem. The main, initial difference is easy to spot and has an enormous impact: its starting point is not an hypothesis to verify, formulated in a more or less diligent way, but the original problem itself. The proof starts by identifying an unknown $\boxdot$ which stands, not for a number as students are used in school mathematics, but for an order relation. Then it proceeds by the identification and application of whatever known properties are useful in its determination. The whole proof, being essentially syntax driven, builds intuition and meaning.

$$\ln(2) + \ln(7) \boxdot \ln(3) + \ln(5)$$

(function $\ln$ distributes over multiplication)

$$= \quad \ln(2 \times 7) \boxdot \ln(3 \times 5)$$

(routine arithmetic)

$$= \quad \ln(14) \boxdot \ln(15)$$

($14 < 15$ and $\ln$ is strictly increasing)

$$= \quad \boxdot \text{ is } <$$

Empirical evidence gathered within MATHIS1 suggests the systematization of such a calculational style of reasoning can greatly improve on the way proofs are presented. In particular it may help to overcome the typical justification for omitting proofs in school mathematics: that they are difficult to follow for all but exceptional students.

The MATHIS project is 'refactoring' several pieces of school mathematics, systematically introducing this sort of *proofs by calculation*. Although it is too early to draw general conclusions (preliminary results, however, appeared in [7] and [8]), this effort shows how the formalization of topics arising in different contexts results in formulae with the same *flavour*, which can be manipulated thereafter by the same rules of the predicate calculus, without reference to a 'domain specific' interpretation of such formulae in their original area of discourse. This is the essence of formal manipulation, and yields proofs that are shorter, explicit, independent of hidden assumptions, easy to re-construct, check and generalize.

Conclusions and Future Work

Understood, more and more, as a condition for democratic citizenship in modern Information Societies, mathematical literacy has to be taken as a serious concern for the years to come. From our perspective this entails the need for a systematic (and, given *l'esprit du temps*, courageous) *unveiling* of Mathematics. That is, to make mathematical reasoning explicit at all levels of human argumentation and develop, through adequate teaching strategies, the skills suitable to empower correct reasoning in all sorts of social, cultural or professional contexts.

This paper focused on two main issues in this process: empowering *mathematical argumentation*, by developing adequate teaching strategies, and *proof*, made simpler, easier to produce and more systematic through a new calculation style which has proved successful in reasoning about complex software. The latter may be, so we believe, a contribution of Computing Science to reinvigorating mathematical education.

A final word is in order on the above mentioned relationship of Mathematics and Computing. Actually, the latter is probably the paradigm of an area of knowledge from which a popular and effective technology emerged long before a solid, specific, scientific methodology, let alone formal foundations, have been put forward. Often, as our readers may notice, in software industry the whole software production seems to be totally biased to specific technologies, encircling, as a long term effect, the company's culture in quite strict limits. For example, mastering of particular, often ephemeral, technologies appears as a decisive requirement for recruitment policies.

This state of affairs is, however, only the surface of the iceberg. Companies involved in the development of safety-critical or mission-critical software have already recognized that mathematical rigorous reasoning is, not only the key to success in market, but also the warrantee of their own survival. With a long experience in training software engineers and collaborating with software industry, the authors can only claim the need for a double change:

- in the Mathematics *middle school curriculum*, in which the notion of *proof* and the development of argumentation skills are virtually absent;
- in a popular, but pernicious, technology-driven computing education which fails to provide effective training in tackling rigorously the overwhelming complex problems software is supposed to solve.

Future research, specially in the context of the MATHIS project, goes exactly in this direction. In particular, we are currently working on strategies for developing argumentation and calculational proof skills in probabilistic reasoning. As researchers in Education and Computing Science, respectively, the authors see their job as E. W. Dijkstra once put it, *We must give industry not what it wants, but what it needs*. Mathematics should not, definitively, remain hidden.

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Mathematical Modeling in Industrial Engineering: a isolated activity does not change a educational structure

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Abstract In this paper we present some reflections about Brazilian educational structure that has remained the same with guided curriculum in many disciplines, not enough time for them to be thorough and each of these disciplines being under the responsibility of one teacher, making it difficult to produce significant changes in students' education. We have been dedicated to research in Mathematical Modeling in Education with empirical data for most of these studies obtained through teaching experiences at all levels of schooling; among the experiments performed were in the various disciplines of mathematics in Industrial Engineering. As example, we present a Modeling' experience we did in the first semester of 2009 in the discipline of Operations Research. By making use of modeling as a method of teaching, we seek to develop mathematics that provides students the opportunity to familiarize themselves with issues that they will handle in the future and especially promote collaboration between the various courses of the discipline. Although we do not underestimate the importance of mathematical modeling some aspects should be checked as to not stress too much emphasis, forgetting the limitations that produce the educational structure for both the teacher and the students.

1. Presentation

Brazil, a country of continental proportions, has broad industrial development in almost all areas of trade and service. Brazilian production does not only serve its own market (180 million) but also exports to several countries. Collaborating on this are several research centers of the industries themselves, in partnership with research centers at universities or subsidized by federal and state governments. For example, one of the major research centers is the EMBRAPA (Empresa Brasileira de Pesquisa Agropecuaria), which, in the last 30 years, has transformed food production in Brazil: improving the quality and variety, among other aspects, allowing the entire population to have good food at great quantity and minimum prices.

In this social system and Brazilian production is required of professionals: general and specific knowledge, ability to apply this knowledge and a keen sense in decision making. Part of this knowledge is in mathematics. This is because there are "intimate connections between mathematics and industry" [...] "in generating and solving problems associated with the development of humankind, economically and socially" (Damlamian and Sträßer, 2009, Discussion Document). However, most of these professionals become able to develop their functions only from the experience and training/courses conducted by the company for whom they work. Therefore, a significant number of entrepreneurs say of recent graduates they do not know how to apply the knowledge acquired in school. *What are the reasons why most students, particularly at the undergraduate level learn when they go to work, regardless of the number of class hours and years of study in college? How can this situation be changed that has existed for decades both in Brazil and in several other countries?*

A Brazilian study of the productive sector is as significant as the research on educational issues. A major focus of research in education is about training teachers. Many studies indicate that the teacher must find ways to make lessons more interesting and motivating, proposing strategies and methods for teachers to integrate into the syllabus for the students' reality, making use of technological resources, etc. Regarding the teaching of mathematics that has been done "to find ways to make the theoretical content transparent and communicate to the students the end-user perspective of mathematical knowledge" (HEILIO, 2009).

In the area of mathematics education, even with significant research and restructuring of the curriculum, the teaching of mathematics, unless on the level of individual experiments, doesn't provides the student with enough skill to interpret and solve problems. Occasionally they are given problem-solving situations that require reading and interpretation, then, a formulation and explanation of context. Without this experience, this ability is lost. Mathematics in general, is treated to be impervious, not related, nor between the mathematics itself. *How difficult is it for teachers to free themselves from this practice despite the criticism received because of poor performance of students?*

Over the last two decades we have been dedicated to research in Mathematical Modeling (MM) in Education with empirical data for most of these studies obtained through teaching experiences at all levels of schooling: primary to Bachelor's and Postgraduate. Among the experiments performed were in the various disciplines of mathematics courses in Engineering (Electrical, Civil, Chemical, Industrial). Among the different aspects analyzed in this research, one shows that MM contributes not only to the students learning the mathematical content, but also mainly to formulate, solve and make decisions (Biembengut and Hein, 2007).

Although these studies have contributed to MM, as stated in official documents of Brazilian education, many training courses for teachers include the discipline of MM into the curriculum, and national conferences on MM have begun to occur (since 1999), with a significant number of research and participants, little has changed in practice in the classroom. In fact, what has changed in schools and universities has been only the physical structure (modern furniture, computers, internet etc.). But the educational structure has remained the same with guided curriculum in many disciplines, not enough time for them to be thorough and each of these disciplines being under the responsibility of one teacher, making it difficult to produce significant changes in students' education. We cannot confront the evidence that most teachers are looking for effective ways for students to learn. The teaching resources vary according to the subject, using methods that are thought of as appropriate to promote learning. However, this structure without teachers of related disciplines coming together to organize an effective proposal and each teaching the content in his own design, does not help students understand the reality, interest in environmental issues, express proposals, present a new creation. To what extent can we change the educational structure? These are some issues with which we deal.

2. Example of Mathematical Modeling at Industrial Engineering

In order to justify what we have seen advertised for more than a decade - the need to change the educational structure, we present one of the works we did in the first semester/2009 (February-July) in the discipline of Operational Research involving 23 students of Industrial Engineering. One of the challenges of the industrial engineer is to develop and implement procedures and methods for the manufacture of consumer items on a large scale, quickly and above all with minimal impact on nature.

The course in Industrial Engineering from the Regional University of Blumenau (FURB) has at least 3762 hours/classes, grouped into four areas: common (general education and basic), *professional* (specific training and specific options); *stage, elective courses* and *work* at the end of the course. The subjects of the common area, according to the specificity, are the responsibility of other departments, such as: Pedagogy, Administration, and Mathematics. The disciplines of mathematics, under the responsibility of the Department of Mathematics, occupying 684 h/a (about 20%) of this course, divided into: Differential Integral Calculus (I, II, III), Linear Algebra and Analytic Geometry (I and II), Numerical Calculation, Descriptive Statistics and Probability and Statistical Methods, Financial Mathematics and Investment Analysis and Operations Research (I and II). In general these courses are taught by different mathematics teachers, some beginners, others senior. That is, different concepts and mathematical training. Each semester, the department indicates that teachers do not always teach the same classes in subsequent semesters. We have identified that these disciplines are still prioritized by techniques over theories, despite the mathematical software, and applications are restricted to 'classic' that are part of the textbooks, often written decades ago.

Although there is a discipline called Modeling and Simulation (54 h/a), engineering courses, in general, are reworked examples of textbooks as well. The problem-solving situations that appear in these books have all required data and have at least one of these situations resolved, as an example. In general, the teacher presents the models or examples and then offers students to resolve issues by applying the data to the model (mathematical formula). The application is made, often by rote, without the students really understanding the issue, nor assessing the validity of the result. To perform modeling it is necessary to study and interpret a subject of some area of knowledge, raising issues that answer the problem or requires formulation to solve them. To use MM in teaching, teachers need to know how to do modeling and also to adjust some models relevant to the course that allows them to develop the syllabus and arouse students' interest to learn.

As stated earlier, since 1990 we have been using MM as a method of teaching various mathematics disciplines in courses, particularly engineering at FURB. Courses in engineering due to existing structure (half-yearly, credit system, disciplines, time) and the difficulty for a mathematics teacher in a short time, become aware of an engineering problem which mathematics is an 'instrument', we did some redirections in this method for teaching math. This method is guided by the *development of program content* from mathematical models applied to different areas of their engineering and at the same time guiding the students to research — *modeling* (Biembengut, 1997).

In the first semester of 2009, to develop program content in the discipline of Operations Research II (Course of Industrial Engineering), we chose the subject: the acquisition of an area for building 620 homes for people who had become homeless due to an environmental disaster in the city of Blumenau (the location of FURB) in November 2008. This tragedy, according to international reports, led to the deaths of hundreds of people and left thousands homeless. To develop the topic we worked in six steps, not necessarily disjointed: (1st) exposure of the subject; (2nd) survey of issues and data; (3rd) development of mathematical content and presentation of similar examples; (4th) formulation and resolution of issues proposed; (5th) analysis of the model.

1ST STEP: We started by refreshing the students about the *flood* and *landslides*, and better explaining the possible causes and *presenting the subject* to be modeled.

> The city of Blumenau (SC) located in southern Brazil, is positioned geographically in the southeast, being cut by the Rio Itajai Açu, running downstream to the east. The currents of cold air from the South Pole are in the orthogonal direction to the valley, besides making the land wet when they meet with the currents of warm air from the Amazon, which rotate over Brazil in a counterclockwise direction, causing extreme weather tragedies which kill people and cause considerable economic losses. It also helps the altitude of the city center in relation to sea level of 14 m effect of the tides of the Atlantic Ocean acting on the flow of water. Add to all these weather phenomena, are the effects of global warming. The rains began in August in volumes never before listed and on November 23rd and 24th caused flooding and landslides from the hills, causing a tragedy with deaths exceeding one hundred and economic losses exceeding three billion dollars. The high number of homeless forced government offices to relocate people. There was a problem for government agencies to locate a suitable area to create housing. For this, the industry work from the municipal government formed a team composed of engineers, geologists, meteorologists and technicians responsible for the financial area of the municipality.

2ND STAGE: In order to *raise the issue*, and inform students that the problem in the choice of area to be acquired was set within the environment of discreet multi- criteria decision. There was a conflict between the objectives: to minimize the cost, to maximize the extent of the share of flood, to minimize the number of nearby streams and to minimize the slope of the area. The aim was to get a cheaper area, free from floods, free from torrents and landslides.

> What is the area that best met these goals? To answer the question, we obtained data from the sector of the local government team who was already dealing with the mat-

ter and presented it to students. According to data obtained, there were three proposals for the construction of 620 dwellings. All had the minimum size required for the project. Possible areas proposed were located in remote regions, and not completely free from floods, torrents and landslides. The characteristics of the areas in question are in table 01, that take into account the criteria analyzed.

Criteria-Regions	Cost (R$)	Flood* (m)	Torrent** (n°)	Landslide*** (°)
A	2.100.000,00	8,9	3	5°
B	1.900.000,00	14,2	0	8°
C	1.550.000,00	11,7	5	23°

Table 01—Technical data of each plot

Source: Department of Public Works PMB

*Quota-free flood

**Number of streams nearby

***Maximum slope of the land

3RD STAGE: The development of the mathematical content of the course (OR) and presentation of similar examples began at this point, but occurred in several stages whenever necessary.

To get an answer to this question, we used the Method of Hierarchical Analysis proposed by Saaty (1991). Saaty starts with a fundamental scale, from the matrix of preferences and the definition of each. The established hierarchical structure takes place compared to a pair- of each alternative within each criterion to the next higher level, that is, for each criterion which is related to the alternatives properly applied in the verbal scale presented. The court verbal team decision becomes a scale of values.

4TH STAGE: The wording of the question allowed several discussions and suggestions. We reviewed and formulated with students the criteria: cost, risk of flooding, flood and landslides.

These criteria were not the same scale nor directly proportional. Therefore we had to establish preferences that divided the problem into hierarchical levels: the criteria and preferences, individual and collective presented. Thus, we established the preferences by consensus, following a scale of 1 to 9, a scale that follows the “psychological threshold”, the human being can judge. As table 02, following:

Value	Importance	Description
1	Equal importance	Alternatives also contribute to the goal.
3	Little importance	Experience or other favor one alternative over another.
5	Great importance	Experience or other strongly favor one alternative over another.
7	Highly important	An alternative is strongly favored over another. It can be shown in practice.
9	Absolute importance	Evidence favors one alternative over the other, with a high degree of security.
2,4,6,8	Intermediate values	Used when seeking a condition of compromise between two definitions.

Table 02—Scale of preferences
Source: Survey data

The use of this scale was applied to the definition of the area to be acquired. However, the use of this model was not the only decision tool used in the purchase process. Assessing the preferences as to costs, we got a matrix of preferences. The cost of each site compared to itself received support 1, showing the same degree of importance. This preference value from R$ 1,550,000.00 in the area C was four times more attractive to R$ 2,100,000.00 in area A and three times more important to R$ 1,900,000.00 in area B. Therefore, comparing inverse preferences found that the area A was the least preferred because it was valued at one-third (1/3) in respect to area B and one quarter (1/4) in respect to area C. Finally, area C was evaluated as being twice as preferable to area B, so the land was valued preferably in a medium (1/2) in relation to C. The other matrices were thus evaluated, preferably:

Flooding	A	B	C	Torrent	A	B	C	Landslide	A	B	C
A	1	1/4	1/6	**A**	1	1/3	4	**A**	1	2	8
B	4	1	1/3	**B**	3	1	1/7	**B**	1/2	1	6
C	6	3	1	**C**	1/4	7	1	**C**	1/8	1/6	1

Table 03—Matrices of preference as to the flooding, torrents and landslides

Taking the matrix of preferences of the areas in relation to landslides, we see that the best land was assessed. But, how much better was it compared to others? We had to establish a new scale that not only confirmed the position, but also assessed the position of each area within each criterion. To this we added the values of each column and divided each cell by the sum obtained in the column. The area had the better valuation. The arithmetic mean of each line down the rankings as the criterion evaluated (see Table 4). The final rankings for each criterion are in Table 05.

Landslide	A	B	C	Mean
A	8/13	12/19	8/15	0,593
B	4/13	6/19	6/15	0,341
C	1/13	1/19	1/15	0,066
Sum	1	1	1	1

Table 04—Standardization and ranking criterion area
Source: Survey data

Area	Cost	Flooding	Torrent	Landslide
A	0.123	0.087	0.265	0.593
B	0.320	0.274	0.655	0.341
C	0.557	0.639	0.080	0.066

Table 05—Final ranking
Source: Survey data

The issues of cost, likelihood of flooding or landslide were most desirable in land A. In the criterion of torrents area B was preferable .Using data from financial experts about the value of the land as impact and frequency that each disaster has on society, we found another array of preferences regarding the criteria for evaluation.

Preferences	Cost	Flooding	Torrent	Landslide
Cost	1	3	2	2
Flooding	1/3	1	1/4	1/4
Torrent	1/2	4	1	1/2
Landslide	1/2	4	2	1

Table 06—Preferences regarding the criteria for evaluation
Source: Survey data

Thus, the cost of land was at the top of the ranking with 0.398 points, likelihood of flood second with 0.085 points, torrent third with 0.218 points and landslides fourth with 0.299 points. Multiplying the matrix of land for ranking criterion, by the matrix of ranking criteria, it was possible to establish the final ranking as an aid to the trial purchase of land for the construction of housing for the homeless in the disaster of November 2008.

$$\begin{bmatrix} 0.123 & 0.087 & 0.265 & 0.593 \\ 0.320 & 0.274 & 0.655 & 0.341 \\ 0.557 & 0.639 & 0.080 & 0.066 \end{bmatrix} \times \begin{bmatrix} 0.398 \\ 0.085 \\ 0.218 \\ 0.299 \end{bmatrix} = \begin{bmatrix} 0.265 \\ 0.421 \\ 0.314 \end{bmatrix}$$

5TH STAGE: We checked that area B was preferred, in analyzing the validity of the model, followed by area C. However, we learned that the team's decision-sector of the city opted for the land of lesser value, paying little attention to the criteria of "flood". That is, changing the situation, team, values and risks, there may be a reordering.

It is worth noting that throughout the process these students participated actively. While not directly a question of industrial engineering, this tragedy reached most of them, directly or indirectly. Directly to those whom had some damage to their physical assets and indirectly because the tragedy brought problems to all industries and commerce of the region; sectors in which the majority worked. Apart from developing the program content, we proposed that they gather in groups (3 students each) and elect topics that could work in MM. Their work did not meet our expectations. The reasons given by them: (1) parallel, they had to assist other disciplines who were enrolled in the semester and (2) the time was insufficient to account for the disciplines and data collection since most of them worked.

3. Final Considerations

The work summarized above has been a constant in all our activities in education in any discipline. At the end of the course, such as Morrison (1991), students experience the elements of science education: facts, abstractions and comparison of facts with abstractions. They come to understand when facts are reduced to abstractions, the abstractions manipulated to make predictions, and predictions compared with the facts. However, this type of pedagogical action alone, even when the expectations are met by a group of students, is not enough to break with the fragmented training that these students receive from the earliest years of schooling and therefore does not cause any change in the training process of these professionals. For example, in the month of August/2009, we asked the students in class to give their opinion about the method we used. Most said it was great and that all teachers should do the same, but it does not occur. We have heard similar views since we began using Modeling in Education.

By making use of modeling as a method of teaching, we seek to develop mathematics that provides students the opportunity to familiarize themselves with issues that they will handle in the future and especially promote collaboration between the various courses of the discipline. Although we do not underestimate the importance of mathematical modeling as a method of teaching and learning, some aspects should be checked as to not stress too much emphasis, forgetting the limitations that produce the educational structure for both the teacher and the students. For example: the number of subjects per semester, together with the lack of continuity of some students in the same class and different teachers (in the same area), makes it difficult to work on research; alongside this is the long-term absence of interaction between teachers of the basic subjects (algebra/mathematics, physics, chemistry) and those of other disciplines of the courses; the teacher stops to meet the specific needs of each area, which contributes to the repetition of topics and/or neglects some essential material. In a temporary difficulty of transforming the current educational structure, the proposal is that we research professors continue to search for paths, processes and methods needed to one day make a change in the educational structure that actually brings the necessary and sufficient training to students and teachers at any stage of schooling.

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How it is possible to make real-world mathematics more visible: Some results from two Italian Projects

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Abstract In this contribution we will present some results of two ongoing Italian projects, the first aimed at primary schools, the second at high schools. The first project is articulated in some teaching experiments using suitable cultural artifacts, interactive teaching methods and new socio-mathematical norms in order to create a substantially modified teaching/learning environment. The focus is on fostering a mindful approach toward realistic mathematical modelling and applications of mathematics, as well as a problem posing attitude, even at the primary school level. The second project is part of a national project launched in 2005 sponsored by the Italian Ministry of Education, in cooperation with the Faculties of Science of the Italian Universities and the Confederation of Italian Industries. Its main purpose is to encourage high-school graduates entering Universities to apply for degrees in the "hard" sciences, including Mathematics, Physics, and Chemistry.

Mathematical modelling and problem posing

The first project described in this contribution regards the primary school level and is articulated in some teaching experiments aimed at showing how an extensive use of suitable artifacts could prove to be useful instrument in creating a new tension between school mathematics and real world with its incorporated mathematics. The teaching/learning environment designed in these teaching experiments is characterized by an attempt to establish a new classroom culture also through new socio-mathematical norms, for example norms about what counts as a good or acceptable response, or as a good or acceptable solution procedure, are debated (Bonotto, 2005). The focus is on fostering a mindful approach toward realistic mathematical modelling, mathematics applications and also a problem posing attitude, even at the primary school level.

The term mathematical modelling is not only used to refer to a process whereby a situation has to be problematized and understood, translated into mathematics, worked out mathematically, translated back into the original (real-world) situation, evaluated and communicated. Besides this type of modelling, which requires that the student has already at his disposal at least some mathematical models and tools to mathematize, there is another kind of modelling, wherein model-eliciting activities are used as a vehicle for *the development* (rather than the application) of mathematical concepts (Greer, Verschaffel & Mukhopadhyay, 2007). This second type of modeling is called 'emergent modeling' in Gravemeijer (2007), and its focus is on long-term learning processes, in which a model develops from an informal, situated model ("a model of"), into a generalizable mathematical structure ("a model for").

Although it is very difficult, if not impossible, to make a sharp distinction between the two aspects of mathematical modelling, it is clear that they are associated with different phases in the teaching/learning process and with different kinds of instructional activities (Greer et al., 2007).

We deem that an early introduction in schools of fundamental ideas about modelling is not only possible but also indeed desirable even at the primary school level. We argue for modelling as a means of recognizing the potential of mathematics as a critical tool to interpret and understand reality, the communities children live in, and society in general. An important aim for compulsory education should be to teach students to interpret critically the reality they live in and understand its codes and messages so as not to be excluded or misled (Bonotto, 2007).

As regard the problem posing, this process is of central importance in the discipline of mathematics and in the nature of mathematical thinking and it is an important companion to problem solving. Recently many mathematics educators realized that developing the ability to pose mathematics problems is at least as important, educationally, as developing the ability to solve them and have underlined the need to incorporate problem posing activities into mathematics classrooms (e.g. English, 2003; Cristou et al, 2005).

Problem posing has been defined by researchers from different perspectives (see Silver & Cai, 1996). In this contribution we consider mathematical problem posing as the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems. It, therefore, becomes an opportunity for interpretation and critical analysis of reality since: i) they have to discern significant data from immaterial data; ii) they must discover the relations between the data; iii) they must decide whether the information in their possession is sufficient to solve the problem; and iv) to investigate if numerical data involved is numerically and/or contextually coherent. These activities, quite absent from today's Italian school context, are typical of the modelling process and are similar to situations to be mathematized that students have encountered or will encounter outside school, for examples in the work situations.

In our approach in and out of school mathematics, even with their specific differences, in terms both of practices and learning processes, are not seen as two disjoint and independent entities. Furthermore we think that the conditions that often make out-of-school learning more effective can and must be re-created, at least partially, within classroom activities. Indeed, though there may be some inherent differences between in and out of school mathematics, these can be reduced by creating classroom situations that promote learning processes closer to those arising from out-of-school mathematics practices (Bonotto, 2005).

The classroom activities present in the teaching experiments we have conducted fall under the type of "rich context", in the Freudenthal sense, i.e. context which not only serves as the application area but also as source for learning mathematics, and in particular for the emergence of a mathematical modelling disposition (Freudenthal, 1991).

About artifacts

The artifacts introduced in our teaching experiments (for example receipts, advertising leaflets containing discount coupons for supermarkets and stores, the weather forecast from a newspaper, a weekly TV guide, an informational booklet issued by "Poste Italiane", and so on) are materials, real or reproduced, which children typically meet in real-life situations and then are relevant and meaningful. In this way we offer the opportunity of making connections between the mathematics incorporated in real-life situations and school mathematics, which although closely related, are governed by different laws and principles. In this way we present mathematics as a means of interpreting and understanding reality and increase the opportunities of observing mathematics outside the school context. The usefulness and pervasive character of mathematics are merely two of its many facets and cannot by themselves capture its very special character, relevance, and cultural value. Nonetheless, these two elements can be usefully exploited from a teaching point of view because they can change students' common behaviour and attitude (Bonotto, 2005).

Furthermore the use of suitable artifacts allows the teacher to propose many questions, remarks, and culturally and scientifically interesting inquiries. The activities and connections that can be made depend, of course, on the students' scholastic level. These artifacts contain different codes, percentages, numerical expressions, and different quantities with their related units of measure, and hence are connected with other mathematical concepts and also other disciplines (chemistry, biology, geography, astronomy, etc.).

Furthermore by asking children i) to select other artifacts from their everyday life, ii) to identify the embedded mathematical facts, iii) to look for analogies and differences (e.g. different number representations), iv) to generate problems (e.g. discover relationships between quantities) the children should be encouraged to recognize a great variety of situations as mathematical situations, or more precisely mathematizable situations. A "re-mathematization" process is thereby favoured, wherein students are invited to unpack from artifacts the mathematics that has been "hidden" in them, in contrast with the demathematization process in which the need to understand mathematics that becomes embodies in artifacts disappears (see Gellert & Jablonka, 2007). In this way we can multiply the occasions when students encounter mathematics outside of the school context, "everydaying" the mathematics (Bonotto, 2007).

Some results

The results of this project (which involves about 20 primary teachers and about 400 children) show that, contrary to the practice of traditional word problem solving, children do not ignore the relevant and plausible aspects of reality, nor did they exclude real-world knowledge from their observation and reasoning. They confronted with this kind of activity also show flexibility in their reasoning processes by exploring, comparing, and selecting among different strategies (see e.g. Bonotto, 2003b and 2005). These strategies are often sensitive to the context and number quantities involved, and closer to the procedures emerging from out-of-school mathematics practice; so mathematical reasoning needed in extra-scholastic contexts, for example in work places, is favoured (see e.g. Bonotto & Basso, 2001; Bonotto, 2003a and 2005). Finally also creativity and a problem critiquing process is favoured; the children attempted to criticize and make suggestions or correct the problems created by their classmates or the results obtained (see e.g. Bonotto, 2006 and 2009).

Regarding the mathematical content we also laid the basis for overcoming some conceptual obstacles, for example the misconception that multiplication always produces a larger result than the factors (see Bonotto, 2005). This confirms the hypothesis that this kind of classroom activities can give support for accessing more formal mathematical knowledge and promote a process of "abstraction-as-construction", in according to the "emergent modelling" perspective (Gravemeijer, 2007).

The positive results obtained in this project can be attributed to a combination of closely linked factors: a) an extensive use of suitable artifacts that, with their incorporated mathematics, played a fundamental role in bringing students' out-of-school meaningful reasoning and experiences into play, and allowed a good control of inferences and results; b) the application of a variety of complementary, integrated, and interactive instructional techniques (involving children's own written descriptions of the methods they use, work in pairs or small groups, and whole-class discussions); c) the introduction of particular sociomathematical norms that played an important role in giving meaning to new mathematical knowledge, in reinforcing previous knowledge and in paying systematic attention to the nature of the problems and the classroom culture; d) an adequate balance between problem-posing and problem-solving activities, in order to promote also a mathematical modelling disposition.

But is there a reverse of the coin, if the word "reverse" can be used? On the basis of the experience of our studies, we entirely agree with Freudenthal (1991), that the main problem regarding rich context is implementation requiring a fundamental change in teaching attitudes. The effective establishment of a learning environment like the one described here makes very high demands on the teacher, and therefore requires revision and change in teacher training, both initially and through in-service programs.

We do not suggest that the classroom activities present in the teaching experiments of this project are a prototype for all classroom activities related to mathematics, although we think that the presence of realistic mathematical modelling activity, as well as of problem posing activity, should not emanate from a specific part of the curriculum but should permeate the entire curriculum.

The project aimed at high schools: Progetto Lauree Scientifiche

The second project described in this contribution is part of a national project launched in 2005. The Italian Ministry of Education, in collaboration with the Faculties of Science of the Italian Universities and the Confederation of Italian Industries (Confindustria) started a special project (PLS = Progetto Lauree Scientifiche) whose main purpose was to encourage high-school graduates entering Universities to apply for degrees in the "hard" sciences, including Mathematics, Physics, and Chemistry.

This project turned out to be successful and presently represents the most effective initiative of collaboration between high-school students and teachers, university teachers, and Industries, over the last two decades.

The project management was divided into geographic regions, dealing separately with Mathematics, Physics, Chemistry and Science of Materials.

Procedure

With regard to Mathematics in the Veneto region (Venezia, Verona, Padova, Vicenza, Treviso, Rovigo, Belluno), over the past 4 years PLS has involved about 30 University teachers, 80 high school teachers and over 1000 students. The Director was the prof. Benedetto Scimemi of the University of Padova.

To start the project activity, in the Fall of 2005 the Project coordinator selected in the Region 15 public high schools, each enjoying a good reputation as educational institution and having a reasonable amount of math in its curriculum (mainly "Licei scientifici"). The choice was made so that all the provinces were represented.

In each of these institutes a reliable math teacher was asked to join the Project and to enrol two more colleagues of his school. The proper choice of these teachers was crucial for the Project success. In fact, most of them had had previous contacts with the Universities, having being either former students or participants of teacher-training activities. These 3-4 people in each school would be in charge of selecting a number from 15 to 30 volunteer students of their school (17-18 years of age), who would be willing to attend supplementary classes in Mathematics, 5-6 times in the afternoons, 2-3 hours each.

At the same time, within the Universities of Padova and Verona, the Project Coordinator contacted a number professors and assistants, about 25 people, whose competence and dedication to teaching activities was well known. The coordinators were invited to choose a one of their favorite topics in Mathematics, which would be both understandable by good high school students and suitable for a short term "Math Laboratory". They were invited to prepare a plan for delivering their material within 5-10 teaching hours, followed by an adequate number of practice hours.

These two groups of mathematicians - high school and university teachers - joined in a two-hour meeting in Padova in September 2005. First the university people were asked to briefly describe their proposals; then each high school was invited to choose a subject and its expert. As a result of this meeting and further personal contacts, in each school a "work-group" was formed, made of 3 local teachers plus one or two from the university staff. In later months each work-group met separately to plan the details and decide its own calendar.

A typical schedule would be: for 5 weeks, each Thursday, from 3 to 5 p.m.

In most of these Labs the role of the university teachers was very important during the first meeting, to explain the main theory involved. Then more time was left to the high-school teachers until, in some cases, the final work - normally an application to a real life problem - was entirely made by the students, who often used their computers.

In parallel, a teacher in-service training program [Corso di perfezionamento in Metodologia e Didattica della Matematica, Director Cinzia Bonotto] was initiated. It aimed at providing in-service secondary mathematics teachers with new training in the background (mathematical or not) needed for the applications discussed in the project.

Contents

Various subjects were discussed in the different schools including, when possible, problems that were suggested by local industries, public administrations or other non-scholastic institutions. Various subjects were discussed in different schools, including, when possible, problems suggested by local industries, public administrations or other non-scholastic institutions. In some cases the managers of the firm or institution which had provided the data wanted to take part in the preparatory seminars, either visiting the school themselves or inviting the students to visit the firm offices. At the end, the students' work was reported to the Confederation of the local industries, which also assigned a prize to a school which had best interpreted the spirit of collaboration between school and industry.

Here are some examples of the mathematical themes presented, some of which were later published (see Languasco & Zaccagnini, 2006; Centomo, Gregorio & Mantese, 2007; Carminati, Gheno & Mattarolo, 2006, 2007 and 2008; Chignola et al, 2006; Burato et al, 2007):

- Statistics, in particular, Cluster analysis: data to be treated were collected from a provincial tourist office, a pharmaceutical industry, and an agency specializing in quality control of industrial production.
- Linear programming and operations research: optimization problems were suggested by a garbage-collecting firm, a telephone call-center, a stock exchange trading agency and others.

- Modelling: a model for the growth of tumour tissues was suggested by a biology researcher; a number of mathematical subjects, such as Fourier analysis, were suggested by the cardiology department of a public hospital.

Other subjects, although not directly suggested by local industries or agencies, were also chosen with special attention to applications (Cryptography, Dynamics of populations, Dynamical systems).

The mathematics themes chosen seem to please the audience much more than the traditional subjects of a standard school program. However, in addition to these choices of topic, an important role was surely played by the selected audience, all students attending the seminars having indicated a potential interest in mathematics.

Some results

In reviewing the activities carried out by the various schools under the Project, the following examples of collaborations with non-scholastic institutions may be considered specially interesting:

a) at Padova (Liceo Cornaro) a number of statistic elaborations were made by the students, regarding the therapeutical efficiency of some drugs produced by a national pharmaceutical industry (Sigma-Tau). A scientific consultant of this industry visited the school and explained how they test their drugs, then made comparisons with the students' methods.

b) at Belluno (Liceo Galilei) more statistical studies were made on the wild animal populations in the near region. The lecture of a scholar from the University of Vilnius (Lithuania) permitted the students to learn and use a special free software (called R) to elaborate their data for finding the principal components.

c) at Thiene (Liceo Corradini) a class visited the local Hospital, where a special computer program, produced by the national industry Exprivia, was currently used for cardiological purposes. The students' work contributed to improving this mathematical program, which was then adopted. This school was the winner of a special prize for the most interesting PSL activity of the year.

d) at Bassano (Liceo Da Ponte) a local firm in charge of collecting public garbage was looking for criteria to optimize the place and time schedule of their collecting cars and trucks. The students studied Linear programming and the Simplex method, enough to produce a final proposal. The firm sponsored the publication of a book describing the whole content of this Math-Lab (see Carminati, Gheno & Mattarolo, 2006).

Final comments

In conclusion, PLS can be considered a very successful project, in terms of both encouraging mathematics teachers and increasing enrolments into scientific faculties: The number of mathematics students at the University of Padova has doubled in the last five years. Beyond this target, un unexpected result was a mutual appreciation of the competence and dedication of University and High-school teachers, which in some cases even produced scientific collaboration.

The Veneto branch of the Confederation of Italian Industries welcomed the work done within the project. Indeed, it established a prize for the participating schools that most actively collaborated with local industries in applying mathematical techniques to one or more problems of interest to the industries themselves.

The PLS project has recently been extended for another next two years.

Acknowledgments

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The project "Ways to more MINT-graduates" of the Bavarian business association (vbw) with focus on the M (=Mathematics) at the University of Augsburg, Germany

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Abstract In order to reduce the college dropout-rate in mathematics, a project that is financially backed by the Bavarian business association was initiated at several Bavarian universities. At the University of Augsburg we follow an incremental-evolutionary approach to change the math students' and teachers' attitudes and beliefs by setting up several actions that interact in a synergetic way in the form of an integrated approach: attraction of more students to study mathematics (at school) and reduction of the dropout-rate (at university).

The Project

The project "Ways to more MINT-graduates" was launched by the Bavarian business association (vbw) in January 2008 to attract more students to the subjects mathematics (M), informatics (I), natural sciences (N) and technology/engineering (T). In the last year the Bavarian economy lacked of 7500 jobs for engineers; and while the demographic change makes this situation worse, the college dropout-rate of more than 30% in the subjects mentioned has to be reduced.

At the University of Augsburg the project is concentrated on the attraction and fostering of students in mathematics (project title: "Studying Mathematics!"). Because of its success in the meantime, the project in Augsburg was honoured by a visit of the Bavarian minister for science in May 2009.

Empirical facts to college dropouts in Germany

In order to reduce the dropout-rate the first step is to analyse the situation. According to Heublein et al. (2008), the current dropout-rate averaged over all subjects in Germany is 21 percent, but in MINT-subjects it is often beyond 30 percent. A dropout in this scenario is a former student, who left the system of higher education without a first degree. Students interrupting their studies or changing university are not included. The biggest problem results from the fact that universities neither are allowed to follow the course of the students' studies, nor to give this information away. So the numbers concerning the dropout-rate are only based on estimation and can only limitedly differentiated.

The causes for a college dropout are regularly evaluated by the Higher Education Information System (HIS) in Hanover. The results of the last surveys were published in Heublein et al. (2003) and Heublein et al. (2009). In figure 1, taken from Heublein et al. (2009) p. 153 and translated into English, the crucial causes for a dropout in the subject group mathematics and science at universities are listed in comparison for the two evaluated years 2000 and 2008.

Most problems mentioned by the students can not or hardly be influenced by the universities. As the most promising fields remain problems of performance, problematic study conditions and exam failure. As systematic offers concerning the first two moments, especially knowledge based supporting methods are necessary to help the student getting along with the amount and standard of the learning matter.

According to figure 1, the step to the bachelor/master-system in the context of the Bologna process has increased the dominance of problems concerning performance requirements

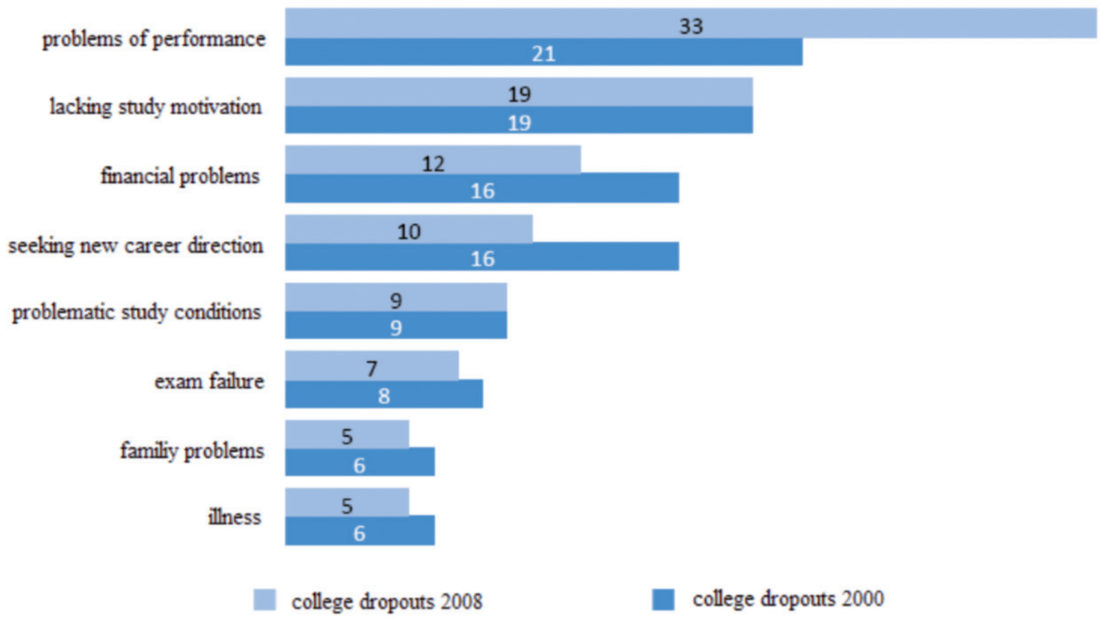

Figure 1—crucial causes for a dropout in the subject group mathematics and science at universities (in %).

severely. Together with exam failure, the value for dropping out because of insufficient performance is two fifth of all exmatriculated students without an exam. The challenging study and examination requirements already in the first semesters are experienced as a performance concentration that is hardly manageable without adequate support.

Besides these factors a lacking study motivation may also be a starting-point for the university to act. The lack of interest on the study subject can be the result of insufficient information about the content and the requirements while still attending school. So a broadening of the range of information possibilities seems reasonable.

Students' opinions about a dropout in mathematics

To adapt the project "Studying Mathematics!" at the University of Augsburg best possible to the situation, we did an additional survey among 83 students of different math study courses in December 2007. The students were asked to give free answers to the following questions:

—*When is the most possible time to drop out?*

Answer: 82 out of 83 respondents referred to the first three semesters, which corresponds to the official dropout statistic at the institute for mathematics at the University of Augsburg, where the main loss of maths students is observed in just this period.

—*Why are there dropouts?*

ANSWERS: There were different aspects in the students' answers that can be grouped according to "Insufficient vision of the kind what studying mathematics is like", "Large discrepancy between mathematics at school and at university", "Unfamiliar extent of personal responsibility and self-organisation" and "Barely recognizable relation to a subsequent profession".

Incremental-evolutionary changes on the meta-level

In order to achieve the aim of attracting more students to the subject math, the chair for the Didactics of Mathematics follows an "integrated approach" that is based on insights from the theory of cybernetics.

According to the definitions given in (Malik, 1992) or (Vester, 1999) mathematics education in Europe and even mathematics education at a concrete school must be seen as "complex" systems. Such are networks of multiply connected components. One cannot change a component without influencing the character of the whole system. The same holds for the variety of industry itself and the interface between the system of mathematical education and the subsequent industrial employers.

With reference to (Malik, 1992) two dimensions of steering complex systems can be distinguished. While the first dimension concerns the manner, the second one deals with the level of steering activities (see figure 2, taken from (Ulm, 2009)).

Hierarchical-authoritarian systems, for example, are founded on the method of *analytic-constructive* steering. This principle needs a controlling authority that defines ways for reaching certain aims. However, complex systems are defined as a network that can potentially be in so many states that nobody can cognitively grasp all possible states of the system and all possible transitions between the states. So this first approach fails by the fact that it would afford information about the system that cannot be gained in reality.

On the other hand one can try to focus on the natural growing and developing processes and claim that the changes in complex systems only result from those. This incremental-evolutionary steering tries to influence these systemic processes by accepting the fact that complex systems cannot be steered entirely in all details. Instead, this approach is satisfied with only little steps, i.e. incremental changes, in promising directions. And as to the metaphor of the butterfly's wings that may change the weather far away, every small step may

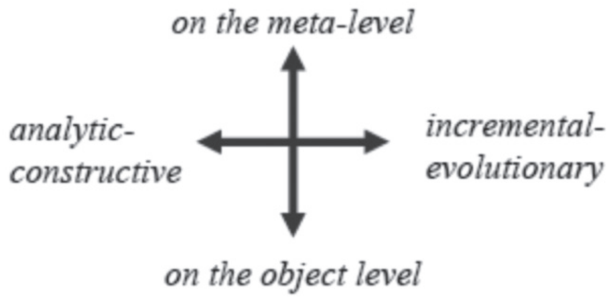

Figure 2—Steering of complex systems

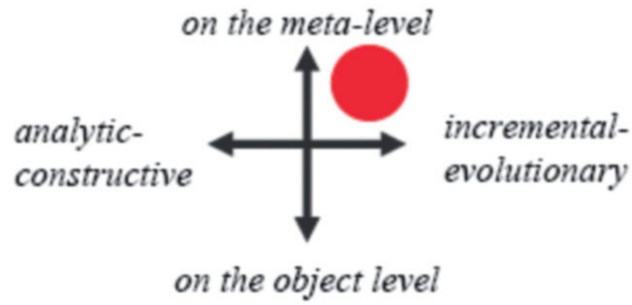

Figure 3—Innovations in complex systems.

cause unpredictable consequences. So with respect not to endanger the soundness of the whole system, only small changes are essential.

Perpendicular to this dimension, figure 2 illustrates the dimension that distinguishes between the object and the meta-level. As the name implies, the object level consists of all concrete objects of the system. In the system of higher mathematical education these would be the teachers, the students, the materials as books, computers, software, the buildings and so on. In contrast, the meta-level comprehends organisational structures, social relationships, notions of the functions of the system etc. In the educational system e.g. notions of the subject mathematics and beliefs concerning learning and applying the subject matter are on this level.

For the question, how substantial innovations in the complex system "mathematics education with respect to the interface maths - industry" can be initiated successfully, the theory of cybernetic says: attempts of analytic-constructive steering will fail in the long term, since they ignore the complexity immanent in the system; changes on the object level do not necessarily cause structural changes of the system; in contrast, it is much more promising to initiate incremental-evolutionary changes on the meta-level of beliefs and attitudes of the learners, see figure 3. On the one hand these are in accord with the complexity of the system and do not endanger its existence, and on the other hand they can cause substantial changes within the system by having effects on the meta-level, especially when they work cumulatively (see Ulm, 2009).

Integrated approach

As a consequence of the integrated approach with respect to incremental-evolutionary changes on the meta-level of beliefs and attitudes of the students, the project "Studying

Mathematics!" is based on two columns that interact in a synergetic way:

1. Attraction of more students to study a MINT-subject (at school)
2. Reduction of the dropout-rate (at university)

COLUMN 1:

- information concerning studies and vocational field of mathematics for high school students

  The answers of the survey in December 2007 showed that students do not have appropriate ideas about studying mathematics – neither with regard to contents, nor to organisation. Hence, we offer and perform information events concerning both studying mathematics and working as a mathematician in industry and economy for students in upper secondary school. These events, which up to now always had a positive resonance, serve as a motivational help for occupational orientation and science propaedeutics.

- "early studies" at university for highly gifted high school students

  In order to foster mathematical gifted students of upper secondary school, we offer free "early studies". These students take part in regular lectures and the credits earned can be transferred to regular studies after finishing high school. Hereby potential later students of MINT-subjects are lead to the university in a very early stage.

COLUMN 2:

- the one-week-lecture "Preparatory course for mathematics" as a bridge between high school and university

  Within a one-week-lecture prior to the first semester the characteristics of studying mathematics are presented: less calculation of routine tasks (that's what often happens in maths lessons at school), more developing of and working with theories. Concrete contents are "What is mathematics?", symbolism, mathematical texts, logic, proofs, set theory, mappings and relations, construction of the number system.

  The course was started in 2008 and regularly has over 100 participants each year that evaluated it very positively.

- information concerning the vocational field of mathematics for students at university

  Students often have problems seeing the relevance of study contents for jobs aimed at. The consequence is a lack of motivation or a "crisis of meaning" while studying. Therefore we offer in collaboration with the centre for further training and transfer of knowledge (ZWW) at the University of Augsburg regular talks by employed mathematicians as well as excursions to companies that employ mathematicians.

- development of learning environments addressing mathematical issues between high school and university

  In cooperation with high school teachers we develop learning environments for students that aim at two things: first, the methodological concept is such that it fosters self dependent, autonomous and cooperative working of the students; second, contents are chosen with respect to building bridges between mathematics in high school and university. By developing the aforementioned skills at high school already, students will do much easier at university level later on. Several examples (as Brandl M. (2008) or Brandl M. (2009)) were published on the internet at the online portal "Begabte fördern" (i.e. Program for Gifted) at www.lehrer-online.de.

Further specifics of the project framework

In order to provide a potential success of all the small actions taken, to alter the students' opinions in a positive way there are several supporting structures of the projects, which are

- Academic support services by the Bavarian State Institute for Higher Education Research and Planning (Bayerisches Staatsinstitut für Hochschulforschung und Hochschulplanung, IHF)
- Support of the participants by workshops such as construction of questionnaires or project management
- Documentation of the experiences and results by the project members themselves and the IHF in order to collect a pool of good and bad practice examples for other interested universities
- Gaining knowledge by integration; realized by biannual network meetings and internal newsletters

Positioning within the EIMI-Study

So for the aims of the ICMI Study 20 on "Educational Interfaces between Mathematics and Industry" (EIMI) in (Damlamian & Sträßler, 2009), i.e. among others

- to attract and retain more students, encouraging them to continue their mathematical studies at all levels of education through meaningful and relevant contextualised examples, and

- to improve mathematics curricula at all levels of education

we strongly refer to the research questions in paragraph 9 Teacher training (ibid.), for example,

- "What are good practices that support this new direction in teacher training?" and
- "How to implement these changes in an efficient way?",

but also try to answer the questions

- "How can we attract more students to study mathematics?" and
- "How can we efficiently reduce the drop-out-rate in mathematics?"

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Engineering, Mathematics communication and Education: reflections on a personal experience

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1. Introduction: communication of mathematics, a personal journey

My personal experience in the communication of modern concepts of Mathematics for a mathematically knowledgeable audience began in earnest in 1995, when I was invited to write a regular column on the general subject of the popularization of Science for the professional journal of the Portuguese Engineering Society, Ingenium [1]. Being a mathematician, these essays gradually morphed into short pieces, sometimes with a slight journalistic flavor, on Mathematics proper.

Conceiving these essays was always a rewarding challenge. As opposed to strictly journalistic pieces, they had the enormous advantage of addressing a mathematically educated audience potentially very interested in learning more about mathematical content, albeit in an informal setting. The questions addressed had a range, from Wiles' proof of the Fermat theorem to exotic n-spheres, from check digit schemes to the mathematics of Sudoku, which would be impossible to reach with a more generalistic audience. The emphasis is always to convey some meaningful mathematical content through "real-world" problems which would appeal to an audience consisting mostly of engineers.

Reactions from the readers were extremely encouraging from the very beginning; I started receiving very lively correspondence either related to the subject matter or as request for expert advice into specific mathematical problems. Indeed some past columns were extremely popular; in one of them I showed that the Portuguese ID card incoporates a check digit explaining its algorithm, thus resolving to wild urban myths then circulating about "the extra digit in the ID card". In another pair of columns, when the Euro started circulating, I proposed to find the check digit scheme behind the Euro banknotes and enlisted my reader to send me Euro banknote numbers. I had hundreds of replies (and the answer as well: it is simply "casting out nines"!).

At some stage the interest on these publications overflowed the intended audience. Clearly there was an interest in seeing Mathematics in action in real world problems, in contexts where we would least expect to find it – be it our ID cards, our Euro banknotes, the Sudoku puzzles in the paper, the graph of the Internet to coding theory and CD scratches. I started having lots of requests for authorization for classroom use by high school and college teachers. Many readers suggested enthusiastically that these essays should be published in book form.

In 2001 I published my first book thus derived, *O mistério do BI e outras histórias* [2] with Gradiva, the foremost Science publishing house in Portugal. The title comes precisely from the piece on the check digit of the ID card, about which I still receive questions today. It was a best-seller by Portuguese standards, standing today at its 11th edition and over 10.000

copies sold. It was followed in 2003 by *Da falsificação de Euros aos pequenos mundos* [3] and in 2007 by *O fim do Mundo está próximo?* [4]. The number of original papers in *Ingenium* is by now over one hundred.

These books had great impact in the educational community, since they conveyed, in clear but rigorous language, examples of use of Mathematics to solve real-world problems. Examples of this impact in schools are many and, in some sense, unusual. I would briefly mention the following:

1. Utilization of some of the problems described (ID card, Monty Hall problem, Euro banknotes ...) in College-made exhibitions of Mathematics (e.g. IPL Leiria, 2003).
2. Dozens of yearly and standing invitations to speak on Mathematics in high schools, technical schools, Colleges and Universities all over the country. Between March and May alone I receive about a dozen speaking invitations in high schools.
3. Proliferation of Java applets in the Web with the algorithms I describe, with reference (see e.g. [5]).
4. Many high schools had students elaborate school works on some of the topics covered, and more than one chose to profile me as a mathematician.

2. First case study: the Portuguese ID card and check digit schemes.

I would like to mention two especially significant episodes arising in my efforts for communicating Mathematics. The first one is the case of the check digit in the Portuguese ID cards.

Sometime in the early 1990s the governmental agencies decided to include an "extra" digit following the national ID card number, in an isolated box. Obviously, this was a check digit, in most likelihood a checksum digit.

However, care was not taken to explain what the new digit was about. As a consequence, by the late 1990s the wildest urban myths about this recently introduced extra digit were floating around. The most popular one was that it would represent the number of people with the same name as the card bearer. This urban myth was extremely widespread at the time (and in fact still survives). But there were others, ranging from the number of outstanding traffic tickets (a reporter, interviewing me in 2000, said "and in my case, it is correct!") to the number of family members having been jailed during the pre-1974 dictatorship.

In the meantime, a fellow mathematician at the University of Coimbra, Prof. Jorge Picado, had gone to work on disproving this myth. He collected the ID numbers and corresponding check digits of a few dozen persons and, under the assumption that the check sum algorithm was similar to the ISBN algorithm, programmed his computer to reverse engineer the problem and discover the checking algorithm.

Some surprises were in store. First of all, only 10 digits were used as checksums: 0 through 9. It is well-known (see e.g. [5]) that the additive algorithms require a prime number of check digits; thus for instance the ISBN algorithm uses the set of 11 digits {0, 1, ... 9, X}, since 11 is the smallest prime number greater or equal than our usual numbering base 10. Now Prof. Picado's data included only a set of 10 check digits; there was no 11^{th} digit. This might lead us to think that non-additive algorithms, such as non-commutative algebraic codings based on dihedral groups, like the Verhoeff scheme adopted by the Bundesbank ([5], [6]) might be in use.

Secondly, a very strange thing happened. When Prof. Picado ran the algorithm-detection code, the program did not converge, whatever the supplementary heuristic hypotheses applied. This seemed to indicate either an inconsistency in the data or a faulty error-detecting algorithm (e.g. one which would not injectivity of the check digit).

In fact, the data revealed an obvious anomaly: the relative frequency of the digit 0 was double the frequency of the other digits. This suggested that something could be wrong with the check digit 0. In fact, simply eliminating all the 0s from the data and running the algorithm-detecting code ensured immediate convergence and allowed Prof. Picado to discover the identification system used by the Portuguese agency.

This discovery was a double-edged sword. On the one hand, it was shown that the identification scheme is (almost) precisely the ISBN scheme. On the other hand, since there is no 11^{th} check, the scheme cannot work efficiently.

Indeed, what was observed is that the Portuguese ID card uses a version of the ISBN identification scheme with a *mathematical bug*: the non-existent 11^{th} digit for checksums is replaced by a second (false) zero. Thus the identification system is not injective, and 1/2 the occurrences of the digit 0 as a checksum are false.

This is a rather embarrassing situation, since it implies that indeed the check digit cannot be relied upon for error-detecting purposes, therefore defeating the purpose of its own introduction and rendering it useless! In actual practice, no official agency using the ID card number ever bothers asking for the check digit – not even the agency which issues the cards themselves.

Things get even more curious. Although the whole ID card was recently changed, with the issue of a *Citizen card* incorporating the most modern biometrical and physical technologies, *the ID numbers did not change, neither did the check digits*. Thus the ID card bug propagated to the new identification scheme. Moreover, the exact same identification scheme is used in the Fiscal ID number – and thus the exact same mathematical bug occurs.

In May 2000 I published a column in Ingenium about this question, *O mistério do Bilhete de Identidade (The mistery of the ID card)* which instantly became my most widely read article. In a short span of time I got reactions of awe, disbelief and the article was widely reproduced in the media and the Internet, where it still can be found today. As mentioned above, in 2001 I published a book, mostly a collection of some the columns I had publishes in *Ingenium* [2].

Its publication had a large impact and made the material available to a very different audience, the general public. The title of the book, the problem itself and the question of the bug in the identification system led to a large impactideas in the educational community: everybody has an ID card and number, and it is easy to explain and implement the mathematics and the algorithm behind the ISBN scheme.

The ramifications and implications of this work are quite surprising. Many high schools and Colleges had students doing school works about identification numbers and the ID card. Many students, either in Colleges or Universities, developed computer applets to compute the ID card number (as well as the Fiscal ID number). One College (ESTL, at Leiria) developed an exhibition of Mathematics for high school students based partly on material from my book. Both me and Prof. Picado were contacted very frequently to give talks on these (and other subjects in schools. In fact, even to this day I am asked to give talks in high schools *specifically on this subject*.

And, last but not least, Both me and Prof. Picado were contacted by the financial branch of a large multinational corporation because of our "expertise about the identification problem in Portuguese ID cards" (we did not, however, become private consultants)!

This shows quite clearly how communicating Mathematics to an Engineering audience can have an "overspill effect" which quickly crosses the interface to the outside world and has a real impact on the educational community. Indeed, it is quite likely that without the original article in *Ingenium* Prof. Picado's discovery might still be largely unknown to the world outside mathematicians, with a loss both for Mathematics Education and the general public (the urban myths about "the extra digit" still exist to this day!).

3. Second case study: the Euro banknotes.

As a second case study, it is relevant to mention the Euro banknotes. With the introduction of the Euro in 2002, the European Central Bank faced the obvious problem of facing counterfeiting of banknotes in the biggest market in the world.

The most advanced technology was used in the banknotes, employing special paper end silver bands, lasers and holograms; at least two of the anti-counterfeiting measures were kept secret by the ECB. One could expect the identification scheme used for Euro banknotes would be equally sophiscated, truly XXIst century, as it was neing equally kept secret.

After the circulation of the Euro banknotes on 1st January 2002, I undertook a somewhat quixotic personal project: to discover the mathematical algorithm of identification of Euro banknotes. I started collecting and annotating all the banknote numbers which passed thorough me, and recruited some friends as well. In the process of programming and entering the data, I noticed the structure of the numbers: they were

L-DDDDDDDDDDD

where L stands for a letter of the alphabet from J to Z and D stands for a digit from 0 to 9, so the number is a alphanumeric string made up of one letter and 11 digits.. However, a strange thing happened: *the last digit is never allowed to be* 0. This alone indicated a mathematically special role for the last digit as a check digit.

At that point I published an article in *Ingenium* explaining the problem, giving preliminary results, and inviting readers to submit their recorded data. Response was enthusiastic: with this collective effort, I gathered thousands of data. In the process of programming and entering the data, I realized experimentally the following.

If a fixed numerical value is attributed to the letter L at the start of the identification number, then the (last) control digit is simply determined by imposing that the sum of all the 12 numerical values of the digits is congruent with 0 (mod 9). Or, in plainer terms: the identification scheme of Euro banknotes is simply the thousand-year old process of *casting out 9s*!

This is all the more surprising since it is an extremely inefficient error-detecting algorithm, as we all know from elementary school (success rate < 90%). So the banknotes with the most advanced physical systems against counterfeiting had also, embarrassingly, the most mathematically unsophisticated algorithms for error-detection!

The reader can check this for him/herself by pulling out a Euro banknote. All that is required is to know the correspondence between the leading letter L and its numerical value. This is given in the table below.

The numerical values of the letters are simply 1-9 in ascending order (again with no attribution of 0, since this is arithmetic (mod 9).). Moreover, a letter has no deeper meaning than simply identifying the country of origin of the banknotes. In fact, even though the UK, Denmark and Sweden are not (yet?) part of the Euro zone, they already have allotted places by this process, should they wish to join!

Letter	Value	Country	Letter	Value	Country
J	2	UK	R	1	Luxemburg
K	3	Sweden	S	2	Italy
L	4	Finland	T	3	Ireland
M	5	Portugal	U	4	France
N	6	Austria	V	5	Spain
O	7	—	X	6	Denmark
P	8	Netherlands	Y	8	Greece
Q	9	—	Z	9	Belgium

This collective quest with the readership of *Ingenium* was a great success. We vene managed to identify what may be called a second "mini-code" on the flip side of the notes, in very small print, consisting of L-DDD-LL in very small print. This was an even bigger mathematical disappointment: it is a mere serial number from the typography where the banknote is issued and has no mathematical content at all. All this information is gathered in book form in [3] and, to the author's knowledge, nowhere else.

This makes for wonderful material for communicating mathematics in popular lectures. Everybody (in the Euro zone) handles banknotes, even small children. So it is possible to tell a story with meaningful mathematical content, by adapting the theoretical aspects of coding theory and identification systems to the audience, but always keeping in mind a very real example – the banknotes people handle.

In this way a very unlikely interface between Mathematics Education and Engineering was created. The audience of mathematically educated Engineers provided the data. A mathematician discovered the solution. This solution is ideally suited for communication and popularization of Mathematics. A very unlikely connection indeed!

4. Some conclusions from an unexpectedly useful interface between Engineering and Mathematics.

Admittedly, my experience as a Mathematics communicator is not at all typical (if such a thing exists at all). Starting from writing about Mathematics for the professional society of Engineers, it quickly spread into the educational community and thus established an unexpected interface between Industry (in the broadest sense of the term) and Education. Although this has been a very rewarding journey, it could hardly serve as a model. However, there are some useful lessons I think can be drawn from this personal experience.

A first point which professional mathematicians are very prone to miss: the educational community is in fact eager for meaningful examples of real, everyday-world applications of Mathematics satisfying both the following conditions:

(1) The problems they describe should be easily understandable, namely by high-school students preferably through some everyday-life appeal;

(2) They should have real mathematical content.

These kind of problems cannot in general be supplied by high-school teachers. The Mathematics they learned are generally either superficial, outdated or both. In fact a cursory browse through high-school textbooks shows a very limited set of more or less applications of Mathematics to the real world – Fibonacci sequences in pine cones and the golden ratio is probably a universal example.

It has become clear to me that problems with significant mathematical content above the more or less trivial must be supplied by mathematicians or scientists willing to invest the necessary time, work and effort. Mathematics changes. Technology changes. Innovation occurs, and Mathematics a crucial part of it. The uses of Mathematics in technology, be it in cell phones, DVDs or ATM machines, changes. It seems clear that the communication of emerging applications can only be made by those in the know – mathematicians, in particular – and there is no reason why we should expect high school teacher should even be aware of these emerging subjects. In hindsight, it is obvious that it is the mathematical community which should have the energy to make the outreach effort.

Both case studies above illustrate this situation quite clearly. They deal, each in their own way, with the same kind of mathematical problem – identification systems and check digits. This is a problem with which everyone can be assured be acquainted in one form or another: it arises in contexts from national ID cards (mandatory in Portugal) to credit cards, from Internet passwords to bar codes, from cell phones to DVDs, and increasingly so in the past 20 years. Moreover, it satisfies both criteria stated above (understandability and mathematical content) as characterizing problems which should interest the educational community, so it seems ideally suited to establish such an interface.

However, it is not only unrealistic but altogether absurd to expect the typical high school teacher to have any knowledge at all of the mathematics involved in identification systems, or even to be aware of such questions.

So the first lesson is that mathematicians, or at least some mathematicians, should be actively engaged in outreach activities.

A second interesting point, which acts somewhat as a counterpoint to the first one, is as follows. Although Mathematics Departments in general recognize that outreach is part of their mission, in practice this is not much more significant than lip service. Somewhat paradoxically, generally the attitude of academic peers to the personal investment of time and effort into outreach activities and communication of Mathematics is not necessarily positive.

There exist quite gratifying exceptions, but generically the average mathematician ranks as a first-rate activity scientific research and as a second priority teaching duties and academic chores. Outreach activities and communication of Mathematics to the general public generally are not considered "important" or, for the most fundamentalist people, even "serious" work, and in terms of the spectrum of scholarly activities its consideration lies somewhere between the "not serious" and the "waste of time".

It is not hard to hard to imagine that the tension between (1) and (2) raises some very real and serious issues for the future of Mathematics Education. On the one hand, we want to show young students that today's Mathematics has real impact and meaning for today's world, and this can only be effectively done by professional mathematicians. On the other hand, there are no formal or informal reward mechanisms in the academic world to perform outreach activities – a situation that sometimes even perversely acts as a *negative* incentive, which can become at times quite frustrating.

There is clearly a tension in this interface that should at some stage, in the interest of all parties, be resolved.

I think there is a larger issue involves that is not necessarily related specifically to Mathematics. The articulation between in fact, the whole of Science) and society at large has probably never been a more pressing issue. Paradoxically, we have never lived in a society more dependent on advanced technology, with generalized reliance on energy, cars, computers or cell phones; and at the same time never before has such a society been in such a state of generalized dependence on Science (and, in particular, Mathematics) for this technology and thus for its own functioning.

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Tackling the challenges of computational mathematics education of engineers

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Abstract With the presence of powerful simulation tools in today's engineering practice, the challenge in designing an introductory course in computational mathematics is twofold: students must be convinced of the necessity of opening some of the black boxes, and the mathematics education they receive must prove useful in controlling the solving process and use of the tools. The paper provides guidelines and suggestions to help meet these challenges.

Introduction

We live in a world of black boxes, all of us do,
however well we may be educated.

Isaac Asimov, 1967

As part of an effort to reduce the perceived gap[1] between engineering education and real-world demands on engineers, a growing number of engineering schools are integrating an industrial practicum into their curriculum. This certainly provides an early and valuable encounter with the physical and organisational complexity of the problems engineers are now required to solve, along with the elements of the working environment (resources, teams, rules and regulations) which now shape the engineering practice, but it also tends to create the perception that there now exists a wide array of software tools to do the job, that one of the main objectives of engineering education should be to master the interface to these tools, and that there is little need to know what lies within.[2]

In fact, one may say that engineers live in an increasingly vast and thick world of black boxes: their work relies more and more on simulation software tools which are based on sophisticated models and state of the art numerical algorithms that only experts on these topics fully understand. These tools (e.g. FLUENT, ANSYS, COMSOL, MODFLOW) can typically solve most of the problems engineers will submit to them, but there always comes a time when a problem no longer fits the necessary conditions for the software to provide the solution, or requires a careful selection and customisation of the “advanced features” which may look enigmatic to the neophyte.

In large corporations, this may not represent an issue, as there is often a division of modelling and simulation experts who can act as consultants for their generalist colleagues. But in smaller engineering firms, such consulting services may only be found outside the company, at a prohibitive cost — unless a kind university professor is willing to help. From our own experience and perspective, the need for consulting experts in modelling and simulation only seems to be growing, just as the need for statistical consulting services rose significantly when statistical software tools became more accessible and widely used by researchers from all fields (Hodgson, 1987).

In this context, the challenge in designing an introductory course in computational mathematics is twofold: students must be convinced of the necessity of opening some of the black boxes, and the mathematics education they receive must prove useful in controlling the solution process and the use of the tools. The following provides suggestions to help meet these challenges. Although these suggestions stem from our computational fluid dynamics background, we believe they are applicable to any field of engineering where computational mathematics is used.

Addressing motivation

Getting from A to B

When talking about software tools or technology devices, people often compare them to cars, as "just a means for getting from A to B", and this comparison seems to relieve from the obligation of learning about their inner workings. Although one could argue that there is a minimum of mechanics and technology that still needs to be learned for driving a car safely and efficiently, what makes this statement particularly inappropriate for describing the use of specialised software tools for simulations is the difficulty of deciding whether "B" has been reached. After all, this is why we do simulations: to get a better appreciation of where we might end up going. And even when we have the knowledge to determine that an improper "B" has been reached, what may still remain unclear is why we got there in the first place and what we could do differently to get to a more "realistic" destination.

Saying that by no means guarantees buy-in from students. But a couple of pictures may help – especially if they were obtained using commercial software. The following graphs (Figure 1) represent the flow within a channel with a square obstruction; they were produced with the COMSOL software for the same boundary conditions and the same physical parameters. Which view (1 or 2) is right? Can you explain the difference?

Actually, view 2 was obtained simply by deactivating the default numerical stabilisation method. The choice of the stabilisation method depends on the partial differential equations, their type and order, the interpolation of the variables considered, and the physics of the problem (which can be reduced to the Reynolds number in this example). Stabilisation is not always required and users may not be always aware of the options or default values used by the program to favour robustness over accuracy. By smoothing the solution, stabilisation favours convergence, but this is sometimes done at the cost of hiding the symptoms of an inadequate modelling decision, the choice of a linear interpolation scheme in this case. Such an example provides a clear illustration of the need to get a little deeper in order to assess the validity of a solution.

Social accountability

News headlines may also provide additional incentive for learning about the inner working of simulation tools and their sensitivity to modelling decisions. The following case could

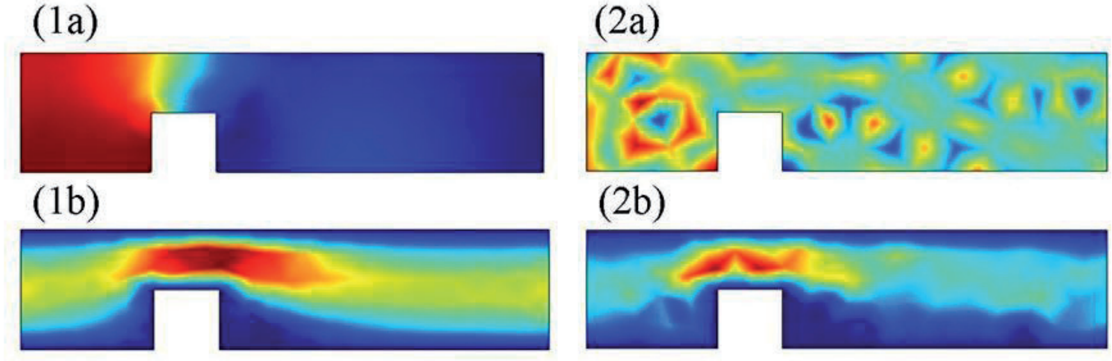

Figure 1.—Distribution of pressure (a) and norm of the velocity field (b) for two different simulations (1 & 2) of the same situation

announce the emergence of a new trend in public movement, especially when environment, health and safety are concerned.

In Canada, the choice of a particular landfill site in the Simcoe County (Ontario) was the object of debate for some time and received wide media coverage on a few occasions. The county had been promoting and developing a new landfill site ("Site 41"), with the approval of the Ministry of the Environment and despite the strong opposition from residents in the surrounding communities, concerned with the risk of contaminating the groundwater beneath the site. What is rather exceptional here, but could become more typical in the years to come, is the fact that a group of citizens, through one of its members, asked access to the hydrogeological model and input data which had been prepared and used by the county's external engineering consultant for simulating groundwater flow in the proposed site. The consultant had performed these simulations with open-source software (MODFLOW[3]), documented the results into a report submitted to the county, and this report contributed significantly to obtaining approval for the landfill development project. An order, issued in June 2009, from the Information and Privacy Commissioner to the county to provide the requested access gave no results, as the engineering firm refused to provide the County with the model and data, and the County felt in no position to claim them for the following reasons: this information was not part of the agreed upon "deliverables" with the firm, such a request would fall outside the "custom in the trade", and the county did not see any basis upon which it could take legal action against the engineering firm. Although an appeal from the Commissioner and the active resistance from citizens finally led the county councillors to vote in September 2009 for a permanent moratorium on Site 41, some elements of the story continue to raise interesting questions.

First, there is the question of ownership of the model and data, where the development perspective and commercial interest may come at odds with the public's right to access to information. On this question, we may witness in the near future some evolution of the legal framework and implications, especially when models are used to help make decisions with potentially significant social impact; it would only make sense from both a scientific and an ethical perspectives. But irrespective of whether or not direct access is given to the model, the fact that the software used is publicly or commercially available makes it possible for a third party to come up with different simulation results, based on a different model or simulation parameters. As a consequence, engineering firms that deal with sensitive matters such as the environment, where public awareness and concern are only growing, can now expect a greater pressure to document and justify their assumptions, modelling decisions and solving parameterisation. Relying too much on black boxes and standard procedures could only expose them to the risk of revealing themselves as the blind trying to lead the blind. Without great surprise, this would go against some of the principles of ethics and conduct established by the Canadian Engineering Qualifications Board (2001) with respect to the environment:

> Engineers must recognise and respect the boundaries of their competence and must only undertake the environmental evaluation of those aspects within this competence.

The question now becomes: how can computational mathematics education significantly contribute to developing a competence to undertake sound evaluation of the performance and impact of engineering development? How should we define competence in computational mathematics for engineers?

Developing competence in computational mathematics

Computational mathematics typically encompasses both the solving of mathematical problems by computer simulation, and the numerical methods which can be used in these simulations. Yet, as we will see in the following, computational mathematics in the engineering practice can hardly make abstraction of the real world context from which the problem emerged, as it can have a direct incidence on the method to be selected and used.

In an introductory course in computational mathematics, the numerical methods that are taught typically deal with relatively simple mathematical objects and algorithms. Rather than focusing on the mathematical content, we will direct our attention to some of the general ideas and attitudes which we believe should be developed even when solving problems with elementary numerical methods, so as to favour adequate transfer to the use of the more sophisticated numerical methods that are used in solving today's complex engineering problems.

To illustrate the necessity of taking into account the real world context in computational mathematics, we select as example the classical Poisson equation:

$$\frac{d^2T}{dx^2} = 1$$

This equation is often used in introductory courses to illustrate the process of discretisation through which a continuous differential equation is transformed into a system of algebraic equations.

Faced with such an equation, the students who have learned to discretise first-, second- and third-order differential operators using forward, backward or central finite-difference schemes, systematically reproduce this process to any given differential equation, without questioning the assumption of continuity. To avoid such pitfall, our didactical orientation makes us look for a situation where the need for revisiting this assumption will naturally emerge.

The Poisson equation can be seen as a one-dimensional simplified version of the Darcy equation used in the MODFLOW software. A more general version of the Poisson equation could read:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = q$$

In the context of groundwater flow modelling, T would be the pressure head, k the hydrogeological diffusivity which can vary in space, and the right-hand side would represent either a source or a sink within the hydrogeological system being studied.

To solve this equation, the natural tendency of students is to rewrite it so as to show explicitly the differential operators on the primary variable T and proceed from there to the application of the familiar finite-difference schemes. Their initial transformation thus takes the following form:

$$\frac{dk}{dx}\frac{dT}{dx} + k\frac{d^2T}{dx^2} = q$$

Such transformation is valid only if k is a "sufficiently continuous" function since this indirectly determines the continuity of the primary variable and its derivatives, but this condition is often overlooked by students. Interestingly enough, a typical hydrogeological structure falls outside the domain of validity for this condition, as it is composed of different layers of soil, where k is discontinuous from one layer to the next. Trying to solve the discretised version of the transformed equation on such problem, a student will be faced with a solution that strongly oscillates near the interface between two layers of soil. As the student

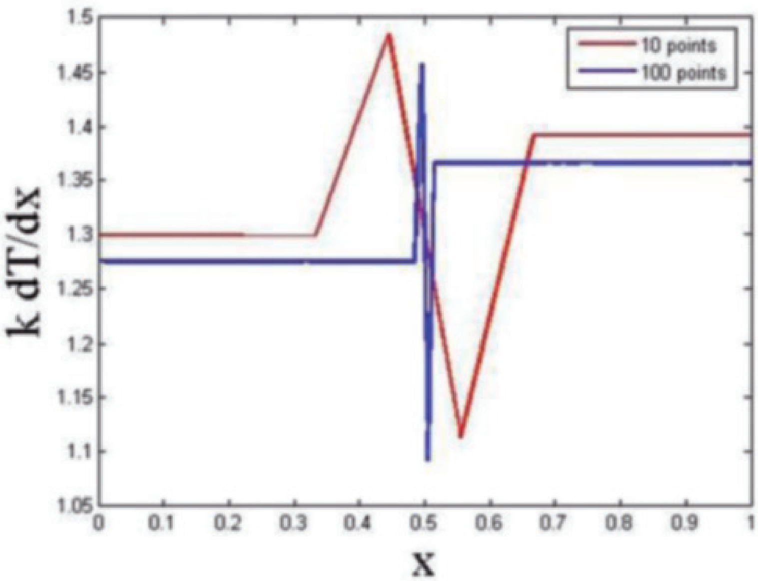

Figure 2.—Effect of grid refinement on the simulated flux

tries, unsuccessfully, to overcome the problem by refining the discretisation (Figure 2), he must figure out that the problem must lie with the mathematical model used at the interface between two layers of soil, where the conservation of flux has been lost.

The problem is that a typical numerical introductory course focuses on the differential operators and rather tends to overlook the underlying physical principles; but principles such as conservation laws can and should guide the selection of variables for which the necessary continuity conditions are met, thereby enabling the application of finite difference formula. In this example the flux kdT/dx is continuous over the computational domain, whereas dT/dx is not.

In designing the numerical model, we must be coherent with the physical problem. More important than the perceived accuracy of the numerical model is its capacity to reflect in its transfer of information the physical principles that govern the situation (e.g. transport, diffusion). This knowledge should determine the appropriate choice of the discretisation schemes, interpolation polynomials and stabilisation methods.

The role of analytical mathematics

The discretisation targets not only the equations but also the computational geometric domain since the primary variables are solved for a finite number of grid points chosen to maximise the accuracy. The application of the discretisation schemes over this grid results into an algebraic system of equations. In an introductory course, students are often confronted to the difficulty of reasoning the equations in terms of discrete variables; it is as if the time spent in calculus in making h tend to zero acts as obstacle to their capacity or acceptance of applying the finite difference schemes to a set of pre-determined points.

This argument, combined with the foreseen mathematical practice of engineers, could militate in favour of earlier exposure to numerical methods in calculus courses for engineers, in the direction proposed by Enelund and Larsson (2006). With this approach, the numerical methods are typically used either to illustrate some of the mathematical concepts being taught, or to tackle more complex problems than those that can be solved symbolically. Yet, trying to teach simultaneously analytical and numerical approaches does not come without risks. Depending on the choice of examples and tasks, a student can either develop a false sense of security with regard to the use of the numerical methods or be completely discouraged from their use. But it is mainly the wide domain of applicability of numerical methods that tends to turn students away from analytical solutions, despite the precious information these solutions can provide in appreciating and anticipating the physical properties and solving challenges of the modelled phenomenon.

From our perspective, one of the key challenges in the mathematics education of engineers actually resides in the articulation of the two paradigms (analytical and numerical) to take advantage of their respective pragmatic and epistemic contributions in tailoring the solving process to a given engineering field of application. Rather than aiming for a systematic concurrent exposure of the two approaches in any course of mathematics, we propose that the courses that mainly deal with analytical calculus go deeper into the assumptions upon which some of the analytical methods are based and the properties[4] that they may bring to light.

This rather bold position may be perceived as a way to rehabilitate a more theoretical orientation to the mathematics education of engineers. But it actually comes from the necessity, for both academia and industry, to provide verified and validated engineering results. And, as shown by the Site 41 story, this may be further stressed by growing public awareness.

Verification and validation as a framework for designing the computational mathematics curriculum

Requirements for verification and validation have been specified by ERCOFTAC[5] (2000) and AIAA[6] (1998). ERCOFTAC defines verification as the "procedure to ensure that the program solves the equations correctly" and validation as the "procedure to test the extent to which the model accurately represents reality". Incorporating verification into computational mathematics education of engineers thus requires to go beyond the sole use of numerical methods as these come with intrinsic uncertainty.

A first step in verification consists in verifying that "the code is capable of achieving correct mathematical solutions to the governing continuum equations in the limit of $\Delta \rightarrow 0$, and the order of convergence is verified at least for well-behaved problems", typically nonlinear problems with an exact closed-form solution (Roache, 2008). This requirement clearly points to the necessity of integrating an analytical approach to computational mathematics.

A second step in verification typically makes use of numerical techniques such as Richardson's extrapolation. This technique is usually taught in introductory numerical courses as a recursive means to improve the accuracy of a solution. For instance, its application to the trapezoidal rule leads to the Romberg integration. But what must be made clear to the students is that a sufficiently improved solution is an acceptable surrogate to the closed-form solution, which can be used to derive a reliable error estimator; the latter takes the form of the Grid Convergence Index (GCI) as proposed by Roache (1998). We believe this is where the focus should be when teaching about these techniques: as they allow for error assessment, they can serve as powerful tools to ensure that the engineer remains in control of the numerical solving process. Learning activities should be designed to hone skills and expertise in using these techniques for assessing the quality of the solution. Abstract mathematical concepts and techniques (e.g. vector norm in Sobolev space) naturally reveal their usefulness in such learning context.

In contrast, validation is defined as the "procedure to test the extent to which the model accurately represents reality". This can only be done after a thorough and successful verification process. It is usually accomplished by measuring the difference between simulation results and experimental data. As such, the results produced for meeting the second requirement of verification can be used to provide "discretisation-error free" simulation results, in an attempt to isolate the modelling error.

Conclusion

We have provided in this paper some general orientations and more specific suggestions for designing an introductory course in computational mathematics for engineers. Acknowledging the presence of powerful simulation tools in today's engineering practice, we aim at developing in students a sense of control over the solution process and the use of these tools, while nurturing their motivation to go beyond the user interface. In particular, it appears clearly to us that to teach computational mathematics effectively to future engineers, elements of physics specific to their field, mathematical analysis and engineering practice must be used to articulate within the lectures and tasks, the ideas, skills and attitudes expected from an engineer.

Acknowledgments

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Notes

1 This perception has led to the CDIO Initiative, an educational framework developed by engineering schools around the world with input from academics, industry, engineers and students. It aims at providing students with an education stressing engineering fundamentals set in the context of Conceiving – Designing – Implementing – Operating real-world systems and products. http://www.cdio.org

2 These perceptions can be inferred from students' internship reports.

3 MODFLOW is an open-source software that was developed by the U.S. Geological Survey (USGS), which is a science agency within the U.S. Department of the Interior.

4 One such property often overlooked is the linearity of the solution which, when it applies, can allow the breaking of a problem into its elementary components and can help reduce the cost of a solution.

5 This was done by the Special Interest Group on "Quality and Trust in Industrial Computational Fluid Dynamics" within the European Research Community on Flow, Turbulence and Combustion.

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Improving the industrial/mathematics interface

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Abstract Mathematicians and industry can mutually benefit from a closer collaboration. There are numerous examples of advantageous cooperation around the world, a good example being study groups with industry. The concept of modelling is central to the practical applications of industrial and applied mathematics. The refocussing of our mathematical resources in mathematical modelling is desirable, essential even, for the regeneration of economic success. So why are mathematicians who participate in industrial mathematics in the minority and what can be done to improve the situation? The answer must lie in the training: mathematics curriculums at all levels need to be redesigned to reflect the ever-growing interest in mathematical modelling and provide our students with the basic skills necessary to become real modellers.

Introduction

Mathematics is ubiquitous in the sciences and engineering. Arguably a science is considered to have come of age when it has become sufficiently mathematical as illustrated by the burgeoning areas of mathematical biology and mathematical finance. The concept of mathematical modelling is central to the application of mathematics in the sciences: practitioners may wish to predict the price of shares, the weather for the next few days, the number of faulty products in a given production line or the efficiency of a new drug.

Despite all these potential applications, some mathematicians have moved away from industry and from real applications. From the late 1960s under the influence of the Oxford group (Alan Tayler and Leslie Fox, see Ockendon (1998), Tayler (1990)), interest in modelling real industrial problems has steadily grown. This discipline is called industrial mathematics, though it is also referred to as applied mathematics or mathematical modelling. The meaning of these terms will be discussed in more detail later but at its simplest we can define industrial mathematics and its near synonyms as being problem driven mathematics for the sake of the sciences while pure mathematics may be regarded as being mathematics for its own sake. In this context 'industry' is interpreted in a very broad sense: the remit of these groups includes more than collaboration with industry (problems may come from anywhere in the sciences e.g. mathematical biology, mathematical finance, the environment). Collaborations between 'industry' and mathematicians can prove extremely successful; study groups with industry are an excellent example. These are typically week long intensive sessions involving mathematicians and industrialists/scientists who propose the problems. But in order to fully exploit industrial activities such as these, with the intention of improving the interface between industry and mathematicians, we must introduce modifications in the training of mathematicians. We suggest that there should be an element of industrial mathematics in every third level mathematics group.

We will discuss the development of such activities in the context of experience with the 'Mathematics Consortium for Science and Industry (MACSI)', a network of applied mathematical modellers across Ireland, centred at the University of Limerick on foot of a Science Foundation Ireland grant of Eur 4.34M in 2006. This group has over twenty five active collaborations with industry, (e.g. electronics, pharmaceutical, food, finance), engineering and the sciences.

What is industrial mathematics?

The terms 'industrial mathematics', 'applied mathematics' and 'mathematical modelling' are often used as near synonyms to make the distinction with pure mathematics whose central tenet is formal proof and which is not generally concerned with real problems arising outside of mathematics. An applied mathematician is a kind of lapsed pure mathematician in the sense that (s)he would like to prove every result formally but is sometimes unable to do so and must make intuitive leaps in the search for understanding. Mathematical modellers excel at paring problems down to their essence and using mathematical tools to discover the essential mechanisms which govern processes. Otherwise, for example, the flow of air over airplane wings would not be understood. Classical applied mathematics is associated with names such as Archimedes, Newton,

G.I Taylor, Stokes, Reynolds, Kelvin all of whom regularly used mathematics to understand phenomena in the physical world, essentially operating as mathematical modellers. Of course, in their times there was no serious industry requiring mathematical modelling on a large scale but if there had been there is little doubt that they would have been involved in it. It was only late in the nineteenth century that pure mathematicians, who do not usually seek inspiration from the world in which they live, emerged.

The Mathematics Subject Area Group (2002) was unanimous in identifying three skills which it believes every mathematics graduate should acquire: the ability to conceive a proof, to solve problems using mathematical tools and to *model a situation*. In point of fact, many groups do not recognize the latter as a key skill. (We elaborate upon this in the next section). The concept of modelling is central to the practical applications of industrial and applied mathematics. The U.K. Smith Institute (2004) identifies mathematical modelling and simulation as the core outlet for applied mathematics in Europe. Furthermore this document states that: ‘Mathematics now has the opportunity more than ever before to underpin quantitative understanding of industrial strategy and processes across all sectors of business. Companies that take best advantage of this opportunity will gain a significant competitive advantage: mathematics truly gives industry the edge’. The coordination of our mathematical resources in this way can only aid the regeneration of economic success.

Historically, the investigation of real problems, has been the mechanism for some of the most significant mathematical developments. The exposure to industrial problems naturally leads to associated theoretical problems which, de facto, involve the areas which are necessary to give industry and science an edge. Industrial mathematics groups thus include in their mission the wider aim of building up expertise in the core mathematical areas of most relevance in industry and science. Nowadays, applied mathematics lies at the intersection of a wide number of subjects. This includes pure mathematics of course but also physics, chemistry, biology, finance, engineering, economy, business or social sciences to mention but a few. In all these subjects, practitioners in academia or industry model phenomena to analyse the present and prepare for the future. Applied mathematicians therefore occupy a unique position at the border of theoretical science and practical applications as they have a wide expertise easily applicable to industry. Original ways to develop links between applied mathematics and industry have been developed over the years. The most spectacular interaction occurs during workshops called study groups with industry which we discuss below.

The time is now ripe for a realignment in mathematics. Perhaps every mathematical group should have at least some industrial mathematics activity. (This is certainly not the case in Ireland where there is only one such group). There is a feeling of change in the air: applied and industrial mathematics are being ever more prevalent. The European Science Foundation is funding a study called ‘Forward look at mathematics in industry’, the European Consortium for Mathematics in Industry is growing in influence, the MATHEI (Mathematics European Infrastructure) project is about to begin. Under a call from the European Commission, this latter project aims to improve infrastructures for Mathematics in Europe and its interfaces with science, technology and society at large. Perhaps it is even time to relabel applied and pure mathematics as real and non-real (imaginary?) mathematics respectively.

Evolution of the training

Many mathematics courses do not include a real modelling component and this neglect continues through to postgraduate level and beyond. Of course, many courses claim to have a significant modelling component but this is often achieved by relabelling an old course (typically a differential equations course). Unfortunately, some mathematicians pretend to be modellers and to solve real problems in order to obtain research funding.

In fact, they pick out problems which they can deal with but are typically of no interest to the industrial collaborator. The reality is that employers seeking mathematical graduates would like them to have genuine modelling skills, to really understand the discipline which they are modelling, to care about the significance of the results which they obtain and hence to have a broad scientific background. It is important that students gain exposure to suitable texts e.g. Fowler (1997), Ockendon et al. (1999), Mattheij et al. (2005), which espouse the real philosophy of modelling. Modelling starts at the application level and a key part of the process is the formulation of the mathematical problem (Fig. 1). Very few undergraduate third level courses even touch on this extremely difficult topic. Most mathematics students are used to being presented with nicely formulated mathematical problems, which have equally elegant solutions. In reality, the hardest part of the process often involves asking the right question! Once the problem has been formulated and solved (hopefully), just

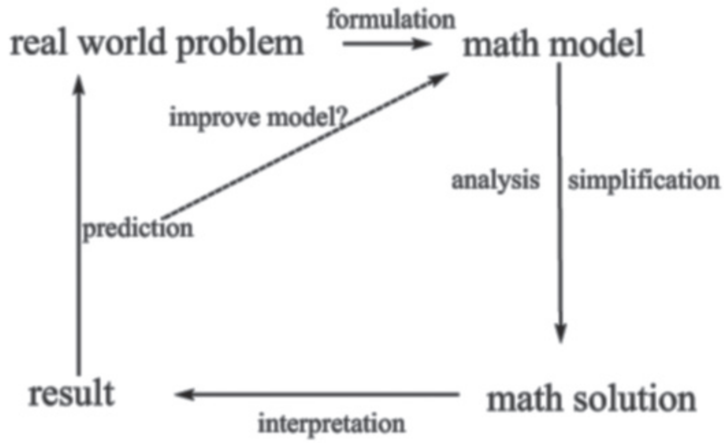

Figure 1—How mathematical modelling works.

as important is the interpretation of the solutions for the benefit of the industrial/scientific partner. What do the solutions really mean? How can the knowledge gained be put to use?

Genuine applied mathematical modelling deals almost exclusively with problems which arise outside of mathematics: in physics, chemistry, biology, finance, economics and industry in general. In many traditional pure mathematics undergraduate courses the emphasis is placed almost completely on mathematical rigour and technique with no genuine attempt to develop physical intuition and a feeling for real problems. In effect, the current generation of pure mathematicians is training the next generation to be like itself, to be logically rigorous and to prove theorems. In this philosophy, mathematics is closer to logic than science or engineering. There is nothing wrong with this: such mathematics involves significant intellectual activity. However, what industry needs is mathematicians who are genuine scientists and who are interested in solving real problems. In the controversial words of Dr. Bernard Beauzamy (2002), trained to doctoral level in pure mathematics, and later to set up his own mathematical consultancy company, 'it (pure mathematics) brings solutions which nobody understands to questions nobody asked'. As already mentioned, applied mathematics is organically linked with the sciences, engineering and industry. Strengthening links with industry requires a well designed training: while providing students with a fundamental mathematical education (logic, rigour etc.), courses can also provide them with a set of practical skills which are useful in the real world. This means, naturally, that courses must provide them with more genuine mathematical modelling skills and at least introductory courses in the connected disciplines. The degrees should involve industrial internships and even exposure to experimental work. Students must learn that the genuine application of mathematics involves starting with a real problem (physical, chemical. . .) and will usually mean communicating with engineers, physicists or any other scientist with practical knowledge of the problem, translating that problem into mathematics, solving the mathematical problem and interpreting the solutions in such a way that is meaningful for the practitioners at the source of the problem. Educating present and future mathematicians about the close links between applied mathematics and industry is not limited to the classroom. This important aspect can take many forms. A natural route is to set up an industrial mathematics group in every mathematics department (Friedman et al. (1993)). This has been the approach at the university of Limerick, where MACSI (www.macsi.ul.ie) has developed a range of activities to stimulate links between present or future mathematicians and industry. These include:

- For the younger students and the general public, outreach lectures are organised to stimulate interest in industrial applications of mathematics. During these events, the importance and ubiquity of mathematics in everyday life is presented. The basic message is that mathematics is the underpinning technology for modern society. Without mathematics, there would be no computers, no aeroplanes, no space programme. There would be no internet credit card transactions, no Google. There would be no computer games, even if there were computers to run them on. Even if airlines ran hot air balloons, there would be no airlines, because they were would be no way of scheduling arrival and departure of the balloons. Nor indeed would there be train companies or truck companies, not because there would be no trains or trucks (although there wouldn't be), but for the same reason.

  There would be fishing, but no longer any fish without the control of stocks, there would be no Intergovernmental Panel on Climate Change, indeed global warming would be unknown. There would be no weather prediction, no water supply, no North Sea oil, indeed no oil recovery.

  There would be some science: geography, geology, zoology, in some form but no engineering. There would be no scientific prediction at all, and hence no understanding of the world. Without mathematics, we are in the middle ages, where there is consequently no chemistry, no physics, no astronomy, no geophysics, no cosmology. There would be some surgery and possibly some prescribed drugs, but their use would be uncontrolled and dangerous. And although our financial systems may have recently collapsed, without mathematics, there is no financial system at all.

- A summer school is organised every year to introduce secondary school students and first year university students to modelling concepts. The goal of this activity is to show students that they are in a position to solve some practical problems even with the rather rudimentary tools at their disposal. The topics are extremely varied and the week always includes a site visit to one of our industrial partners.

- Undergraduate students may follow a summer internship programme in MACSI. Here, they are confronted with the reality of applied mathematics and its industrial applications. They work in conjunction with an experienced researcher on a topic suitable for their background. This experience has had a profound effect on several students who have admitted that they never realised the power and usefulness of mathematics.

- Modelling workshops are organised on a regular basis. A typical session lasts up to a day and involves an industry representative, post-graduate students and research staff. This format is very close to a study group but on a much shorter time scale. These meetings are generally extremely productive and they provide an initiation into the activities of the industrial mathematician in an informal environment. It is important to realise that much mathematical modelling begins in informal group discussions around a table.

- Finally, modelling classes given by experienced practitioners occur several times a year. The topics can range from standard modelling techniques like asymptotics and non-dimensionalisation to the presentation of study group reports and examination of real scientific problems.

Coupled with an adequate and varied under-graduate curriculum, these activities familiarise mathematicians of all levels with real life problems and promote long term links with industry.

Study groups with Industry

Study groups with industry are organised on a regular basis on all continents. In the European context, study groups with industry (as initiated in Oxford University in the 1960s and continued under the umbrella of the European Consortium for Mathematics in Industry (ECMI)) are week long meetings where groups of industrialists, mathematicians and other scientists work intensively on problems proposed by the industrialists.

Study group format

The study group format is standard. Mathematicians and other scientists gather for approximately a week with industrial collaborators to find solutions to a set of problems proposed by the latter.

- The first morning, industry representatives present the problems. One must appreciate that the problems presented are usually not mathematical problems to begin with. Typically they are descriptions of a complicated industrial process which is not well understood from a scientific point of view. Usually, there is a specific question of the type 'How might we prevent this happening? 'Sometimes, the request is vaguer to the effect that if we can help to model the situation, something useful may come from the mathematical solutions. When all problems have been presented, the academic/ scientific participants select the problem(s) they would like to work on.

- The first afternoon, subgroups of the scientific participants meet with each industry representative and ask far more detailed questions. Ideally, at the end of the day, the team should have defined in broad terms the approximate goals for the week. It is important to realise that in some cases, a successful outcome at the end of the week may be a properly formulated mathematical problem (i.e., the correct mathematical question).

- During the rest of the week, the group works on the problems and progresses towards a solution. Participants may choose two strategies. Either they focus on a single problem all week or they may decide to hover between different problems. It is really a matter of taste. Some people like to work intensively on one problem: other prefer to make smaller contributions to a number of problems. The industrial partner may or may not attend all sessions. (S)he also should be easy to reach if more information is required.

- A mid week progress report may be required when the groups present their results.

- On the last day, all groups present their results to the industry representatives and the other academics.

- A report describing the work of the group is written in the weeks following the study group and given to the industrial partner.

In a study group one is confronted with real applications of mathematics and must focus on getting significant results quickly. However the first study group one attends can be an overwhelming experience, particularly for students. For this reason, specially designed students sessions are often organised. The goal is to familiarise students with the concept of a study group and with some of the standard techniques and ideas which commonly occur. To achieve this, before the study group, an experienced mathematician runs a session where (s)he uses one or several past (and generally simplified) study group problems with the students. (S)he plays the part of the industrial representative and presents the problem to the students. Using the solution previously obtained, (s)he can guide the students and help them rediscover the results. Working in groups, the students all contribute to the final result and this constitutes an excellent first contact with industrial problems.

Study group topics

MACSI organised two study groups, ESGI62 and ESGI70 in January 2008 and June 2009 respectively. Topics included

- Fluid mechanics: spin coating and self assembly of diblock copolymers, initiating Guinness,
- Chemistry: improvement of energy efficiency for wastewater treatment,
- Biology: lubrication of an artificial knee,
- Electronic: blowing up of polysilicon fuses, the effect of mechanical loading on the frequency of an oscillator circuit,
- Engineering: polymer laser welding, polishing lead crystal glass, solar reflector design,
- Environment: designing a green roof for Ireland,
- Finance: on the estimation of the distribution of power generated at a wind farm using forecast data; uplift quadratic programming in Irish electricity price setting.

These examples reflect traditional subjects arising in study groups. Optimisation, population modelling and medical problems are other traditional topics. One of the most unusual problems was submitted to mathematicians in Bristol in 2003. Participants were asked to study the artificial incubation of penguin eggs. (In the end, this involved modelling the fluid mechanics inside the eggs.) The problems submitted to study groups are extremely varied but they reflect the skills that are expected from a mathematician doing modelling: although daunting at first, they may be split in a series of sub-tasks and interim models which are much easier to tackle.

Study group benefits

A successful study group is extremely beneficial to all the parties involved:

- Mathematicians and scientists are introduced to new and original problems. A single experience at a study group is enough to convince most mathematicians that industry is an extremely fertile source of interesting problems. They get the opportunity to apply their skills to new exciting problems and get the opportunity to see if the mathematics really works.

- Industrial collaborators obtain direct access to academic expertise. Universities, among other things, are repositories of knowledge and excellence but too often these resources are not easily available to the people who need them most. In the short term, a problem is usually at least partially solved for the industry (or at least some insight has been gained) and this often encourages them to look at their problems from a new perspective and to look for newer and innovative solutions. In the long run, the research carried out during a study group can deliver real solutions for the industrial partners and may lead to patents and genuine financial gains. Potential clients of the industrial partner are often impressed by the improved scientific approach.
- Study groups are unique scientific occasions and are more productive than traditional conferences. Excellent work relationships often develop leading to long-term collaborations.

Study groups are a very important way of improving the industrial/mathematics interface. If study group participants are completely alien to most of the concepts necessary and do not have the appropriate scientific training to solve industrial problems, this sort of initiative is bound to fail. The interface between industry and mathematics can only improve if appropriate training is offered to young (and older) mathematicians.

Conclusion

Developing a successful interface between mathematicians and industry requires much effort but the rewards are tangible. Industry provides new and interesting scientific problems; mathematicians provide insights which allow the industrialists to improve their products. We note in passing that many scientific councils are now including economic relevance among the key requirements in proposals for research funding.

Mathematical modellers hold a unique position in the scientific world with their ability to interact with practitioners in so many different areas in industry, the sciences and engineering. In our opinion, to date this advantage has not been exploited to its fullest extent, partly because of the divide in the mathematical world between pure and applied mathematics. It is our thesis that the way forward involves changing mathematics curriculums and placing more emphasis on real mathematical modelling. Study groups with industry are clearly one of the most significant ways of strengthening the industry/mathematics interface but such interactions can only further develop if the mathematicians develop a set of skills more suited for real industrial problems.

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Numeracy at work – policy and practice

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Abstract Lack of numeracy skills may negatively affect quality of life, labour market possibilities and participation in lifelong learning for the adult population. In the context of the EIMI study, it is highly relevant to discuss and reflect on strategies for education also on a basic level. In Norway, there is significant political interest in basic skills. The long-term goals are avoiding social exclusion, improving chances of entering into or staying in the labour market, and building stepping-stones enabling the individual to go on to more education. We will present examples of good practice and policy initiatives aimed at securing the quality of provision and contributing towards increased awareness and improved accessibility for all adults into numeracy education.

Background

In the ALL survey[1], numeracy was defined as "the knowledge and skills required to effectively manage and respond to the mathematical demands of diverse situations". Results from this and other surveys have shown that a high percentage of the adult population in Europe lack sufficient numeracy skills, which may affect negatively their short-term and long-term possibilities. Taking into account the changes in the labour market and the specific needs of adults, there is an obvious need for good education strategies in this field.

Like many other countries, Norway is facing rapid changes in the labour market, characterised by less manual work and a transition from production to services, implying more complex and information-based work and increasing demands due to decentralised responsibility. Together with the demographic change seen throughout Western Europe, all this enhances the need for flexibility and competence development in working life.

In recent years, giving priority to providing better educational opportunities for adults with low basic skills has been a political focus. The *White Paper No. 16 to the Storting (2006–2007) Early Intervention for Lifelong Learning* states the Government's wish to strengthen the focus on adult learning by introducing several measures, among others the development of a framework with supplementary guidelines for basic skills for adults. Also, a funding programme was set up to make it financially viable for private and public enterprises to start up basic skills development schemes for their employees and for job-seekers.

With the introduction of the latest school reform in 2006, the Knowledge Promotion, the focus on basic skills has been strengthened also within the educational system. In this reform, all levels have been provided with new curricula which clearly state competence objectives and emphasise the role of basic skills in all subjects.

Strategic measures

So far, the education that has been on offer for adults with low basic skills is a full (condensed) primary school course. For many adults this has proved neither useful nor necessary. What they need, is a possibility to attend flexible education geared specifically towards basic skills training, be it numeracy or other skills. To realise the wish to provide better opportunities for adult basic skills education, some strategic tools are necessary to support policy. A major objective has been to improve the quality of the provision and ensure that education options accessible for all are established.

The BCWL Programme

One such option is the Basic Competence in Working Life (BCWL) Programme, which is a national programme directed at funding basic skills training at work. The programme is administered and monitored by Vox.

The programme mainly funds enterprise-based courses on basic skills, but projects organised outside workplaces can also receive funding, provided the objective is to prepare people for working life. The aim of this programme is to give adults the opportunity to get the basic skills they need to keep up with the demands and changes in modern working life and civil society, and encourage them to achieve further educational goals.

Along with giving adults the opportunity to improve their basic skills, the programme aims to increase the quality of the provision by strengthening education providers' ability to offer education adapted to the needs of enterprises and individuals. It is also a programme goal to increase awareness of and open-mindedness towards the need to strengthen basic skills in the adult population, thus reducing stigma, and to increase knowledge of the barriers and success factors. The overall goal is to prevent exclusion from working life because of insufficient basic skills.

Any enterprise in Norway, private and public, can apply for funding from the programme. The following criteria are emphasised: The learning activity should be combined with work, and basic skills training should preferably be linked to other job-relevant learning; the skill levels aimed for correspond to lower secondary school; and the courses should strengthen the participants' motivation to learn.

Since the beginning in 2006, the total funding has increased each year, and the programme is a prominent political issue as part of the government's focus on basic skills in the adult population. The programme has been evaluated twice, and the results have contributed to the development of the programme and an increase in national funding. In 2008, 96 enterprises received 30 million NOK (approx 3,5 mill EUR) for basic skills training through this programme. Due to the financial crisis and increased unemployment, the Government has strengthened and widened this programme in 2009. 50 percent of the unemployed in Norway have not completed their basic education (up to the level of completed upper secondary education).

The Framework

As part of the governing documents for the BCWL Programme, the Framework for Basic Skills for Adults has been created by Vox by mandate from the Norwegian Ministry of Education and Research. From 2008, applications for funding have to relate to the Framework.

The main objective of introducing the Framework is to increase the quality of teaching and ensure that the individual can get education adapted to his/her needs. The Framework is first and foremost a set of tools for adult learning; an attempt at a universal description of the basic skills, as a point of reference for teachers, learners and stakeholders in general, and a set of supporting measures and materials.

The Framework levels correspond to the levels of primary and lower secondary formal education under the Knowledge Promotion,[2] but focus on the basic skills only, i.e. they are subject/curricula independent. The Framework levels may be said to be equivalent to Level 1 and 2 in the EQF.[3] Its central elements, approved by the Ministry in 2007, are sets of competence goals for literacy, numeracy, digital competence and oral communication, adapted to the contexts and needs of adults. The Framework also comprises educational and professional resources in the form of guidelines for providers, screening tools, tests, didactic models and teaching materials, and teacher training schemes.

The competence goals

The competence goals describe learning outcomes for each of the selected basic skills, divided into three levels. Each level is described in detail in the form of intended learning outcomes, and includes an extensive range of examples of practical application from different arenas.

As for numeracy,[4] the competence goals on level 1 describe the minimum competence needed to understand basic concepts and symbols and perform simple mathematical tasks in concrete and familiar contexts. On level 2, the adult can respond actively to mathematical information and can follow children's schoolwork up to 4^{th} form. Level 3 describes a more independent attitude; the adult understands, uses and responds critically to more complex mathematical information in the form of numbers, symbols, graphs, figures etc. On this level, the adult can follow children's schoolwork up to 7^{th} form. The examples on all levels are chosen to emphasise the connection to everyday life and working life. In the Norwegian description, the concept "everyday maths" is used to underline the practical approach to this subject.

Guidelines for teachers

To support teachers and providers in their work with adult basic skills education, guidelines have been developed, to be used in combination with the Framework. The first part of the guidelines describes the background for the Framework and details key issues associated with working with adults and their basic skills. The second part consists of subject-oriented chapters showing how the competence goals in the Framework can provide support for preparing and implementing training programmes for each of the various skill types.

It is essential to communicate to all users that the Framework is a tool for flexibility – not a "set curriculum" in any way. The goal of facilitating for better and more accessible basic skills education is twofold – both increased employability and more opportunities in working life, *and* personal development, increased participation and empowerment for the individual. Realising the individual's potential as a learner can happen in many different ways, and it is vital to take a learner-centred approach when planning the provision.

Mapping tools and tests

As for adult learning, very little weight is traditionally placed on measuring the progression of individuals with low-level qualification. The main emphasis has been on documenting subjective results like higher level of self-esteem and further motivation for learning. No specific qualification system has been created to give formal accreditation for attained levels of basic skills according to the Framework. So far, the emphasis has been on formative assessment, e.g. methodology built on portfolio work and "Can do" statements (self-evaluation).

However, a battery of tests to measure learning outcomes in relation with the Framework, as well as screening tools and tools for formative evaluation, is currently being finalised. Vox is responsible for the work, and projects funded in 2009 through the Basic Competence in Working Life Programme (BCWL) will be required to use these tools to document results.

Using tests in a basic skills setting has been much debated among educators, fearing that participants might feel uneasy in a testing situation and that an experience of being under pressure might lead to an increased drop-out rate, or that a "teaching-to-the-test" practice will establish itself. To ensure the appropriate use of test tools, Vox organises day seminars for teachers and providers, with opportunities to discuss pedagogical use of various forms of evaluation. Also, in order to be certified as a test administrator, attending a special day course is obligatory.

Professional development for teachers

A major strategic measure in the work to improve the quality of the provision is to make sure adequate training for teachers is made available. It is important to bear in mind that there is very little systematic training in teaching adults in the course of the general teacher education. A lot of learning about adult learning is voluntary, supplemented by one university only offering a master's degree. Most of the public discussion about teaching and education focuses on children and youngsters, so training for adult educators so far has been very much left to individual initiative.

Teaching basic skills can be a challenge for teachers, depending on their background and experience. Additional competence may be needed to motivate participants who lack schooling or have previously experienced failure. Also, tailoring the content and organisation of learning to the individual, perhaps aided by mapping tools and interview techniques, may require re-thinking practice and acquiring new competence for the teacher.

Workplace-based basic skills training differs from "ordinary teaching" in many ways, and teachers used to classroom situations often need a change of attitude and adequate PD before they can efficiently implement workplace-based training. Adapting teaching to the various arenas can be challenging on many levels and in different phases. For example, planning also means getting to know work cultures, languages and competence needs, and gathering job-relevant material. Even the practical organisation in itself might be an obstacle, and the time needed for information, organisation and consensus building among employers and employees regarding the implementation of workplace-based training is often underestimated.

In 2008, Vox was assigned the responsibility to design a model for teacher training customised to the needs of teachers who teach basic skills to adults. The model is currently being implemented as a pilot in co-operation with teacher training institutes at universities and university colleges. In addition to this, Vox organises annual series of one-day PD seminars, aimed at training providers in the practical application of the Framework, and covering a wide range of methodological topics. All training is voluntary for the teachers.

Formal teacher training – pilot courses 2009

In this initial phase, the pilot comprises formal PD for teachers working in the implementation of the BCWL Programme only. The training started in September 2009, and access is based on pre-requisite teacher education or validation of prior learning and practice. 53 students are attending the courses this term. The courses are organised as part-time distance education, with 5–6 face-to-face seminars of 3 days each throughout the course period of one year. The courses add up to 30 ECTS credits in total. The model for these courses, as well as content plans, has been worked out in close co-operation between Vox and the higher education institutions responsible for the provision. The model can be adapted and realised in different ways to fit the students and staff.

The pilot training course for teachers teaching adults numeracy / everyday maths is organised in co-operation with Vestfold University College, which provides the course at their Centre for Teacher Education. The 30 credits are distributed evenly over the course period, with 10 credits for a general andragogy part (5 per term) and 20 credits for maths (10 per term). The content of the first term covers the following topics: Theoretical foundations for everyday maths for adults; mapping, assessment, math difficulties; and the connection between literacy problems and numeracy problems. In the second term, the topic is facilitating for learning. The students are assigned obligatory written tasks during the course, and there is a final exam consisting of portfolio assessment and a half-hour oral exam.

Examples of good practice

In this section, three examples of good practice will be presented briefly. The competence goals of the Framework were used in all the projects described, and they all received funding through the BCWL Programme.

Project "New possibilities"

The project *New possibilities* illustrates the challenges and possibilities as for young adults against the backdrop of high drop-out rates in upper secondary education. Receiving funding from the Programme for Basic Competence in Working Life in 2007-2008, the project aimed at facilitating access to working life for young adults who had dropped out or failed in upper secondary education.

The main goal of the project was to motivate and recruit young adults with insufficient educational background and low basic skills to enter working life. After having finished basic skills education in literacy, numeracy and ICT, combined with vocational training and general life skills training in the workplace, the participants were intended to be able to enter into an apprenticeship agreement and to be attractive as an apprentice.

The group of participants in *New possibilities* consisted of young adults aged 20 – 25 with very varied qualifications and backgrounds. Of the nine participants – all male – there were five belonging to ethnic minority groups. Many had had a negative experience at school, and some of the immigrant participants had been subject to traumatic incidents in their home countries.

Working effectively with this group of young adults demands co-operation on various levels and among many stakeholders. *New possibilities* was initiated by a large industrial processing enterprise (Xstrata Nikkelverk AS), and the work was carried out in co-operation with the regional vocational training office for technological subjects in the county and a pedagogical consultant. The local labour and welfare administration office actively took part in the process, especially in recruiting participants and in financing the participants' subsistence during the project.

The education was organised as a full-time provision over 12 weeks in spring 2008. Four days a week the participants were out working in local enterprises, combined with vocational and general life skills training; on the fifth day they gathered in a course room at Xstrata Nikkelverk AS for basic skills education built on their work experience.

In 2009, one year after the project was finished, six of the nine participants have signed apprenticeship contracts, one has passed his journeyman's certificate, and one is employed in other work. These are outstanding project results for a group of people who saw little or no hope to begin with.

"New possibilities" continued

The results of the project *New Possibilities* have been promising, and have lead to a decision from the Norwegian government to continue exploring possibilities for this group. In 2009, more money has been allocated to nine new projects, aiming to get more examples of good practice concerning how to get young adults not in education or employment back on track. The projects are in an initial phase.

Good practice in this case means the organisation of the work done by the stakeholders and the co-operation between them, as well as good teaching methods and pedagogical approaches. The subjects that are emphasised are reading, writing, basic ICT skills and numeracy.

Numeracy provision in a building supply store

During spring 2007, employees in a building supply store in Bergen took part in a numeracy course aimed at raising the skills needed to perform better in the job. The initiative was taken by Bergen municipal adult education centre who contacted the store. The company management experienced a need for upgrading the numeracy skills of the workers, and the co-operation was established. The adult education centre provided the numeracy education, and tailored a course that was based on the competence goals[5] and adapted to the needs of the employees in their daily work.

To get an insight into the numeracy-related tasks in the job and to get an understanding of which numeracy skills were required, the teacher went for several study visits to the enterprise. She had the opportunity to follow the employees in their work during the day so she could observe the use of numeracy. This was crucial to get an understanding of the different tasks and to get ideas about what kind of material to make for the course. It was also of great importance when it came to the mapping of each participant's numeracy skills. By observing and by asking the employees about the numeracy tasks they had to carry out, she got an idea about their skills. Later on this was done more thoroughly by using a mapping tool.

Based on these preparations, and in agreement with the participants, it was decided that the main focus of the course would be measurements, calculation of area and volume, percentages, geometrical figures and shapes and also dealing with algebra and equations.

The information from the study visits and the mapping constituted the basis for the course. The teacher made material with a practical angle, closely connected to tasks the participants carried out in their daily work. The course content was not fixed, but became dynamic in the way that the teacher paid attention to what the participants had in mind and wanted to bring into the lessons. She then shaped and adapted the course on the background of the ideas she got. In this way the participants had real influence on their own learning.

This approach to tailoring a numeracy course for employees in an enterprise makes learning math relevant for the participants. The way the teacher worked beforehand; visiting, observing and talking to the participants as well as the management, involves every level of the enterprise in the process. This is important for the motivation to attend the course.

These preparations also contributed to raising awareness about the mathematical content in the job and about what kind of math skills the participants needed to upgrade. Being aware of their own need could also be a motivating factor for the participants.

Finally, the way the participants themselves had an influence on the course content made it meaningful and relevant to attend and to learn more mathematics.

Numeracy for cleaning staff

This project was set up for minority language speakers employed as cleaners. The participants had very little schooling; one of them had no schooling at all, and they had poor knowledge of Norwegian. Thus they were in great need of improving their Norwegian language skills; meaning understanding, speaking and writing.

The group was already involved in a course at an adult learning centre aimed at raising their language skills, with the intention of handling their jobs better. Having already signed up for this course, the participants were offered numeracy tuition in addition.

Mapping the participants' knowledge of numeracy could not be done by using the available mapping tools. Because of their poor knowledge of Norwegian the participants could not take a written test. Instead the teacher presented a selection of tasks that was carefully picked to be suitable. She also had to assist with reading texts and clarifying problems when it was needed. Some in the group were not tested at all, when the tasks were regarded as too difficult. All participants had an individual dialogue with the teacher, whether they took part in the testing or not.

Together with mapping the knowledge of numeracy the teacher also had meetings with the employer to find out what kind of math was relevant for the work. On this background a course was created that responded to the personal needs of the participants, which they in turn saw the immediate use for. This was of great importance for the motivation to join tuition.

To make a numeracy course at this level is a lot of work. The course focused on relevant tasks and subjects for the workplace and on Norwegian terms used in mathematics. This kind of tailoring implied that the teacher had to make all the material herself in the context of the daily work of the participants. The aim of the course became upskilling the participants from a zero level (more or less) to a level one corresponding to the competence goals[6] for adult numeracy.

As for methods, extensive use of dialogue and one-to-one tuition was seen as helpful. An interesting observation was that numeracy teaching also lead to better Norwegian skills. By learning about mathematical terms and talking about their jobs, the participants also learned the language better.

Notes

1 The ALL (Adult Literacy and Life Skills) survey report was presented in 2005.

2 The school reform as mentioned on p. 1.

3 It should be noted that the process of discussing how to implement the EQF in Norwegian education is still in progress.

4 An English translation of the competence goals for numeracy can be downloaded from www.vox.no/english.

5 As described on pp. 3–4.

6 As described on pp. 3–4.

Mathematics in a safety-critical work context: the case of numeracy for nursing

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Abstract This paper draws on our interdisciplinary research on the relationship between mathematics and nursing, a safety-critical work context in which mathematics really matters - for nurses, their patients, their employers and the public at large. There is a growing international literature indicating the problematic nature of this relationship in many countries. Scare stories of mathematical errors by nurses dent public confidence and may impede recruitment into nursing from those who, with the right education and continuing professional support and development, have the potential to become good nurses. Against this background, we are working on the teaching, learning and assessment of numeracy for nursing. In one project we aim to create an evidence-based benchmark in numeracy for nursing, capable of being operationalised as part of nursing students' preparation for professional practice and used in qualified nurses' continuing professional development. In this paper we outline our work towards the creation of such a benchmark, focusing on our study of the assessment of student nurses' skills in medication dosage calculation against our proposed benchmark, undertaken with the aim of ensuring that all concerned may have confidence in nurses' ability to manage the mathematical demands of this key area of nursing safely and effectively.

Background

Mathematics matters in nursing: to patients, nurses, their employers and to nurse educators. Successive studies reveal a lack of proficiency amongst both student and registered nurses (Sabin, 2001) and alarming headlines periodically fuel public fears (Hall, 5th August, 2006). The development of appropriate mathematical competence by healthcare staff and students is a key area for concern but there is no consensus on the nature and scope of what is usually termed numeracy for nursing, nor on ways of improving the situation. The relationship between mathematics and nursing — the scope and nature of numeracy for nursing and how to ensure that nurses are well-prepared for and periodically updated on the mathematical demands of their work — is still poorly-understood (Coben, Hall, et al., 2008). The EIMI Discussion Document statement that "there is a need for a fundamental analysis and reflection on strategies for the education and training of students and maybe the development of new ones" certainly applies to numeracy for nursing. This need is made more urgent by the safety-critical nature of nursing generally (Cooke, 2009), particularly with respect to aspects of nursing involving mathematics (e.g., ISMP, 2008). For example, nurses need to be able to calculate drug dosage, estimate a patient's fluid balance and nutritional status and interpret and act appropriately on data shown by machines used to monitor a patient's condition or dispense treatment: a mistake in any of these could be life-threatening for the patient and end the nurse's career. Through our work on numeracy for nursing we are working to reduce this risk.

We decided to focus initially on one area of nursing responsibility that is high risk and closely associated with the use of mathematics: medication dosage. Medication errors have been highlighted recently by the National Patient Safety Agency (NPSA) in England and Wales and targeted for remedial action (NPSA, 2006, 2009). The number of injuries and deaths attributable to medication error in the National Health Service (NHS) in the UK is unknown but 9% of incidents reported to the NPSA in its 2003 audit involved medicines (NPSA, 2003), a figure consistent with historical data. The extent of calculation error encompassed within medication error is not specified in the report but the NPSA state that

> Miscalculation, failure to titrate the dose to the patient's needs, miscommunication and failure on the part of all team members to check the dose before dispensing, preparing or administering a dose are the most common factors contributing to dosing errors. (NPSA, 2009, p. 19)

The Department of Health report on *Improving Medication Safety* (Smith, 2004) highlighted inadequacies in the education and training of both doctors and nurses as contributory factors in medication errors. In response to growing concern, from September 2008 the body regulating the nursing profession in the UK, the Nursing and Midwifery Council (NMC), has required students to be assessed in numerical competence in the practice setting with 100% passmark before being allowed to register as nurses (NMC, 2007).

However, there are currently no national standards for teaching or assessment of numeracy during pre-registration nurse education and, in the absence of a robust evidence-based standard (a benchmark), a relativistic position has emerged with a multiplicity of tests, processes and criteria being developed and deployed locally. We have recently investigated this situation with respect to the assessment of numeracy for nursing in one university in England (Coben, Hodgen, Hutton, & Ogston-Tuck, 2008).

We argue that without such a benchmark, any measure of numerical competence is:

> ... in the eye of the recipient of evidence of that competence, be it higher education institutions, regulators, employers or service users. (Hutton, 2004)

Given the link between the required competence and its public consumption, the nature of any benchmark requires not just an assessment method which is reliable and valid in educational terms, but one which directly and authentically represents the purpose and context in which it will be performed. Such a process cements the relationship between the desired expectation of competence in the workplace and the governance of its development and subsequent performance.

Once established, a robust assessment benchmark provides not just some assurance of baseline professional standards but for the first tme an opportunity to move away from relativistic interpretations of mathematical competence in relation to nursing and explore the relationship between entry qualifications, in-programme preparation, placement experience, remedial support and subsequent achievement. Thus, it would be possible to work backwards from the benchmark to key stage indicators and the probability of achieving the required standard in the time available. Such data would be extremely powerful in supporting subsequent education. It would also allow us to look forward to subsequent stages in the nurse's career where either higher level skills are needed or a refreshing of core skills can be facilitated by recourse to the core standard.

Developing an evidence-based benchmark in numeracy for nursing

Against this background, NHS Education for Scotland (NES) brought together an interdisciplinary group of subject experts to explore the key issues associated with determining the achievement of competence in numeracy for professional practice in nursing[1]. This followed a review of relevant literature (Sabin, 2001), a consultation on healthcare numeracy (NES Numeracy Working Group, 2006) and the subsequent strategy developed by NES (Sabin, 2006a, 2006b).

The overall aim of this work is to develop a proposed benchmark assessment for numeracy for nursing in Scotland. In the first instance we propose the establishment of a benchmark at the point at which students become registered nurses.

In our first-stage project towards this end: 'Benchmark assessment of numeracy for nursing: Medication dosage calculation at point of registration'[2] (Coben, et al., 2010), funded by NHS Education for Scotland http://www.nes.scot.nhs.uk/, we aimed to evaluate empirical evidence of the reliability and convergent validity of a computer-based learning and assessment tool of medicine dosage calculations by comparing its outcomes with the outcomes of a practical activity requiring the same calculations and to determine learner-perceived acceptability of the assessment tools in relation to authenticity, relevance, fidelity and value.

Our research questions were:

1. What is the internal consistency reliability of the computer-based assessment and the practice based assessment?
2. What is the criterion-related validity of the computer-based assessment and the practice-based assessment?
3. How acceptable to nursing students are the assessments in terms of authenticity, relevance, fidelity and value?

The research began with preliminary work undertaken between November 2006 and April 2007 (reported in Coben, Hall, et al., 2008). This comprised:

- The adoption of a definition of numeracy applicable to nursing and capable of being operationalised in our research, as follows:

  > To be numerate means to be competent, confident, and comfortable with one's judgements on *whether* to use mathematics in a particular situation and if so, *what* mathematics to use, *how* to do it, what *degree of accuracy* is appropriate, and what the answer *means* in relation to the context. (Coben, 2000, p. 35, emphasis in the original);

- The development of evidence-based principles for numeracy for nursing;
- The development of criteria for a benchmark assessment of numeracy for nursing;
- The development of an evidence-based prototype benchmark assessment tool covering practical knowledge in medicine dosage calculations.

Having reviewed the literature, we developed the following criteria, recognising that an authentic numeracy assessment tool should be:

- Realistic: Evidence-based literature in the field of nursing numeracy (Hutton, 1997; Keith W. Weeks, 2001) strongly supports a realistic approach to the teaching and learning of calculation skills, which in turn deserve to be tested in an authentic environment. Questions should be derived from authentic settings.

- Appropriate: The assessment tool should determine competence in the key elements of the required competence (OECD, 2005; Sabin, 2001).

- Differentiated: There should be an element of differentiation between the requirements for each of the branches of nursing (Hutton, 1997).

- Consistent with adult numeracy principles: The assessment should be consistent with the principles of adult numeracy learning teaching and assessment, having an enablement focus (Coben, 2000).

- Diagnostic: The assessment tool should provide a diagnostic element, identifying which area of competence has been achieved, and which requires further intervention (Black & Wiliam, 1998).

- Transparent: The assessment should be able to demonstrate a clear relationship between 'test' achievement and performance in the practice context (Keith W. Weeks, Lyne, Mosely, & Torrance, 2001).

- Well-structured: The assessment tool should provide a unique set of questions with a consistent level of difficulty and a structured range of complexity (Hodgen & Wiliam, 2006).

- Easy to administer: the assessment should provide the opportunity for rapid collation of results, error determination, diagnosis and feedback (Black & Wiliam, 1998). (Coben, Hall, et al., 2008, pp. 96-97)

In response to these requirements, an evidence-based benchmark computer assessment tool was designed by Weeks and Woolley to develop nurses' medication dosage calculation skills. It uses a framework derived from the Authentic World® computer programme based on Gulikers et al's framework for authentic assessment (Gulikers, Bastiaens, & Kirschner, 2004). Graphics and participant interaction create close proximity to real world practice. It provides the full range of complexity of dosage calculation problems which Adult Branch nurses are likely to meet at the point of registration. This includes unit dose, sub- and multiple-unit dose, complex problems and conversion of *Systeme Internationale* (SI) units (Keith

W. Weeks, et al., 2001). The 28 practical simulation items were a subset of 50 computer simulation items selected on the basis of their high internal consistency reliability in an earlier study (Clochesy, 2008; K. W. Weeks & Woolley, 2007).

The practical (real world) simulation assessment was designed and developed to mirror the computer-based assessment.

A full account of the research design, development of key concepts, the research process and its outcomes is given in the project report (Coben, et al., 2010). For the purposes of this paper, we report our findings, as follows.

We found high congruence between results from the two methods of assessment in the tablets and capsules section, suggesting that for determination of calculation competence in management of this type of prescription, an authentic computer assessment is equivalent to an assessment through practice simulation.

In assessing calculation of liquid medicine doses, both the authentic computer environment and the practice environment assessments facilitated detection of technical measurement errors associated with selection of inappropriate measurement vehicles and measurement of incorrect liquid doses. The definition of numeracy which we have used includes competence in knowing what the transfer of a calculated answer to a technical measurement vehicle means in practice but the computer model did not allow for technical measurement errors such as failing to displace air from syringes, an error manifested by several students in the practice environment. This was a measure of numeracy competency which has not been widely considered in the literature and was apparent in all the sections of the practice assessment which involved liquids. Some students made arithmetic or computational errors but the majority of errors were made in practical technical measurement and were apparent regardless of the accuracy of the original calculation of dose. This element was appropriately identified in the practice simulation assessment, but would not have been detected via the computer assessment. However, this competence could be assessed as a practical skill with *any* prescription requiring liquid medicine without recourse to repeated measures across the range of complexity and, if coupled with the authentic computer assessment would be adequately assessed. The same argument would apply to prescriptions for injections.

We conclude that for calculation of medicines dosage, the major advantage of the authentic computer environment was to provide prescriptions covering the full range of calculations likely to be met in practice. It allowed easy assessment of the mathematical element of these calculations with large numbers of students in a short time and, given that marking and feedback generation was entirely automated, the process was quick, easy and totally objective. In assessing nurses' calculation of medicine doses, an authentic computer model that presents dosage problems within an agreed rubric is invaluable in providing assessment of the full range of calculations likely to be met in practice as a newly qualified nurse.

In the practice context it would be impossible to ensure that all third year student nurses encountered and were reliably assessed in the full range of dosage calculation problems they might meet as a qualified nurse. However, the assessment of the numeracy element of a nurse's competence in medicines management needs to include assessment of both the full range of calculations likely to be required *and* the measurement vehicle manipulation and measurement skills available in most clinical settings and/or able to be simulated in a practical environment.

We propose that if used together, the assessment tools and processes identified within this report provide a robust form of assessment that meets the needs of regulators, educators, employers, practitioners, students and public in reliably identifying conceptual, calculation and technical measurement competence in the context of medicines administration.

Further, we propose that this research provides a benchmark against which other researchers and interested stakeholders can measure the impact of other innovations in learning, teaching and assessment strategies, and of recruitment, development and support/retraining strategies.

Summary of our findings

The main overall focus of this study was to determine the validity of the computer simulation format of delivering dosage calculation problems. The validity of the computer simulation format was tested against the gold standard practical simulation format. The underlying rationale was that the practical simulation format is not feasible for mass testing, particularly across the full range of question type (tablets, liquid medicines, injections, etc.) and complexity (single unit, multiple unit etc) whereas computer simulation enables testing across the range of question type and complexity of large numbers of people at the same time, in remote locations, with limited costs. If computer simulated testing could be shown to operate similarly (provide similar results) to practical simulation testing, this would validate the use of computer simulated testing for future research.

From the results presented above, the criterion-related validity of the computer simulation format has been supported, both in terms of putting participants in a similar order of competence and in terms of participants obtaining similar absolute results (getting the same number of questions correct on the computer simulation as they would on the practical simulation). These results supplement Weeks' (2001) more detailed item-by-item comparisons, which produced similar results in terms of confidence in the computer simulated testing format as a substitute method for practical assessment.

Some caveats remain, however:

- computer simulation does not test certain elements of the real-world dosage calculation problem (e.g., technical competency)
- these conclusions should only be applied to similar situations, populations, and constructs.

Conclusion and thoughts on wider implications for the educational interface between industry and mathematics

In conclusion, we offer this account of our ongoing research as a contribution to wider discussions on the educational interface between industry and mathematics, in the form of numeracy for nursing. Though mindful of the second caveat above, we think there is at least one lesson to be learned from our research for other fields (including other healthcare fields) in which mathematics plays a key part. This is the prime importance of authentic teaching, learning and assessment of mathematics for industrial purposes. Where mathematics is situated in professional/vocational practice it should be taught, learned and assessed in relation to that practice, both directly *in practice* and through authentic and comprehensive *simulation of practice*; the latter enables individuals to be exposed to the full range of problems associated with the use of mathematics in their professional practice, something which may be impossible to do safely, comprehensively and effectively in real world, real time contexts.

Notes

1 The interdisciplinary team comprises: Professor Diana Coben, Professor of Adult Numeracy, King's College London; Dr Carol Hall, Associate Professor, School of Nursing, University of Nottingham; Dr Meriel Hutton, Senior Visiting Research Fellow, King's College London; Dr David Rowe, Reader in the Department of Sport, Culture and the Arts, University of Strathclyde; Dr Keith Weeks, Reader of Health Professional Education, Faculty of Health, Sport and Science, University of Glamorgan; Norman Woolley, Head of Learning and Teaching; Associate Head Dept of Professional Education and Service Delivery, Faculty of Health, Sport and Science, University of Glamorgan. The project is sponsored by Michael Sabin, Learning Teaching and Assessment, Programme Director, NHS Education for Scotland, currently on secondment as Nursing Officer Workforce Development, Chief Nursing Officer's Directorate, Scottish Government.

2 The project report and associated materials are online at http://www.nursingnumeracy.info/index.html. This paper draws on this report and other publications from the work of the project listed on the website.

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Error correcting codes in secondary mathematics education, a teaching experiment

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Abstract Making industrial mathematics more visible makes people better aware of the importance of mathematics in modern technologies. We present a teaching experiment in which secondary school students are introduced to a topic in industrial mathematics, namely error correcting codes. The experiment started in 2007 and is still going on. The teaching sequence, designed by students from teacher education, has been tested on a small scale in the academic year 2008-2009. It was revised during the first semester of 2009-2010 and is tested again on a larger scale during the second semester. It connects abstract algebra (examples of finite fields) to interesting applications in technology: the (7,4) Hamming code and the (6,4) Reed Solomon code.

Introduction

The discussion document for the ICMI/ICIAM International Study concerning Educational Interfaces between Mathematics and Industry (Damlamian, & Sträßer, 2009) mentions that "people are often not aware of the importance of the role of mathematics in modern technologies. Many people have a restricted view of what mathematics is and does" and that "[w]e need to make the use of mathematics in modern society more visible" (p. 527). In fact quite often the mathematics is hidden inside and is used in practice in an unnoticed way, e.g. typically 1/4th of the bits stored on a CD disk are there for the mathematics and not for the music or the software. In question 1 of paragraph 2 (p. 527) the discussion document asks "[h]ow [...] mathematics, especially industrial mathematics, [can] be made more visible to the public at large". This paper presents a teaching experiment in which a topic in industrial mathematics, namely error correcting codes, is made clear for various applications in information and communication technology (ICT). Examples of applications of mathematics in industry are introduced more often in higher education than in secondary education. But there is no obvious reason for not introducing such applications in the secondary school. So our teaching experiment was carried out with secondary school students.

The teaching experiment was set up not only to make industrial mathematics visible to students, it also intended to make mathematics more appealing and exciting to students, which refers to question 2 in paragraph 2 of the discussion document (p. 527). In fact, the subject of error correcting codes provides opportunities to connect abstract and modern mathematics to exciting applications, thus simultaneously opening two completely new worlds for secondary school students.

The official curriculum in Flanders (Belgium), where the teaching experiment was held, stresses the importance of applications. More and more applications of mathematics are found in school books and in class rooms nowadays. However, many of the problem situations are not 'really real'. One of the reasons to construct the teaching sequence presented in this paper, was to raise the level of authenticity of problems used in the class room.

Not only the subject is new to the secondary school students (and their teachers) but to a certain extent also the teaching methods and the assessment system. Students had to study the subject in groups guided by a text and they received not too much help from the teacher. The assessment was based on a test of students' knowledge of the subject at the end of the teaching sequence and on the process of studying the text in groups.

The teaching experiment was and still is prepared and carried out by student teachers, first of the KU Leuven and later of the Universiteit Antwerpen. The second named author is one of the student teachers involved in the experiment. The other authors supervise the experiment.

We would like to thank here the teachers who were prepared to let the experiment happen in their class rooms, especially Rita De Ridder and Ann Vanderkimpen, who were engaged in the small scale testing of the first version, and Isabelle Poels, who drafted a preliminary version of the text used during the experiment.

Description of the teaching experiment

The experiment started in the academic year 2007–2008 when a student teacher from the KU Leuven wrote a first draft of a text for secondary school students about error correcting codes. This text was based on Rousseau and Saint-Aubin (2008), a book which is intended for tertiary education. During the academic year 2008-2009 another student teacher of the KU Leuven revised the text, in collaboration with two secondary school teachers who used the text in their classes. During the academic year 2009–2010, a third student teacher, now from the Universiteit Antwerpen, continues the experiment. During the first semester, she revised the text again, based on the experiences with secondary school students in the previous year. A group of 12 teachers will use the revised text in their classes during the second semester.

We discuss the text in more detail in the following paragraphs. Now we only give a short overview. The text (more specifically: the revised version used during the academic year 2009–2010) consists of six chapters. The first chapter presents examples of situations in which error correcting codes are used and gives some general information concerning error correcting codes. In the second chapter arithmetic modulo 2 is introduced. This is applied in the third chapter, which is dedicated to the (7,4) Hamming code. Chapter four deals with modular arithmetic in general. Examples of finite fields with a prime number of elements are introduced and used in small applications like International Standard Book Numbers (ISBN) and EAN product codes. In chapter five, two examples of finite fields whose number of elements is a power of a prime number, are constructed using modular arithmetic with polynomials. The final chapter uses the knowledge from chapter five to study a simplified version of the (6,4) Reed-Solomon code.

The text is meant to be studied more or less independently in small groups of students with only few interventions of the teacher. Short pieces of information (definitions, examples, ...) alternate with exercises that stimulate students to study the text actively. At the end of each chapter some more challenging problems are given.

Two teachers were involved in the small scale testing during the academic year 2008–2009. Teacher A had a class of 18 students in grade 11 and 12, all of them being relatively good in mathematics (top 20%). The class of teacher B was in the same range of age, but much more mixed in terms of ability in mathematics (top 50%). It contained 10 students. Teacher A and teacher B spent 6, respectively 10, lessons (of 50 minutes) to the topic. Unfortunately, due to a lack of time, it was not possible to deal with the last chapter. The student teacher observed the class and helped the teachers in supervising and giving feedback to the work of the students.

The larger scale testing during the academic year 2009–2010 involves 12 teachers and around 150 students. Again, students are in grade 11 and 12. Most of them belong to the top 20% in terms of ability in mathematics. The student teacher will observe a number of classes. Moreover, she will record conversations of students in groups while working with the text, she will interview some of the teachers and administer a questionnaire to the participating students.

Overview of the text

Here, we give a more detailed description of the text used in the teaching experiment (more specifically: the revised version used in the larger scale teaching experiment during the academic year 2009-2010). In the following paragraphs, then, we show a series of excerpts from the text, giving the reader a more detailed impression of the material covered and the teaching method.

The first of the six chapters of the text starts by an introduction to the binary number system and its use in the transmission of information. Next, it describes two situations in which error correcting codes are used: transmitting of images and sound (CD). This introduction intends to motivate the students. Then the key idea of error correcting codes is introduced. When transmitting information in real life, errors are inevitable. In order to be able to detect and possibly correct certain of these errors after transmission, redundant information is added in some intelligent way. During the whole teaching sequence, this idea will be illustrated by a number of concrete examples. Generalization to a more abstract theory of error correcting codes is not the goal here. In the first chapter, the examples are very simple: the NATO phonetic alphabet, repetition codes and parity codes. Finally, in the first chapter, some terminology is introduced: alphabet, code word, one error detecting code and one error correcting code, information rate, ...

The (very short) second chapter introduces a system for calculations which is new to the students: modular addition and multiplication with the two elements 0 and 1. For future use, vectors and matrices based on this system are introduced as well.

The new mathematics introduced in the second chapter is immediately applied in a technological context in the third chapter, where it is used to explain the principle of the (7,4) Hamming code. In this way, mathematics is connected to the real world right from the start, with the intention of enhancing the motivation of the students to study the abstract mathematics. The Hamming code itself is motivated by remarking that it is a code with a better information rate than the codes in the first chapter. The (7,4) Hamming code is defined and it is proved that it can detect and correct one error. Then it is shown how to use Venn-diagrams and/or matrices (generator matrix, parity-check matrix) to encode and decode information.

The fourth chapter generalizes modular arithmetic with the two elements 0 and 1 to modular arithmetic in general. In some of these new systems for calculations, the basic properties of the traditional systems of calculation are conserved. This leads to the concept of the finite fields $\mathbb{F}_p$ where p denotes a prime number. At the end of this chapter, a number of smaller scale applications are treated: (Belgian) bank account numbers, International Standard Book Numbers (ISBN) and EAN product codes.

Now that students have seen that new systems of calculation are very useful in a technological context, they are introduced to yet another (and more complicated) new system for calculations in chapter five. Now, fields whose number of elements is a power of a prime number are constructed. The elements of a field with p^n elements are polynomials with coefficients in the field $\mathbb{F}_p$ and degree at most $n-1$. Addition is the normal addition of polynomials, adding its coefficients modulo p. Multiplication is more complicated. The normal multiplication of polynomials may yield a polynomial having degree n or more. Therefore, the product of two polynomials in the new sense is the remainder of the 'normal' product after division by a given irreducible polynomial. We do not study finite fields in general, but treat two examples: $\mathbb{F}_8$ and $\mathbb{F}_{16}$. The elements of the fields are represented in different forms: as a polynomial with restricted degree, a number in decimal form, a number in binary form or a power of a primitive element. Some representations lend themselves better to do additions, whereas others are better suited for multiplications.

Again, the abstract mathematical constructions and calculations are applied in a technological context as soon as possible. The final chapter deals with a simplified version of the (6,4) Reed-Solomon code, which makes use of calculations in the field $\mathbb{F}_{16}$. Encoding and decoding are discussed in detail using concrete examples of messages in which errors are introduced during transmission. The techniques that are needed give the occasion to enlarge and revise the meaning of classical mathematical objects (numbers, polynomials, vectors, matrices ...) and calculations by using these objects in decoding information that is stored or sent.

Examples from chapter 3: The Hamming code

The (7,4) Hamming code adds to a set $a_1a_2a_3a_4$ of four information bits three control bits c_5, c_6 and c_7 defined by

$$c_5 = a_1 + a_3 + a_4,\ c_6 = a_1 + a_2 + a_4, \text{ and } c_7 = a_1 + a_2 + a_3,$$

(where addition is modulo 2) resulting in code words of seven bits. The text introduces students to this definition and they do some exercises (e.g. calculating the control bits for a message, finding the number of possible code words) to make sure that they understand the rule. Then it is shown that the Hamming code is able to detect and correct one error, based on a property from the first chapter: if different code words differ in at least three positions, the code is one error correcting. The case of messages differing in one position, giving rise to code words differing in three or four positions, is demonstrated in the text. Our first excerpt (translated from Dutch to English) from the text is problem 27, in which the students are asked to deal more independently with the case of messages differing in two positions.

> PROBLEM 27. Assume that two messages differ in two positions. Show that the corresponding code words differ in at least three positions by considering the following cases:
> a. a_1 and a_2 are different;
> [...]
> f. a_3 and a_4 are different.
> HINT: Some of the cases are analogous!

Finally, the text finishes the proof.

We show a second, more practically oriented example from this chapter. In the part concerning decoding, students receive the following (simple) problem.

> PROBLEM 34. Use the definition of the Hamming code to find the error in the received word 1010000.

Students can consult model solutions for the problems. Here, the model solution goes as follows.

> We verify the defining equations of the Hamming code:
>
> $$1 + 1 + 0 = 0?\ ,\ 1 + 0 + 0 = 0?,\ 1 + 0 + 1 = 0?$$
>
> and conclude from this that the second equation is not fulfilled. Changing the second control bit from 0 to 1, gives three fulfilled equations. Hence, the code word is 1010010.

Students also learn to encode and decode using Venn diagrams. If they have already studied matrices in their mathematics course, they are also introduced to encoding and decoding using matrices.

The final problem in this chapter gives the students the opportunity to run through the whole process of encoding and decoding.

> PROBLEM 39. Draft a message and translate it into a sequence of bits using the ASCII-code in table 1.1. Encode this message using the (7,4) Hamming code and send it to one of your fellow students after inserting a few errors. You will receive a message from one of your fellow students. Decode this message.

Examples from chapter 4: Prime fields

The start of this chapter is purely mathematical in nature. Students meet examples of calculation systems based on modular addition and multiplication. They experience that this leads to a system sharing the basic properties of the real number system in some cases (i.e. arithmetic modulo a prime number), whereas the resulting system lacks some of these properties in other cases. This is shown in the following problem, which comes after the text has treated the non-existence of a multiplicative inverse for the number 4 in $\mathbb{Z}_6$.

PROBLEM 49. Fill in the table below

Number	Calculation	Conclusion The multiplicative inverse in $\mathbb{Z}_6$
1	...	...
2	...	...
3	...	...
4	...	does not exist
5	...	...
6	...	...

At the end of this chapter, we added a number of smaller applications in the realm of error correcting codes. We show one example, concerning the (10-digit) ISBN.

> PROJECT 2.V. A common error when typing in an ISBN is that two successive digits are interchanged, for example: 0–1311–0362–8 is typed in as 0–1311–0326–8. Show that detection and correction of this type of error is possible.

Example from chapter 5: Extension fields

The whole chapter is of a purely mathematical nature and is moreover rather difficult for the students. Two examples are treated. The construction of the field $\mathbb{F}_8$ of eight elements is explained in detail to the students, while they do a number of smaller exercises to ensure that they understand the construction. Then, the students more or less independently construct the field $\mathbb{F}_{16}$ of sixteen elements, guided by a number of questions, as is shown in the problem below.

Problem 63. Construct the field $\mathbb{F}_{16}$ by copying the construction of $\mathbb{F}_8$.

a. How many elements has the field $\mathbb{F}_{16}$? Describe all elements using their polynomial, binary and decimal notation.

[...]

c. How do we multiply in $\mathbb{F}_{16}$? Like in $\mathbb{F}_8$, we experience the problem that multiplication in the ordinary sense may yield a polynomial of a too high degree. Now, the polynomial used to reduce the degree, is $X^4 + X + 1$. Work out the following multiplications:

1. $(X^2 + 1) \cdot (X + 1)$;

2. $(X^2 + X) \cdot (X^2 + X + 1)$;

3. $(X^3 + X^2 + 1) \cdot (X^3 + 1)$.

d. The polynomial X is a primitive element in $\mathbb{F}_{16}$ as well. Show this by calculating the following powers of X:

[...]

f. Do the following multiplications, rewriting the polynomials as powers of X first:

1. $(X^3 + X^2 + 1) \cdot (X^2 + X)$;

2. $(X^3 + 1) \cdot (X + 1)$.

Examples from chapter 6: Reed Solomon code

The final chapter of the text is dedicated to the study of a simplified version of the Reed Solomon code. Using the knowledge from chapter 5, students are able to understand the principles of coding and decoding following this code. We show an example of the work that we expect students to do while studying the principle of encoding messages using the Reed Solomon code. It is explained that messages now have the form $a_1a_2a_3a_4$, where now each of the four 'symbols' a_i is no longer an individual bit, but an element of $\mathbb{F}_{16}$, i.e. messages consist of four times four bits. To these information symbols, two control symbols c_1 and c_2, also elements of $\mathbb{F}_{16}$, are added such that

$$a_1 + a_2 + a_3 + a_4 + c_5 + c_6 = 0 \quad (6.1)$$

$$a_1X^5 + a_2X^4 + a_3X^3 + a_4X^2 + c_5X + c_6 = 0 \quad (6.2)$$

Next, the text asserts that c_5 and c_6 can be solved from these equations, producing the following expressions:

$$c_5 = a_1X^6 + a_2X^{12} + a_3X^{10} + a_4X^4 \quad (6.3)$$

$$c_6 = a_1X^{13} + a_2X^{11} + a_3X^5 + a_4X \quad (6.4)$$

Then students are asked to verify this, guided by a number of questions, in the following problem.

PROBLEM 65. Show that the expressions for c_5 and c_6 given in 6.3 and 6.4 indeed follow from the equations 6.1 en 6.2. Do this along the following lines:

1. Solve c_6 from equation 6.1.

2. Plug your expression for c_6 into equation 6.2.

3. Use the power of X notation to work this out.

4. Find the multiplicative inverse of X^4 in $\mathbb{F}_{16}$.

HINT: The inverse of X^4 is x if and only if $X^4 \cdot x = 1$.

5. Solve c_5 from the equation found in question 3 while making use of what you obtained in step 4.

6. Plug your expression for c_5 into the expression for c_6.

After having learned also the decoding procedure, students have to do some practical work, as is shown in the following example.

PROBLEM 70. Decode the following message:

010001100100010111010100010101010100101111101100.

Use the ASCII-code.

HINT: The message consists of two code words.

Of course, the text ends with a problem in the style of problem 39 discussed above, in which students can run through the whole process of encoding and decoding using their own message.

Experiences

First, we report on the experiences with the small scale testing during the academic year 2008–2009. The version of the text used then differed from the revised text discussed in the previous paragraphs. The main difference is that in 2008–2009 modular arithmetic modulo another number than 2 was treated before Hamming codes. The participating students filled in a questionnaire when the chapter on the Hamming code had been treated in class. Hence, students had not studied extension fields and Reed Solomon codes yet. We discuss some of the results here. Moreover, we give a number of remarks made by the teachers and the authors concerning the text and the experiences in class.

The two application oriented chapters, the introductory chapter and certainly the chapter concerning Hamming codes, were considered the most interesting ones by the students. They reported that they had learned a lot in these chapters. Moreover, they appreciated that these chapters were practically oriented, entirely new to them, and that they contained information about real world applications of what they studied. Students also appreciated the fact that these chapters gave them the opportunity to do a lot of exercises. They especially liked to decode coded texts by themselves. They found these chapters not too difficult.

The introduction to modular arithmetic and finite fields having a prime number of elements, was considered the most difficult part of the text. Students found this part the most complex one and had the impression that it might have been explained in a simpler way. Moreover, this part had a high level of abstraction, according to the students.

In general, students appreciated the topic of error correcting codes and more than half of the students reported that it had changed their opinion concerning mathematics.

Concerning the teaching method, students having a better background in mathematics liked working independently in groups. They appreciated having the opportunity to struggle with problems themselves and finding alternative solutions. They reported that they were more concentrated during the lessons. Students whose background in mathematics was poorer, showed considerably less appreciation for the teaching method. One possible explanation, according to the teachers, is that the text may have been too difficult for them.

As to the teachers and the student conducting the experiment, the main point concerning the teaching method, is that even though the text is self-contained, the role of the teacher remains very important. For both classes, the teachers wrote a detailed set of instructions, indicating the parts of the text to be studied, the exercises to be done, the homework to be done, ... Moreover, at the beginning of the teaching sequence and of each chapter, a stimulating introduction by the teacher is needed. Reviewing and summarizing important elements in the text, dealing with some of the more difficult passages in the text, correcting misconceptions, ... are other important tasks for the teacher.

According to the supervisors, a point that had to be considered when revising the text after the small scale experiment in 2008–2009 was the position of modular arithmetic using other numbers (and more specifically: primes) than 2. In the original text the first part mainly focused on calculations modulo 2, whereas in the second part calculations modulo a polynomial are needed. There is a huge difference in level of difficulty between these two parts. We felt that, for didactical reasons, it was important to give more attention to calculations modulo numbers distinct from 2. This is realized indeed in the revised text used during the academic year 2009-2010 and discussed above. The (limited) material on arithmetic modulo a number distinct from 2 has been removed from chapter 2, where it did not receive enough attention and was not used in the application (Hamming code) that followed, and it has been inserted in a slightly expanded form into a separate chapter (chapter 4) in the new text. This chapter deals with modular arithmetic in general and discusses some short applications. We feel that the text is more coherent now and hope that students are better prepared for calculations modulo a polynomial. At least, the first reactions from the teachers are positive.

Conclusion and discussion

As the teaching experiment is still going on, we can only present provisional conclusions here. One of our aims was to make the role of mathematics in modern technologies more visible. In this respect, a majority of students reported that the teaching experiment had changed their view concerning mathematics. Concerning our focus on the secondary school level, the devised text and the first experiences show that error correcting codes and the necessary (non-trivial and abstract) mathematics can indeed be presented and understood on secondary school level, at least if the students belong to the top 20% range of ability in mathematics. Moreover, students reported that they were really interested in the topic of error correcting codes, showing that this topic may help to make abstract mathematics more appealing and exciting, as was intended. Finally, it was seen that students were able to deal with the topic more or less independently guided by a text, but that the role of the teacher remains very important.

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Mathematical modeling and technology as robust “tools” for industry.

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Abstract This paper reports the applied mathematicians’ descriptions of modeling and reflections on the use of technology in applications and modeling. The study was conducted over a period of twelve months in a university setting. Mathematicians were first contacted by email and the interview schedules fixed on individual basis. Ten mathematicians were interviewed but only four interviews are reported in this paper. All the interviews were videotaped. Four main modeling competencies/skills are identified from the study: finding similar examples or phenomena; connecting physical phenomena with visual concepts; building models from the ground up; and communicating broader contexts. The paper argues that these skills are also very much needed in industry.

Introduction

Mathematical modeling has a strong historical link with industry[1]. The teaching of modeling began with concerns from the work place about the undergraduate preparation, who after University training were later required to solve real problems, often collaboratively, (Mclone 1973; Davis, 1994; Challis, Gretton, Houston & Neill, 2002; Houston, Galbraith,& Kaiser. 2007).Training through modeling was a response by mathematics educators to the demands by industry for well prepared graduates Today modeling and applications is an expanding discipline in mathematics education. But modeling has also had different interpretations and meanings in different communities, partly because of how it is conceived. The paper adopts Blum et al's (2007) view of mathematical modeling. Particularly, the duality of mathematical modeling as a means, or an end for educational goals is underscored. That means modeling may serve either as a medium for teaching and learning mathematics, or as a product of mathematical learning.

With respect to the applications of mathematics in industry, there is relevance in Burkhardt's reflection,

> ... there is no point in educating human automata; they are losing their jobs all over the world. Society now needs thinkers, who can use their mathematics for their own and for their society's purposes. Mathematics education needs to focus on developing these capabilities (2006, p.183).

To address Burckhardt's concerns, modeling has to be applied more as a product of mathematical learning. Burkhardt is certainly not advocating for the withdrawal of humans from industry. Rather his reflection, which in this paper is called "Burkhardt's paradigm" underscores the central role humans play in industry, for example in innovations, product designs, and analyses of processes, which roles machines in themselves cannot play. I will return to the paradigm at the end. But without modeling competencies, it will be difficult to address Burckhardt's point. However, teaching and learning through modeling at school might contribute to developing mathematical modeling competency of students and preparing them for industry.

By mathematical modeling competency, I mean (Blum et al 2007),

> "... the ability to identify relevant questions, variables, relations or assumptions in a given real world situation, to translate these into mathematics and to interpret and validate the solution of the resulting mathematics problem in relation to the given situation" (p.12).

A range of conditions which should be met for students to successfully engage in modeling tasks are documented in many studies (Verschaffel, 2002; Blomohoj & Jensen, 2007; Henning & Keune, 2007; Singer, 2007; De Bock, Doreen & Janssens, 2007; Houston, 2007; Usikin, 2007). The findings indicate that modeling should be properly incorporated in the curricula, and should start in the early years of school, taking into account the appropriate mathematical disposition of students. Modeling of physical or social phenomena should be incorporated in the training because they provide authentic links between school mathematics and lived experiences of students (Verschaffel, Mukhopadhyay & Greer, 2007). These studies provide helpful information on what is needed for students to benefit from modeling.

The current study builds on the above ideas but asks different sets of general questions. "What do the expert modelers actually do, and say about modeling and application? "What might we learn from the experts? The purpose is to make this information available to young modelers and to industry in general.

The study is on applied mathematicians' modeling practices, and use of technology, and the specific questions addressed are: 1) what are some of the mathematical modeling competencies relevant to industry, and how should these competencies be developed at tertiary education? 2) from the perspective of applied mathematicians, what role should technology play in the teaching and learning of mathematical modeling at tertiary education? The proposition is that training in mathematical modeling, with efficient use of technology at tertiary education, provides experiences and practices very much needed in industry.

The Study

Only a small portion of a larger study on the role of dynamic mathematical representations and dynamic thinking in mathematics across different levels is presented here. The larger study involves use of dynamic models (such as the dynamic number line and dynagraphs) and their impact on students' reasoning. Details of that are not presented here in the current paper.

The specific portions of the study reported here are the applied mathematicians' descriptions of their modeling activities, and reflections on technology use in application and modeling.

The study took place over a period of twelve months in a university setting. Mathematicians were first contacted by email and the interviews arranged on individual basis. The interviews took place in the individual mathematician's offices to give the interviewers a feel of the modeling environment the applied mathematicians work in. (Drawings and sketches on the white boards were seen in offices, not to mention tons of reference materials on modeling).Ten mathematicians were interviewed, but as mentioned before only four are reported.

In the interview, the mathematicians were first asked to describe their areas of teaching, research, and any applied problems they were working on. They were asked to describe their mode of working on the applied problems: did they use computers, paper and pencil, or drawing? If so what did they find most helpful? Did they use mathematical concepts differently from their counterpart mathematicians? Then in the next phase of the interview the mathematicians were introduced to dynamic models (dynamic number line, dynagraphs, matrix transformations), and asked to interact with the models and provide feedback on their experience. Each interview lasted approximately one hour. All interviews were videotaped.

Results

The results are summarized into four major themes. The four themes are also the modeling skills or competencies identified from the study. They are: *finding similar examples or phenomena; connecting physical phenomena with visual concepts; building models from the ground up; and communicating broader contexts*. Each of these is discussed through question and answer by the mathematicians J, K, L, M, and the interviewer I.

Finding similar examples or phenomena

1. I: What does the process of modeling and application involve?

2. J: Part of it is having an encyclopedic collection of things that are important. You know that this thing has been done for these classes of problems so you can kind of extract things that are similar.

In industry, "encyclopedic collection" is probably attributed to long service and experience gained on the job. However, from a modeling perspective, other competencies such as problem tackling, representing and communicating also fit in the encyclopedic collection. When asked what else was included in the "encyclopedic collection" of things,

3. J: in this process [they] have to identify what the tools are, and then they also have to know how to use the tools once they have done the identification of the problem.

The encyclopedic collection seems to include a package of experiences, and "tools" to draw from in the modeling process. But often there is no clear way of doing this. Mathematician K brings in practical experience in modeling how,

4. K: the data was very suggestive of things that we'd seen in other contexts, so we had to think about what types of mathematical objects would give rise to such pictures.

Referring to "things seen in other contexts", [line 4], K seems to confirm J's notion of the need for the encyclopedic collection of experiences [line 2] in modeling,

5. K: that was much harder and I have to confess that the model was cobbled with different terms that each individually described certain aspects.

The complexity of a typical modeling process "cobbled together in different terms" could be compounded by the difficulty of connecting a mathematical model to a "real object". To do this, a modeler has no specific reference point, except to resort to experience, similar patterns, behavior or theory, to develop the model.

CONNECTING PHYSICAL PHENOMENON WITH ABSTRACT CONCEPTS.

This is again one of the challenging questions in practice,

6. I: How do you mobilize the mathematics in solving a modeling problem?

7. J: the basic tools are, how can I write something down which describes certain aspects of what I'm interested in?

8. L: Well, in modeling you have to have some physical model of what the object is, its structure, shape. So you have to write down the equations that actually describe that.

9. L: Then you somehow have to program that up in an algorithm so that the computer can actually simulate ...

The word *somehow* [line 9] may have been used deliberately by J to point to the complex nature of modeling. In this case, students and other prospecting modelers would be encouraged to learn that "the" solution to a modeling problem hardly exists, but a relative solution can be improved upon with more input or generally more data. The good news is that modeling also provides opportunities for students to collaborate and work together on problems, thus developing the necessary experience and skills for industry.

10. I: Having had some experience with our dynamic models, what ideas do you have about the use of technology in modeling?

11. J: I think that anything that " kids" do in some way of playing when they are building things up is good for sustaining interest. I think about building the model piece by piece, and that is the tool to give students.

12. M: When I look at a tool like this one, my first question is what would use it for? I'm always keen on anything that gets people to play, anything that brings a sense of discovery.

It is interesting that both J and M independently mention "play" in modeling. The reason given by J is to sustain interest when doing modeling, M links play with discovery. M extends the use of technology to the question "what for?" or for instance, why use this technology and not the other one? In education, the choice is often made by the teachers. But as discussed earlier, what important is to choose the right tool for the right question. The second insight on use of technology is for meaningful exploration. "Play" according to M seems to imply a game with a purpose, some exploration aimed at bringing about a sense of discovery. Unfortunately, the word "play" is not always synonymous with industry. But, for teaching and learning, it seems important to engage students in some playful learning with technology. How can this be done?

M again on play,

13. M: Well, there is certainly a lot of test cases and playing around, prototyping small cases, "what if I had to fit a square into a circle, what would really happen?

So one option is to allow students use technology to try out modeling activities on their on, and see if they can describe what is going on in their own words. There is evidence to support the use of technology for exploration in mathematics education, with substantial success (Papert, 1980).

Communication

Two levels of communication are reported:

Communication between applied mathematicians and students; and communication in the larger community.

i) *Communication between applied mathematicians and students*

14. I: What do you say about communication involved in mathematical modeling?

15. L: after working their solutions, some [students] will say, "okay, that is the solution", even if they have made a mistake. You look at the solution and it can't be anything physical. The physical world doesn't work that way.

16. L: but they [students] are resistant that they might be able to apply their intuition about the real world to the solution of the model.

The apparent resistance to apply intuition to real solution is not necessarily a problem in itself. There are times when intuition is clearly unreliable, particularly in dealing with stochastic processes. However, even then one has to be clear when to use intuition and when not to. L describes the resistance [by students] to relate their mathematical solutions to reality, as *lack of physical intuition* or unwillingness to use intuition. "Physical intuition" in this context is described as a disposition to translate mathematical modeling solution to reality. And this seems to be an issue even at the undergraduate level,

17. L: This is really a big problem, whether it hurts [them] or not I'm not so sure because a lot of that is just [like] turning the crank and getting results.

Communication is clearly an important factor in modeling, particularly across disciplines. Gone it seems, are the days, when scientists, and engineers did not need to say they were doing because they simulated computers. Now the computers are doing the calculations, and the roles are shifting towards "Burkhardt's paradigm", introduced at the beginning of this paper. It would seem that communication is at the centre of all these.

ii) *Communication to the larger community*

18 I: Is there a different culture of communication in applied mathematics?

19. J: In applied math, there is always a lot of communication among the people who are working together on a problem, so a lot of writing, starting right away with something broader, with a lot more background ...

Communication is not only important among the professional applied mathematicians themselves, but also with the general public and industry.

Ultimately this is important for the expansion and growth of application and modeling as a discipline. The implication is that students will need to be helped to develop quite different kind of communication skills to prepare them for industry. They should communicate in ways that provide contexts and motivation to a broader audience.

Summary

The goal of this paper was to highlight applied mathematicians views on mathematical modeling and some skills relevant to industry. Previous studies have reported on modeling mainly from elementary to secondary school settings. This study focused on the instructors at the tertiary level and identified four modeling competencies at undergraduate: finding similar examples or phenomena in a modeling process; connecting physical phenomena with visual concepts; building a model piece by piece; and communicating broader context for the benefit of a larger audience. Technology plays a big role in fostering exploration towards discovery, also in sustaining interest in the modeling process. It is argued from that data presented that all the above skills are relevant to industry.

The paper is motivated by Burckhardt's thoughts, which here I call "Burckhardt's paradigm". The paradigm is an important challenge to mathematical modelers to stretch our minds beyond the machines, encourage growth of new ideas. Hopefully the interactions among experts in many different disciplines and the improvement in communication will contribute to better solutions to real problems for industry.

Notes:

1. Industry means "...any activity of economic or social value, ...in the private or public sector" (OECD 2008, p.4).In this paper industry is also understood broadly to mean work place.

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Perceptions of middle school children about mathematical connections in a robotic-based learning task

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Abstract The paper reports about a case study on robotics-based learning conducted with 45 Middle school students from a small French Canadian rural community. We were working collaboratively with teachers and technology mentor to collect a variety of data. In this paper, we report how students, interviewed about their experience with robotics-based tasks, perceived connections to mathematics. Our questioning was focused on the use of mathematics to do robotics and use of robotics to build better mathematical understanding, as well as a possible increase of students' motivation and social engagement in meaningful and authentic problem solving related to the real life. The themes emerged from our data on the use of robotics-based open-ended and context-rich tasks show connections to measurement, use of proportion and work with angles.

Context and problem statement

A view of mathematics teaching and learning in 21st century society goes beyond traditionally taught techniques and procedures towards development of more complex set of abilities helping young students to solve real-life related problem situations. These situations allow students to employ diverse strategies, more sophisticated mathematics reasoning and communication skills and making intra-, inter- and transdiciplinary connections (NCTM, 2000). Following these common trends, new reform-based curriculum in New Brunswick, Canada aims to develop innovation-minded active citizens able to learn to learn in order to fully contribute into the process of collective complex problem solving that is a common characteristic of any changing and creative society. Those outcomes include building strong communication, critical thinking, technology, work organization skills along with good personal and social development habits as well as sense of belonging to the community and its culture (Ministère de l'éducation du Nouveau-Brunswick, 2009).

While curriculum states clearly the importance to establish real-life connections, its realization is still facing several obstacles such as compartmentalization of school subjects' structure, lack of interdisciplinary teaching materials, assessment tools and didactical support of new student-centered and diversified environment (Blain, et al., 2007). These findings are intact with those raised by other authors who mention also lack of time and teacher training as important factors of success of new approaches (Holcomb, 2009).

New Brunswick is a bilingual (French and English) Canadian province with a dual (according to the language) educational system. The French minority population faces many challenges related to the lack of human and material resources, demographic decline and survival of language and culture (Freiman & Lirette-Pitre, 2007). A quest for a better education that began in the early 90s with curriculum reform has been supported by international studies in mathematical, scientific and reading literacy showing that 15-year olds New-Brunswickers are trailing their peers from other Canadian provinces (Bussière, et al., 2001). Based on the assumption that these results, at least regarding mathematics, can be partly explained by a gap between school curriculum and a real-life needs, government encourages teachers to look for innovative ways to relate mathematics and industry thus bringing additional motivation to learn mathematics and enabling development of critical and reflective thinking in all students disregarding their abilities and skills.

New mathematics curriculum implemented in New Brunswick's French Canadian school system in the past decade aims to bridge a gap between school and real life that helps students build more explicit intra-, inter- and transdisciplinary connections. From the point of view of school practitioners, more needs to be done to make mathematics teaching and learning meaningful for today's generation of students.

That is why teachers and school administrators discovered a need to work together with researchers and the community to look for innovative ways of teaching and learning, which is one of the reasons used by provincial government to launch the Innovative Learning Fund. This fund supports teachers in their quest for good practices which can subsequently be shared with and replicated in other schools while being analyzed and interpreted by the scientific community (Ministère de l'éducation du Nouveau-Brunswick, 2007).

RoboMaTIC is an example of such initiatives. The teachers from one rural elementary (K-8) school were provided with funds allowing them to use LEGO Mindstormã robots and laptops with their students. Additional time was allocated for professional development letting teachers to learn how to teach in such an environment and to elaborate robotics-based learning scenarios. The research team CASMI (www.umoncton.ca/casmi) from the Université de Moncton was invited to collaborate on implementation of these scenarios with Grade 5-6 Middle School children over two school years. One of the research questions stated by the team was about the connections to mathematics children could discover during experiences with robotics-based tasks.

In the following sections, we will briefly review existing literature on the use of robotics in different educational settings and potential links with mathematics. Then, we will discuss our methodological decisions and present data that support our assumptions about explicit connection children make between robotics-based task and mathematics. However, further analysis is however needed to understand better these links.

Theoretical considerations

Gura (2007) argues that students often see little relationship between the math curriculum and the world in which they live in because they are constantly told to go to school in order to learn what is offered there. This 'misunderstanding' of mathematics as real-life disconnected academic activity may have a negative impact on their achievement. The author advances that robotics involves such mathematical skills and knowledge as measuring, counting, calculating, estimation, algebra and geometry; moreover, the very nature of a robotic activity helps to present these concepts not in a isolated way but as being authentically integrated in a problem solving process.

In order to look in-depth at possible connections between mathematics and robotics, we turned to the results of a few available studies. Lloyd and al. (2005) conducted a naturalistic phenomenological study of mathematical understanding emerging in middle school students engaged in open-ended robotics activities. They found that working with robots helped children understand mathematics but in a different way compared to traditional curriculum. These kinds of activities would exemplify richness, recursion, relations and rigor. The authors suggest the use of robotics activities to address such equity issues as gender, minority status, and learning disabilities (Lloyd, Wilson, Wilkins, & Behm, 2005). However, one question remains open on how this may be accomplished. Our study conducted in a diverse heterogeneous inclusive classroom in a French language minority school setting brings some insight on these issues.

In a project designed to show how mathematical problem solving can occur in a context-rich lesson employing information technology explorations, Williamson & al (2008), showed that 93 percent of students found hands-on activities to be very helpful in learning. In their study, robotics was a part of various on-campus workshops conducted with school math and science teachers (30) and students (60). These authors also argued that their project attracted rural students to careers in science, engineering and technology. Although their approach embeds technology in complex and long-term activity that features group inquiry and problem solving is similar to our study, the school context in which our study was conducted is different (Williamson, ElSawaf, Abdel-Salam, & Mohammed, 2008).

Being inspired by the ideas of authentic learning provided by robotics activities, we developed a model of learning that integrates notions of robotic-based learning and constructionism, problem-based learning and meeting the needs of the Net generation students that have grown up in a digital and technology-rich world. Figure 1 illustrates our model and its conceptual underpinnings are explained in more detail in the following paragraphs.

Robotic-based learning

The use of robotics in the classroom is, by its nature, an exciting technological breakthrough and relatively unused in classrooms today (Williams, Ma, Prejean, & Ford, 2007). Papert, in 1980, started his robotics research using a constructivist approach to give students a chance to interact with new technologies making way to a new educational paradigm named constructionism. This theory emphasizes the active role of the learner in collaboratively constructing public knowledge in a rich context can be used as the theoretical framework for Robotic-based learning (Papert, 1980, 1991; Williams, et al., 2007).

Robotics research has shown a positive effect on science and technology motivation in classrooms (Barker & Ansorge, 2007; Carbonaro, Rex, & Chambers, 2004; Gura, 2007; Nourbakhsh, et al., 2005; Petre & Price, 2004; Williams, et al., 2007). In addition, several authors have mentioned a positive effect of robotic-based learning on the level of collaboration between students (Nourbakhsh, et al., 2005; Petre & Price, 2004), on problem solving and critical thinking skills in children (Mauch, 2001; Norton, McRobbie, & Ginns, 2007; Petre

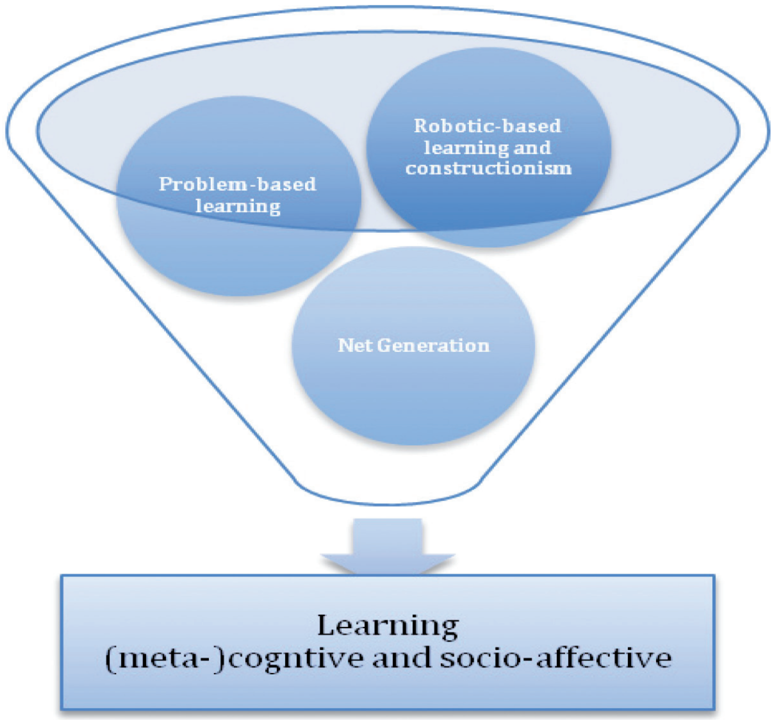

Figure 1—Learning on a cognitive and metacognitive level using Robotic-based learning approach utilizing a problem-based learning strategy for the Net Generation.

& Price, 2004; Wagner, 1998), on the ability to use inquiry skills in classroom (Williams, et al., 2007), on the learning of a programming language (Nourbakhsh, et al., 2005) while being an excellent interdisciplinary agent (Papert, 1980; Rogers & Portsmore, 2004).

In which way are robotics-based tasks suitable to support students making connections to mathematics? In the next paragraph, we analyze literature on active teaching and learning methods that emphasize benefits of problem-based nature of such tasks.

Problem-based Learning tasks

In order to identify learning outcomes in robotics-based experiences of young children, it's important to recast them as active learners developing a wide variety of thinking skills, making them researchers and producers of new knowledge while fostering emergence meaning making skills (Gura, 2007; Sword & Leggott, 2007). It can be achieved with an open-based pedagogy like *problem-based learning*. This pedagogy can be defined as "an inquiry process that resolves questions, curiosities, doubts, and uncertainties about complex phenomena in life" (Barell, 2007).

Problem-based learning is based on challenging students with a problem that they cannot solve straight forward, but need to deal with a doubt, a difficulty, or an uncertainty that requires some kind of complex solution process, including sometimes construction of new means and innovative strategies (Barell, 2007). The authenticity of such learning is seen by Ormrod (2008) as being catalysis of transferable skills and knowledge, as well as support of acquiring of life-long learning experience (Ormrod, 2008).

According to our vision, robotics-based learning using probem-based tasks can not only enchance the development of important cognitive and metocognitive skills and abilities but can also be an important factor of motivating today's young learners. By using technology, these Net Gen may learn mathematics and science in a new way that is more suitable to meet their learning styles and needs while making it challenging and enjoyable experience.

Net Generation

In our previous work (Blanchard, 2009), we analyzed literature decsribing a phenomenon of new 21st century learners forming a so called Net Generation. When teachers address diversity of needs of these learners, they aim to create healthy and creative inclusive environments where students can reach their unique potential. Researchers suggest several pedagogical principles that shoul be respected in building such envrionments that give students freedom of choice, allow expressing their personality using variety of digital communication tools, create opportunities to develop critical judgment and cyberethics, foster integrity and openness, is interactive and motivating, prompts collaboration and socialization, and finally provides with a rapid information and feed-back (Prensky, 2001; Tapscott, 2009). We found that robotics-based learning may create environments that resepect before mentioned principles. Since our study aims to explicit connections that students can make beetween robotics and mathematics, in our analysis we base on social, affective and cogntive aspects of the problem solving in robotics-based learning and teaching environment that is open-ended and complex that may be suitable for today's learners who grow in the digital world and belong to the Net Generation. In the next section, we describe methodological choices while implementing our model with students as well as research tools used to collect data.

Methodology

While all students and all teachers from the school were involved, although in a different way, in the project, we decided to focus our research on two groups: Grade 5 and 6 (N=45). Teachers received one-day training with an expert of implementation of robotics in schools. This collaboration resulted in the conception of seven pedagogical scenarios to be implemented in the grade 5 and 6 classes:[1]

1. Nomenclature and classification LegoMindstorms© kit pieces
2. Robot construction
3. Advancing robot on one meter with RoboLab© programming tool
4. 360-turn task with RoboLab© programming tool
5. Reach three targets from a fixed point
6. React to a frontal obstacle using sensors and retract task
7. Follow a square-shaped path using sensors

During the academic school year 2007–08, the scenarios 1 through 4 were conducted with both Grade 5 and 6 groups and the following year, the scenario 5 was conducted with the students of Grade 5 (who advanced to the Grade 6). Students were working in small groups of 3–4. A team of two university professors and two university students with the help of the ICT mentor collected research data.

During the activities with robots, we used classroom observations, video recording, and interviews with some students and teachers, school blogs written by administrators, mentors, teachers and students. At the end of the project, individual interviews were conducted with each of 21 students from one of two initial groups (Grade 6).

While a detailed analysis of all collected data is still underway and considering limited space for this proposal, we focus the analysis on individual interviews while looking at how children perceive connections between robotics and mathematics and what categories emerge from their discourse.

Results and discussion

During 15-minuts semi-structural interviews, students were first asked about general perception of the project revealing what they particularly liked or disliked. Then, we asked them if they found any connection of robotics to mathematics. Finally, we asked if they saw some utility of the experience for their future life and career, as well as interest to continue the project with robots. The interviews were video-recorded and transcribed.

The analysis was made in three stages. First, we went through the whole corpus finding themes related to mathematics, to robotics and to general appreciation of the project. Then, we sorted the themes according to our theoretical framework for each, (meta-)cognitive and socio-affective aspects (Tables 1 and 2). Then we briefly discuss findings from both, tables and corpus. Table 1 summarizes categories emerging from interviews with participating children with regards on cognitive and metacognitive aspects. Table 2 summarizes categories emerging from interviews with participating children with a regard on socio-affective interpretation.

Categories	n = 21	Categories	n = 21
Measurement of angles	16	Use of numbers	1
Proportions (scale)	1	Cartesian coordinate system	1
Rotation (number or time)	12	Percentage	1
Circumference	2	Symmetry	1
Convert units	2	Trial and error	4
Precision	2	Making mathematics more concrete	8

Table 1—Summury of emerging cognitive and metacognitive categories.

Regarding cognitive aspects relating robotics and mathematics, we learned from the interviews that many children recognize utility of mathematical concepts and procedures in a problem solving process. Namely, using rotations (number or time), measuring angles, circumference, and calculating with proportions (scales) to convert units were mentioned the most often. Inversely, children see that robotics does create opportunity to apply mathematics and eventually lead to better understanding of concepts and precise measurement. They tell in interviews that robotics task does provide them with explicit application of mathematics making its learning more concrete. For example, one child said that she learned how concept of wheel rotation may be applied to make robot move.

However, some children shared that mathematics remains too abstract to them to see any connection to the real life. Others were talking about difficulties with certain tasks, especially one that required the robot to pick up three targets placed in vertices of a triangle. While concrete experience showed that robot was not moving in a right direction, some children told they were unable to fix it using mostly trial-and-error strategy.

While many participants said that tasks were enriching and going beyond traditional curriculum, few children found problems not challenging enough and some would program more complex and real-life related tasks, as making robot throwing a ball, feeding animals, or washing a dish. One child mentioned that the triangular path was too easy and suggested pentagon- or ellipse-shaped circuits.

On the affective side of the experiences with robots, children shared with us common opinion that despite some hard time with certain tasks, they liked the experience as it allowed them to 'build robots', to 'use technology', to 'do programming' and 'see the utility of mathematics'. From the social point of view, many children spoke about positive experience of working in teams (learning community), helping each other and distributing tasks according to strengths and interests of each team member. Another aspect was mentioned is re-

Categories	n = 21
Appreciate the manual aspects of robots	3
Appreciates the programming aspect of the robots	6
Sees robotic helping in future careers, jobs, or leisure	14
Help in general	1
Help in labor work	3
Help in agriculture	1
Help in architecture	2
Help in mechanics	1
Help in sports	3
Help in computer work	3
Interest in robotics and learning more about robotics	15
Learning community	5

Table 2—Summary of emerging socio-affective categories.

lated to the utility of robotics in their future carrier. While students could not affirm their particular interest of working directly with robots in the future, they believe to acquire intra-, inter- and transdisciplinary skills and abilities, among them mathematical, which would help them to succeed in school and in everyday life.

Conclusion

Our results confirm rich potential of the robotics-based learning to develop more explicit connections between mathematics and real life in young students, as it gives concrete examples of application of mathematical concepts and procedures. Moreover, they see potential utility of robotics in their future life and therefore may influence their carrier choices. It is not surprising that many mathematical concepts become better understood by students since they can do measurements and calculations and then validate the results by manipulation with robots. This direct and immediate feed-back helps to engage students into more complex problem-solving process, in which they can appreciate challenge and team-work.

While trial and error remains the most often strategy used, students may eventually develop some more effective cognitive and meta-cognitive tools as they find themselves in a supportive and risk-free learning environment. How to guide them towards the emergence of higher-order thinking still remains unsolved pedagogical task. Our participation at the ICME study could help us to advance in our theoretical, methodological and practice-oriented research.

Notes

1 http://cahm.elg.ca/archives/2009/02/participation_d.html

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Two examples of a program ‘Mathematics for Industry’ at the master’s level in a French University: Université Pierre et Marie Curie-Paris 6 and Université de Pau et des Pays de l’Adour

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Abstract Over the years, the range of applications of mathematics has been progressively enhanced and applied mathematics, modeling and simulation have taken an increasing role. This motivated the French Department of Education to launch a Diploma in applied mathematics at the master’s level thirty years ago. The two first examples of such programs were created in different contexts, one in Paris, one in a smaller provincial town. These programs have evolved according to the European implementation of the Bologna process; they have taken into account the new range of applications, for instance in Finance, and profited from the spectacular development of computer science.

1. Introduction

History: DESS (*Diplôme d'Etudes Supérieures Spécialisées*) in applied mathematics were created in French Universities in the late 70' (see CNE (2002)): they correspond to the 5th year of higher education, the graduate program is chosen only in the fifth year, but coherent with the four previous years. The first diploma Dess in applied mathematics was created in *Université Paris 6* (the largest scientific French university), now called *Université Pierre et Marie Curie-Paris 6* (or UPMC), in 1977. It was then called *Dess de mathématiques appliquées* and was backed up by the competence of the French greatest laboratories in applied mathematics (numerical analysis, probability and statistics) and continuum mechanics. It appears that the second one was created in 1979 in *Université de Pau,* now *Université de Pau et des Pays de l'Adour* (UPPA), a smaller multi-disciplinary University in the southwest of France. The name of this last program changed in 1998 in *Dess Ingénierie Mathématique et Outils Informatiques* (mathematical engineering and computer tools) in order to enhance the close link between mathematics and computer sciences at industry level. This University benefits from a favourable industrial environment thanks to the presence of important industries: Elf Aquitaine (now Total) in oil exploration and Turbomeca (now in Safran group) in aeronautics (helicopter engines).

The French organization of higher education, following the Bologna declaration on the European space for higher education in 1998 has undergone a change to implement the Bologna Process and passed to the so-called LMD frame (for Licence – Master – Doctorat). All courses are organized in compliance with the European Credit Transfer System (ECTS) of credits accumulation. Hence the *Dess de mathématiques appliquées* in UPMC has become in 2004 a Master's degree level program *Master Ingénierie mathématique, Mathématiques Pour l'Entreprise* (MPE) which exactly means 'Mathematics for Industry' in the sense given in this study. The MPE program is specialized in a second year (M2) of a master's program which lasts two years, the first year (M1) beeing in common with other programs. In the same way, the *Dess de mathématiques appliquées* in UPPA has become in 2004 a Master's degree level program in Applied Mathematics with two possible curricula: Mathematics, Modelling and Simulation *Mathématiques, Modélisation et Simulation* (MMS) and Stochastic and Computational Methods for Decision making *Méthodes Stochastiques et Informatiques pour la Décision* (MSID). This change of name reflects a greater awareness in the teaching teams of the importance of identifying a 'core curriculum of mathematics for industry' or, at least, the teams wanted to make the program 'more appealing and exciting to students and the professionals in industry'.

The situation in France is specific because of the coexistence of two separate tracks for the training of students: Universities and Schools of Engineering (*Écoles d'ingénieurs*). While in

Schools of Engineering, less and less mathematics are being taught (outside Mathematical Finance) since more time is given to management and economy, in French Universities the traditional high level of training in mathematics has been more or less preserved and is appreciated in industry, so that French students following this path still find good job opportunities thanks to their specific skills. However France is presently facing a debate on education at all levels (school, training of teachers, universities). Several reform projects may have a deep impact on our programs.

2. Presentation of the two programs

As already said, the two universities UPMC and UPPA are very different in localization, size, organization. However the two diploma share many common points.

2.1. Master 'Mathematics for industry' – UPMC

The professional Masters degree *Ingénierie mathématique* at UPMC offers presently two possible curricula: 'Mathematics for industry' and 'Financial engineering and random models'. In these two curricula, students have little choice and benefit from a supervised training which does not allow to have a job outside. However, some students need to earn their living and face difficulties at beeing present all week long. For some other students, the supervision is too heavy, and they need more freedom. There was for 4 years a third curriculum OML (*Outils mathématiques et logiciels*) which was meant for these students. It was stopped because too few of them were able to manage the two activities at once. For the few who succeeded in carying on this program, it was a good program, and we still think that if more students were interested, we could open the curriculum again, with little additional forces. The present consequence is that a few low income students, those who do not benefit from a grant, do not succeed in getting their master's degree in engineering mathematics, which we deplore. The first curriculum exists since 2004, and truly since 1977 since few changes were brought with the transformation from a Dess to a master's degree. This does not mean that it did not evolve for thirty years, but the first principles which lead to its creation are still believed to be active. For what concerns the second curriculum, *Ingénierie financière et modèles aléatoires*—IFMA, it was open later, in 2006, to face the increasing demand of both students and the banking and insurance industries (more precisely capital markets) for high level formations in applied mathematics including stochastic modelisation and simulation using numerical probability methods. We will not analyze this program which is the object of another study on Mathematics and Finance by Gilles Pagès. Note that in the past few years, more students enrolled in mathematics at the master's level attracted by the positions and salaries offered by the later though unemployment does not affect the former program. But traditionnal manufacturing industries did not communicate enough on the need of well trained engineers or of hiring mathematical scientists.

The study will focus on the first curriculum, the *Mathématiques Pour l'Entreprise* or MPE-program. The program covers a wide range of applied Mathematics: numerical analysis, partial differential equations, scientific computing, and either probability, statistics and mathematical finance, or mechanics. Graduates of this program will have a dual profile of mathematician and engineer, able to model various phenomena and to develop new methods of numerical simulation. Careers are possible in a number of sectors using scientific computing, mechanics, stochastic modeling or statistics: R&D departments in public organisations or industry, software developers and service providers, studies and estimates departments in banking and insurance.

Two curricula are possible with a common core syllabus in numerical analysis and scientific computing (whose courses are in numerical analysis and computation, mathematical software, advanced programing and algorithms):

- probability – statistics, with an initiation to quantitative methods in finance
- mechanics of continuous media (fluids and solids), jointly with the Mechanics and Energetic specialization of the Engineering Science master (concerns few students).

Each curriculum comprises elective courses in the first semester and ends with a practical training period, an internship in industry (usually six-month long), completed by a formal report and an oral defense. Conferences introducing companies and career guidance workshops are organized. Some training in English is also provided, with the possibility of passing the Toeic (Test of English for International Communication) to certify language skills and enhance international competitiveness. The precise list of courses can be found on the website www.ann.jussieu.fr/MPE

The list has evolved with time, for example courses on matlab, C++, MPI, Java, VBA, Fluent, GPU, ... , and new optimization algorithms (genetic) were gradually introduced, maintaining the highest level in mathematics, with reinforced ability in computer science together with good control of programming language and common software. Some data are given in Table 1. One can observe that the number of students coming from a first year (M1) of the master's degree at UPMC is increasing. This comes naturally from the passage from *Dess* to LMD, students who used to move to follow our program at the beginning of the year of *Dess* now rather move at the beginning of the Master's degree (M1). The fluctuations in the 'proba' and 'mechanics' columns, concerning students who have chosen either a course in

Dess/ MPE	total	girls	proba	mechanics	from UPMC	% from UPMC	success	gave up	repeat	answer quest	found job	follow another program	among which PhD
2004	30	9	18	12	14	47%	28	2		12	8	7	3
2005	24	8	19	5	9	38%	24	0	1	18	12	6	4
2006	17	5	6	11	9	53%	17	0		13	9	4	3
2007	24	6	11	13	11	46%	20	2	2	13	6	4	3
2008	23	6	15	13	17	74%	19	2	2	14	11	2	1
2009	22	4	20	9	17	77%	18	0	4	10	7	2	1
2010	36	15	20	2	28	78%							
total	176	53	109	16	105	60%	126	6	9	80	53	25	15
%		30%	62%	39%	60%		90%	4%	6%	57%	38%	18%	11%

Table 1—Statistics for the MPE curriculum from 2004 to 2009

probability and statistics or a course in mechanics, may be explained by specificities of other programs in UPMC 'competing' with MPE. The figures also indicate a very good success ratio (90%) thanks to a selective entry and a close training, following individually each student. Note also a percentage of students undergoing a doctoral program. The doctoral positions offered after a professional program are often funded thanks to a collaboration between an enterprise and the French ministry of research (so called CIFRE *conventions industrielles de formation par la recherche*). In the 'found job' column, we have all kind of (sometimes short time) positions found by our students in the year after the program. Employment concerns mostly service industry: finance, insurance, CAD, aircraft design, automotive engineering, civil engineering, petroleum engineering (oil exploration), energy, image and signal processing, telecom, networks, communication industry, possibly pharmaceutical industry ... There are broad possibilities, in both lowand high-technology industries. It is difficult to have a precise employment rate on a longer period because of the lack of data, students requested to stay in touch with the faculty forget to do so; after some time, few of them answer our request. The two statistics/mechanics curricula yield similar rates, the first more naturally in studies and estimates departments in banking and insurance.

2.2 Master Mathématiques et applications – UPPA

The organization of the former DESS and present Master in UPPA has been conceived taking into account the already mentioned industrial environment in petroleum engineering (Total) and aeronautics (Turbomeca). Specific required skills have been identified, leading to the definition of appropriate courses. However, a broader range of applied mathematics is covered so as to ensure more professional prospects to the students. There are two possible curricula: *Mathématiques, Modélisation et Simulation* (MMS) or Mathematics, Modelling and and Simulation, and *Méthodes Stochastiques et Informatiques pour la Décision* (MSID) or Simulation and Stochastic and Computational Methods for Decision making.

The MMS curriculum: we do not give details here since the content does not differ much from that of the above MPE diploma. Note however that it may also lead to a doctoral program so that the training periods may take place in a University laboratory, which is not possible for a professional program. Moreover in collaboration with the university in Zaragoza (Spain), the program may be joint to a Spanish one (*Licenciatura de Matemáticas*) and become a Spanish-French diploma for students spending part of their training in each university.

The MSID curriculum: trained in the statistical and computing methods of treatment and data analysis, in the advanced programming and in the management of the computer systems, the students stemming from this program will have acquired the skills required to exercise the professions where the administration of databases and the statistical methods generally are indispensable to the production of syntheses of relevant data for the decision-making.

Main topics are: statistical modelling, data analysis, experimental design, use, development and adaptation of software for the IT and statistical processing of data, conception and administration of databases and information systems, conception of internet applications for the management, the administration and the data processing, running and management of projects in research and development.

Professional prospects are in industry and services (development, planning for local government, environment, biology and medicine).

The theoretical and professional courses are gathered as follows:

- computing: Computer systems, Databases and advanced Programming
- geomatics: geographical Information system — stochastic methods: applications of the stochastic simulation, statistical methods, datamining and image processing.

Context of the setting-up of the curriculum in statistics

The curriculum in statistics at graduate level has been set up at UPPA as a response to the combination of the following facts. In the early 1980's, the number of graduates was low

	2005	2006	2007	2008	2009	Total
MMS	7	9	12	14	13	55
MSID	13	19	18	20	8	78
total	20	28	30	34	21	133
girls	6	11	8	17	8	50 (38%)
M1 degree out of Pau	9	5	5	7	5	31 (23%)
from UPPA	11	23	25	27	5	102 (77%)

Table 1—Graduates per year since 2005

relative to the need of statistical well trained collaborators in the companies; many mathematical sciences departments in the French universities had a PhD program in statistics but there were no programs specifically devoted to the training in applied statistics (statistics for business and industry) at graduate level. The specific situation at the university in Pau where most of the faculties of the mathematics department maintained close relations with companies at local as well as national level in the field of industrial mathematics; the statistics group of this department was not strong enough to support a PhD program in statistics but its skills enabled it to consider setting up training in applied statistics at graduate levels. Besides, the debate within the French professional societies for statistics had shown the necessity to strengthen the teaching of statistics at a high education level and create professional-oriented training in statistics; moreover there were government incentives which lead to the creation of Dess.

Student flows in UPPA since the beginning of the Masters level program.

The data in table 1 show a regular growth of the number of graduates from 2005 to 2008 and then a sudden drop. Such a fall is a common phenomenon for science studies in French universities. However, the number of students interested in our programs is quite good compared with programs in other domains of mathematics in France, due to the teaching of mathematics for industry in our curricula. Indeed, the number of students preparing for the Masters degree is greater in 2010 than in 2009. Altogether, there are 27 (14 for the MMS, resp. 13 for the MSID curriculum). As the success ratio is close to 100% thanks to a selective entry, it is expected more graduates in 2010 than in 2009 and therefore a stability of the flows.

MMS	Petroleum Industry 21%	Aeronautics 15%	Other industries 10%	University laboratories 54%
MSID	Industries 40%	Banks 10%	Administrations 50%	

Table 2—Statistics for the MMS and MSID curricula from 2005 to 2009

Industries	Administrations	others	teaching	research	unknown
31%	6%	10%	10%	22%	20%

Table 3—Global statistics for the two curricula from 2005 to 2009

Practical training period

The main industrial sectors receiving students for training periods are: petroleum engineering with industries such as Total or research organisms such as IFP, aeronautics with industries such as Turbomeca or research organisms such as CNES, computer engineering, banks and insurance companies, food processing, administrations (regional and local authorities) (see table 2). Students generally do their period of practical training in the region; due to an additional financial cost only 2 or 3% do it in the Paris region. In table 2, for MSID, the Industries concern petroleum, food processing, computer engineering companies.

Careers of graduates from 2005 to 2009 for the two curricula

The graduates finding a job in Industry (31%) work in the industrial sectors listed above; some of them stay in the company where they did their training period. Other graduates (22%) study for a PhD doctorate, among them 30% are sponsored by an industrial company (see table 3).

2.3 Important features for both Paris and Pau programs

Let us emphasize some of the main lines that guide our teaching projects:

- all the faculty involved in teaching courses are specialists of their domain (applied mathematics, computer and information science statistics or mechanics) and have contact with research in industry using applied mathematics. Some of them even work in a company but have a high level training (for instance PhD) in applied mathematics.

- Emphasis is laid both on analysis' (theory with axioms, definitions, theorems, programming languages, algorithms...) and some use of 'black boxes' (students may learn to use them by just knowing the great line of inner workings).

So the program lies somewhere between edutech (teaching and learning with technology) and indutech (using technology in industry). Since it is impossible to provide a relevant program with depth in all area using applied mathematics, a few of them only have been chosen, all of them for their relevance (they are used in the applications, though not always at this level). For instance modelisation with pde (partial differential equations), numerical analysis of some methods and simulation are in the core of the MPE-UPMC or MMS-UPPA programs; this is linked to the talents of the local teaching team.

The idea is that students having been well trained on these chosen topics ought to

- learn quickly many other topics in applied mathematics
- understand a problem modelized with a meaninful mathematical formulation
- have ability in new computational implementations
- know that black boxes are built with something inside, which they could understand
- be able to analyze results obtained even with black boxes.

At least that is what is expected, and what is indeed observed for a large part of our students. To be honest, the program prepares them more at applying *existing* analytical tools and computational techniques than at innovating or discovering *new* tools and techniques. However, in a world of growing complexity and economic competitivness you can hardly innovate from a low scientific level, having learnt just the first basic rudiments. We do think it is usefull to have (and hope our best students are left with) sufficient and broad enough scientific bakground to be able to benefit from a specific professional formation in the industry and then enhance innovation if their company aims at beeing innovative. This does not concern all our students, some of them find jobs which do not need such competencies, but the latter are appreciated, besides their computational skills, for instance for their good organization skills, reliability, analytical minds, qualities often associated to a training in mathematical science.

We are also conscious that more qualifications are needed for a job in industry, for instance communication skills and ability to work in a team. Not minimizing the importance of these, our belief is that our competence lies in high level applied mathematics and that this training can help them discover latter new tools and techniques. Thus, the way we choose to prepare them involves a lot of up-to-date high level classrooms. Besides the long training period gives our students a good opportunity to get some of these skills. Eventually, our main justification is that the program is professionnaly recognized. The interest of companies for the students from these programs is their complementary training with a solid grasp of theory and excellent computing skills.

2.4 Contacts with industry

Contacts with industry are crucial for both students and faculty members.

For the students: during the first semester in university, career guidance workshops are organized which help them to writing curriculum vitae and cover letters, preparing to ask for positions. Some meetings or professional seminars with alumni, research engineers, professionals are held about once a week. If not numerous, these first contacts are important and valuable and make a difference with classical academic programs. Besides a few courses are taught by professionals: for UPMC, a research engineer introduces to parallel computing, another initiates students on Code_Aster (which is an Open Source software package for numerical simulation in structural mechanics). The program is completed by a long training period. For the professional programs all the internships are immersed in industry, most often in a private company or in some public research institution. The student receives some allowance for his work. An advisor from the company has defined some assignment and supervises the student's work. The student must write a master's 'thesis' or memoir, in which he or she describes his activity in the company, the tasks he has fulfilled and possibly adds some material concerning the mathematical modeling involved. At the end of the period, he gives a formal talk to present his work to the faculty committee at the university, in presence of his company advisor. The evaluation is done in concert, taking into account several features among which not only the academic skills but the student's involvment and his willingness to understand the company's requirement.

For the professors: every year, contacts with all the companies where the students have done their training period improve their knowledge of the employment market and the needs from industry. For instance a course on softwares (Fluent, Python) used in the companies receiving students has been introduced in the MMS program.

3. Critical evaluation and long-term analysis

Before we go into analyzing our programs, let us emphasize that they may evolve greatly in the forthcoming years. Indeed the situation may change because of several factors: the recent LRU act, the reforms of the school system and the debate linked to the coexistence of two different programs for the training of students, Universities and Schools of Engineering (in French *Ecoles d'ingénieurs*) and among them, the elite colleges called *Grandes Écoles*. The present situation is indeed specific in several respects.

3.1 A specific French background

Ecoles d'ingénieurs

Ingénieur is a specific diploma delivered only by the *Ecoles d'ingénieurs* together with the grade of master. It is better valued by industry than a corresponding university diploma and considered as a key to well paid jobs. The training of engineers may be provided as part of a component of a university, as in the Polytech network; even in this case, there are few connections between these specific components of a university and the classical curriculum (master's degree) taught at university. The training supposes the acquisition of generic knowledge of the kind that permits later changes of career or the pursuit of studies 'throughout life' more easily than the more specialized master's degree in science and technology. Besides, if passing the competitive entrance exams of *Grandes Écoles* requires years of preparation and knowledge of the French education system, they are seen as the country's premier path to prosperity and power. Students from working class or immigrant families often lack this knowledge and thus follow more frequently university programs. Indeed, the democratization of the French university even if partial and limited, is far greater. In France, university is open to nearly every one, with little filter at entrance besides *baccalauréat* and offers very low cost programs for students, while giving them the opportunity of achieving a high level training as our Master's program do. If more low-income students can attend *Grandes Écoles*, they will be taken among the universities' possibly best students. It is already partly true since *Grandes Écoles* try to attract the students with good potential and allow them to enter their program in their second year of training. It will be even truer if the government forces its most prestigious schools to be less elitist about the students they let in, which is the object of an actual debate. The debate might also accelerate the merging between the two different programs.

LRU

The French Universities are affected by the recent LRU, i.e. the Liberties and Responsibilities of Universities law *Loi relative aux libertés et responsabilités des universités*. Aimed at radically renewing French universities, the LRU act passed in 2007 and established the right of universities to become autonomous in budgetary matters within five years and its implementation began in earnest only in 2009. It already affects deeply the French educational system and this will be even more so in the forthcoming years. Moreover, many initiatives came lately trying to gather University and *Grandes Écoles*, in particular because research is much more active in the university laboratories than elsewhere. This is true also in Applied mathematics. In order to favor the creation of 'poles of excellence' that are aimed at improving the ranking of French universities, the LRU reform encourages competition between public institutions of education and research. The two universities in Paris and Pau may thus evolve quite differently.

Reforms

The French school system is again the subject of reforms that may have a deep impact on the level of our students. For instance, the number of hours taught in mathematics in *Lycée* could still decrease; the last year before baccalauréat it has already decreased from 9 hours in the eighties to 6 hours in the nineties and is presently 5 hours and a half (plus 2 hours optional which few students take). In primary school too, the number of teaching hours in school has lately decreased by 2 hours.

The ministry of education is also modifying deeply the structure of degree programs that prepare secondary school teachers and the content of competitive exams required for teaching in primary and secondary education; the training the teachers would receive in their original area of interest (thus in mathematics as far as we are concerned) could suffer. Last but not least concerning higher education, the reform of *agrégation*, which many students aiming at an academic career used to prepare, by postponing by one year the date of the competitive exam could dissuade the PhD students from preparing the *agrégation de mathématiques* which is a very competitive exam as well as leading the future teachers away from PhD programs in mathematics. Some fear that these reforms will have a negative impact on the French mathematical school and more generaly for the school system.

No doubt that the long range consequences on our programs will be important.

3.2. Critical evaluation

The study makes us ask ourselves some questions:

- are we (all teachers) innovative?
- do we teach the right maths?
- is it worth asking for so much work?

since there is a gap between the (high) level of mathematics taught in the program and the industrial practice (low level used in most companies). The naive answer to all questions is NO. However it might as well be YES, because that is the way we are working and cannot be too severe with ourselves! Moreover, if the answer is inbetween, it is not so easy to see how to improve on either point.

The programs that we want are first and foremost for our university students, but also suppose they fit into the faculty members' competence. As far as possible we are innovative, at least for updating the content of some courses in computer science or numerical methods. For instance, the UPMC program introduces very innovative computer tools such as GPU (Graphics Processing Unit) or recent methods such a genetic algorithms. A course in mathematical modelling could be developed, however, there are not so many scientists to teach such a course, they are already hired in other programs. For other courses, it is easier to reproduce what we have learnt or are good at, which we think is not so bad, than to spend time at trying to innovate when the usual courses are sufficiently appreciated. However, to answer the first point, we might try for instance teaching more interactively.

For what concerns the third point, about ten per cent of our students go on directly with a doctoral program. Another small part will undergo a PhD program later, once in the company. It seems worthwhile for them to ask for a competitive training. For nearly half of the group it is not, so why not start two programs differering on their ambition? However, some students who finally go on in research programs were not meant to, and would have enrolled in the less ambitious program, not having yet discovered their vocation; we are somehow proud of this democratization that the university can offer. Moreover, the percentage of students who really suffer is small, on the contrary most of them appreciate our strictness and investment, and tell us so, at least after they get their diploma.

3.3. SWOT analysis

It seems natural in the context of relation with industry to evaluate the Strengths, Weaknesses, Opportunities and Threats of our educational projects.

Strength

- high level training
- the programs have been present for over 30 years
- most of the students are satisfied after their training period, and find a job
- most training periods are well appreciated by the companies, when they can, they hire

Weaknesses

- Fluctuations in the number of students applying from one year to the other; for instance for the UPMC program, in 2008-09, only 2 students followed the course in mechanics while they were 16 of them in the following academic year. A course in computational science taught in the first year (M1) of the program, has decreased dramatically from over 80 students to about 20.
- The program is specific only in second year which somehow comes too late, some training should begin at the M1 level (internship, guidance workshop,)
- Lack of resources, concerning both teachers and means: innovative courses need time, faculty members are often asked to spend a lot of their time in administrative or practical tasks
- for the UPMC program, students have a small computer room (with AC problems) but no place of their own for working or relaxing
- if no opposition, little help from the colleagues of the math department, few people capable of managing such a program are hired
- to ensure a good running relies heavily on the 'good will' of the involved teachers, there is too much work for them
- university administration: few secretaries, 'professional insertion' is not well organized in the university (often disconnected from the teaching)
- it is difficult to have a precise employment rate because of the lack of data and of means to get them.

Opportunities

There are more and more initiatives to stimulate the interaction between mathematics and industry, in France, in Europe and worldwide: 2007 OECD report, the launching of this EIMI study … Let us list a few specific opportunities concerning France:

- Maths a venir 2009 (see http://www.maths-a-venir.org/2009/). The conference took place in decembre 2009 in Paris, over 700 participants attended the conference which was meant for non-mathematicians and aimed at making mathematics more visible and creating a stronger foundation in public awareness than before for the role and importance of mathematics. Some speakers from industry claimed a greater awareness of the need to hire students with a good training in mathematics.
- with LRU, universities will be allowed more personal initiative so that, together with a greater awareness in the teaching community of the interest of training students in

applied mathematics, and an increasing recognition of teaching commitments, more positions might be opened

- jobs should open in risk management in the context of security, industrial processes, financial portfolios, actuarial assessments, or public health and safety, environment.

- INSMI, the Institute for research in mathematics and applications at CNRS (French National Center for Scientific Research) has launched a specific program of 'maths for industry' http://www.cnrs.fr/insmi/spip.php?article234 which is an incitement to inscribe the French mathematical science community in this action while contributing to an augmented sensitization by decision-makers in industry, notably of small and medium-sized companies, of the role of Mathematics for innovation. All areas are concerned.
- For what concerns the UPMC master's degree, an initiative of Laboratoire Jacques-Louis Lions, trying to launch a project on the Franhaufer ITWM model might increase the attraction of this program (www.itwm.fraunhofer.de/). At long range, it might have a much broader impact.
- the French Applied and Industrial Mathematics Society (SMAI) has contributed lately to increase the visibility of doctoral studies in applied mathematics within companies.

Threats

- expect hard times during the forthcoming years with respect to economic returns
- lack of specialized faculty — lack of well trained students, of students enrolling in mathematical science — competition with engineering schools.

4. Conclusion

The faculty staff is passionate about the need to create a supportive environment for students. Our experiences have demonstrated that it is a demanding, maturing, and enriching process for our students. Our students need to be recruited, encouraged and mentored to be able to develop to the point of being able to successfully engage in a professional project. The last sentences mostly taken out of http://compmath.wordpress.com/ concern 'research in scientific computing in undergraduate education' but applies very well to our graduate programs, perhaps because the corresponding undergraduate programs are not demanding enough.

We believe applied mathematics is essential to increase competitiveness and innovation. We hope the numerous initiatives to promote applied mathematics at all levels and in industry will converge in particular to an increasing number of students enrolling. Concerning our programs, we think they have been successful programs, and hope they will progress. While they have evolved continuously in two different locations for thirty years, taking into account local constraints, they still look much alike and are coherent. However the above mentionned reforms in France make difficult a prospective analysis. Let us take a date and compare them in ten years!

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Educational interfaces between Mathematics and Finance: the French context

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Abstract Since the beginning of the nineties, Mathematics, and more particularly the theory of probability, have taken an increasing role in the banking and insurance industries. This motivated the authors to present here some interactions between Mathematics and Finance and their consequences at the level of training in France in these domains as well as at the level of research.

This contribution is devoted to the connections made since the mid-1980's (in France) between Finance (in fact investment banking) and the field of (not so) Applied Probability Theory. This led on the one hand to the development of "Mathematical finance" a recognized academic field of Applied mathematics and, on the other hand to the opening of many graduate programs (Master 2) in both universities and in engineering schools. To some extend it is probably the first time that an economic sector (investment banking) produced such a demand for graduate students with a high level in applied Mathematics (mostly "Stochastics", but not only). These students aim at becoming *quantitative analysts* (*Quants* in short as they are sometimes called in Europe). So, we will seek to understand these connections, mainly at an education and training level. This connection between, say in short "Finance" and "mathematics" has even become a deeply controversial debate inside and outside the mathematical community (see [PA07], [JA08]) since the subprime crisis has spread all over the economy of the world (see [PR08]).

In a first part, we will briefly describe how the connection between mathematics and Finance took place, mainly based on the so-called derivative products. In a second part we will describe how deeply it has modified the training landscape in scientific universities and engineering schools in the last ten years in France, more to some extend than in business schools.

1. Mathematics and financial markets.

The development of financial markets dedicated to derivative products in the seventies and eighties, as well as the recent emergence of energy markets, first in the United States and now in Europe, have greatly contributed to the blossoming and development of several branches of applied mathematics, and in the first place, of probability theory and stochastic calculus. The major crisis that started in 2007 and came to light with the bankruptcy of Lehman Brothers has raised many questions about the use of applied (and not so applied) mathematics in the field of Financial markets (see [EKJ08], [PR08]). However several signals emitted by the major actors of investment banking industry during the last months suggest think that if the crisis may change the way mathematics will be used, it will not have a significant consequence on the quantity of high level applied mathematics involved in this activity: switching from sophisticated hybrid derivative products to algorithmic high frequency trading, requires more insight on the recent developments in Statistics and Optimization than in Probability Theory *stricto sensu*. It remains, that fundamentally the demand for high level graduated students in Applied mathematics, specialized in "Stochastics" seems still very strong.

But let us come back to the (slightly less) recent history of the relationship between financial markets, investment banking and (applied) mathematics. The pionneering work in Mathematical Finance goes back to Bachelier in his PhD thesis “Théorie de la Spéculation” defended at the Sorbonne in 1900. However, one may reasonably consider that the true and operating connection between mathematics and financial markets really started in 1973 with the opening of the first organised market of options opened on the Chicago Board of Trade (CBOT) while Black, Scholes (and Merton) published their celebrated pricing formula. Even more important was the fact that this formula was coupled with a “dynamical hedging formula” which, roughly speaking, induced a daily need for efficient quantitative engineers from the birth to the maturity of a derivative product. This new feature is probably crucial with respect to the risk diversification through static portfolio optimization as devised by Markowicz in the 1950’s.

In this respect, the most striking fact appears to be the sudden shift of Brownian motion, Itô’s formula and stochastic differential equations from the closed circle of the purest of probabilists ... to business schools! As far as Mathematics is concerned, such an example is not unique but, in the recent era, it is particularly spectacular. Another notable specificity is related to the very nature of financial markets: it is a human activity which evolves in constant urgency and is in permanent mutation; there, modelling has, at the same time, a central and volatile position: what is true today may not be so tomorrow. The mathematician, who, by nature, is eager to solve problems, may find interesting questions there whereas, on the other side, every financial analyst is eager to obtain solutions! However, both groups may face some disillusions for, when the mathematician shall be attached, above all, to finding a rigorous and exhaustive resolution of the problem he was asked for, the financial analyst will prefer the “interpretability” of models and of their parameters (in order to get a mental representation of the universe of possible decisions) and above all the ease of implementation (explicit formulae, numerical performances,..) which is the only way to preserve his reactivity in the midst of delicate transactions (where the unity of time is the second).

The field where the interactions with Finance have been the strongest is clearly that of probability theory: stochastic calculus and Brownian motion in a first period, notably with the emergence of exotic options. Then, little by little, the increasing complexity of products, the escalation of models, the multiplication of indicators necessary to delineate the sources of risks, have led to situations where explicit computations need, at least partially, to give way to numerical methods. Two large families of methods are available, those originating from numerical analysis and those from numerical probability. Each of these two disciplines may be summarized in almost one word: partial differential equations for one, Monte Carlo method for the other (that is, the computation of a mean by massive computer simulation of random scenarios). Numerical analysis, the historical pillar of applied mathematics in France, found a new source of problems where its methods, with well-tested efficiency, might be implemented. On the other hand, numerical probability, under the impulse of quantitative Finance, has undergone an unprecedented development notably with the methods of processes characterization (in particular with the role of D. Talay in Inria). Most of the essential domains of probability theory are put to contribution, including the Malliavin calculus (the calculus of stochastic variations), which came recently to play an important, although in some respect, unexpected, role. Other domains of probability theory have known a truly rejuvenating period, in particular the optimal stopping theory via American options, or optimization theory, which plays an overwhelming role, from the Föllmer-Sondermann mean-variance hedging theory to the many calibration algorithms. However, the development of numerical probability and of simulation was not detrimental to other more theoretical aspects of probability theory since, during the last ten years, jump processes, which have been more usually associated with queuing and networks problems, have also been used today massively in financial modelling, generally in their most sophisticated aspects (the Lévy processes, see for example [CT04]).

Finally, as is often the case in this type of interaction, financial modelling led to the emergence of new problems which developed essentially in an autonomous manner within probability theory: this is notably the case for questions arisen from the generalization of the notion of arbitrage, either to spaces of more and more general processes, or to more realistic modelling of markets activities (taking into account the bid-ask spread on quotations, and discussing various bounds about managers' margins of action, and so on).

2. Training for quants in France.

The quick development of Mathematical Finance in the 1980's has had a strong impact in terms of training in applied mathematics, mainly in probability theory, under the initial impulse (among others) of N. Bouleau, N. El Karoui, L. Élie, H. Geman, J. Jacod, M. Jeanblanc, D. Lamberton, B. Lapeyre. Since the end of the eighties, the first courses of stochastic calculus oriented towards Finance appear, notably in the École nationale des ponts et chaussées, then quickly in École Polytechnique.

It is remarkable that universities also took an important role in these developments, in particular on the campus of Jussieu, as at the same time, courses specialized in Finance were taught within the probabilistic DEA's (= graduate studies courses) of the Pierre et Marie Curie and Denis Diderot universities. Success has been immediate, and so remained throughout the years: while the first promotion of the specialty Probabilités & Finance within the

DEA of Probabilités et Applications of the Pierre et Marie Curie University (in collaboration with École Polytechnique concerning the Finance theme) numbered only 5 graduates in 1991, after 2003 each promotion numbers in general more than 80 graduates. A similar dynamic is observed within University Denis Diderot. Meanwhile, the "old" terms of DEA or DESS have been changed into Master 2 (= second year of Master studies), with respective qualifications of "Research" and "Professiona". Today, if we consider only the extended Paris region (= Ile-de-France), and only these trainings, three other Masters oriented towards Mathematical Finance have developed successfully: the DEA Mathématiques Appliquées aux Sciences Économiques of Paris-Dauphine (which has become Master 2 MASEF for Mathématiques Appliquées aux Sciences Économiques et Finance) and the Master 2 Analyse & Systèmes aléatoires (orientation Finance) at the University of Marne-la-Vallée, the Master 2 Ingénierie financière at Evry-Val-d'Essonne.

The students engaged in these different trainings benefit from the strong points of the different local research teams (modelling, stochastic calculus, numerical probability, econometry, statistics ...). One may estimate roughly that between 150 and 200 students graduate each year via these "Paris region" trainings (which often work in partnership with Engineering and/or Business Schools and welcome many students from these Institutions, who are looking for a top level training in Mathematical Finance). One may thus evaluate that, just before the burst of the crisis, about 25% of the students from École polytechnique access the different professions of quantitative Finance via their mathematical knowledge and ability. Beside École polytechnique, many specialized trainings in Mathematical Finance are blossoming within Engineering Schools, such as ENPC, ENSAE, Sup'Aéro in Toulouse or Ensimag in Grenoble, which is very much ahead in this domain. More generally, most engineering schools, and (applied) Mathematics departments in the scientific universities launched some mathematical finance programs. One could have had the feeling that it was the only way to attract the students or at least to prevent them from leaving ...

Further than their specificities and their professional and/or academic orientations, these trainings are a must for the future quant ("analystes quantitatifs" in French). They are organized around three main directions: modelling (based essentially on stochastic calculus), probability and numerical analysis, optimisation, algorithmic and computer programming (see [EKP04] for further details). One must take into account that the research cells of banks in which many quants begin their carriers, function generally as providers for other services of their institution (trading room, manager, ...). Thus, they work (almost) like a "PME" (acronym for French: Petites et Moyennes Entreprises — small businesses). This is all the more true in institutions with relatively small sizes (managing societies, funds, etc). Thus, there is a real necessity to be versatile.

The impact of Mathematical Finance may also be observed within fields which are not primarily mathematically oriented. This is notably true in older training programs which generally end up studies in economics or management (DEA Banques & Finance in Paris 1, 203 in Paris-Dauphine, actuaries trainings as that of Lyon II or ENSAE, ...); likewise, in Business Schools such as HEC or ESSEC, Mathematics for Finance often takes a significant place. This illustrates the position taken by the applied mathematics culture in a priori less "quantitative" domains, such as management or sales, trading of assets or financial products.

If France takes an important position in the training of "quants", which is due in a large part to the importance traditionally given to mathematics in the training of young French people, the employment possibilities in Market Finance are obviously directly related with the importance of financial places. Today, the Europe of Finance and the employment which goes with it develop essentially in London where, each year, an increasing part of young graduates will put in practice there the theoretical know-how acquired in the Hexagone. London is not, far from that, their only destination: many of them do not hesitate to answer the call of the wide world, and go earn their spurs in New York or Tokyo, ...

The attraction capacity of these trainings for talented foreign students is quite obvious, given this most favorable context. However, it may be slowed down due to the language barrier and some adaptability to the French system, notably concerning the mode of evaluation, which, quasi exclusively, remains the resolution of problems in limited time, a criterion which is far from being universally adopted throughout the rest of the academic world, and for which foreign students are often ill-prepared. Conversely, the almost cost free of the French university system should, if it lasts, constitute a major advantage. This is a French specificity, which may seem quite astonishing in comparison with both Great Britain and United States similar trainings, for which the cost often reaches the equivalent of several tens of thousands of euros.

In a more developed version of this contribution, we will provide some examples of Master 2 programs and how academics and professionals interact to smoothen and optimize the transition from one world to another, through specific courses but also using some internships of high mathematic level. We will also spend more time on a second aspect (or level): the PhD, which clearly has not the same status in France as concerns professional insertion that it has in most other countries, especially in the Anglo-saxon world. This is partly due to a French specificity, the "Grandes Ecoles" and we will see the kind of unexpected behavior it may induce in terms of brain drain (see [PAG07]).

As a conclusion, let us note that the message has diffused among the upcoming generations and that one observes more and more undergraduate students, and students in Engineering Schools which take up — sometimes with difficulty — advanced mathematical studies with the unique goal to access the jobs of Market Finance. Whether one rejoices or deplores this fact, stochastic calculus, and by extension, probability and applied mathematics have become, during the last fifteen years “the” access road in the scientific field to the jobs of Market Finance. At the moment, it is still a “highway”, future will tell ...

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Modelling with students – a practical approach

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Abstract Nowadays mathematical modelling becomes more important at all levels of education. Students and teachers as well are engaged to treat real-life problems where the mathematical component is anything but clear at first glance. These kinds of problems play a major role in many interdisciplinary disciplines and provide a successful tool to sensitize students to mathematical issues and to convey and revise lessons. In this article, we reflect our own experiences in conducting modelling courses with students (especially secondary level and undergraduates) and describe how practical implementations can be performed. Since modelling with students has a long tradition at the University of Kaiserslautern, we thoroughly comment on the established concept and make possible forecasts. Furthermore we report on the integration of teachers into modelling projects in on-the-job training as well as fixed modules during their tertiary education.

Introduction – an overview

Mathematical modelling is one of the most important areas in mathematics education in recent years. The hope is to develop mathematical competences beyond simply applying algorithms and fixed schemes. The purpose of all modelling teaching activities is to offer students interdisciplinary lessons combined with a focus on real-life applications. Typical questions arise not only from natural sciences and engineering disciplines (such as physics, biology, chemistry, and engineering) but also from social sciences (such as economics, sports, society and politics). The latter are deliberately formulated in a more open way such that the underlying model allows for different modelling approaches.

There are a number of books and articles published that offer modelling tasks at different levels of complexity. Many suggestions and a plenty of exercises at secondary level I can, for example, be found in [Greefrath 2006, Hamacher et al. 2004, Herget 2005, Maß 2007]. Advanced literature including interesting problems suitable at secondary level II as well as undergraduate studies provide the books of [Kiehl 2006, Pesch 2002, Sonar 2001]. Unfortunately, to my knowledge, the recommended material is only available in German. However, modelling tasks usually assigned at univerity seminars will be in English since they are often based on real-world research projects proposed by a sponsor from industry or a national lab. More detailed information upon request.[1]

Since 2004 the individual states in Germany have been started with the inclusion of mathematical modelling activities within the school program. In this connection, mathematical modelling is a broad term that covers a wide range of applications during the educational process. The implementation of mathematical modelling concepts pursue the target to develop and use a range of skills that are necessary for treating real situations and facts. On an abstract level, from a mathematically didactic point of view, the well-known modelling circle (Figure 1) has been established (see [Kaiser 1996, p. 68 and Blum 1996, p.18]).

The arrows symbolize the individual subskills in the modelling process: (a) idealize, structure, simplify, (b) mathematize, translate into the language of mathematics, (c) mathematical work, operate and (d) interpret and validate. If the result is not satisfying, the steps will be iterated again — if necessary, the whole circle will be passed once again. Of course, this circle does not allow for a uniform definition of modelling skills but there is consensus that modelling competencies embrace both, the abilities as well as the willingness to convert those into action.

Nevertheless, throughout this article, the practical implementation of modelling events will be of major interest. Particularly three different modules will be introduced. The presented concepts are either already tested in practice or in preparation.

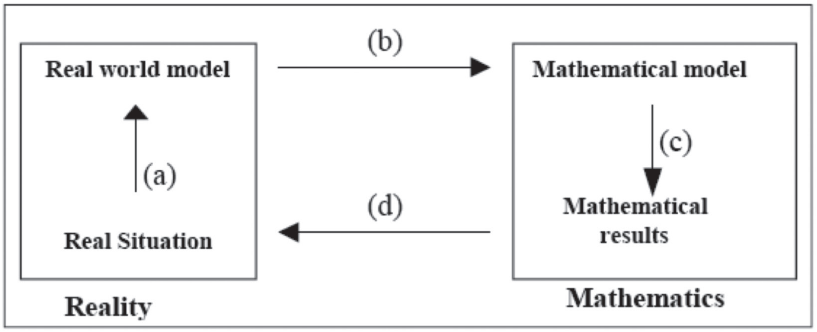

Figure 1—Modeling circle.

The overall aim in all modlues is to create a stimulating challenge so that pupils, students and teachers get motivated to start working on diverse interdisciplinary problems. We like to communicate another view of mathematics — away from being boring and incomprehensible. We want to encourage young people to recognize mathematics as an interactive science by exploiting their mathematical knowledge.

The remainder of this article is organized as follows: Section *Compact courses — a progress report* is a summary on modelling projects with pupils and students with particular emphasis on feasibility. Then, in section *Modeling lessons at schools — a proposal*, an example is shown how a practicable integration of modelling seminars into the school curriculum can be realized. Section *Special trainings for teachers — future work* deals with several ideas to offer special modelling events for teachers only.

Compact courses – a progress report

The Department of Mathematics at TU Kaiserslautern[2] has been organized so-called modelling weeks for students (secondary level) for more than 15 years.[3] Furthermore, university lessons in modelling are nowadays an inherent part in the Bachelor curricula (Bachelor of Science as well as Bachelor of Education). All modelling problems — independent of the level of difficulty — provide a wide a range of examples — from industry to society. Recent projects have included how to water a garden, how to climb a mountain, optimization of roundabouts, population growth of two-spotted ladybirds, and lighting of soccer fields. This is just a sampling of different types of projects assigned to the students. Due to good contacts to research institutions and companies, projects will be renewed at regular intervals.

Basically, we distinguish two different ways of modelling courses in Kaiserslautern: seminars for bachelor and master students for one, and modelling projects at schools for another. Both approaches have in common that students have to independently deal with an everyday life problem which seems initially far away from mathematics. Honestly, one should mention, that in the context of generating new practically relevant problems, we greatly benefit from the close cooperation between the University of Kaiserslautern and the Fraunhofer ITWM[4], also located in Kaiserslautern.

The modelling projects we employ are designed in a way, that students shall first familiarize themselves with their assigned problem and collect information and data as much as possible. Because of the intensive communication with *users* (in most cases the responsible lecturer or instructor), the students learn to specify the kernel of a problem and to gather the needed mathematical knowledge. They are able to create a mathematical model and to solve it with available skills. To interpret and to validate the solution in a generally understandable way, finally completes the so-called modelling circle (cf. Figure 1). From an educational point of view, the students develop their ability to work in a team, improve their time and project management and learn to lecture.

The first modelling seminars in Kaiserslautern have been conducted since the mid 1980s. University modelling projects are organized as teamwork (3–5 students form a group) in the following sense: The Bachelor of Science students are mixed with the Bachelor of Education students. This successful concept implies several synergy effects such as the exchange of sophisticated mathematical tools including advanced software engineering aspects as well as the interpretation of mathematical results. This way of cooperative learning has potential to be a crucial factor during the education of young teachers. Such seminars shall be installed in all curricula and particularly adjusted to the teacher trainees needs. Think of designing new modelling problems which are suitable for lessons at schools. Subsequent practical implementations represent a first elementary step towards the integration of modelling activities at schools. The inhibition threshold of dealing with modelling problems will be consequently dropped at an early stage. Apparently, self-generated performances will possibly allow for adequate passing grades later on.

Modeling weeks (or short-time school projects) have been held since 1993 (modelling weeks) and 2000 (school projects), respectively. The only differences are duration and venue. So-called modelling weeks last for one week (from Sunday to Friday, cf. table 1) and take place at youth hostels. On the contrary, short-time school projects last up to three days and are locally organized. In both cases, as one might expect, the help of teachers concerning organisatorial questions is absolutely necessary.

The purpose of such a modelling event is to offer students (and teachers as well) an intensive workshop on real world problem solving. For instance, in case of modelling weeks, the school pupils and teachers (all coming from different schools) are divided into mixed teams

Weekday	Activities
Sunday	Arrivals and presentations by instructors
Monday–Thursday	Group discussions and excursion on Wednesday afternoon
Friday	Pupils presentations

Table 1—Schedule of a modelling week.

of five pupils and two teachers. The pupils are allocated to teams on the basis of their areas of interest and mathematical expertise. The teachers are responsible for the management within the group, are a reliable partner regarding mathematical difficulties, and build a bridge to the guiding academic staff members. The week starts with a brief presentation of sevaral problems where the underlying industrial or scientific background is highlighted and motivated. Questions can be posed and every participant is allowed to select a problem of personal choice (that works incredibly well). To identify and understand the real problem may take some time. Also the development of a mathematical model of the typically non-unique mathematical problem is not that easy. Usually, to get a solution, computer programmes (Excel or more sophisticated ones) must be applied. If the simulation results do not provide a satisfactory answer to the original problem, the model has to be slightly refined and the modelling iteration process starts circling. At the end of the week, the team prepares a final presentation about the results of their work, whereby each team member has to make a personal contribution. Last but not least, the pupils produce a written report that will be published among each other. Finally, it should be noted that for Bachelor and Master students it is even possible to participate in international modelling weeks. A pioneer in this area is the European Consortium for Mathematics in Industry (ECMI),[5] followed by the Institute for Pure and Applied Mathematics (IPAM)[6] and the Mathematical Contest in Modeling (MCM)[7] — only to mention a few.

Modeling lessons at schools – a proposal

A first step towards the practical implementation of modelling courses at schools is presented. This is an in process project which will be started in the upcoming school year. The project seminar, shortly P-seminar, is recently integrated into the Bavarian curriculum.[8] You should know that the school system in Germany is decentralized, i.e. the responsibility for the German education system lies primarily with the individual states. The P-seminar is an essential component of the reform of German upper school-level that is compulsory for all students beginning with the 2009/2010 school year. This seminar emphasizes the particular importance of early and extensive vocational guidance and practical occupational support in secondary schools. Therefore focus is on a project-oriented style of learning geared towards methodological and social skills. The students should have the opportunity to get in touch with the working world and gain practice-oriented skills. Simultaneously, dealing with the concrete professional world provides impulses for their own professional career. Possible project partners are companies, public institutions or associations. In total, the P-seminar offers a unique opportunity to work on real-world projects, to acquire first professional experience, and to strengthen the cooperation that have emerged as a result.

The precise implementation could rely on two modules that are spread over 1.5 years, see table 2.

Special trainings for teachers – future work

As mentioned above, there are possibilities for teachers to get in contact with mathematical modelling.

For two years, there have been successful attempts to integrate mathematical modelling lessons into the education of prospective teachers (at least at TU Kaiserslautern). This is done either in form of seminars or as a degree thesis. With the help of these teaching methods, a clear impulse is given that mathematical modelling knowledge ought to be conveyed in schools at an early stage.

At TU Kaiserslautern, many staff members are voluntarily engaged in modelling activities — although there is no Mathematics Didactics group. But the positive feedback encourages to carry on activities with a more intensive focus on special trainings for teachers.

At present, special teachers training courses are offered to give educators and mathematics teachers more insight into the world of mathematical modelling: Either teachers take part in modelling weeks as active members (learning by doing principle) or, alternatively, workshops, talks and additional camps for teachers are organized. Naturally, we hope that the active participation in a modelling week will be officially recognized as further training soon.

In the near future, we intend to set up an innovative teacher training program. One way might be to extend the relation to the Fraunhofer Institute by launching workshops with teachers and practitioners. Thereby teachers would have the chance to get some impressions of current practical problems where mathematical know-how is combined with industrial applications.

Period	Activities
11/1	1. Vocational experience • develop professional profile • career-related excursions • prepare an application
11/2	2. Project work • presentation of possible projects • introduction to a programming language, e.g. MATLAB[8] or FREEMAT[9] • select project and start research
12/2	2. Project work • check quality of results • refinement of model • final presentation and discussion

Table 2—The modelling P-seminar.

Conclusion

Summarizing, there are plenty of ideas and ways to perform mathematical modelling, but how they should be implemented? One solution would be to install a platform — something like a highly interconnected network of schools, universities and *modelling experts* — to promote the active exchange of experiences, to communicate, to inform and so forth. Mathematical modelling is an ongoing process which solely thrives and grows through interdisciplinary cooperations. Let us all work together to that end!

Notes

1 Please send an email to goettlich@mathematik.uni-kl.de

2 http://www.mathematik.uni-kl.de/CDindex.html

3 http://wwwagtm.mathematik.uni-kl.de/agtm/home/modelling.html

4 http://www.itwm.fraunhofer.de/

5 http://www.ecmi-indmath.org/edu/index.php

6 http://www.ipam.ucla.edu/rips/

7 http://www.comap.com/undergraduate/contests/mcm/ 8 http://www.gymnasium.bayern.de/gymnasialnetz/oberstufe/seminare/p-seminar/

8 www.mathworks.com

9 Free of charge alternative, see http:// freemat.sourceforge.net/

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Mathematics Education and the Information Society

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Abstract Point of departure for this paper is that societal changes ask for adaptations of a foundational mathematics curriculum for all. This paper especially looks at the effects of information technology and globalization on the job market and employability. Here, economists observe a shift from routine tasks to non-routine tasks and a growing demand for problem solving and communication skills. But next to the need for a change towards a more problem solving and interaction, changes in the content of the curriculum will be needed as well. It is argued that the latter may concern topics such as reasoning with (models of) relations between variables, and variability. In addition it is shown that information technology may offer the means to tailor education towards these goals.

Introduction

"Where are the jobs?" asked Businessweek on the cover of its March 24, 2004 issue. At that time, the economy bounced back from a small recession, but although the economy recovered, employment did not rise. An explanation for this paradox was given by the Levy and Murnane (2006), who did an empirical study on how the job market in the U.S. developed between 1960 and 2000. Their analysis was that a crucial factor in the development of the job market is whether a task can be turned into a routine. Tasks that can be broken down into repeatable steps that hardly vary, will be consigned to computers, or will be handed over to lower-paid workers outside the U.S.[1] They speak of a routine task, when a task can be carried out by a machine, which follows formal rules that can be implemented in a computer program. This definition applies to a lot of production work, which in fact already has been taken over by machines. Mark, that the divide between routine and non-routine jobs, does not coincide with little or much education. Not all jobs that require a small amount of education can be computerized. Guiding a car through the traffic, for instance, cannot be computerized (yet).[2] In contrast, many tasks that well educated accountants and computer programmers carry out can be entrusted to dedicated software.

The issue that we want to address in this paper is, what implications such changes in the (future) job market have for a foundational mathematics curriculum. We will first look more closely at the changes in the job market under influence of globalization and informatization, how they influence the requirements for future employability in a more general sense. Then we will investigate what the implications are for the content of the mathematics curriculum by trying to get a handle on what mathematics is required in the modern workplace. Finally we will link these (preliminary) findings to ways in which information technology can help achieve those goals.

Jobs of the future

To answer the question, what capabilities will be important for future employment, we have to look at the kind of jobs that will be offering good prospects. These jobs concern non-routine tasks, which are tasks that require flexibility, creativity, problem solving skills, and complex communication skills. Autor, Levy, and Murnane (2003) refer to examples such as reacting to irregularities, improving a production process, or managing people. Empirical research in the US shows that employment involving cognitive and manual routine tasks has dropped between 1960 and 2000, while employment involving analytical and inter-active non-routine tasks has grown in the same period. This change especially concerned industries that rapidly automatized their production. Parallel to the development in industry, similar changes occurred in other areas where a strong computerization took place. This change happened on all levels of education. Jobs with a high routine character are disappearing. The jobs of the future are the ones that ask for flexibility, creativity, lifelong learning, and social skills. The latter are jobs that require communication skills, or face-to-face interaction — such as selling cars or managing people. These changes do not only affect the decline or rise specific jobs, existing jobs are changing as well. Secretaries, and bank employees for instance have got more complex tasks since word processors and ATM's have taken over the more simple tasks.

It shows that the effects of computerization and globalization overlap and reinforce each other. Routine tasks can easily be outsourced, while information technology enables a quick and easy worldwide exchange of information. The latter also makes it possible to outsource business services, such as call-centers, or the work of accountants and computer programmers. Another effect of globalization is that it forces companies to work as efficient as possible. This requires companies to immediately implement computerization and outsourcing, when it is economically profitable and strengthens the market position of the company. It also demands of the company to be on the lookout for opportunities to improve efficiency. As a consequence, working processes will have to be adapted continuously. This in turn, put high demands on the workers, who have to have a certain level of general and mathematical literacy to be able to keep up.

Educational change

Already in 1996, Kenelly pointed at the need for educational change:

> Business is a network of word processors and spreadsheets. Engineering and Industry are a maze of workstations and automated controls. Our students will have vastly different careers and we, the earlier generation must radically change the way that education prepares a significant larger part of the population for information intensive professional lifes. (Kenelly, 1996, 24)

Since then many reports and books have appeared that point to the need for educational change in response to the informatization and globalization of our society. They bear sweeping titles as, 'The Achievement Gap', or, '21st. Century Skills'. The gist of these publications is that the current education in the US does not prepare students for the competition they are going to face in the global economy. Firstly, future employees will have to compete with colleagues in other countries with similar skills who work for lower wages. Secondly, the skills that the current and future jobs require differ significantly from what even the best education offers. CEO's of large companies stress that they look for employees, 'who ask the right questions' (Wagner, 2008). In line with the findings of Levy and Murnane (2006) problem solving and communication skills come to the fore as what is asked for, while schools focus on standard procedures and conventional skills.

If we look at these developments from the perspective of mathematics education, it shows that in the aforementioned documents much emphasis is put on general skills, while little attention is given to the content of the mathematics curriculum. One of the reasons may be that one of the effects of computerization is that mathematics becomes invisible. Mathematics is disappearing in black boxes, which makes it difficult to get a clear view on what mathematical knowledge is applied at the workplace. The mathematics is hidden in integrated systems, such as spreadsheets, automatics cashiers, and automated production lines, and people who use these systems are expected to make decisions on the basis of the output of the hidden mathematical calculations. Levy en Murnane (2006, 19) point to the significance of this development: "Because of computerization, the use of abstract models now permeates many jobs and has turned many people into mathematics consumers". They mention the manager of a clothing store who uses a quantitative model to predict the future dress demand, and a truck dispatcher who uses a mathematical algorithm to determine delivery routes, as examples. Another example concerns the employee of a bakery who monitors the production of bread by means of digital data instead of the smell or looks of the bread. They point out that the computerized equipment often does the actual calculation in such cases. However, they go on to say, if the decision maker does not understand the underlying mathematics, he or she is very vulnerable to serious errors of judgment.

If we follow this line of reasoning, we may discern two seemingly conflicting tendencies. On the one hand, we appear to need less and less mathematics, since various apparatus take over a growing number of mathematical tasks. On the other hand, we develop into 'mathematics consumers', who become increasingly dependant of the quantitative information and mathematical models, which we ought to understand.

This brings us to the central question of this paper, what are the implications of the increasing informatization and globalization of our society for mathematics education? Or more specifically, which goals and contents should we aim for if we want to prepare students for the information society? In this paper, the focus is on employability. Firstly, because the impact of computerization and globalization appears to be the biggest for the workplace and employability. Secondly, because we may assume that what one needs in the workplace will encompass what one needs for everyday personal life. In this respect, we will follow Levy en Murnane's (2006) notion of mathematics consumers that ought to understand the mathematical models they base their decisions on. We may add to their point of the risk of errors of judgment, the importance of innovation — in the light of global competition, and in response to societal problems such as global warming.

Techno-mathematical literacy's

When trying to relate the goals of mathematics education to the requirements of the workplace, an additional complication is that the mathematics used at the work place is strikingly different from conventional mathematics (Hoyles & Noss, 2003; Roth, 2005). To describe this specific kind of mathematics, Hoyles and Noss (2003) coined the term 'techno-mathematical literacy's' — or TmL's for short. Those TmL's are defined as idiosyncratic forms of mathematics that are shaped by work-place practices, tasks, and tools. Acting successfully at the workplace is dependant on a combination of mathematical knowledge and contextual knowledge. They identify competencies such as,

> seeing the need to quantify, identifying and measuring key variables, representing and interpreting data (Bakker, Hoyles, Kent & Noss, 2006, 355-356),

and,

> reasoning about the models embedded in the IT system, in terms of the key relationships between product "variables" (...) and their effect on "outputs" that are visible to the customer (...) (Kent, Noss, Hoyles & Bakker, 2007, 80).

Bakker, Hoyles, Kent en Noss (2006) stress that the role of TmL's does not simply boil down to applying mathematical knowledge. In contrast to mathematical modeling — where contextual aspects are considered 'noise' — contextual knowledge is an essential element since it gives meaning to the decisions that are being made. Albeit, neither yields, that conventional mathematics is not needed or that common sense would be sufficient. For, a lack of conventional mathematical knowledge makes it hard to build, or expand, Tml's. However, students will have to be able to flexibly adapt their existing mathematical knowledge or adopt new knowledge. This asks for experience with a variety of non-canonical forms of mathematics. We may observe, however, that there is a tension between the benefits of canonical forms of mathematics for a longitudinal learning process, and the need for exploring varied and informal forms of mathematics to help students develop a kind of flexibility that may needed to develop TmL's. In a similar vein Steen (2001) observers that the work place asks for sophisticated use of elementary mathematics, while school mathematics focuses at elementary use of sophisticated mathematics. In addition, he observes a bias in favor of algebraic formulas as the preferred style of mathematics, instead of graphs, computers and the like. In connection to this he advocates, among other things, more emphasis on data analysis and geometry.

A quantitative perspective

The calculation power of computers makes it possible to process quantitative data from various sources. As a consequence, the increased use of information technology led to an increasing role for quantitative information. We may argue therefore, that students will have to be made familiar with a quantitative approach of reality. A quantitative approach requires one to discern and properties and to quantify them. This asks for understanding of what it means to measure and a notion of what a variable is. As is elucidated by Jones (1971, 335-336):

> Note that what is measured is not an *object* but a *property* or *attribute* of an object. One does not measure a table, but one may measure a table's length (...). By attribute or property, one refers to a recognized characteristic on which the objects of a set can vary. Each object in the set may be assigned a specific *value* of the attribute. The term attribute, as used here, is synonymous with the term *variable* in common scientific usage. Weight in pounds is an attribute of students. This *variable* takes on different *values* for different students--for Tom, 137; for Jeffrey, 154; for Kathleen, 118.

An important aspect of the latter is that quantitative data are understood dynamically. On the one hand this concerns 'if-then' relations, such as, 'If the room temperature rises above 20° C, the thermostat will turn off. On the other hand, this concerns dynamic dependencies between variable magnitudes, for instance, 'the heavier an object, the more difficult it is to move that objects. In addition to knowing what direction the dependency has, it often also is important to know what the co-variation looks like. Another aspect of a quantitative description of reality is that there will always be some inaccuracy and uncertainty involved. As there will be measurement errors and variation. In general students do not realize that there will always be some measurement error, and that a repeated measurement may result in a different outcome. Nor do they realize that variance is part of industrial production, even though they will be aware of the phenomenon of natural variance in nature.

We would argue that it is important for students to develop this kind of understanding, for many of the numbers they will have to work with are in fact averages, which are often the result of sampling. Furthermore, students will have to come to grips to the idea that uncertainty and inaccuracy itself are predictable — to a certain measure.

In conclusion, we may distinguish competencies, such as:

- looking though a quantitative lens
- seeing measures as values on a variable

- interpreting a situation dynamically, and considering the dependencies between the variables involved
- considering the consequences of a given intervention
- not just looking at the general tendency — increasing of decreasing — but also watch the pattern of the dependency
- basic understanding of variance, uncertainty, and measures of central tendency

We may add that information technology offers eminent educational tools for this purpose.

Information technology and instruction

A point which is, for instance, made by Kaput (Kaput & Schorr, 2007), who argues that information technology allows for new ways to come to grips with what he calls the *mathematics of change*. Modern information technology, he argues, allows for dynamic representations. Computers can show the numerical results or graphical representations of measuring activities real time on a screen, but also offer representations of simulations that can be manipulated at will. These dynamic representations in turn allow for more qualitative ways of reasoning about change, which do not require algebraic calculations. This qualitative approach may become an alternative for conventional calculus for large groups of students. In this way we may be able to make reasoning about variables accessible for those students, for whom the algebraic calculations the calculus requires constitute an unsurpassable stumble block. For a select group of students, calculus will of course keep its value. But as far as basic education for all students is concerned, this more qualitative approach seems to offer unique possibilities.

Exploratory experiments in the Netherlands showed that mathematics education along those lines might already start in primary school (Galen & Gravemeijer, 2008). This can be illustrated with the computer game 'Train Driver'. This game shows a dynamic picture of a rail track, on which a train is visualized as a moving red dot (see fig.1 in the next page).

The students can speed up or slow down the train with the arrow buttons. The speed of the train is shown by the vertical bar below right. As the train moves, the computer produces a crude graph of the speed of the train by adding a copy of that small bar every second. Thus each bar in the graph signifies a separate measurement and there is a direct connection between the speedometer and the graph. This supports the students in reasoning about change in a very direct manner. Observations showed that many Grade 5 and 6 students could reason sensibly about acceleration by referring to the differences between the bars, and about the relation between speed and distance covered by referring to an addition the

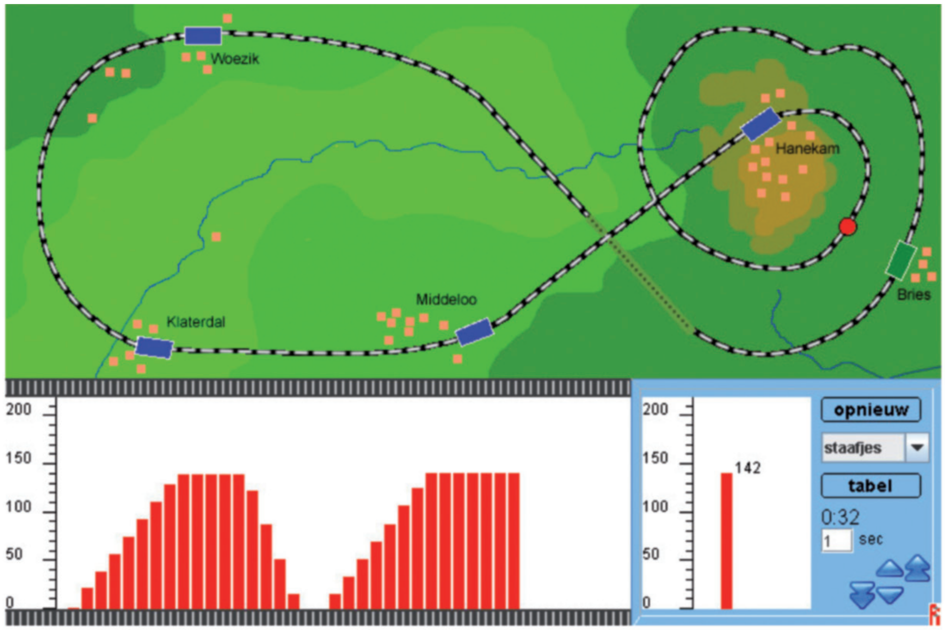

Figure 1—Train Driver.

lengths of the bars. We realize this type of reasoning is still informal and qualitative, and that the graph is not a canonical speed-time graph, but what is important here is that the students could use this dynamic representation to reason about speed, time and distance in a manner that was grounded in the experiential reality created while working with the computer game.

We see this as an example of the *mathematics of change* Kaput talks about (Kaput & Schorr, 2007). It shows how the dynamic representation in which the graph emerges out of instantaneous pictures of the 'speed bar', supports the students' reasoning. We would argue that these experiences with the train simulation also help student to come to grips with the phenomena speed and acceleration. Similar representations can of course be used to explore a variety of situations. Design research by Cobb & Gravemeijer (2006) showed that, information technology could be used for a qualitative introduction in exploratory data analysis (see also Bakker & Gravemeijer, 2004). Mark that information technology plays a double role here, on the one hand, mathematics education has to prepare students for the use of information technology, on the other hand, information technology offers the means to do so.

Conclusion

In this paper we investigated how a foundational mathematics curriculum would have to be adapted to the effects that informatization and globalization will have on employability.

An important factor appears to be a distinction between routine and non-routine tasks, as the former will be consigned to computers, or handed over to lower-paid workers. We further observed that employees exceedingly become mathematics consumers, and we argued they ought to understand the mathematics on which they base their decisions. We further touched upon the relation between mathematics in school and mathematics at the workplace, which may be described as techno-mathematical literacy's. We finally concluded that in a foundational mathematics curriculum that prepares for the future, next to problem solving and communicating, mathematical contents, such as measuring, reasoning with (models of) relations between variables, and variability, would be important. We closed by indicating how information technology might be employed for these very goals.

Notes

1. The extent to which outsourcing is a significant factor will vary in other countries; we do assume however that the transfer of jobs to computer will be a universal phenomenon.
2. In relation to this the Levy and Murnane distinguish between deductive rules that are easy to implement in a computer program, and inductive rules or pattern recognition, which is much harder to capture by computer programs.

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Mathematical modelling and tacit rationality – two intertwining kinds of professional knowledge

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Abstract In this paper the view that mathematical practice always is applications of mathematical models is problematized. We argue that there is a tacit rationality, with a broader qualitative mathematical essence, that has another origin, and another character and function. This kind of rationality is bound to personal practice and our bodily interactions with the outside world. It is also bridging the gulf between the generality of the applied general mathematical model and the specific concrete acting in the material situation.

For validation of the skilled and experienced worker the traditional school based tests of mathematical knowledge are therefore misleading and often contra-productive, and a depreciation of a workers professional skills.

Validation settings in order to evaluate for instance vocational education and adults further education efforts should have a much broader repertoire to make this kind of knowledge visible and evaluable.

There is also a need to problematize the definition of mathematical acting. By school tradition the pencil and paper activities has been the only “legal” means to show your mathematical capacity. This poor environment has been supposed to be context-free, but we argue it has its own context, and is often a doubtful preparation for future working life.

Background

In January 2007, the National center for mathematics education (NCM), at the University of Gothenburg, Sweden, was commissioned by the Government via the Validation Delegation to initiate work on the validation of adult's proficiency in mathematics.

Given the complexity of validation, and specifically its application to adult mathematical proficiency, and a great need for analyses and knowledge reviews, the work in the first instance focused on documenting the problem areas which would then provide the foundation for more operationally oriented initiatives in the future. This article builds on and develops further our analysis made in the report (Gustafsson & Mouwitz, 2008). Our analysis is relevant for a discussion of relations between formal mathematics and "mathematics" (in a broad sense) at the workplace, not least the well known difficulties students experience in their first workplace trying to apply school knowledge. This phenomenon — *the transfer problem* — is well documented in research (Evans, 2000; Wake & Williams, 2001) as well as among practitioners. Our contribution might also throw some light on issues connected to the construction of syllabuses in vocational education and training. These three problem areas are just now of extreme importance for the Swedish industry and its relation to a future formal educational system — and also, we think, for the knowledge society as a whole. We have tried to achieve a synthesis of some ideas and aspects essential for a deeper understanding of relations between formal mathematical knowledge taught and learned at school and - using a term which will be more elaborated later on — a *tacit rationality*, an important, and often overlooked, part of the professional proficiency in focus. Our discussion builds on research findings from some seminal theoretical approaches, and on policy documents concerning adult learning, as well as many years of personal experience from adult education. The synthesis reflects an approach to work as an imbedded responsible and intricate knowledge depending activity, and thus also has an ethical and epistemological dimension.

Our work has as its forerunners three commissions carried out and documented by the authors. These documents are: *Adults and Mathematics — a vital subject* (Gustafsson & Mouwitz, 2004) (commissioned by the Swedish Government); *Validation of adults proficiency — fairness in focus* (commissioned by the *Validation delegation* — a national authority established by the Swedish Government in 2004); and finally *Matematiken i yrkesprogrammens kursplaner* (*Mathematics in syllabuses for vocational training*; commissioned by Swedish National Agency for Education).

The role of mathematics at the workplace and in vocational proficiency has been a subject for research during the last 25 years (Noss, Hoyles, & Pozzi, 2000; Pozzi, Noss, & Hoyles, 1998). This research is still only scratching the surface of the intricate relationships between mathematics (in a broad sense) and its use at the workplace, and has only a minor place in mathematics education research community so far. Yet what emerges from these studies is that the relation between school mathematics and its use at the workplace is much more complicated and problematic than normally assumed (Hoyles, Noss, & Pozzi, 2001; Pozzi, et al., 1998).

In school settings there is a strong belief in the value of formal mathematics and its transferability to diverse contexts outside the school context. At the workplace however, it is not unusual, even though mathematical knowledge is highly regarded per se, to hear that school mathematics plays only a minor role in practising diverse work tasks. This peculiar paradox is a gateway to our contribution. A rather extreme and fascinating example of the impact from more informal kinds of knowledge was presented by (Cooley, 1985):

> At one aircraft company they engaged four mathematicians, all of PhD level, to attempt to define in a programme a method of drawing a afterburner of a large jet engine. This was an extremely complex shape, which they attempted to define by using Coon's Patch Surface Definitions. They spent some two years dealing with this problem and can't find any satisfactory solution. When, however, they went to the experimental workshop of the aircraft factory, they found that a skilled metal worker, together with a draughtsman had actually succeeded in drawing and making one of these. One of the mathematicians observed: "They may have succeeded in making it but they didn't understand how they did it. ... All their knowledge of the physical world about them acquired through years of making things and seeing them break and rupture, is regarded as insignificant, irrelevant ... " (Cooley, 1985; p. 171)

How, indeed, was it possible for the metal worker and the draughtsman to solve the problem? What kind of knowledge did they use? And what lessons should be learned about how to validate the professional knowledge of a skilled and experienced worker? And who should have the "definition power" to decide what "understanding" is?

One lesson we can learn at least is that there is a lot of knowledge transfer outside the school system, and also a lot of knowledge production outside the academic institutions. This a bit disturbing fact about the importance of informal learning, outside the classroom, has been highlighted by some researchers the last decade, for instance (Feutrie, 2007):

> The main problem is that this [informal] learning is:
>
> - not formalised, less codified than traditional knowledge not organised as traditional knowledge in disciplines, domains, ...

- rather unconscious hidden in action
- contextualised, attached to a specific environment
- built of elements more or less coherent, specific to an individual

Indeed this kind of knowledge is contextualised and specific to an individual. But what about school mathematics, is it free from this kind of "burdens"?

Theory and practice

Introduction

A common view is that practice is applied theory. This, however, is only one special type of practice, typical of science and technical applications. At least equally common is that practice lives its own life, without the support of scientific models and formalised language. Scientific application is, in addition, dependent on the adaptation and implementation of "silent" practice which is not formulated in the model itself.

Models and applications

The knowledge ideal of our era is theoretical and intimately related to the concept of a model. A model is an abstraction formulated with the support of one or more examples. The modelling phase involves removing the concrete which is considered to be unnecessarily complex and also the distracting deviations. A model thus always represents a *loss of reality* and implicitly a *choice of view* since it is based on a view of what exists, its structures and connections, how these interact, and what is considered to be essential. For instance are both Newtons and Einsteins physics mathematical models, but the view mediating between mathematical theory per se and reality are quite different.

A model is perceived as general, i.e. it claims applicability to a variety of new situations, it should be able to approximate reality, and be applicable to the complexity of specific cases in the future, as well as able to explain or forecast concrete future events and the success of constructions. A model is explicitly formulated, e.g. mathematically formalised, to enable its meaning to be communicated through education.

Since the model should be able to explain the complexity it has been extracted from, some practical complications arise. In many cases, successive adaptation between the model and the individual case is required for its application to be possible. Sometimes a real situation creates such intensive "resistance" that the model must be revised or rejected. If the problem has to be resolved quickly, the model must be replaced by the hands-on knowledge and skills of the labour force, as when an unanticipated error suddenly occurs, for example, with a nuclear power facility. One example is a rather serious "incident" at a Swedish nuclear power setting some years ago, that had not been foreseen neither by the formal operator educators nor by the simulator devise, and therefore also nothing in the instruction books how to handle it.

Practice and analogy

Practical proficiency developed through more or less practical actions does not fit comfortably into the model domain. The boat builder, the carpenter and the sheet metal worker do not develop theoretical models in the same way as a physicist, chemist or economist does. The proficiency that a tradesman possesses is of a more analogical type. The analogies are not abstract, but rather consist of a number of concrete examples, analogically connected with each other from the practitioner's repertoire of past experience. In this way knowledge becomes highly personal and often unformulated. The examples may retain their complexity. Each new concrete situation is compared and related to earlier concrete examples and these examples have more or less general applicability without claiming the generality of a model. Analogical proficiency is related to the person, situation and complexity, and is not easily transferred through formal education. Instead it must be demonstrated rather than formulated, and the classical method of conveying such proficiency is through a master-apprentice system (Polanyi, 1998).

Theoretical knowledge is largely dependent on practical proficiency, formalised knowledge comes to life and becomes meaningful in the confrontation with the concrete. On the other hand, practical proficiency can to some extent be transformed into theoretical models or even be replaced by them.

Use of mathematics in the school world

In school situations, attempts are often made to solve mathematical problems of a purely theoretical nature, i.e. they deal with a world of mathematical concepts. Sometimes attempts are made to link these to reality, but these become "cosmetic" since not only the starting point, but also the aim, is to solve what is essentially a mathematical problem. Practical reality is thus used to illustrate a *theoretical* problem, as opposed to formulating a practical one. It is also common that focus is put on pupils demonstrating a particular theoretical method of solving a problem, a method which in the problem context is perhaps unnecessarily advanced and cumbersome.

It is also worth considering that this practical reality does not exist as such, but only as an artificial interpretation, a virtual and theoretical "school reality". The pupils' real practice is to sit at their desks, and satisfy the specific requirements on how theoretical knowledge should be presented *orally* and in *writing*. Sometimes concrete aids such as plastic or wooden cubes are used. The aim of this, however, is also abstract, and you only "touch down" in the concrete world for a brief landing. Concrete "tools" are not used to process a concrete reality, but instead to represent a theoretical activity. The actual arrangement of the cubes has no practical relevance, and they are discarded as soon as the theoretical problem is solved.

Other types of aids such as calculators and computers are different in kind, as they are not used to illustrate theoretical reasoning, but as tools to replace the *person* making the calculations. Computers can with greater speed and precision in their calculations carry out theoretical work that earlier required significant brainpower. In such cases, however, the theoretical school domain also determines the nature and purpose of the activity.

Despite claims for generality, it can be argued that theoretical mathematical education is just as context dependent as other mathematical activities in vocational life. A strong indication that this is the case is the relative helplessness that people with purely theoretical backgrounds initially demonstrate at a workplace. New aims, methods, strategies and evaluations of results must be identified and internalised, much that was valued and encouraged in the school environment lacks to varying degrees immediate relevance at the workplace.

The role of mathematics in practical proficiency

The aim of mathematics in practical proficiency is more instrumental. Now mathematics is the intermediary: not just the problem, but also its solution is of a practical nature. From this, it follows that the mathematical reasoning most often carried out is in the form of simple rules and approximations. The heights of mathematical theory, advanced use of methods or extreme precision is of little relevance or value in solving practical problems. Here there is a *rationality of practice* which is just as effective as the theoretical rationality used for theoretical problem solving. In many cases, problems are solved on the spot immediately, in physical interaction with the surroundings. Withdrawing to a different setting to carry out calculations becomes both cumbersome, time- consuming, costly and unnecessary.

In the practical application of mathematics, there is no need for mathematical proof or internal theoretical consistency, practical usability is the criterion for "truth" and relevance of the methods used.

In addition to mathematics in the form of rules and methods in practical proficiency, there are also mathematical models incorporated in e.g. computer programs. One example mentioned[I] is how a successful sheet metal worker today must both master a long established trade tradition, and at the same time understand how to handle a computer and various drawing and spreadsheet programs. Usability in this context is also the point, not the underlying mathematical theory in the software.

Both as regards validation and instruction of adults with vocational experience, it is important to take into account different types of practical mathematical proficiency. Here we provide some examples:

- Knowledge of the rule of thumb type, which from a mathematical viewpoint can be regarded as a special case or an approximate application of theories or methods, but which historically very likely has another origin.
- A general belief in the usability of mathematics for practical problem solving.
- The ability to make judgements and realistic requirements for precision in the use of numbers and forms.
- Qualitative competencies, e.g. the ability to reason using scales and proportions, or the capacity to represent figures in three-dimensional form.
- Knowledge of calculations and algorithms, i.e. knowledge of how to handle formulae and carry out calculations.
- Knowledge of how mathematical models can be applied, and the calculations for using these.
- Knowledge of different software with mathematical content, and how the programmes can be used e.g. CAD and calculation programmes.
- Knowledge of different forms of representation and how they are to be interpreted, e.g. linear and pie-charts, tables and formulae.

In many cases instruction can be linked to the proficiency the adult already possesses; for instance working as a carpenter with the number 1.414[2] "opens the window" to Pythagoras' theorem, linear models and irrational numbers in a possible theoretical education process.

Another form of "window opening" is when a dilemma occurs, which ordinary rules of thumb cannot resolve, but where a more general method could provide a solution (Noss, 2002). Practical proficiency, however, is most often intimately related to its practical application and traditional theoretical validation takes the adult back into a "school context" which can be both confusing and somewhat humiliating.

As said practical proficiency is person-related and often forms a part of the adult's identity. Knowing one's job is a source of self-esteem and vocational pride, and leads to the desire to do a "good job" which has both an aesthetic and ethical dimension: taking responsibility for ensuring that the result is good, and corresponds to the customer's or employer's quality expectations.

Loosing one's job can lead to an identity crisis, which is further aggravated if the adult's vocational proficiency is not identified, validated and taken advantage of in future educational or vocational situations.

Much practical work is carried out in teams where communication and the ability to cooperate is an important competence. Sometimes joint initiatives are taken putting high demands on discipline, planning and coordination. It is also possible to see that the tools used "speak to" the user and vice versa. The tool becomes an extension of the body in a continuous interplay with the work situation. In some industries, there is also a master-apprentice trainee period, or where a new employee merely functions as an observer, and the person with experience demonstrates and talks whilst the trainee imitates and puts questions. The above are important aspects of practical proficiency, often involving some mathematical content, aspects which have very low priority in "school mathematics". In practical application in the real world, thought and action, tools and materials, quality and responsibility, identity and co-ordination together form an integral whole.

Propositional knowledge and praxis knowledge

Background

In Arbetslivscentrum (The Swedish centre for working life) during the 1970s, there was an intensive discussion on the meaning of vocational proficiency in relation to contemporary research into working life at that time. The latter basically focused on research into qualifications, i.e. research into the qualifications an individual needed to be able to carry out a specific work task. In the first instance, the findings showed that vocational proficiency appeared to be an application of specific advanced theoretical knowledge. Attempts to theoretically describe different work tasks produced, however, only marginal success, as they were often misleading or counterproductive.

The skill and familiarity typical of well-established vocational proficiency appeared to be quite different from what could be "caught" in theoretically formulated models and rule systems. This insight gradually led to the development of a completely new research area, Yrkeskunnande och Teknologi (Skill and Technology) under the supervision of professor Bo Göranzon at KTH (The Royal institute of technology). A number of philosophers, amongst others, Bengt Molander, Tore Nordenstam and Kjell S. Johannessen at the same time worked on trying to analyse the underlying theoretical knowledge complex. The latter in particular has had a major impact on this research and this section is primarily based on this analysis (Johannessen, 1988, 1999). It may be worth mentioning that other research environments, which from completely different theoretical and methodological starting points, focusing on the relationship between theoretical and practical proficiency, have come to conclusions which essentially coincide with the analysis we present here. Examples of this are narrative research, activity theory, situated learning and socio-cultural theories.

Praxis knowledge and mode of articulation

Fundamental to the analysis of vocational proficiency is the concept of *praxis*, which was inspired by the later work of the philosopher Wittgenstein. Based on this there is, in this tradition a discussion on how concepts are formed, used and transferred. The primary means by which vocational proficiency is expressed is through its practical application and not through description. This is why it is sometimes tempting to claim that vocational proficiency is "tacit" and thus necessarily "owned" by a specific individual and not transferable to others. But some elements of vocational proficiency can be articulated through language, and others by modes other than the purely verbal.

To demonstrate an appropriate practice of a profession is the specific mode of articulation for vocational proficiency. Proficiency can thus be handed down from one generation to another, i.e. transmitted, but in the first instance this does not take place theoretically. Instead it occurs in concrete working situations where the expert through concrete examples shows how the work is to be carried out. A typical form for such generational transmission is the traditional master- apprentice relationship.

The more abstract and formalised a language is, the more inappropriate it becomes as the mode for articulating descriptions of practical vocational proficiency, which in the first instance are based on personal experiences and examples from concrete and complex working situations. Since the mathematics taught in the formal education system in school and university has this characteristic, validation thus faces a special set of problems. If the validation criteria are based on such a view of mathematics, mathematical praxis knowledge will remain invisible.

Tacit knowledge is thus not absolute, its scope is dependent on which mode of articulation is considered legitimate. In a society increasingly permeated by abstract formalised language, praxis knowledge tends to be marginalised, not only is it "silent", but it is also "silenced". There is an undoubted risk that mathematical proficiency embedded in praxis is not recognised. We thus raise the question of whether such mathematical proficiency could possibly be articulated and validated by means of modes other than traditional mathematics tests in school. Such a validation instrument would be of great value both from an economic perspective and an individual perspective, in terms of the individual's self-esteem and vocational pride.

Propositional knowledge, practical knowledge and knowledge by familiarity

In connection with the use of language in science and bureaucracy, and its application in the formal education system, knowledge has increasingly come to be identified as what is expressed through language. The requirement that knowledge can be formulated linguistically has become a necessary element in all assessments of knowledge. Showing that one "knows" has become equivalent to formulating statements that can be verified. This is in contrast to praxis knowledge where proficiency is demonstrated by carrying out a practical task leading to the desired result.

Propositional knowledge transmitted via verbal communication, in writing and orally, dominates the formal education system. As a contrast praxis knowledge requires other modes of articulation and is transmitted by other forms of human interaction.

Praxis knowledge itself can be viewed from two different perspectives: as *knowledge by familiarity* and as *practical knowledge*. The former is about the degree of familiarity the individual has with the nature of the environment, and the latter is about the individual's capacity to act successfully in this environment, e.g. a carpenter's familiarity with materials and tools and skills in applying these in practice. Praxis knowledge has also aesthetic and moral elements, in addition to the more factual and functional, e.g. the carpenter's desire and ability to do a "good job" which should also look "nice". Praxis knowledge is closely related to self-esteem and vocational pride, which further strengthens the humiliation that many vocationally experienced adults feel when tests are used to measure "impersonal" propositional knowledge.

Tacit rationality as a non-formal intentional acting

Praxis knowledge has indeed many rational elements. It relies, for instance, on doing a set of adjustments in the right order, to put together a complicated machine with all parts in the right place, or connecting an electrical network correctly. In new situations this can't be a routinized behaviour. Instead it's a kind of rational not formalized, intentional, and often unconscious, acting. We have coined the term *tacit rationality* to pay attention to this particular form of knowledge.

If we focus on a test situation in school contexts, the approved mode to articulate an underlying rationality is traditionally restricted to the "tools" pencil and paper (and some times orally), but in working life settings rational acting is expressed through a broad repertoire of tools and materials. If the "answer" is wrong the costs will often be extensive in many aspects. Indeed the poor context in school settings is possible as an ideal because the educational system has by tradition the authority to define the nature of and to decide what counts as knowledge (FitzSimons, 2002; Zevenbergen & Zevenbergen, 2009). Behind this idea of the almost empty classroom is often a non-formulated idea that a poor context is good for abstraction. Due to its character as not being clearly worded, tacit rationality becomes invisible and therefore neglected in school contexts. The test situation accepted reveals indirectly also the paradigm defining knowledge as such, and how knowledge should be visible and possible to learn.

Tacit rationality includes, of course, some general cognitive abilities. Partly as a consequence of its embeddedness in traditions, culture, artefacts and action it's constituted by our experiences and manifests itself as *knowledge by familiarity* and as *practical knowledge*. Some of these manifestations are rather mathematics-alike, but not as mathematical knowledge in a traditional sense. The craftsmen that tesselated the walls and ceiling of Al Hambra where not mathematicians, and did not use mathematics, but their craftsmanship can be thoroughly analyzed from a pure mathematical perspective. In *Finding moonshine: a mathematician's journey through symmetry* Marcus Du Sautoy (Du Sautoy, 2008) gives a striking example of the power of praxis knowledge. In Al Hambra there is 17 different kinds of symmetries in the tilings. No less and no more. It took the science of mathematics 800 years to developed methods by which it was possible to strictly prove that there theoretically couldn't be more than these 17 different underlying symmetries. Another, perhaps even more striking, example is the discovery of quasi-crystalline Penrose patterns in Arabic tilings 500 years before they were discovered and described by mathematicians in the West (Lu & Steinhardt, 2007).

Finally it's important to emphasize the fact that tacit rationality isn't restricted only to traditional trades. It's, as mentioned earlier, equally significant, in highly specialized and academic occupations as well(Roth, 2003; Vergnaud, 2000).

Propositional knowledge in formal education

Typical of propositional knowledge is that terms are defined by using other terms. Ultimately, one must move beyond the language aspect, and language then represents a special form of action in a broader context of other actions, which provide the specific language utterance with meaning and relevance. Such knowledge statements are thus context dependent, the meaning of the statement is dependent on non-linguistic foundations. Trying to hand over, or transfer propositional knowledge is thus in practice much more difficult than the formal educational system is prepared to recognise: the result is that the newly qualified engineer is generally quite helpless when starting at a technology intensive workplace despite having a solid grounding in mathematical-technical education.

As has already been mentioned it is not the case that knowledge in the formal educational system is "without context" and thus applicable everywhere, instead this knowledge is permeated by a specific school context which gives the propositional knowledge meaning and relevance. Statements of a mathematical nature learnt in a school environment also derive their meaning from this environment, e.g. depending on how tests in mathematics are designed, what examples textbooks highlight, the particular interests of the teacher and how grading is carried out. Students who take these preconceptions of mathematics to their first workplace remain virtually helpless until their preconceptions take on a new meaning in the praxis of specific workplaces. Parts of the propositional knowledge acquired may remain meaningless in their new context, and there may also be substantial praxis knowledge which remains "tacit" since it is not covered by the propositional statements learned. Moving from a theoretical education to vocational praxis can thus be as precarious as taking the other route.

Model thinking and analogical thinking

Modern vocational life often consists of a mixture of propositional knowledge in the form of models and praxis knowledge with a more traditional trade background. One example is that of sheet metal work, where on the one hand, work is carried out using software programs for designing different constructions, and on the other hand the use of praxis knowledge transferred over many generations. In the first case, the starting point is a mathematical-geometric model, and practice involves the application of this model. In the second case, the thinking is more analogical: in the absence of an explicitly formulated model, the work is instead guided by experience from the use of earlier examples, which are assumed to have analogue structures. Designing a bend in a ventilation duct for a specific building is a unique task, but earlier designs for other houses may be sufficiently similar to provide guidance in the new task.

Model thinking and analogical thinking represent two thinking styles which can very well come into conflict with each other. In many cases model thinking is the winner in such conflicts, and this can lead to a loss of praxis knowledge in e.g. a company. This applies particularly where there is a generational change and the importance of hidden praxis knowledge becomes evident. A new group of practitioners, even though highly educated, may not be able to replace the many years of praxis knowledge accumulated over time.

Model thinking represents a form of propositional knowledge which itself must be based on praxis. For example, we can take a software program that produces drawings of what the parts in a non-linear ventilation duct should look like. The results from using the programme must be interpreted, modified and supplemented in the light of the concrete situation, e.g. depending on the characteristics of the sheet metal to be used, access to usable tools and the conditions specific to the construction of the building.

Mathematical proficiency at the workplace

It is evident that the use of mathematical models in the form of programmes or formulae requires a degree of mathematical proficiency. This proficiency is similar to propositional knowledge, but also contains elements of praxis knowledge. Propositional knowledge is by definition formulated in abstract terms, and a special form of praxis knowledge is required for its interpretation and application in specific practical situations. In such contexts, mathematics is usually viewed as instrumental, i.e. the focus is not on mathematics as a subject per se, but only as a tool for solving practical problems.

An interesting question is the extent to which mathematical proficiency is also embedded in the analogical thinking that typifies the more trade-like aspects of vocational proficiency. The ability of a sheet metal worker to recognise that a desired construction is similar to something he has done before requires some form of ability to understand similarities and differences between two geometrical structures in three dimensions. Such recognition gained through experience should be of great relevance, not just in the sheet metal trade, but also in many other occupations with similar demands.

Essentially the vocationally active person thus has, or needs, three types of mathematical proficiency:

- Propositional knowledge enabling formulae and software to be used correctly in mathematical terms, e.g. solving for a specific variable from a formula. This knowledge is similar in nature to "school knowledge", but often has a highly instrumental orientation.

- Praxis knowledge of how programmes with extensive mathematical content and formulae should be handled, interpreted and applied in relation to concrete situations particularly in one's own occupation. This praxis knowledge creates the necessary bridge between theory and practice, and gives propositional knowledge its meaning and relevance. This knowledge is developed in praxis and usually acquired at the workplace.
- Praxis knowledge representing analogical trade thinking. The knowledge is based on seeing analogies between different examples. This type of knowledge must also be developed in praxis at the workplace, and often contains hidden mathematical proficiency.

As regards validation and also the formulation of qualification requirements, focus is usually put on the first type of mathematical proficiency i.e. "school knowledge". The two types of praxis knowledge — practical knowledge and knowledge by familiarity — are on the other hand often neglected, this is partly due to the fact that they are not formulated and are perceived as having lower value. If conceptual tools do not exist for making a deeper analysis of knowledge, then praxis knowledge will also remain invisible and unknown. Praxis knowledge is also in contrast to propositional knowledge personal, whilst instruments for validation and assessing qualifications are generally of a more abstract and impersonal nature. Vocational proficiency is closely connected with questions about identity and self-esteem, and an impersonal test that only recognizes school knowledge may have an overwhelmingly negative impact on a person's desire and ability to develop and advance.

Concluding remarks

Mathematics as a subject provides for many a highly emotive experience connected with different types of negative experiences from the school environment. At the same time mathematics functions as an instrument for society to determine access to many advanced vocational education programmes with initial knowledge requirements based on school mathematics. A more flexible and multifaceted view of what counts as mathematics could in this context serve the dual purpose of not only recognising an individual's praxis knowledge in mathematics in terms other than school mathematics, but also serve as an instrument for society to formulate more realistic and specific initial knowledge requirements in many occupations and vocational education programmes.

Notes

1 (Gustafsson & Mouwitz, 2008; p. 12-13).

2 (Gustafsson & Mouwitz, 2008; p. 5)

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Linking professional experiences with academic knowledge
The construction of statistical concepts by sale managers apprentices

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Abstract As it builds a partnership between the school and the firm, the French alternance system helps us to link mathematics to students' professional experience. In this paper we will describe a pedagogical device we experimented with manager apprentices in order to make them confront the different conceptualisations they built through their multiple experiences, at school and at work.

To learn between school and workplace

A major well-known difficulty in vocational education is to help students to link professional experience to theory learned at school (see for example Hahn, 2000). As the constitutive role of cultural practices on cognition is now widely recognised (Hatano and Wertsch, 2001), in order to enhance learning the aim is to confront students with epistemologically rich problems. These problems should be not only inspired by "real" situations but also familiar, part of their field of experience (Boero and Douek, 2008), and related to the community of practice (Lave and Wenger, 1991) at work.

However, linking disciplinary knowledge with work practices is not an easy task as work situations are always multidisciplinary. This is a major problem encountered by teachers, especially in higher education where the use of sophisticated technology makes mathematics mostly invisible (Strässer, 2000). Although referring to different theoretical frameworks, some authors agree that learning appears through a dialectical process — between conceptualisations in action, embedded in the setting in which they occur and theories or "scientific" concepts — whether they stress the continuity between them (Noss and Hoyles, 2000) or the discontinuity (Pastré, Vergnaud, Mayen, 2006). A dialectical learning process implies the construction of an internal space where knowledge of different levels of generalisation play/work/compete together (Brossard, 2008). From our point of view, this implies that the learner should be involved in the construction of the problem. But, on the other hand, how can we be sure that a problem enacted from the learner's personal experience would fit with the school's aims and help the learner to construct the knowledge that s/he is supposed to learn? In vocational and professional school education, curricula often force teachers to follow a prescribed pathway, leaving little room for such activities, when in the field of adult education, the problem differs: the pressure of curriculum is less important and learners usually already have work experience.

French Alternance system

In this system 16 to 26 years old students sign an apprenticeship contract with a firm and study part time at school in order to prepare a vocational degree. This concerns all types of qualifications from lower levels up to highly skilled jobs such as engineers and managers. As it builds a partnership between the school and the firm, this system encourages us to consider the relationship between them. These two organisations share the same goal of educating the new generation of professionals, although their cultures and the way they consider knowledge are so different that it is a real challenge to help the learner make productive associations between them.

We have been studying the positive effects of the French “alternance” system on learning for some years now (Hahn, 2000, Hahn et al, 2005). As adult education, it offers the opportunity to link school content to students’ professional experience, and this system creates a two-way relationship, from school to firm and from firm to school. Alternance may be described as a boundary space, an in-between place where sense making should become easier.

Managers and Statistics

Vocational curricula often include statistics because being able to handle data is an important competency in many workplaces and in everyday life. Statistics is supposed to be more easily linked to out-of-school practices. Nevertheless, when we work with our post-graduate business students on problems they have constructed, these problems rarely include a statistical dimension, not even a basic one. In fact, most decisions can be made without considering any statistical methods. This is what usually happens in the field, although statistics certainly provide much insight into many corporate questions and issues (Dassonville and Hahn, 2002).

The question is not only to help professionals to improve their understanding of the statistical tools they use in the workplace (see for example Noss, Hoyles and Pozzi, 2002, Bakker et al., 2008) but also help them to “see” the statistics that could be useful. Therefore, we need to “enculturate” students into statistical reasoning (Pfannkuch, 2005) so that, as managers, they will be able to improve their decision-making processes by using statistical methods.

The decision-making process is not only a question of processing information and finding patterns in observed data. Our rationality is shaped by the values and beliefs that are raised through our participation in different communities and of which we are mostly unaware. Not all of this tacit knowledge can be codified and it shapes not only the means but also the evaluation of ends (Polanyi, 1966). We must also consider that scientific rationality as it is developed at school — i.e. explicit logical reasoning — co-exists with other social forms of rationality.

The way the learner solves a problem depends on what the problem means to her/him. The aim of the experiment we will now describe was to find out how these different forms of rationality shaped a statistical decision-making problem and the use of statistical concepts by students.

The experiment

The device

We designed a 4-step pedagogical device focussing on the concept of variation. This concept is central in management (e.g., to consider investments' volatility in finance, segmentation in marketing, etc.), as in many other fields, and it is claimed that it is very hard to deal with at any age or level (Garfield & Ben Zvi, 2005).

We had planned to study how the device mediated the construction of statistical concepts and how students' personal experiences shaped their decisions. The device was based on a mini case study about a firm ("T" sells office equipment) that is hiring a sales manager. Students were asked to choose which of three sales areas they would prefer to manage. They had to make their decision according to information they were given on a group of customers (businesses) located in each area (different group sizes in each area). This information consisted of an Excel File with a set comprised of one qualitative variable (date of first purchase) and five quantitative variables: previous year's amount of sales (with the client), distance (from the client to "T" location), staff (of the client), evaluation of commercial relation (a grade from 0 to 10), number of different items (sold to the client last year).

First, each student was provided individually with the distribution of one variable from one sales area, and was subsequently asked to write a brief summary of the information he or she received (step 1). Next, we formed groups of three students, with each of them having studied the same variable in a different area, and we asked each group to summarise the information it had received by comparing the three distributions of the same variable in the three different samples (step 2). Therefore they were able to consider two types of variability, within a group and between groups (Garfield & Ben Zvi, 2005). Then we built new groups of 6 students, each of them having different information about one variable (among 6) in all three sales areas[1] (step 3). Last we asked the students (in groups of 3, as in step 2) to make a final decision about the area they would choose by analysing all available data simultaneously (step 4).

We assumed that step 1 and 2 were closer to school practice and step 3 and 4 closer to professional practice (step 3 was more typically a situation faced by a salesperson, step 4 a situation faced by a manager). We also expected that passing from step 1 to 2 but also from step 3 to 4, would lead students to move from a local (data seen as a collection of individuals) to a global point of view and thus to the construction of the concept of distribution (Makar & Confrey, 2005).

The population

The device was first tested with 36 postgraduate students engaged in a 3-year master's level program including several periods of internship. Most of these students (n = 34) completed a 2-year commerce degree prior to entering the masters program. They all had previously undertaken at least a basic statistic course and had work experience, most of them as a salesperson.

Two questionnaires were given to the students at the beginning of the school year by the teacher in charge of the Business course. The first questionnaire focussed on their former experiences and professional project, the second on their statistical knowledge: given a list of statistical concepts they were asked if they had learned these at school and if they knew how to use them.

The experiment took place during the two first sessions (three hours each) of a compulsory statistics course during the first year. We split the group into 2 subgroups located in 2 different classrooms. Each subgroup followed the same procedure. At each step the students were able to use their personal calculator or computer.

The debates between students during steps 2, 3 and 4 were audio-taped; in addition we took field notes and collected reports written by the students at each step. Students were told that we wanted to keep track of the discussions in order to help to adapt the course to their needs, what I actually did. They were allowed to stop the recorder if they wanted. Some did occasionally during breaks.

Results and discussion

We will describe some results concerning the quantitative variables (30 students studied these variables, 6 per variable at the first step). Here we will mostly focus on the use of the mean, median and standard deviation. Table1 compares, for 30 students out of 36[2], answers to the questionnaire (what students claim to know) and observations we made at step1. What students claimed seems coherent with they were able to do — although they seem to underestimate their capacity to calculate a mean and a median.

Although very few students calculated or used variation indicators, many of them expressed an intuitive conception of variation as evidenced by their references to the shape of distribution. As we suspected, as it is a natural process (Hammerman & Rubin, 2004), many students divided the data into subgroups. Nevertheless we found two different types of strategies. At this stage, 19 students out of 30 built subgroups based on the distribution (use of mean or median, of discontinuities in the data set) and 10 built subgroups referring to a "so-

Table 1	Questionnaire					Step 1	
	Learned at School	Met out-of-school	Know how to calculate	Know how to get the result from spreadsheet	Know how to use it out- of-school	Calculated (without mistake)	Calculated (wrongly)
Average	30	16	16	20	18	24	2
Median	18	5	6	7	6	10	5
Standard deviation	24	2	1	4	2	1	3

Table 1

cial norm": the decimal system (hundreds), economic typology (size of firms), or a "school norm" (a good grade must be 5 or above).

Among the hypotheses formulated from our literature review, we forecasted that the students would refer more to school knowledge at step 1 and 2 than at step 3. Indeed, at step 1, many students tried to apply the statistical knowledge learned at school and calculated as many indicators as they could. Nevertheless, many of them already integrated elements of their commercial experience at this stage. Strategies seemed to depend on context and not only on the distribution of numbers: similar strategies were used for the same variable (for example all students who dealt with sales and distance calculated percentages for subgroups). We mostly found references to the context for sales and distance (most important for a salesperson, according to our interviews with professionals).

Our second hypothesis was that steps 2 and 4 would help them to move from a local to a global conception — in particular by using multiplicative strategies (as sample sizes were different). That was obvious in step 2: all students who had made lists or ranking of customers abandoned them. This seems to indicate a shift to a global point of view, although they used few indicators: they kept indicators when they were able to agree on a common interpretation. They dropped indicators that they could not make sense of. This is coherent with previous observations that students have difficulties with spontaneous use of indicators (Konold & Pollatsek, 2002) and, when they calculate indicators, they do not use common sense in solving the problems (Bakker, 2004).

Here is an extract of the discussion in one of the two groups dealing with grade:

Student 1: you did not calculate the average[3] for your area? For your 20 customers, how many?

S2	I told you that there were 10 [customers who gave a grade under 5] out of 20
S1	Yes, but the total average?
S2	but I told you, it is 10
S1	but the average grade, how much?
S2:	I told you
S1	you did not calculate it
S3:	the addition of grades divided by the number of grades
S2:	oh this, I did not do it.
S3:	the average is 6.76 in my area
S3:	Is this good or not?
S3:	This is not so simple ... the average is 6.76 ...
S2:	But how many have a grade above 5, this I am sure you did not do it?
S3:	no, I did not
S1:	in my area, there are 31 customers, the general average is 5
S2:	exactly 5?
S1:	yes those whose business relationship is under 5 are 13, that represents 42%, those whose relationship is above 5 are 18, that represents 58%, then the end result is positive but not good enough.
S3:	In my area, 11 customers reach average 5, then we must improve commercial relationship and try to find out during appointments what they really need and adapt commercial policy to improve their satisfaction. [...]
S1:	standard deviation that means the repartition of grades, if they are close or far, is 2,006
S3:	That's nice what you did, because me I did not think about standard deviation, no I did not think about it but, standard deviation uh
S2	better not think about it because there is no use for it

We could claim that the 3 students were at different stages of understanding but it seems to us that they are not solving the same problem. The problem is shaped by the objective they set themselves: student 1 seems to plan a comparative study in order to understand differences between areas, student 2 is solving a school problem to answer the teacher's request and student 3 wants to answer the question "which is the best area?" at this stage already. Then of course strategies differ. Student 1 referred to what seems to be a scientific rationality: to compare distribution and use statistical concepts. Student 2's actions are based on technical rationality (Schön, 1996): he applied techniques learned at school but does not know how he can use the result to answer a question. Student 3's reasoning is pragmatic and he used a simple intuitive strategy.

The second group dealing with grade followed a similar path, although they all calculated the mean at step1:[4]

S1: For area A, the mean is 4.3

S2: Forme,5

S3: And for me 6.76

S2: Let me do the calculation again ... yes it is 5, Ok

S3: Ok, then what do you want to do?

S1: The number of customers under 5

S2: It is important?

S2: V ery important

S1: But you must understand! There are 20 customers in my area, amongthem, 10 are not satisfied at all

S2: Ok

S3: They do not reach average

S1: Not average = not satisfied

S2: Ok

S1: Then this is global

[...]

S2: [*he gives the standard deviation of the distribution of grades in sector A explains how to calculate it and calculate it for sectors B and C with the spreadsheet as the two others did not do it*]

S1: what is the need finally?

S2: see, we found about the same value

S1: yes because we do a calculation but we do not know what it is for? If you got a big standard deviation, what does it mean?

Table 2 (10 groups)	sales	staff	items	distance	grade
Average	3	3	4	4	3
Median	1	1	2	2	1
Standard deviation		2	2		
Use of effective commercial context	5			5	5

Table 2

S2: [*he kept silent*]

We can see that, finally, the same procedure is chosen by both groups: consider the number of values under 5 and above 5. and do not compare the means. They used an anchoring strategy (Tversky & Kahneman, 1974), comparing the values to midrange 5 which they called "average": it is "global", as all data can be compared to it when the other means are only "local". It is known that students have difficulties with considering the mean as a good representation of a distribution (Konold & Pollatsek, 2002). But when both groups dealing with sales compared means, only the groups studying grade used the midrange. This is probably connected with the magnitude of grade values. But we suspect that is also linked to a strong social/school norm in France: "to obtain average"[5] means "reach midrange value" which is usually the passing grade. If we refer to Vergnaud's theory of conceptual fields (Vergnaud, 1990), it seems that students built a concept-in-action "average as middle value" linked with the theorem-in-action "a grade above 5 is good" and activated the scheme "compare number of data above and under 5".

In both groups, students who referred to technical rationality switched easily to the strategy of comparison to 5, while students who wanted to study the distribution more deeply (student 1 in group 1 and student 2 in group 2) seemed more reluctant.

At step 3, when groups of 6 students had to draw a conclusion about one area from their individual study of each of the variables, we noticed that the use of indicators was less frequent and, as in step 2, dependent on the context (see table 2). They indicated standard deviation for 2 variables only, those for whom we found no occurrence of commercial comments. It seems that they calculated this indicator for variables that made no sense for them to consider the context. One group went back to a local strategy by numbering customers.

At step 4, when groups of 3 were supposed to decide on the area they would like to manage, we found almost no occurrence of the use of indicators: only two groups (out of 10) used in their argumentation the average for distance and grade. Their argumentation was based on commercial arguments and mostly built on comparison of percentages within sub-groups.

Considering the references to the context, sometimes even at step 1, it seems that students very quickly built a representation of the problem as a commercial problem. As they moved forward in the experiment they left behind their statistical knowledge. They found it not practical enough for the objectives they had set. Pragmatic rationality prevailed.

The difficulty in moving from a local to a global point of view seems somewhere to reflect the difficulty in moving from a salesperson identity to that of a sales manager, because a salesperson deals with his customers more individually. And this implies the need to mobilise more statistical knowledge.

Conclusion

We used this experiment as a basis for the statistics course that year. The course was unusually successful. Of course we could not evaluate on a scientific basis whether the experiment helped the students to change their views about statistics, but many of them mentioned in the evaluation that they became more aware of the utility of statistical methods.

This experiment is going to be extended to a larger population through a Computer-Supported Collaborative Learning system based on the same device (we have added 4 variables to the file). We plan to use this system with students from our 5 European campuses.

Notes

1 They were not able to use what they had done at step 2

2 The six remaining students dealt with the qualitative variable

3 In French there is only one word for 'mean' and 'average', and we decide to translate it by 'average' in the dialogue

4 I numbered the students from 1 to 3 in both groups but of course they represent different students

5 Translated from the French "avoir la moyenne"

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Learning Conversation in Mathematics Practice – school-industry partnerships as arena for teacher education

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Abstract On-going research from the research project Learning Conversation in Mathematics Practice is reported. The paper gives an overview of issues within this project that relate to the EIMI-study. School-industry partnerships enable the study of pupils and student teachers participating in an educational setting which includes an industrial environment. Two examples are given. The first example illustrates how an assignment made possible by the industrial context influences the intentionality, functionality and empowerment of pupils and student teachers. The second example is concerned with mathematical modelling in this context, especially with regard to the development of critical democratic competence.

The on-going research project *Learning Conversation in Mathematics Practice* (LCMP)[1] focuses on communication and learning in the field of mathematics. Its goal is to develop the notion of learning conversation as a didactical concept and tool for describing and facilitating learning processes. The project collects research data from schools that have established partnerships with industrial[2] companies. An aim of these partnerships is for pupils[3] to learn mathematics through experiencing and discussing how mathematics is applied and used at work. This paper gives two examples of research areas within the LCMP-project that are made possible or enhanced by the school-industry partnerships. These examples are connected with several of the themes outlined in the Discussion Document of the EIMI-study, including teacher training, examples of practice, modelling and issues related to communication and collaboration. How the school-industry contexts influence students will be a main theme. We want to elaborate on the research context and the questions and discussions that are potential as part of the LCMP-project. We do not aim to give answers to these questions in this paper or at this stage.

A school development initiative "Real-life Education"[4] enables the LCMP-project to study industry as part of the learning environment of pupils and students. In this development initiative, lower-secondary schools have established school-industry partnership agreements. This agreement entails that the learning and teaching of mathematics is organised sequentially in industrial and school environments. Students from Bergen University College participate in the "Real-life Education"-initiative through their practice teaching5 The experiences gained, by both pupils and students, through the "Reallife Education"-initiative are conjectured to impact their attitude towards mathematics and the way mathematics is communicated. The LCMP-project collects and analyses data from three different layers6 (Johnsen-Høines, in print):

- the school development initiative, where the research focus is on pupils' ability to communicate and learn mathematics
- the professional development of student teachers engaging in the school development initiative, where the research focus is on the students' communication related to their professional development as mathematics teachers
- the collaboration between didacticians, schoolteachers and students, where the research focus is on the communicative learning processes that develop between members of the learning community.

Central to the project is the passage between different spaces of learning. This can be illustrated by learning loops that depict different contexts where pupils and students participate (Johnsen-Høines, 2009). Pupils and students participate in the different practices and move between them. The learning this involves is an object of study in the project (cf. Lave

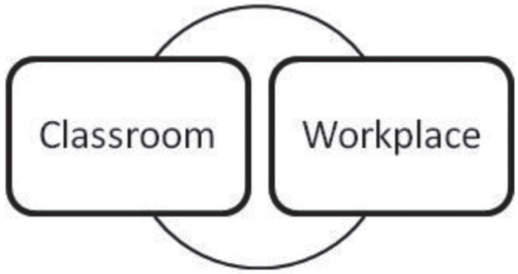

Figure 1—Learning loop – pupils

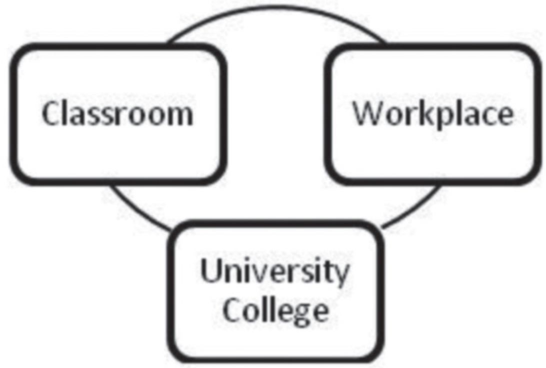

Figure 2—Learning loop – students

(1999) and Dreier (1999), who argue that learning is development through participating in and between different practices).

We investigate what kind of meta-learning the pupils' learning loop facilitates. We envision that pupils and students can develop a meta-reflection where the differences between mathematics at a workplace and in the classroom are problematised and reflected on. For example, the tools used by the company for solving mathematical problems (e.g. different technological tools) can be in tension with the school's intentions of what should be learned and how. In other cases can different use of language and tools be an enriching source for new learning. It is of interest to study how the pupils' knowledge about the company and the company's praxis characterize the conversation on mathematical topics and how the pupils' experiences from school mathematics influence their conversation when visiting the company.

The intentionality inherent in learning and teaching is influenced by the resources of intention available. The workplace context makes resources of intention available that are not necessarily available in the school context. Alrø & Skovsmose (2004, 154163) considers intention as basic to participation in a dialogue. As such, the workplace context may enable involvement in dialogues for which the school context, for some pupils, does not give the necessary resources of intention. The *intentions-in-learning* and *intentions-in-teaching* of students having their practice teaching connected to a school-industry partnership can also be influenced in a likewise manner. In the first example it is shown how the context influences the intentions-in-teaching of a task given by students to pupils.

The *functionality* of knowledge is essential for its use and application. Our consideration of the "functionality of knowledge" is inspired by Mellin-Olsen (1989) and Skovsmose (1994). The contexts given by the school-industry partnerships enable a widening of pupils and students functional understanding.[7] Experiencing mathematics functioning in the industry context will hopefully develop their understanding and view of specific mathematical topics, as well as of mathematics as a whole.

Example – workplace assignment

The group of four students, a school teacher and a didactician are preparing for the next teaching period. The school had a partnership agreement with a company producing valves for the oil industry. The students decided on statistics to be the mathematical focus for the pupils. As the students anticipated the necessity of the pupils having some rudimentary knowledge of statistics before they met the people at the company, the students developed a short module containing what they considered to be the most fundamental statistical concepts. However, they did not succeed as desired, and commented that "the pupils did not get engaged, they did not show interest in the learning even though they were told that these concepts would be helpful when working in contact with the industry". The discussion that followed made it clear that the students wanted the pupils to "work on issues they saw as important"; they wanted them "to experience knowledge as useful and necessary"; they wanted to organize the activities in a way that "stimulated the pupils' independence and motivation". These discussions are to be seen as a background when the students decided to challenge the company leaders to request the pupils to carry out an assignment. "The pupils should be requested to do a job that the company really needs done," the students said.

The director welcomed the pupils to their first visit at the company by saying: "Statistics is the Alpha and Omega for us, we can just close down if we buy too little metal of the different sorts needed for producing the valves that are ordered or are expected to be ordered. We must have an overview of our production and stock. [...] We have expanded so fast and we really have not developed good enough systems." The manager addressed one group of pupils by saying: "We want you to help us!"[8] He gave them a register of unsorted orders.[9]

The pupils worked on this assignment in the classroom. The register seemed cryptic and chaotic for the pupils as well as for the students (and for the teacher and the didacticians). The pupils needed to understand the codes used for labelling different metals and products (i.e. valves) made, and contacted the company several times in order to get more information. Additionally, tools for sorting were needed. The students were used as experts when the pupils learned to use Excel to sort. The pupils made new registers that gave a better overview of the register.

The student teachers decided that the pupils were to be viewed as a "consultancy firm".[10] The pupils should ask for information when this was needed, they should present the results when the work was done, and they should challenge the company to discuss consequences if they saw any. The company cooperated on these ideas, as can be seen by them informing the pupils that they would pay for the job — if it was well done. The pupils invited people from the company to a meeting at the school, and presented their result. They told that they had experienced the importance of accuracy, and gave examples of choices they had made in order to obtain an overview they predicted to be sufficient. They discussed the result, and addressed the company by asking critical questions. The presentation and discussion was seen as important in a pre-perspective: The fact that this was part of the task given gave motivation for and direction to the work. However, it also turned out to be evident that the presentation-session became important for the pupils' and the students' subsequent learning. It became a stage to look back at and make reference to, as well as for bringing about reflections for later work. "What have we learnt" became a meaningful utterance to inquire into.

The teaching and learning sequences referred to above illustrate the context in which the LCMP research is situated, and it gives the background to briefly consider some themes of research interest connected with this example. In this paper we restrict ourselves to pointing out a few themes that we have found significant in the chosen example: intentionality, functionality, and empowerment (Mündigkeit). The conversations in the initial module on statistics that was given to the pupils appeared to be very different from those in the further continuation. A central issue for the group to inquire into is then: How can the communication be characterized in these situations, and how do the differences emerge? Further discussions deal with investigating the conditions: What influences the conversations and the learning, what are conditions for developing different sorts of communication in the context of learning?

Intentionality. By studying the learning/teaching sequences and the collaborative communications within the LCMP-group the intentions-in-teaching of the students become evident. They want to[II] organize for the pupils to be enthusiastic learners; they want the pupils to work on issues that they see as "real" and important; they want the pupils to experience mathematics as essential knowledge; they want the pupils to be independent and to see the importance of accuracy. The intentions of the students developed during the process. This development can partly be seen as interplay with the intentions of the manager, the pupils, the teacher and the researcher. The way we are analyzing students' intention here, it can be characterized as intentions-in-teaching. But we find that the students are describing their intentions as intentions-in-learning as well. They see themselves as learners, of this being part of their professionalization as mathematics teachers.

Functionality. The students have inquired into "functionality of knowledge" as part of their study in mathematics education. The discussions in practice challenge the concept of functionality from different angles. The participants are questioning why and how statistics (and also tools such as Excel) is vital as a tool for the industry and for the students' and the pupils' learning of mathematics. The functionality of theories and didactical models is also discussed in the context of developing the teaching-and-learning sequences, as part of the students' professionalization as mathematics teachers.

Empowerment (Mündigkeit). Communication underpins the project; it defines the research focus and is seen as a tool for developing the collaboration and the teaching and learning interactions. Referring to "inquiry co-operation", as developed by Alrø & Skovsmose (2004), it is vital that participants see contributions by themselves and others as important for the community. Thus it becomes relevant to organize for and study the students' development as independent, responsible and collaborative partners. The development of the pupils' empowerment as collaborative learners is seen as important as well, and this is articulated as one of the students' main interests. The collaborative inquiries focused on the empowerment of pupils and of students in the light of one another. The data includes such meta-conversations.

Example – mathematical modelling

Another theme we study is how mathematical modelling can be integrated into education, both in primary schools and in teacher education. Of special interest is to study how students and pupils develop *critical democratic competence*[12] in interactions with such models.

Mathematical modelling is a theme for the students in their course work at the university college. Together with students we have tried to include mathematical modelling in the learning loop of the pupils. This also makes mathematical modelling a theme for the students throughout their own learning loop. Questions regarding mathematical modelling are discussed in the LCMP-group consisting of students, school teacher and didacticians.

A group of students have included semi-authentic models[13] as part of their practice teaching (Hansen, 2009). After a study of different mathematical models at the university college, the students decided to let the pupils in the classroom work with trend diagrams and regression to predict company turnover.[14] Two different models were used. These models gave different predictions. This leads to questions about validity and uncertainty; and reflections about how the results given by a particular model may depend on various input-data. We regard students' and pupils' abilities to raise such questions as important parts of their critical democratic competence. For such questions to arise we believe that some models and pedagogical initiatives could be more appropriate than others. By analysing conversations in the classroom and between didacticians, teachers and students we hope to get further insight into such issues. This is related to the following research questions:

- How can industrial mathematical models be made relevant for pupils' learning?
- How can some models be more appropriate than others for development of critical democratic competence in the field of mathematics?
- How necessary is the use of authentic models to achieve insight into modelling, and thus to the development of critical democratic competence?
- What criteria do we use when we discuss fruitful ways of working with models?
- Assuming that working with mathematical models increases pupils' critical democratic competence in mathematics, how does classroom communication reflect such skill development?

Our belief regarding the first question is that not all models used for industrial purposes are well-suited for classroom use. Authentic mathematical models often hide complex mathematical structures and advanced technology which excludes many models (cf. the notion of "black box"). For use in primary education these authentic models usually need to be simplified, which leads to the use of semi-authentic models. Thus it becomes pertinent to investigate how pupils and students in different learning situations can work with semi-authentic models connected to topics in the school curriculum (e.g. concepts such as functions, equations and linearity). Part of this will focus on how classroom communication may reveal different stages in pupils' development of critical democratic competence.

Final remarks

In this paper we have included examples where students participate, but the focus is also on the pupils' conversations and learning.[15] What characterizes the pupils' mathematical conversations as they move between classroom and company? Studying pupils' mathematical conversations at the workplace and in the classroom linked to pupils' readiness to apply mathematics in new contexts is believed to give new knowledge about pupils' learning through conversation. We investigate whether the inclusion of outside-school mathematics in mathematics teaching facilitate pupils' participation in problem solving, communication of ideas and discussions about strategy use and solutions. The pupils' learning loop makes it possible to mathematize and model real-world problems. Mathematization and modelling require discussion; e.g. simplifications must be done and results must be critically evaluated. Through studying mathematical conversations in and outside the classroom, we will search for insight into the impact mathematical conversations have on the pupils' development of mathematical literacy. The OECD describes mathematical literacy as a preparedness to employ mathematics when one needs it, a critical strength to influence through mathematics, and critically consider mathematical results (OECD 2003; 2006). Mathematization, functional understanding, empowerment and critical democratic competence are all related to mathematical literacy.

We also wish to look for traces of inquiry — whether pupils and students by their own interests together wonder and seek information (Lindfors, 1999) — and how they initiate and invite into searching conversations with an inquiring stance in and about mathematics. An inquiring stance will impact the intentionality and empowerment of the participants.

We have given two examples from the LCMP-project where we see the influence of the school-industry partnership. This illustrates how the context of the project enables us to give insight into the relationship between school and industry. The first example points toward the school-industry context changing the conditions of learning and teaching, and thus changing the intentions of students and pupils. The context gives the knowledge acquired a different functionality and empowers the students and pupils. In the second example the source of relevant mathematical models in the classroom is enhanced by the partnership with a company. The real-world connection may be helpful in engaging pupils and students in discussions that relate to critical democratic competence.

Notes

1 LCMP is financed by the Research Council of Norway (NFR) and Bergen University College. LCMP is part of the research consortium Teaching Better Mathematics, which consists of mathematics educators from University of Agder, Bergen University College, Bodø University College, Oslo University College and Sør-Trøndelag University College. LCMP is lead by Marit Johnsen-Høines. Webpage (in Norwegian): Læringssamtalen i matematikkfagets praksis (LIMP), http://www.hib.no/fou/limp/.

2 Industry is in this paper broadly interpreted to include different types of workplaces where mathematics is used. Different types of companies participate, varying from oil related mechanic industry to shops.

3 In this paper, children in school are referred to as pupils and student teachers as students.

4 In Norwegian: *praksisnær undervisning*. The initiative is administrated by Gode Sirklar AS (http://www.godesirklar.no/).

5 I. e. practicum (student teaching), which is an integrated part of their study in mathematics/mathematics education.

6 Empirical data is collected from teaching and learning sequences in the mathematics classroom and in pre-and post-classroom discussions. Conversations in which mathematical and didactical issues

are discussed are recorded and transcribed. The data collected from teaching and learning sequences and from the collaborative communications in which students, teachers and didacticians participate are analyzed. Students and teachers take part in discussing some of these analyses

7 De Villiers (1994) defines functional understanding as "the role, function or value of a specific mathematical concept or of a particular process".

8 The director's voice acts here as reference for the pupils on the given tasks. The pupils may interpret the challenge given as referring to statistics as important knowledge and an industrial tool, as referring to the importance of this specific job being done and that the director believes that the pupils are capable to do it, and/or as referring to the importance of being done diligently. The director's approach can be seen as influencing the intentions-in-learning and intentions-in-teaching by the way it puts the tasks into the discussion on necessity and functionality.

9 Different groups of pupils got different challenges. Each student teacher collaborated mainly with one group. 10The students defining the consultancy firm role can be seen in the context of note 8, referring to functionality and meaningfulness, and is connected to the students' discussions about ownership and empowerment of pupils using mathematics (intentions-in-teaching).

10 The students defining the consultancy firm role can be seen in the context of note 8, referring to functionality and meaningfulness, and is connected to the students' discussions about ownership and empowerment of pupils using mathematics (intentions-in-teaching).

11 Our interpretation "they want to" is based on transcripts from conversations where the students, together with other group members, are inquiring into what they want to try out in their teaching practice and their rationality for making their choices.

12 The term "critical democratic competence" is used here in relation to mathematics and refers to peoples' ability to remain critical, considering and analyzing according to use of and results from mathematics in society (Blomhøj, 1992, 2003; Skovsmose, 1994).

13 By "authentic model" we mean a model used in real world applications (i.e. enterprise planning of income, oil recovery, climate forecasts etc.). The word "semi-authentic model" refers to a simplified model compared to daily life industrial models.

14 This illustrates how the students move in and between different practices in the learning loop.

15 Parts of the LCMP-project are primarily interested in the pupils (this is the first layer described on page 2). A PhD-student, Toril Eskeland Rangnes, is working on pupils mathematical conversations as they participate in the learning loop, with particular emphasis on mathematical literacy. The questions raised in this paragraph will be part of her study.

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Mathematics in Industry and Teachers' Training

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Abstract Mathematics, modelling and simulation are a vital resource for innovation and development in knowledge based society and competitive industry. This means a challenge for education and mathematics teachers'training. A modern view of mathematics should be reflected in curricula and educational practices. A change should be visible in curriculum development, up-to-date contents, novel teaching methods. Development of modelling education is a crucial part of this endeavour and university pedagogy of applied mathematics. I refer to case examples of industrial math projects illuminating the educational challenge. Implications for teachers training programmes and practices at school level are suggested and analysed.

Introduction

Mathematics, modelling and simulation, so called mathematical technology is emerging as a vital resource to achieve competitive edge in knowledge based industries and development of society. This vision about the role of mathematics has inspired efforts to enhance knowledge transfer between universities and industry. Especially it means a challenge for university education. A modern view of mathematics should be reflected in curricula and educational practices. This has implication for the way how mathematical modelling should be inserted to the curricula at various levels.

The main focus of this article is on undergraduate teaching at tertiary level. However some important implications are suggested concerning the schools preparing students for universities.

Computational Technology

The development of technology has modified in many ways the expectations facing the mathematics education and practices of applied research. Today's industry is typically high tech production. Sophisticated methods are involved at all levels. Computationally intensive methods are also used in ordinary production chains, from brick factories to bakeries and laundry machines. The increased supply of computing power has made it possible to implement and apply computational methods. Terms like mathematical technology, industrial mathematics, computational modelling or mathematical simulation are used to describe this active contact zone between technology, computing and mathematics.

A mathematical model is assumed to represent the structure and the laws governing the time evolution of the system or phenomenon that it was set out to mimic. Once we are able to produce a satisfactory model, we have a powerful tool to study the behaviour and hence to understand the nature of the system. The models can be used to

Industrial Mathematics, Educational challenge

The computer age has generated a need and a window of opportunity for a new kind of expertise. This presents a challenge to the educational programmes and curriculum development. Some universities already offer specialized MS-programs oriented towards the industrial needs. There are excellent programs that equip the students with the skills that are needed in the professional use of mathematics. In general there is still room for improvement.

A good educational package would contain a selection of mathematics, computing skills and basic knowledge of physics, engineering or other professional sector. It would be very important to train oneself to work in a project team, where the interpersonal communication is continuously present. To become a successful applied mathematician ready to tackle the fascinating tasks and challenges, development questions in modern industry, the student need a solid and sufficiently broad theoretical education and operational skills in the methods of applied mathematics.

There is a special need to revise the university pedagogy of applied mathematics. Reflection to the preparatory levels of high school should be elaborated. Regarding the undergraduate programmes at universities the following means should be considered: revision of syllabi and curricula, use of computing experiments, data tools and novel teaching methods. The curriculum should contain knowledge of theoretical mathematics and a collection of applied courses. The students should have also knowledge of some applied science or application fields, knowledge in physics, engineering or other “clientdiscipline”ofmathematics. The quality of classroom examples is important. Problem based learning, topical fresh exercises are called for. Mathematics teachers should have interest for different areas of modern professional life. Modeling examples can be found from work places, hobbies, talking with people from various professions, sometimes reading newspapers with mathematically curious mind. Interestign modelling cases can emerge from bakery, laundry machine, brick manufacturing, fermentation process of food or bioprocesses in gardening.

However, the most important single skill is the experience in projects. For successful transfer of mathematical knowledge to client disciplines the theme of mathematical modelling is a crucial educational challenge. The lectures, books and laboratory exercises are necessary, but the actual maturing into an expert can only be achieved by “treating real patients”.

Industrial Mathematic as Profession

The computer age has generated a quest for new special expertise. Many universities offer specialized MS-programs that equip the students with the skills that are needed in the mathematical projects in the R&D-sections in industry. The job title in industry is very seldom that of a mathematician. It can be a researcher, an engineer, a research engineer, systems specialist, development manager.

Industrial mathematics is teamwork. Success stories are born when a group of specialists can join their expertise and visions together in a synergic manner. The teamwork makes communications skills a necessary matter. It would be very important to train oneself to work in a project team, where the interpersonal communication is continuously present. To become a good applied mathematician one should be curious about other areas as well, to be interested and learn basic facts from a few neighbouring areas outside mathematics.

In the following I describe the increasing sphere of areas where modelling simulation and intelligent systems are a crucial vehicle of development. I refer to case examples of industrial math projects illuminating the educational challenge.

Economics and management. The daily functioning of our modern society is based on numerous large-scale systems. Examples are transportation, communication, energy distribution and community service systems. The planning, monitoring and management of these systems offers a lot of opportunities for mathematical approach. The images below represent time series of electricity price variation and the level of water storage in a certain geographic area of integrated energy market. Modelling attempts are needed to understand the mechanisms leading to these fascinating and intricate stochastic phenomena in energy market.

Flow phenomena. The ability to model sophisticated phenomena, including nonlinear effects, the possibility to solve the equations with advanced numerical methods, combined with the latest visualization tools have created a luxury environment for mathematical engineering. Examples are river flow models that are used for flood control and forecasting, planning of hydropower systems and waterways.

Estuary model (rioght) may be needed to understand how the saline water from the sea is penetrating to river estuary. The model should predict the salt concentration and depth of the penetration upstream.

Systems design and control. The design engineers and systems engineers have always been active users of mathematics in their profession. The possibility to set up realistic large-scale system models and the development of modern control theory has made the computational platform a powerful tool with new dimensions. The illustration below describes measurement data from paper mill where the thickness of a 1500 m long paper web has been measured with high precision. The image is actually a product of a simulation model that mimics the actual performance of the paper machine.

Measurement technology, signals and image analysis. The computer and the advanced technologies for measurement, monitoring devices, cameras etc produce a flood of digital information. Examples of advanced measurement technologies are mathematical imaging applications. Applications range from security and surveillance to medical diagnostics. Modern theory of inverse problems is applied in improving the imaging in dental tomography. So called Bayesian stochastic models are a key to these improvements.

Media and entertainment industry are heavy users of mathematical models. The visualization techniques, special effects and simulated motion of virtual reality are based on multi-

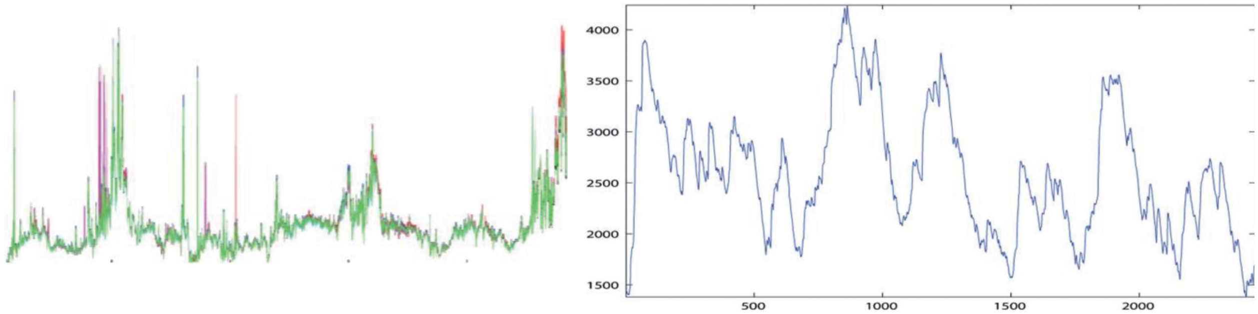

Figure 1—Economics and management

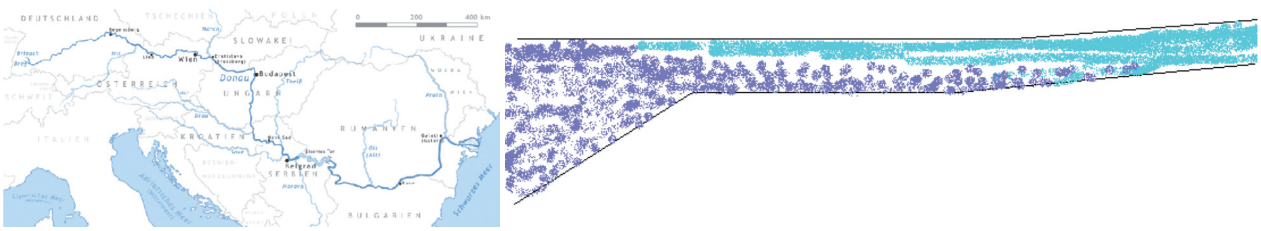

Figure 2—Flow fenomena

disciplinary approach using mathematics, mechanics and computing power. An example could be the sympathetic character of Gollum from "The Lord of the Rings" movie. The odd and alien skin of the character was created by a technique of simulate subsurface scattering, a combination of modelling, physics of light reflection and computing skills.

www.ew.com/ew/article/0,,702019,00.html

Modelling as a Course Subject

Many departments have introduced modelling courses in the curriculum in recent years. A course in modelling may contain study of case examples, reading texts and solving exercises. The actual challenge and fascination is the students'exposure to open problems, addressing questions arising from real context. The real world questions may be found from the student's own fields of activity, hobbies, summer jobs, from the profession of their parents etc. Reading newspapers and professional magazines with a mathematically curious eye may find an idea for a modelling exercise. A good modelling course should

(a) contain an interesting collection of case examples which stir students'curiosity

(b) give an indication of the diversity of model types and purposes

(c) show the development from simple models to more sophisticated ones.

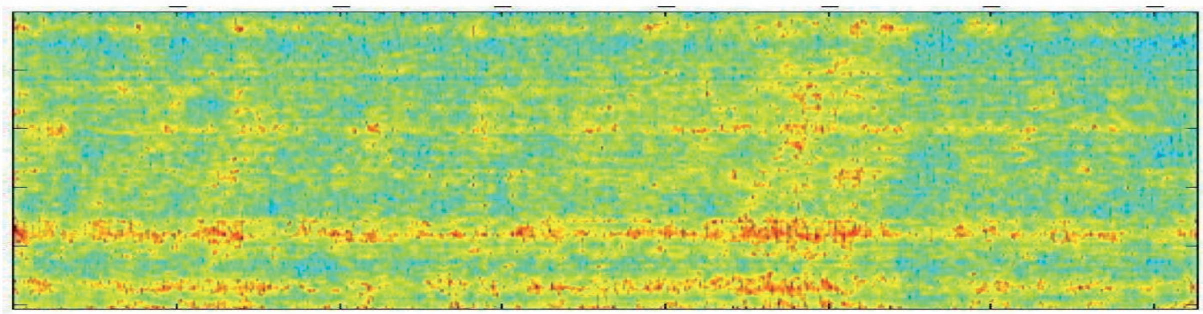

Figure 3—Systems design and control

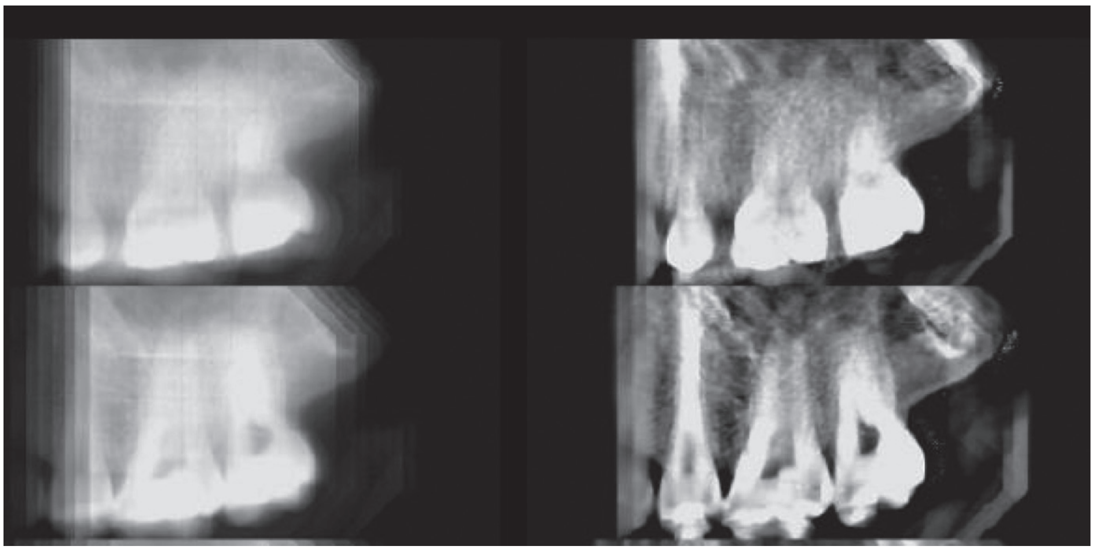

Figure 3—Measurement technology, signals and image analysis

(d) stress the interdisciplinary nature, teamwork aspect, communication skills

(e) tell about the open nature of the problems and non-existence of “right”solutions

(f) bring home the understanding of practical benefits, the usage of the model

(g) tie together mathematical ideas from different earlier courses

The modelling courses have been run in different forms. Traditional lecture course with weekly exercise session is a possibility. It would be important to implement group work mode and PC-lab activities in the course. The most rewarding form of activity might be projects and weekly session where the student report and discuss about their work and progress on the problems.

Figure 4—Media and entertainment industry

Modelling weeks

One of the innovative educational practices introduced in the recent decades is Modelling Week, an intensive mathematical problem solving workshop which simulates the real life R&D procedures. Students from all over Europe come together and work in teams of five or six on modelling real world problems. The cases originate from industry, different organisations or branches of society. The problems are brought in and presented by the problem owners. The teams are guided by a group of academic staff members. The "instructors"usually play the role of the problem owners.

The students are allocated to problem-teams on the basis of their areas of interest and mathematical expertise. The week starts with the problem owner giving a brief outline of the problem, the industrial context and the relevance of the problem for his/her company. The team questions the problem owner about the problem and the expectations.

To identify and understand the "real" problem may take some time. The students must formulate a model and recognize the typically non-unique mathematical problem. The analysis follows leading to analytical studies and efforts to find techniques for numerical solutions. Typically the group arrives at an approximate solution. At the end of the week the student groups have to present their findings in public. Further they are assumed to produce a decent written report, often a short article that will be published in the proceedings of the Modelling Week.

Modelling education, how much and when?

The presence of mathematical modelling on today's technology agenda and in the challenge of knowledge based society indicates the importance of the questions of teaching of mathematical modelling in curricula at various levels of education.

Does this support the idea that modelling education should be promoted for all children at all school levels? How much, in what way and when?

These questions should be discussed seriously in the math education community. The skills of mathematical modelling mean an essential competence which is needed in many science related fields, technology, engineering, economics, biomedical professions etc.

Modelling means a set of specialized science based skills that can be compared with the expert skills of, let us say, airline pilots and brain surgeons. Our society really needs these skills but we do not arrange mini-courses for airline pilots and surgery on primary school levels. This somewhat provocative statement is meant to emphasize the important question of how mathematical modelling should be inserted to the educational system.

The overall understanding of mathematics in today's world should be explained in mathematics classroom. Examples from applications should be used as pedagogical fermentation of the learning process. The joy of problem solving and the use of mathematics for real life situations is an ideal way to build interest and enthusiasm in the classroom.

However the pedagogical challenge must not be overlooked. The phrase of professor Helmut Neunzert at ICTMA14 " ... modeling can be learnt but not taught in a usual way"contains an important message. One should also remember that often educational fashions or New Math reforms tend to drift to overdrive or hype level.

Quality of Classroom Examples

The supply of good classroom examples and case studies from different application areas is a key factor for the development of attractive and inspiring educational modules. We would need a flow of fresh problems. It would be important to maintain contacts to different special sectors, professions, diverse pockets of innovative processes. Good case-histories and exercise material would draw the attention of the students give motivation and make the theoretical ideas transparent. What makes a good classroom problem is not trivial. The examples may be

topical	artificial
fresh	odd relics
witty	dull, mundane
curiosity stirring	perfect, closed
transparent	wizard tricks
realistic	toy-like, concocted
understandable	intricate, complicated

The challenge is, how to carry over the idea of mathematics as a useful and powerful multipurpose environment of problem solving. How to shape the image of a modern brand of expertise, an emerging profession of the future?

Bad Examples

A set of examples is presented below which describe the typical symptoms of poor or questionable ways to create application examples. The problems may hinge upon unrealistic contextual assumptions, model equations dropped from the blue space without a hint of the origin, pure mathematical tricks vested in an application-sounding dress.

Hanging a rope for drying laundry. Derive the equation of the hanging rope. Find the equation in a case of non-homogenous gravity?

Cost-function optimization. The total cost over the life-cycle of a building element is estimated to be

$$P(x) = \int_0^{+\infty} e^{-ct}(a + bt - x)^2 \, dt \, ,$$

where $x =$ *insulator-thickness*. Find the optimal insulator thickness.

Reliability distribution. The following model is used to describe the time-to-failure distribution of an electronic component $\varphi(a, t) = (1 - at)(-0.02t)$. Following data is given [...]. Estimate the model-parameter a.

North Sea fish population. Model for the size of the fish population is given as $dw/dt = aw^{2/3} - bw$. Study the growth of the population.

The goal of modern applied mathematics instruction is to produce genuinely useful mathematical aptitudes — not just to do mathematics. The lecture room presentations and exercise labs should avoid the temptation of well-posed problems in well trodden areas — elegant but unrealistic manoeuvring of computational acrobatics. The aim is to see mathematics in action — to produce benefits.

FP7 programme Science and Society

European Union launched within the 7th Framework programme a call with title Science and Society. Natural Sciences, Mathematics, Informatics, and innovation are vital to increasing competitiveness, enhancing and expanding the economy and improving the quality of life and promote international competitiveness. Europe needs to generate more trained scientists, engineers and researchers to meet the challenges of global competition. This is essentially the content of the Lisbon Agenda.

A key question is the interest of youth in science and engineering education. In the last 20 years an alarming decrease in the number of youths studying Science and Engineering has been observed ["*Science Education Now: A Renewed Pedagogy for the Future of Europe*", Michel Rocard group report on Science Education, 2007]. The trend seems to continue and the phenomenon is known worldwide.

The Universities, as initial and continuous teacher trainers, need to change their approaches to train new and in-service Science teachers. Inquiry based and problem based learning strategies are known to develop student attitudes to their studies of the Science such as raised interest, curiosity, commitment, autonomy in study and interdisciplinary and awareness of the context and real life.

This is a challenge for varied local stakeholders: schools, universities, governmental and non-governmental organisations, local authorities, education research laboratories, science centres, libraries, citizens'associations, local media, enterprises and technological parks.

I suggest that a theme *Mathematics and society — real life perspective in mathematics education* should gain attention in this discussion. The attitudes of young people for math and science teaching and learning are affected by

- teaching methods, syllabi and educational content

- teacher's knowledge about their subject and the world around the subject (other sciences, society, industry, technology, R&D activity, including development challenge in public governance)
- organization and administration of education as a national enterprise
- public awareness about the role and significance of science, technology and the science education within general public
- competition in the minds of the students between our subject and many other fascinating topics (entertainment, fitness, environment, peace, democracy, global responsibility, self expression, fashions and tribal cohesion, family values and friendship).

A big question is how the demanding science education can be promoted in the challenging, complex, multifaceted cultural arena. No doubt main objective should be fermentation of interest, curiosity, commitment and diligence in the minds of students.

Teacher training is a key question. The new generation of math teachers should have a sound overall understanding about the important role of mathematics in today's world so that he/she can bring to the classroom a flavour of the fascinating special skills of mathematics, modelling, simulation and computing. The children hopefully will become aware of several new professional career opportunities in the field of science based professions where mathematical models are the modern space-age toolkit.

The teachers should be able to bring to the classroom the enthusiasm and vision of mathematics at the service of society and the advancing front of development. The idea of computational technology and modelling as a component in creativity and innovation in all science and science based professions should be brought to the attention of 15–18 year old boys and girls by their science teachers backed up by media and other stake holders.

This goal can be achieved only if the curricula in undergraduate and graduate level in universities and teacher training programs are developed. Universities should offer courses, seminars, project experience for student about "Mathematics applied to real world". Mathematics teachers should have been exposed to such material at undergraduate level and in graduate studies.

- general lectures about industrial mathematics
- courses in mathematical modelling
- problem seminars, project work about small modelling cases
- possibility to participate in Contests of Mathematical Problem Solving, etc.

We should apply this viewpoint to the teacher training and re-training programmes, curriculum development at undergraduate and graduate level, development of teaching culture at universities and programmes preparing teachers for high schools and colleges.

Bridging the Gap Between Mathematics and Industry: Master of Science Education in Engineering Mathematics at Lund University

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Abstract In this paper, we present the Master of Science education in engineering mathematics, given since 2002 at Lund University in Sweden. It is a 5 year (300 ECTS) engineering education, leading to a Master of Science degree, with the possibility to obtain a bachelor degree after 3 years. The main motivation for starting the program was to bridge the gap between the need for advanced mathematical methods and knowledge in industry and the educational programs provided by the university. The program is unique by combining advanced mathematical knowledge with engineering topics and several taylor-made courses have been developed for the program. It is also interesting to note that the number of female students are significantly higher than for other similar programs in Lund. This paper is targeted towards topic 7: Teaching and learning for communication and collaboration.

Background

The Faculty of Engineering at Lund University is among the leading engineering faculties in Europe, with more than 7000 undergraduates and 800 postgraduates. Founded in 1961, as an independent institute, it today belongs to Lund University, which is one of Scandinavia's largest institutions for education and research with about 35000 students and 6000 employees.

The first engineering program was in engineering physics and programs in all major engineering disciplines followed. Today there are 15 different engineering programs leading to master of science degrees. They have recently been adopted to the Bologna structure; from 4½ year programs to 5 year programs, with the possibility to obtain a bachelor degree after 3 years.

The programs with the strongest mathematical curriculum has traditionally been the program in engineering physics and to some extent the program in electrical engineering. During the 90's it became evident that the industry demanded engineers with a strong core competence in mathematics, combined with a broad engineering curriculum, as also pointed out in the position document. Based on this observation the program in engineering mathematics started 2002 as the first program in its kind in Sweden, with 40 students/year. The theme of the program has from the start been "*mathematics as engineering practice*".

Program structure

The program consists of a core of mathematical courses, including mathematical statistics and numerical analysis, combined with a broad spectrum of engineering courses. The first three years consists mainly of mandatory courses and in the last two years, the students can select between six different master specialisations.

First year

The first year consists of the following courses:

- Calculus in one variable (15 ECTS)
- Linear algebra (6 ECTS)
- Calculus in several variables (7.5 ECTS)
- Programming, first course (7.5 ECTS)
- Physics (7.5 ECTS)
- Mechanics (statics and dynamics) (7.5 ECTS)
- Mathematical modelling (4.5 ECTS)
- Mathematical communication (4,5 ECTS)

The last two courses are specifically developed for the program and will be described in detail below.

Second year

The second year consists of the following courses:

- Analytic functions (7 ECTS)
- Systems and transformations (7 ECTS)
- Basics in PDE (7,5 ECTS)
- Probability theory and statistics (9 ECTS)
- Stochastic processes (6 ECTS)
- Programming, second course (7.5 ECTS)
- Automatic control theory (7,5 ECTS)
- Introduction to Microeconomics (6 ECTS)
- Project course/Modelling course (3 ECTS)

The last course can be selected among 4 different courses.

Third year

The third year consists of the following courses:

- Matrix theory (6 ECTS)
- Mathematical structures (6 ECTS)
- Mathematical modelling, advanced course (4.5 ECTS)
- Numerical methods of ordinary differential equations (8 ECTS)
- Electromagnetic field theory (7 ECTS)
- Multi-grid methods for differential equations (4 ECTS)
- Algorithm implementation (5 ECTS)
- Signal processing (6 ECTS)
- Environmental Systems Studies and Sustainable development (6 ECTS)
- Biology, Introductory Course (7,5 ECTS)

Master specialisations

During the fourth and fifth year, the student has to select among six different master specialisations:

- Computation and Simulation
- Biological and Medical Modelling
- Financial Modelling
- Environment, Risk and Climate

- Signals and Systems
- Software Development

Each of these master specialisations consists of both mathematical courses and engineering courses within the chosen specialisation. Examples of courses in mathematics are, optimization, linear and combinatorial optimization, non-linear dynamic systems, algebra, image analysis, functional analysis and partial differential equations. Examples of courses in mathematical statistics are time series analysis, markov random fields, Monte-Carlo methods and analysis of survival data. Example of courses in numerical analysis are numerical linear algebra, simulation tools and numerical linear algebra. The master specialisation consists of course also of a master thesis (30 ECTS).

Specific courses

Several courses have been developed specifically for the program in engineering mathematics.

Already from the beginning of the first year, there is a course in Mathematical Communication that goes on for the rest of the first year. The purpose of the course is manifold; introduce the students to some history of mathematics, information search for mathematical texts, writing mathematics in LaTeX and discussing mathematics in smaller groups while solving some smaller exercises. Towards the end of the first year, the students work on a larger project, including a written report and a presentation aimed at a wider audience, providing training in communicating mathematics to a wider audience. Topics for the projects can be selected rather freely, e.g. spherical geometry, option prizing and mathematical games.

There is also a course in Mathematical Modelling in the first year. This is a more compact course focussed on using mathematics to model ordinary and technical phenomena. The course consists of a number of modelling tasks, formulated in non-mathematical terms, the the students have to solve, write a short report and make a presentation.

The project/modelling course in the second year can be selected among:

- ECMI modelling week
- Project in PDE
- Project in Stochastic Processes
- Project in Automatic Control Theory

The purpose of these courses is to further strengthen the modelling competence of the students. The ECMI modelling week is a very popular course and each year about 5 students are selected to attend the course.

International Cooperation

The faculty of engineering at Lund University has agreements with a large number of universities around the world for exchange studies and some double degree agreements with some universities in Europe within the T.I.M.E. Network. The study program in engineering mathematics has of course benefited from these agreements. Because of the special mathematical focus of the program a double degree agreement with University of Kaiserslautern has been obtained and a joint degree agreement with Dresden University is close to be finalized. The program has also been participating in the ECMI network activities, such as modelling weeks and ECMI model master in industrial mathematics.

Outcomes

The program has been running for eight years with 40 new students starting each year, with an average of 30% female students. It is interesting to note that both the number of female students and the geographic spread are significantly higher than for other similar programs in Lund.

So far (end of 2009) 58 students have obtained the master of science degree in engineering mathematics. They have all got jobs in industry, society or academia with a surprisingly diversity of positions, e.g. finance and banking, energy trading, insurance, industrial automation, telecommunication, image technology, software engineering, medical engineering, radio technology, construction engineering, nuclear power engineering and own enterprise/consulting. Out of the 31 students who got their first job in industry, 10 got their first job outside Sweden.

A large number (21) of them have started PhD-studies, in several different areas, e.g. applied mathematics, automatic control theory, bioengineering, biomedicine, building physics, electrical engineering, finance, geology, information theory, material science, medical imaging, signal processing, solid mechanics, mathematics, mathematical statistics and numerical analysis.

The current list of first jobs looks as follows (a few students are not known):

- DTU, Denmark, PhD-studies in material science
- Medviso AB, Lund, Sweden, Medical image analysis

- Building Physics, Lund University, Sweden, PhD-student
- Immunotechnology, Lund University, Sweden, PhD-student
- The Riksbank (central bank), Stockholm, Sweden, Research assistant
- Cybercom, Malmö, Sweden, IT-consulting
- Danske Bank, Copenhagen, Denmark, Analytics
- Simula Research Laboratory, Fornebu, Norway, PhD-student
- TAT (The Astonishing Tribe), Seoul, South Korea (IT)
- Own company (IT)
- Professional poker player
- Cellavision AB, Lund, Sweden, Medical image analysis (2)
- Deloitte, Copenhagen, Denmark, Insurance consultant
- Geneva University, Research assistant (finance)
- Student union, Lund University, Sweden
- Ericsson EMP, Lund, Sweden, Software development
- ÅF, Malmö, Sweden, Industrial automation
- Accenture Technology Solutions, Stockholm, Sweden, System development
- Maurer Söhne, Munich, Germany, Building construction
- Mathematics, Lund University, Sweden, PhD-student
- WWOOF, Argentina, Volontary international aid work
- Illuminate Labs AB, Göteborg, Sweden, Computer Graphics
- Flextronics, Karlskrona, Sweden, Telecommunication
- Cambridge University, UK, Postgraduate in applied mathematics
- Information theory, Lund University, Sweden, PhD-student
- Nordea, Copenhagen, Denmark, System analytics
- Danish Bank, Copenhagen, Denmark, Analytics (2)
- Vattenfall, Stockholm, Sweden, Price analytics
- Sony-Ericsson, Lund, Sweden, Developer, radio electronics (2)
- Ericsson Mobile Platforms EMP, Lund, Sweden, Software development
- Cornell University, USA, PhD-student in finance
- Applied mechanics, Chalmers, Sweden, PhD-student
- Government office of Sweden, Stockholm, Sweden
- Google, Zurich, Switzerland
- Accenture, Copenhagen, Denmark, Consultant system technology
- Clinical physiology, Lund University, Sweden, Research engineer
- Centre for Bioengineering, Trinity College, Dublin, Irland, PhD-student
- Uddcomb Engineering, Helsingborg, Sweden, Scientific computing
- Solid Mechanics, Lund University, Sweden, PhD-student
- Electrical science, Lund University, Sweden, PhD-student
- Matematical statistics, Lund University, Sweden, PhD-student (3)
- Illuminate Labs AB, Göteborg, Sweden, Computer graphics
- Scalado, Lund, Sweden, Image processing
- J.P. Morgan, London, UK
- Axis Communications, Lund, Sweden, Image analysis
- Signal processing, Royal Institute of Technology, Sweden, PhD-student
- Numerical analysis, Lund University, Sweden, PhD-student

- Capital Research AB, Stockholm, Sweden
- Automatic Control Theory, Lund University, Sweden, PhD-student (2)

Conclusion

The master of science program in engineering mathematics has so far been a success story and Chalmers University of Technology has recently started a similar program. It has attracted very good students and they have all got attractive jobs. One interesting aspect is that the unique competence these students have has been very attractive at various different places, in both academia, society and industry.

Using spreadsheets in the finance industry

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Abstract Spreadsheets are major tools in the finance industry. In order to find out how these tools are used, we examined recently published research papers and interviewed five experienced bankers. This examination showed that spreadsheets may be used insufficiently, uncritically, and erroneously. By focusing on the quality of data and models (to be) exploited, this paper presents main details of this business-risky use of spreadsheets. Implications for vocational education are included.

Intoduction

Spreadsheets are in use over a quarter of a century and they are nowadays major tools in the finance industry (Croll, 2005). Most, even the majority of financial documents are found in the form of spreadsheets. A recent survey in December 2007 evidences that Microsoft Excel is used in nearly half of banks in Serbia for recording and monitoring operational risks.[1] That a skilled work with spreadsheets and other business accounting software is vital for a person working in financial industry is evidenced, for example, in the requirements for a position of senior reporting manager in Alpha Bank (see www.alphabank.ro/cariere.pdf).

Spreadsheet models, as other computers models, have a dual role in using mathematics in the workplace: they hide mathematics that is built in those models, but also, through using the software, can help users understand the built-in mathematics (Straesser, 2000). The use of technology to help employees understand (for them) hidden mathematical models has been, for example, examined in project *Techno-mathematical literacies in the workplace*, which found that "calculation and basic arithmetic are less important than a conceptual grasp of how, for example, process improvement works, how graphs and spreadsheets may highlight relationships, and how systematic data may be used with powerful, predictive tools to control and improve processes." (Hoyles, Noss, Bakker, & Kent, 2007; p. 3). However, the assumption that the models to be understood with technology are correct does not always apply. This implies that technology in general and spreadsheets in particular should be also used to test models and data applied (e.g. when using outcomes of what-if analyses and solutions of different scenarios). This relatively neglected afford enables critical evaluations of the results obtained with the software, which is according to the Discussion Document of ICMI/ICIAM Study "Educational Interfaces between Mathematics and Industry" relevant to both industry and education. This opens an important question for the teaching and learning of industry related practice regarding the use of spreadsheets models and the appropriateness of those models that is also acknowledged by this document in a general sense.

The paper examines how spreadsheets are used in the finance industry. This question was answered by using several research papers located through a detailed search of various electronic databases and the Internet. We also used some data gathered in interviews with five experienced bankers. One of them worked at a big software firm that has implemented BI (Business Intelligence) solutions in many banks in Serbia and abroad. Other worked in the Association of Serbian Banks (www.ubs-asb.org), which is a large finance association in Serbia dedicated to improving the business by proposing, among other things, methodological solutions on various financial issues (e.g. operational risk) to be applied in the practice (possibly previously brought into accord with approaches developed at central banks or bank groups[2] and the National Bank of Serbia). The other three interviewed bankers worked at banks.

We found that spreadsheets may be used insufficiently, uncritically, and erroneously. The next section exemplifies such a use of spreadsheets, whereas the closing section briefly deals with some implications for vocational education that would improve the matters.

Inapropriate Practice

Insufficient use

Although a bank may use spreadsheets even for 80% of all its documents (revealed by one interview), spreadsheets may be used insufficiently. Consider, for example, bank branch productivity usually expressed as cost/income ratio (should be less than 80%, for example), which may be obtained though detailed calculations like those at Inter-American Development Bank given at www.iadb.org/sds/doc/IFM%20 Operating%20Manuals%20 Guide%203%20E.pdf . A race among banks to expand into different regions is now taking place in Serbia. Even in small towns, within a circle of few hundred meters in their downtowns, one can find a constant presence of ten or so banks represented by their branches. Despite the facts that their branch productivity is probably low (having in mind such a number of banks represented, town size, and a lack of qualified employees), analyses concerning the branch productivity (or other relevant measures), supported by spreadsheets or other software, seem missing at most banks. This astonishing fact was revealed in one of the interviews mentioned above. The banker clarified that, when he asked colleagues from banks about these analyses, they usually replied that some measures were available at the central bank or bank group headquarters but that they had not yet so far received instructions what to do in that respect. The same situation (expanding of banks into regions without much analysis) applied to the Russian Federation in the early 2000's (see www.strategy.ru/UserFiles/File/presentations/Regional %20 Conference_05oct 2007.pdf). Note that according to a recent report from the National Bank of Serbia (see www.nbs.rs/export/internet/latinica/55/55_4/kvartalni_izvestaj_III_09.pdf), in the first nine months in 2009 just twenty of thirty-four banks in Serbia have operated profitably.

Even when mathematics used in models is quite simple and spreadsheets thus very easy to build, the results generated by them may be questionable. Consider, for example, the profitability of child savings displayed at Screenshot 1 (data are fictional). As it stands, this service is clearly not productive because this bank needs to invest 1.56 € to get 1 €. This outcome may be questionable, however, because, as one interviewed banker underlined, banks usu-

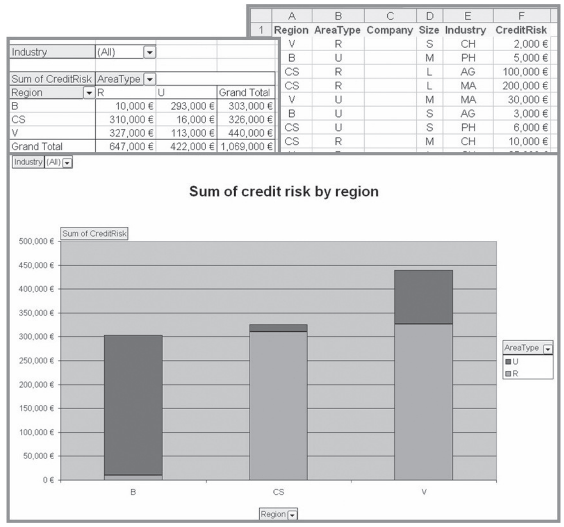
Screenshot 1—Determining profitability of child savings (example suggested by one interviewed banker

	A	B	C
1	Profitability of child savings		
2	**Expenses**		
3	Material costs (saving books and money-boxes)	9,000 €	
4	Marketing costs	7,000 €	
5	Processing costs	10,000 €	
6	Interest paid on savings	24,000 €	
7	Total	50,000 €	=SUM(B3:B6)
8	**Revenue**		
9	Interest received on loans	32,000 €	
10	**Profitability** (Rev/Exp)	**0.64**	=B9/B7
11	**For revenue of 1 €, expenses are**	**1.56 €**	

ally do not have reliable and complete data concerning expenses for some of their services, especially for some of its organizational units. This is an instance of a general business problem. It is widely known that production or service total expenses can be divided into fixed and variable ones, but, as the author's colleagues (an economist) highlighted several years ago, it is not easy task for firm to come up with reliable and complete data regarding these expenses. This opens an opportunity to use spreadsheets to test the validity of the data applied. Processing costs may, for example, be estimated by two spreadsheet models before the relevant number is determined. Also, data for one year may be compared with data for another year knowing that some ratios (e.g. material costs / processing costs) would not vary much by year. Finally, using different scenarios for two kinds of costs in question may indicate problematic data. It should be kept in mind that when some test does not detect problematic data, it may not mean that the data are reliable, but rather that this test gave no information on the issue in question (O'Beirne, 2009). This implies that the quality of data should not be examined by a single test, which brings us to the issue of critical use of spreadsheets.

Uncritical use

Mathematics, and therefore software that makes use of it, may uncritically be applied in modelling. In the case of finance, the sophistication of mathematics used for financial models has increased considerably in the last decades, whereas the common sense of situations being modelled dropped. This situation is nicely illustrated in Wilmott (2000) where a distribution of rate of return (a simple tool) could help Proctor and Gamble (P&G) to analyse the outcome of a financial deal with Banker Trust in a proper context, which eventually caused P&G to lose about $165 million.



Screenshot 2—Credit risk: original data, pivot table, and pivot chart

An uncritical use of software was revealed by the interviews because, as one banker sad, central bank did not explain procedures to be used just asked regional and branch offices to apply them. Another banker said that to his knowledge, despite its importance, critical use of applied financial models had been a neglected issue at the Association (he also added that some bank(s) might take care about this issue). The expert from the above-mentioned software company remarked that just one model was usually implemented per financial issue.

We can, for example, observe an uncritical use of financial models in Serbia in the case of credit risk. Serbian banks seems to use just one model (revealed by the interviews) and it is usually a credit portfolio model called CreditMetrics[3] implemented in Credit Manager software, or by using Microsoft Excel or Matlab tools (Jazic, 2007). As there is no best method for the assessment of credit risk, several methods should be combined (Kalapodas & Thomson, 2006). Such an approach to other risks and other important financial issues would promote a critical use of financial models that seems largely missing at present.

As one interviewed banker said, main reasons for uncritical use of spreadsheets may be found in the fact that employees typically work with pre-defined templates to be filled with financial data as well as they do so in very demanding working conditions (that promptly suppresses employees' creativity if any). Also, an improvement in that respect costs (and bank, on the basis of a cost-benefit analysis if any, may not be willing to invest in training). As psychologically, employees (or people in general) are resistant to changes, implementing new business procedures, even when they ease financial matters, does not go in a simple way. As an example, consider the work with pivot tables and pivot charts.[4] These business intelligence tools promptly and usually effortlessly enable multidimensional views of large quantities of data, enabling answers to questions such as what region has the highest credit risk and in what area type (see Screenshot 2; data are fictional). As the interviewed banker from the software company (the e-banker for short) highlighted, typically just one or two of all managers and analysts that in one bank received training to use pivot tables and charts regularly do so. It may be that, for most users, pivot tables are not effective and easy-to-use interface because they requires, among other things, the understanding of the multi-dimensionality of data being manipulated (O'Donnell, 2005). Also, as the e-banker remarked, the work with pivot tables and graphs differs from the work with spreadsheets and, with employees' frequent resistance to changes, changes are denied and the opportunity to critically examine the data by using several pivot summary representations (that may reveal problematic data) is not used. Thus, in order to attain a critical use of spreadsheets, (some of) employees should use, compare and reconcile the outcomes not only of different models of financial issues, but also of several representations of financial data.

Erroneous use

There are more errors in operational spreadsheets then one would expect and these errors may lead to inadequate or wrong decisions causing (considerable) financial losses. This surprising issue is well documented by, for example, EuSpRIG (European Spreadsheets Risk Interest Group), whose Internet presentation contains many case studies, which reveal that errors are usually found in formulas (see www.eusprig.org/stories). A detailed analysis evidenced that, on average, about 1% of all formulas in operational spreadsheets contain errors (Powell, Baker & Lawson, 2009a). By auditing twenty-five operational spreadsheets, Powell, Baker, and Lawson (2009b) found 117 errors: while 70 errors did not have impact of the output, as many as 27 had an impact of at least $100,000 (with seven being $10,000,000 or more). Among these spreadsheets some were used in a large financial firm that calculated tax liabilities measured in the billions of dollars. As the three authors underlined, "these [tax liability] spreadsheets were astonishingly complex, difficult to understand, difficult to work with, and error-prone." (p. 131) According to Croll (2009), uncontrolled use of spreadsheets in credit derivatives marketplace played a substantional part in the destruction of capital of the global financial system in the period 2007-2009.

In general, most spreadsheet errors can be attributed to the use of wrong data, model, or function attribute (Powell, Baker & Lawson, 2008). They are usually result of (1) applying chaotic spreadsheet design, (2) using numbers in formulas, (3) calculating similar results in different ways, (4) using formulas in a row or a column that change their structure, and (6) using complex formulas. Reasons that prevent their developers to build better spreadsheets may primarily be found in time pressure, unstructured design, changing specification, lack of testing, as well as lack of relevant knowledge and skills (Powell, Baker & Lawson, 2009b).

In classifying credit risk, for example, some banks (and probably most others) in Serbia assumes that an increase in exchange rate of EUR to RSD would be followed by a similar, even higher increase in borrower's salary received in RSD. This means that if the borrower's 200 EUR installment could be monthly covered by 30% of his/her salary in 2009 (a 30% limit is applied for most loans), the same would happen in 2010. Let us assume a fixed salary of 66,000 RSD in 2009 and 2010 (it is slightly above 660 EUR at present, but a 10%-20% reduction may occur as the result of the global financial crisis as did happen in some private firms). Depending on the increase in the exchange rate (e.g. 10% per year), it may happen that this 200 EUR in a year time will consume 35% of the salary or more, compromising the borrower's financial credibility. This undesirable outcome could not be anticipated by using general data a year ago. Of course, credit risk category of each loan is, if need be, updated few times each year, but it is a post-action not a pre-one. However, this individual approach to credit risk may only be relevant to a small number of larger, long-term loans, not justifying (on the basis of an informal or formal estimation, if any) its application to all loans of various types.

Concrete data regarding financial errors due to the use of spreadsheets or other software in some Serbian banks could not be collected, and, although they do occasionally occur as the interviews revealed, these errors cannot, for understandable reasons, be not publicly available. However, it can be said that BI solutions may be questionable if based upon poor data generated by banks themselves, but this outcome banks usually overlook relating errors to the software used. Also, an opportunity to work on financial documents in Microsoft Excel (generated by BI solutions or other employees) by using additional (usually simple) calculations opens a space for introducing errors in these documents.

Although having large and complex spreadsheets that are error free is an unattainable goal, their developers should avoid errors that can be located and corrected, especially those that may cause considerable financial losses. It may be attained with, for example, self-checking built into a spreadsheet (see O'Beirne, 2009, for such checks). If spreadsheet is error free, examining the same financial issue in different ways must produce the same result. This does not happen on Screenshot 3 because the last attribute of function VLOOKUP (omitted at present) must be FALSE (supporting exact match with an arbitrary ordered credit risk categories). The error in this hypothetical example was relatively easy to locate. Its source was suggested by an MS Excel add-in for spreadsheet auditing called *Spreadsheet Professional* (available at www.spreadsheetinnovations.com). Although this tool reported that no general test errors were discovered, it pointed out that, for VLOOKUP function, the range being searched had to be in the ascending order or errors might occur.

Screenshot 3—Securing credit risk according to borrower's financial credibility

C10 =A10*VLOOKUP(B10,$A$2:$B$5, 2)

	A	B	C	D
1	Credit risk category	Part of loan amount to secure credit risk	No of loans by category	Total amount to secure credit risk by category
2	A	1.00%	2	193.46 €
3	B	5.00%	2	568.02 €
4	G	25.00%	1	6,250.00 €
5	D	100.00%	1	4,280.33 €
6		Total	6	11,291.81 €
7				
8	Amount of loan to return	Credit risk category	Amount to cover credit risk	
9	25,000.00 €	G	6,250.00 €	
10	4,280.33 €	D	214.02 €	
11	2,000.00 €	B	100.00 €	
12	12,645.75 €	A	126.46 €	
13	6,700.00 €	A	67.00 €	
14	9,360.38 €	B	468.02 €	
15			7,225.49 €	

Total amount to cover credit risk (amount frosen for bank use; when a loan is returned this total is reduced for the credit risk of its loan that is bank free to use; the amount updated quarterly).

Closing Remarks

Financial industry cannot be imagined without the widespread use of spreadsheets. Because of that, financial institutions should cultivate a critical approach to spreadsheet models and their data (including software tools for auditing too), which questions their appropriateness and correctness as well as compares and reconciles the outcomes of different models and views. Such an approach, as this paper shows, seems missing in most financial institutions at present. As many people who create and use spreadsheets are self-taught (Croll, 2005), this approach calls for the development of an appropriately designed pre-service and in-service vocational courses at secondary and tertiary levels.

According to Hoyles, Noss, Bakker, and Kent (2007), a good learning design promoting techno-mathematical literacies in the workplace calls for authenticy (examine actual workplace events) , visibility (make invisible relationships visible) and complexity (reflect real situations in alternative ways). Inspired with these requirements, promoting these literacies in the context of this paper should require: (1) understanding data and, if needed, questioning and improving their quality, (2) understanding (some of) relationships in models and, if needed, questioning and improving these relationships, and (3) understanding models and, if needed, questioning and improving them.

In the courses above mentioned students may develop various deterministic and non-deterministic spreadsheet models involving optimisations. However, the development and application of simple deterministic spreadsheet models without optimisations should be widely practiced first. Even in this kind of modelling students make various errors that should be carefully pedagogically treated (Kadijevich, 2009), which would promote a critical approach to spreadsheet models and ease the work with more complex models latter.

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Notes

1 At that time, thirty five banks were members of the Associations of Serbian Banks. Thirty two of them return the questionnaire and among them fourteen reported the use Microsoft Excel, with some banks possibly using other software as well (www.ubs-asb.com/s/Download/Rezultati_Upitnika_2.pdf).

2 Eurobank EFG Group, a member of the largest Swiss-based banking groups, is now present in Bulgaria, Cyprus, Greece, Luxembourg, Poland, Romania, Turkey, Serbia, Ukraine, and United Kingdom (www.eurobankefg.rs/upload/ documents/press/2009/fin_results_EnglishFY%202008.pdf).

3 This method is explained in detail at www.riskmetrics.com/system/files/private/CMTD1.pdf (to examine this file the reader need to be a registered user).

4 An account on pivot tables can be found at http://en.wikipedia.org/wiki/Pivot_table, for example. See also numberGo, a free BI tool for multidimensional data views available at www.numbergo.com/index.aspx.

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Authentic Modelling Problems in Mathematics Education

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Abstract The paper presents experiences with modelling activities at the University of Hamburg, in which small groups of students from upper secondary level intensely work for one week on selected modelling problems, while their work is supported by pre-service-teachers. The paper describes the theoretical framework of the activities, the examples dealt with and underlying pedagogical principles. Finally some results of an evaluation are presented, that has been conducted after one Modelling week.

Theoretical framework for modelling in mathematics education

Within mathematics education it is consensus, that applications and modelling shall play an important role. However, how the implementation of these kinds of examples shall take place, which kinds of examples shall be used, how the modelling processes shall be implemented in school is highly debated. Influenced by the goals, which are connected to the teaching of applications and modelling and mathematics education in general, one can distinguish different perspectives of the modelling debate worldwide: there are amongst other perspectives, which emphasise the use of authentic problems — named in a framework developed by Kaiser and Sriraman (2006) as realistic or applied modelling. Other positions emphasise more pedagogical goals such as the development of concepts or the structuring of learning processes. As an overall perspective a meta-perspective is discriminated, called cognitive modelling, which focuses on the cognitive processes taking place during modelling activities. Due to space limitations we refuse to go into more detail concerning the other approaches developed and refer to Kaiser and Sriraman (2006) and the extensive ICMI-study on modelling (see Blum et al. 2007).

The approach described in the following belongs to the so-called realistic modelling perspective. Based on extensive own empirical research (see for example Kaiser-Messmer, 1986) we see the necessity of treating authentic modelling problems, which promote the whole range of modelling competencies and broaden the radius of action of the students. The approach takes as essential starting point that the promotion of modelling competencies needs own experiences of the students carried out with authentic modelling problems. Similar proposals are developed by Haines and Crouch (2006) or already at the beginning of the modelling debate by Pollak (1969).

Authentic problems are defined as problems that are only little simplified and are according to the definition of authentic problems by Niss (1992) recognised by people working in this field as being a problem, they might meet in their daily work. As reference framework of the approach, on which the realisation of modelling weeks are based, the realistic or applied modelling can be stated. Central feature here is the usage of authentic modelling problems in order to implement pragmatic-utilitarian educational goals like understanding of the real world or the promotion of modelling competencies. In consequence, the treatment of modelling examples as a central element in mathematics lessons is followed. These examples then should articulate the relevance of mathematics in daily life, environment and sciences, and impart competencies to apply mathematics in daily life, environment and sciences. The conduction of modelling weeks is considered to be a powerful and effective way to promote these examples.

Structure of the Modelling week

The project 'Mathematical Modelling in School' was established in 2000 by the Department of Mathematics (Jens Struckmeier and Claus Peter Ortlieb) in co-operation with Didactics of Mathematics at the Department of Education (Gabriele Kaiser) at the University of Hamburg. It was originally a university course project with future teachers for upper secondary level teaching and aimed to establish a conjunction between university and school. Within the project the future teachers supervised student groups from upper secondary level (aged 16-18 years) during their modelling activities. It was intended that each group works independently on one modelling example within the regular lessons or in separate activities. The concept has been described at several places and will in the following not be described in detail due to space constraints (cf. Kaiser, & Schwarz 2006).

One aim of the project has been since its establishment that the participating students will acquire competencies to enable them to carry out modelling examples independently, i.e. the ability to extract mathematical questions from the given problem fields and to develop autonomously the solutions of real-world problems. It is not the purpose of this project to provide a comprehensive overview about relevant fields of application of mathematics. Furthermore, it is hoped that students will be enabled to work purposefully on their own in open problem situations and will experience the feelings of uncertainty and insecurity which are characteristics of real applications of mathematics in everyday life and sciences. An overarching goal is that students' experiences with mathematics and their mathematical world views or mathematical beliefs are broadened. This kind of approach can be described as holistic approach, using the terminology of Blomhøj and Jensen (2003), i.e. a whole-scale mathematical modelling process is carried out covering all modelling competencies described above.

The central feature of this project is the usage of authentic examples, i.e. that most of the problems are proposed by applied mathematicians who work in industry and who have met or tackled this problem within their working environment. The problems are only little simplified and often no solution is known, neither to us as organiser of the project nor to the problem poser. For example, the problem of unique identification of fingerprint is not solved satisfactorily until now and the machines are not working on a reliable basis which creates a strong contrast to the widespread usage of machines for fingerprint identification at many places. Quite often only a problematic situation is described and the students have to determine or develop a question that can be solved. The development and description of the problem to be tackled is the most important and most ambitious part of a modelling process, mostly neglected in ordinary mathematics lessons. Another feature of the problems is their openness which means that various problem definitions and solutions are possible in dependence of the norms of the modellers.

The teaching-and-learning-process is characterised as autonomous, self-controlled learning, i.e. that the students decide upon their ways of tackling the problem and no fast intervention by the supervisors takes place (or should take place). The future teachers are expected to offer no more than just assistance, if mathematical means are needed or if the students are heading into a cul-de-sac. With this kind of teaching approach the students experience long phases of helplessness and insecurity, which is an important aspect of modelling and a necessary phase within a modelling process.

We have used modelling activities in the following organisational forms:

- modelling activities during ordinary mathematics lessons spanning over three months, at the beginning of the project, we tackled two problems lasting the whole school year
- modelling activities within a modelling week within the school year

Due to organisational problems caused by the introduction of central school leaving examinations in the core subject such as mathematics in Hamburg, we decided in 2008 to change our organisational concept in the direction of a modelling week. The idea of modelling weeks has been developed at the University of Kaiserslautern by the working group of Helmut Neunzert (for descriptions and examples see for example Bracke, 2010), who are already running modelling weeks for more than a decade and was adopted by other groups in Germany, e.g. at the University of Darmstadt. We have until now carried out modelling weeks two times, in March 2009 the first time with 350 students from 19 schools from Hamburg and its surrounding and in September 2009 with 180 students from 10 schools. It is planned to carry out modelling weeks twice a year. Some of the schools had sent the whole upper-secondary level course for the modelling week, only a few participated with parts of the course. In contrast to other forms of modelling weeks, we aim at working with average students, not especially interested in mathematics.

Within the first modelling activity, which took place within the usual classroom teaching, we have used the following modelling problems

- Share price forecast
- Mathematics in private health insurance
- Prediction of fishing quotas

- Optimal position of rescue helicopters in South Tyrol
- Radio-therapy planning for cancer patients
- Identification of fingerprints
- Pricing for internet booking of flights
- Price calculation of an internet café
- Traffic flow during the Soccer World Championship in 2006 in Hamburg
- Construction of an optimised time table of school

During our recent modelling activities carried out as modelling week taking place at the university, the students dealt with the following four problems, which are already described elsewhere (see for example Kaiser, & Schwarz 2010):

- How can the mixture of chemicals in swimming pools be optimized?
- How can the arrangement of irrigation systems be planned in an optimal way?
- How can the spread of a sexually transmitted disease with ladybugs be predicted concerning the development of the population itself?
- How can Greek land turtles be identified on the basis of firm characteristics of the tortoise shell?

Evaluation of the modelling week

In the following we will report about results of the evaluation of the modelling activities described in the last chapter. We will focus on the central question for our research, whether these kinds of complex problems are feasible for students from upper secondary level, who are interested in mathematics, but not particularly selected.

For the recent evaluation we used a questionnaire with four mainly open questions on the beliefs of students about mathematics teaching and three open questions on the appraisal of modelling examples tackled in the modelling week and five closed questions to be answered on a 5-point-Likert-scale. The questionnaire was filled in at the end of the first modelling week from 289 students. Based on methods of the Grounded Theory (see Strauss, & Corbin 1998) we have used for the open questionnaires in-vivo-codes, i.e. codes extracted out of the text written by the students as verbatim quotation and grouped them to related quotations under a theoretical perspective. In order to finally analyse the answers of the students in the open questionnaires we transformed the grouped in-vivo-codes into theoretical codes. For quality assurance methods of consensual coding were relied on, which means that a coding team consists of two coders who conduct all steps described above together. In the following we describe the appraisal of the modelling week restricted to a few aspects due to space limitations.

Learning outcomes of the modelling week

We start with the appraisal of the modelling examples by the students. The first question dealt with the appreciation of the learning outcomes of the modelling week.

The students answered on the question "From your point of view, what did you learn when dealing with the modelling example?" as follows: 71% of the students answered positively, only 12% did not see any learning outcomes from the modelling week. From the diagram in figure 1 it can be seen, in which area the students see by themselves learning results.

Particularly high learning successes have been achieved in area of the general application of mathematics and working techniques. With regard to applying mathematics, many state that due to working on these modelling problems they understood how mathematics is applied and recognise its practical relevance, and relevance to everyday life. Furthermore, some students say that, in fact, they are able to apply what they have learned in school. In the following we exemplify the reasons by selected verbatim quotations of the students (in italics):

> "… that mathematics not only can be applied in school, but also to specific examples in real life." (Female student, 19 years old)

> "… that mathematics can also be applied to things in everyday life." (Male student, 17 years old)

With regard to working techniques, many students report that they learnt about new strategies in problem solving and now understand how to approach problems which at first seem to be vague.

> "…that it is possible to approach a problem in various ways, based on different assumptions." (Female student, 17 years old)

In addition, many respondents mention that they have improved their ability to work in-

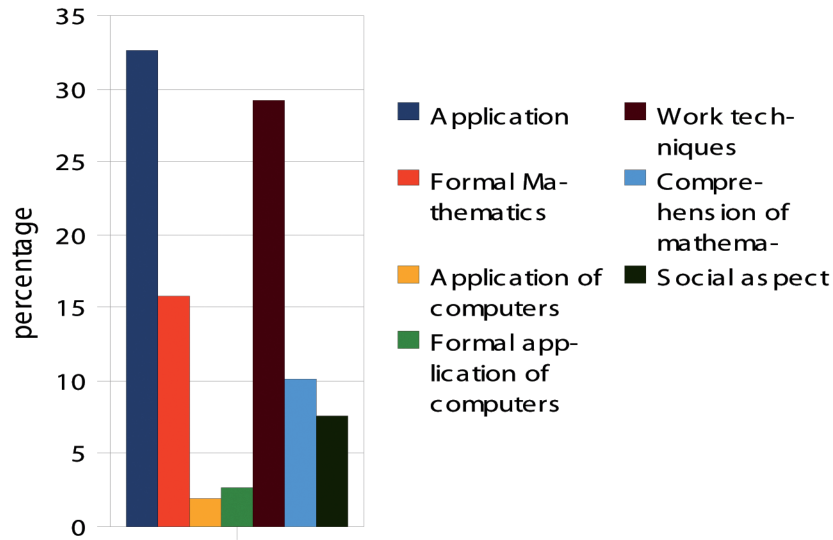

Figure 1—Learning outcomes of the modelling week

dependently, to have more perseverance, and that they now find it easier to structure and simplify.

> "... structured, co-ordinated and deliberate planning and working." (Male student, 19 years old)

> "I learnt to compare different approaches in problem solving and to deal with a problem using quite varied methods and approaches." (Female student, 18 years old)

> "... to solve arising problems autonomously." (Female student, 16 years old)

Other areas where some learning successes were achieved refer to mathematics, and geometrical methods in particular. Besides, many of the students state that they improved on deriving and constructing functions and formulae.

> "... calculating sectors of a circle, combinatorics and approximation of an area for which no specified formula existed to calculate it." (Female student, 17 years old)

> "[...] However, what is most important is that I learnt how to combine the dependencies of different parameters within a function." (Male student, 16 years old)

The field of mathematical understandings comprises changes in how to think about mathematics. Almost 10% of the students state that only because of this week and the intensive work on a quite open topic they realised how complex and wide-ranging mathematics is. Furthermore, they realised how many aspects are integrated in mathematics and what large variety of solutions are available.

> " ... that even in mathematics you do not always have to arrive at a solution in order to solve a problem" (Male student, 18 years old)

> " ... that there is not always an accurate solution to everything, even though it has to do with mathematics." (Female student, 17 years old)

Social aspects are emphasised as well: 10% of the students mention that during the week they not only got to know their fellow students and students of other schools but, most of all, learnt something about teamwork.

> "...to work well/better in groups and how important a good working atmosphere and teamwork is." (Female student, 18 years old)

It can be summarised, that most students describe as their personal impression that they have achieved learning success during the modelling week. Furthermore, the range of aspects mentioned cover all goals connected with the inclusion of modelling problems.

Inclusion of modelling problems in ordinary mathematics lessons

One central aspect is the question, whether the students want to have modelling examples included in their usual mathematics lessons or whether they want to keep these kinds of ambitious examples out of their usual lessons. This aspect is dealt with in the next question, where the students are asked the following question: "Should these examples be increasingly dealt with as part of regular maths classes or would you reject this?"

62% of the students make a plea to include these kinds of modelling examples in their usual mathematics teaching, 28% reject that and 10% do not have an opinion.

An analysis of the negative answers shows the main reasons for this answer (figure 2).

The most important reason for the rejection of modelling examples in ordinary mathematics lessons is the time constraint:

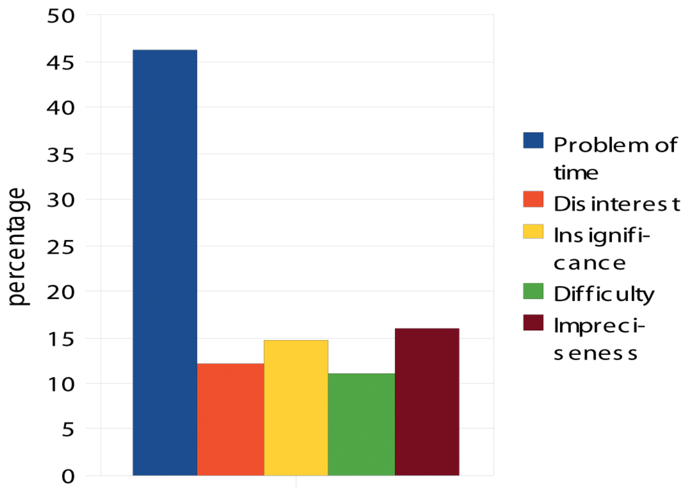

Figure 2—Inclusion of modelling examples in ordinary lessons — negative answers

To my mind this would be too time-consuming. It is more important to master the subject itself." (Female student, 18 years old)

"I think there would not be enough time during maths class to work on problems in such detail." (Female student, 18 years old

The other reasons given by the students such as "not interesting," "not relevant", "too difficult" or "imprecise" are given by similar numbers of students:

"It would not make sense to use this in class, because it has little relation to maths. It is more like something you could puzzle over." (Male student, 17 years old)

"No, because the tasks were not narrowed down and there were too many possible solutions to choose from." (Female student, 17 years old)

The results show clearly the time pressure students are working under in upper secondary level. Furthermore it shows that many students were simply not acquainted with independent work on open problems.

As already stated 62% of the students supported the inclusion of modelling examples in their ordinary mathematics lessons. The reasons for this decision can be separated into four distinct groups, i.e. relation to reality, advancement of working techniques, alternation, sense making. The fifth group of answers express the need for a new way of dealing with these kinds of problems before they are treated in ordinary mathematics lessons.

The following quotation describes the position of this last group:

"Yes, but the problems should be solvable concretely and more structured (problem description, support)." (Male student, 18 years old)

Most frequently the students mentioned as reason for the inclusion of these kinds of examples their relation to the real world. Many students emphasise that only with these kinds of examples they could develop a relation between mathematics and the real world:

> "I think that these examples should DEFINITELY be dealt with in maths classes. Because of these examples one will only realise what mathematics is needed for." (Male student, 16 years old)

> "I think so, because it illustrates the importance of mathematics for everyday life." (Male student, 16 years old)

> "These examples should be dealt with in order to establish a stronger connection between the real life and what is taught at school." (Female student, 18 years old)

Motivation was the second most common reason for advocating the inclusion of these kinds of examples:

> "I think it would be better to introduce less theory but more practical use in class to increase the students' motivation." (Female student, 18 years old)

> "Yes, because this will make class more interesting." (Female student, 17 years old)

Many students mentioned the usefulness of these kinds of problems, being much higher than that of usual examples. Modelling examples are seen to promote complex mathematical thinking, to exercise learned topics, and to allow insight into the work of mathematicians:

> "Such problems SHOULD be dealt with in class, because they will improve the so-called 'competence in problem solving'. Furthermore, it will train the knowledge gained in previous grades." (Male student, 19 years old)

> "I am in favour of dealing with these topics in class, because this will be good practice and they are good exercises." (Male student, 18 years old)

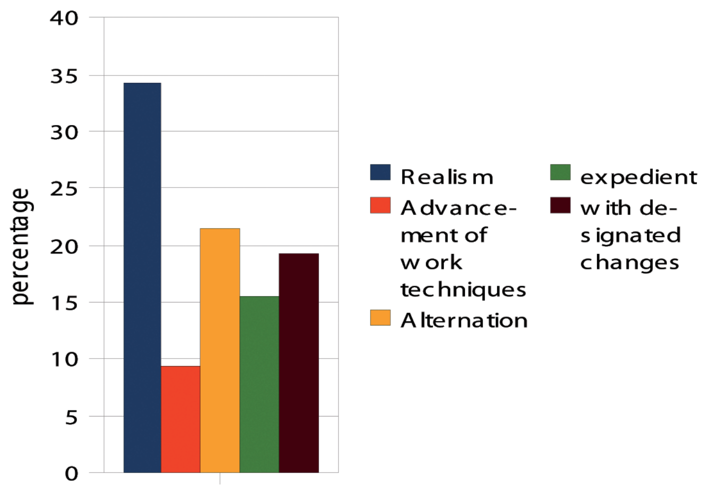

Figure 3—Inclusion of modelling examples in ordinary lessons — positive answers

> “Yes, because one automatically relates more closely to the topic and the relevant example.” (Male student, 18 years old)

The promotion of working techniques is emphasised by 10% of the students, who say that working with modelling examples promotes perseverance, and skills for independent and group work:

„I think it would be good if such problems were dealt with in regular classes, because they support working independently and require teamwork.” (Male student, 17 years old)

> “I should think so, because with something like this you have to be very proactive and think about it yourself quite a lot.“ (Male student, 17 years old)

> “Partially, because on the one hand, it really takes up a lot of time. On the other hand though, the teamwork is beneficial for the working atmosphere in class.” (Female student, 18 years old)

Summing up, a majority of the students are in favour of the inclusion of these kinds of modelling problems and describe as reasons for that many of the goals discussed in the didactical debate.

From our evaluation it is obvious that these kinds of examples can be tackled successfully by ordinary students at upper secondary level. The students describe high learning outcomes that reflect all the goals connected with modelling, ranging from psychological goals such as motivation to meta-aspects such as promoting working attitudes to pedagogical goals, namely enhancing the understanding of the world around us. The strong plea of the students for the inclusion of these kinds of examples in usual mathematics lessons support our position that it is appropriate to include these kinds of problems in ordinary mathematics lessons, clearly not everyday, but on a regular basis.

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The Research of Mathematics Teaching Materials for Senior High School Students Who Want to Become Scientists and Engineers

Links With the Essence of Keplerian and Newtonian Science

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Abstract In addition to the insufficient connection to the university education, an estrangement from science and mathematics is great problem in mathematical education of senior high schools in Japan. So, in order to connect mathematics to the fields of Newtonian science with mathematical modeling, it will be necessary to provide concrete examples such as "Kepler's Laws". Consequently students will realize the necessity of differential equations in order to analyze actual phenomena. This empirical research suggests that the mathematics materials with physical points of view are effective for senior high school students.

1. Introduction

It is great problem that the scholastic ability of mathematics among younger generations is decreasing and the number of senior high school students who want to major in science or engineering in university, is also decreasing. So, The Japanese government revised the national course of study in 2007/08 to solve these problems. However many senior high school teachers think that it is most important for students to succeed in university entrance examinations. They teach the contents required for use in those entrance examinations which don't include any relations between mathematics and science or technology. Therefore many students will lose interest in studying mathematics and science.

The reason why students study mathematics is "to understand its necessity and to learn the concepts". We think that the application of mathematics and the solution of problems by mathematical modelling are necessary to achieve these 2 purposes. Please look at Fig.1. This graph shows the low knowledge of the students to the background of the mathematics theory used for Kepler's Laws. They do not know the deep meaning though they know the formal theory.

Some possible causes

(1) In mathematics and science education in Japan, the curriculum for modeling activities doesn't exist.

(2) "Only studying for the entrance examination of a university is the reason to study mathematics." — Recent educational situation in Japan.

(3) The school system in Japan doesn't permit mathematics teachers to teach other subjects such as physics or information science and processing.

- Students do not study unrelated mathematics for the entrance examinations of universities or senior high schools.
- According to 2003 PISA (Programme for International Student Assessment), the junior high school students in Japan are not good at the analysis of various problems in a scientific situations using mathematics (Less than average score of OECD).
- A lot of senior high school students don't have basic knowledge of the connections between mathematics and other areas (Fig.1).
- Not only the students and the citizens but also teachers lack the recognition of the necessity of mathematics.
- Facing an unknown problem where one should compose a new mathematics model, Students' scholastic ability is low — understanding, transforming knowledge, and

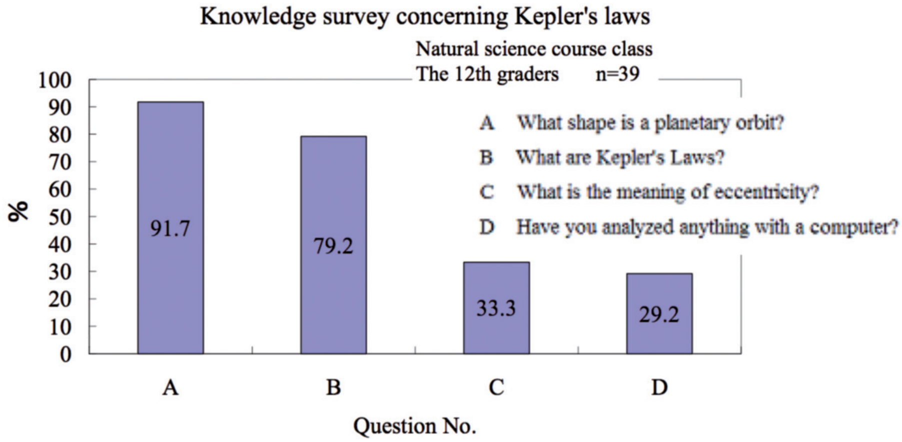


Figure 1—Knowledge survey concerning Kepler's Laws

using expression by using learnt mathematics. They don't study to graduate because there is no graduation examination.

We worry about these situations. We want to change this situation for the better. So, we propose the following mathematical contents and a teaching method to study the relationship between mathematics and science and to foster the scientific spirit needed to become mathematicians, scientists and technologists in the future.

We have developed teaching materials to learn from first order to second order differential equations based on Kepler's Laws.

It is important to prepare the tools necessary for the proof of Kepler's Laws. Fig.2 is the system diagram simplified as much as possible. As for this feature, the simple harmonic motion that appears in the high school physics can be treated by differential equations. And this second order differential equation relates to the law of universal gravitation. Contents of other necessary mathematics are polar coordinates and linear transformations. Details are described in Chapter 2.2.

2. A practice case using this modelling.– The teaching materials using "Kepler's Law" for high school students who want to become scientists

This case is not the only suggestion for modelling. In this lesson, there is the meaning that we should solve the problems of school education from the preceding chapter. We want to recommend the proof of the elliptical orbit of the planet as a mathematics material with physical viewpoints, because the essence of modern science is connected to Keplerian and Newtonian science. Students advanced the following practice by Kawasaki's support.

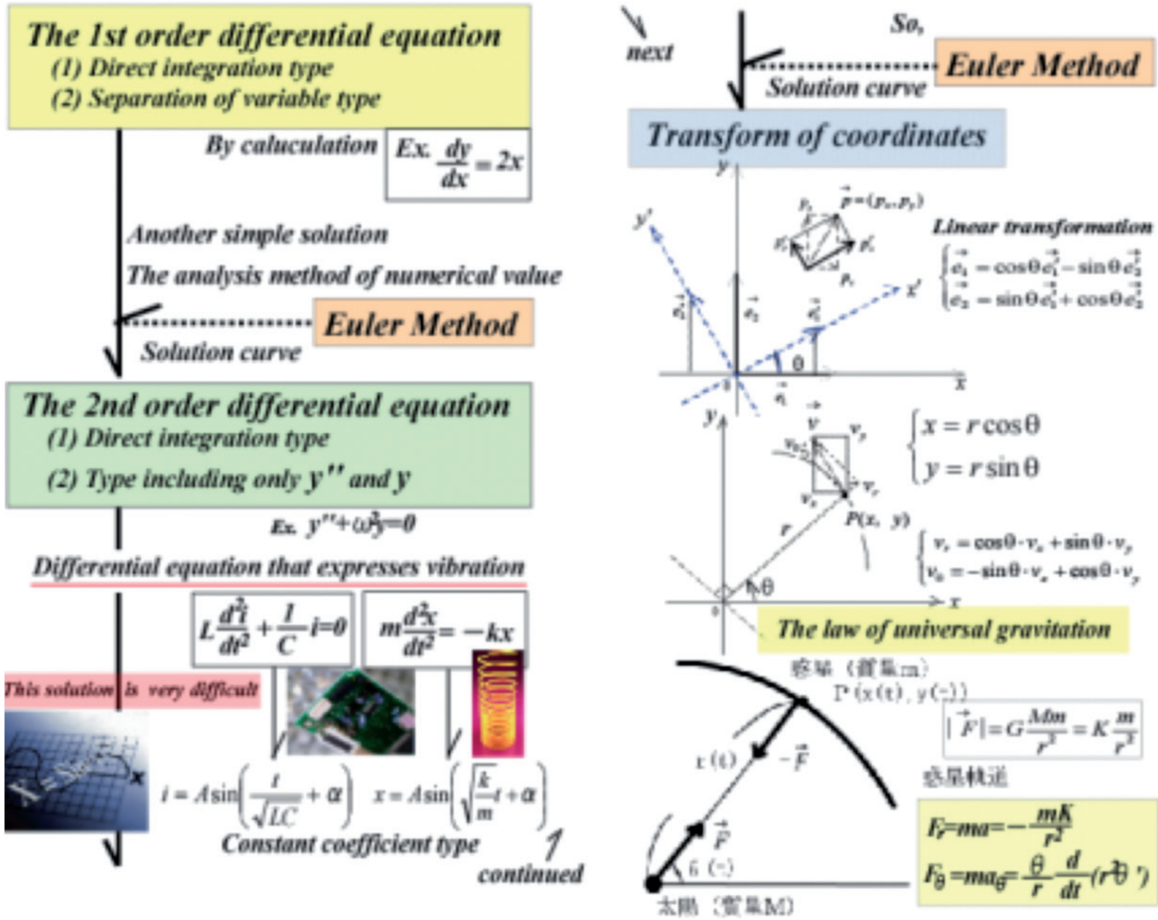


Figure 2—The concept map of teaching material based on Kepler's Laws

2.1. Pre-model

First of all, Students will practice by using the observational data of Mercury. This will be named the elementary model to understand Kepler's Law.

The elliptical orbit of Mercury appears by the crowd of the tangential lines, "Kepler's 1st Law". And, when the length of observational periods are equal, the sectoral areas caused by the segment that connects the sun with Mercury are equal, "Kepler's 2nd Law".

Usually only this model must be satisfactory, and as a result, we think that students will confirm the image of Kepler's Law. However, it is doubtful whether the students understand these rightly. They admitted this when drawing the orbit. I couldn't believe they really understood the movement of Mercury. As written in the preceding chapter, it is difficult for them to develop their past knowledge or convert their ideas. In fact I showed them the simulation of two planetary movements. And, I made them judge which was correct (Fig.5).

Year	Eastern Max. angle (E)		Western Max angle (W)	
1990	April 14	20°	February 1	25°
	August 12	27°	May 31	25°
	December 6	21°	September 24	18°
1991	March 27	27°	January 14	24°
	July 25	27°	May 13	26°
	November 19	22°	September 8	18°
			December 28	22°
1992	March 10	18°	April 23	27°
	July 6	26°	August 21	19°
	November 1	24°	December 9	21°

Table 1—The observational data of Mercury

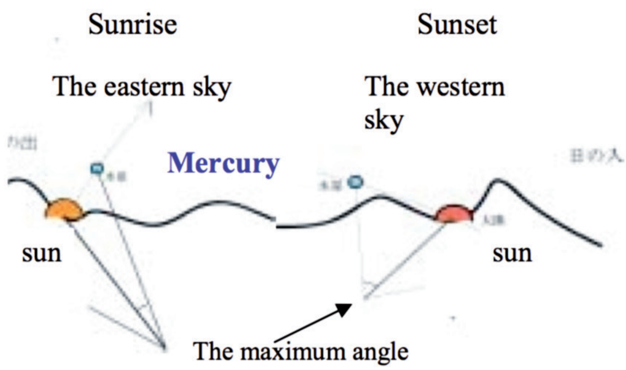


Figure 3—The maximum angle

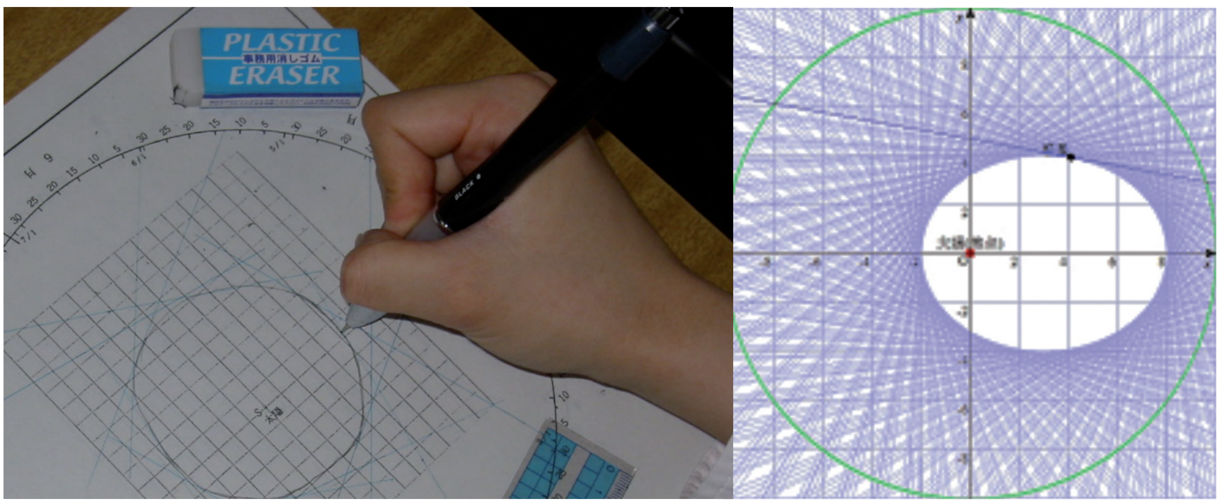


Figure 4—Drawing in Mercury`s orbit by a lot of tangential lines: (left) elementary model, (right)

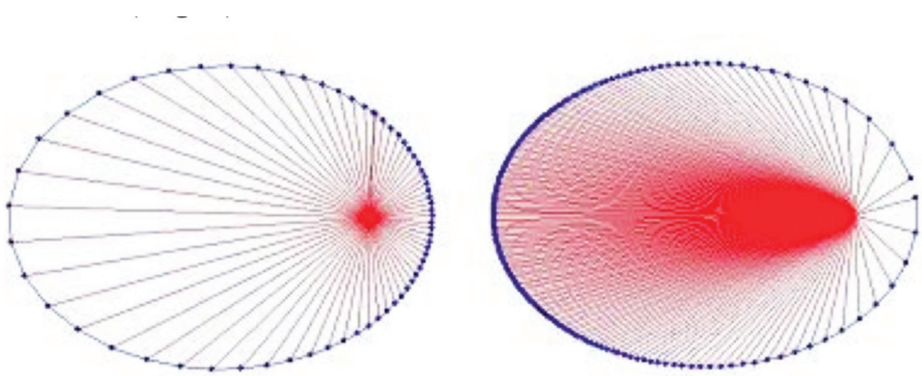
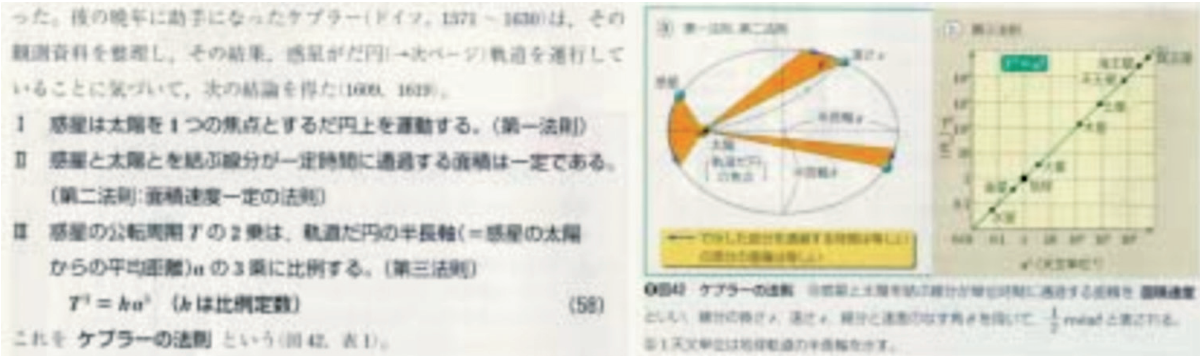


Figure 5—(Left) Uniform motion (Right) Planetary motion


った。彼の晩年に助手になったケプラー（ドイツ，1571～1630）は，その観測資料を整理し，その結果，惑星がだ円（→次ページ）軌道を運行していることに気づいて，次の結論を得た（1609，1619）。

I　惑星は太陽を1つの焦点とするだ円上を運動する。（第一法則）

II　惑星と太陽とを結ぶ線分が一定時間に通過する面積は一定である。（第二法則：面積速度一定の法則）

III　惑星の公転周期 T の2乗は，軌道だ円の半長軸（＝惑星の太陽からの平均距離）a の3乗に比例する。（第三法則）

$T^2 = ka^3$　（k は比例定数）　(58)

これを **ケプラーの法則** という（図42，表1）。

Figure 6—Formal knowledge and the illustration of Kepler's Laws (An excerpt from our high school physics textbook in Japan)

At once students couldn't judge, and worried about the selection. Many students selected the left movement. At this time, I was notsurprised and I became certain of my doubts.

This elementary model is not enough for the students' knowledge to mature. But it has some good results as a model. Before long they will demand a new model. Therefore, we call it the pre-model that prepares for a new model. It is necessary to prepare new mathematical contents. The new model is shown in the next section. We think that it becomes one composition through these two models.

The new model has a means to lead to a new mathematics unit. Students will make the best use of the new model for the next practice.

2.2. Mathematical development model

The differential equation by the law of universal gravitation (Newton's Law) is necessary to prove Kepler's Laws. However, high school physics doesn't prove Kepler's Laws. The illustration of the textbook is only presented (Fig.6). High school math mainly treats the ellipse in orthogonal axes coordinates. High school physics mainly treats uniform circular motion. Both high school math and physics only show formal knowledge. Students will not be able to notice math's necessity or be able to learn correct thought by applying math. We are uncertain that they can gain the spirit of modern science well from the situation of our school education.

Then, we newly took the differential equation. It is necessary to prepare new contents to apply math. And, it becomes an introduction of the differential equation. There are a lot of methods to solve the differential equation. The equation of Kepler's Laws by Newton's Law is a second order linear differential equation with a scalar type constant number. This is very difficult and they can't find the new model from only their experiences. A minimum

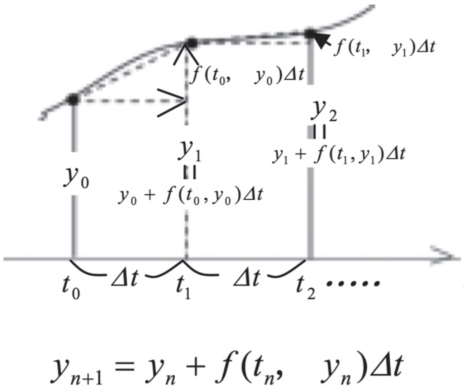


Figure 7—Euler's method

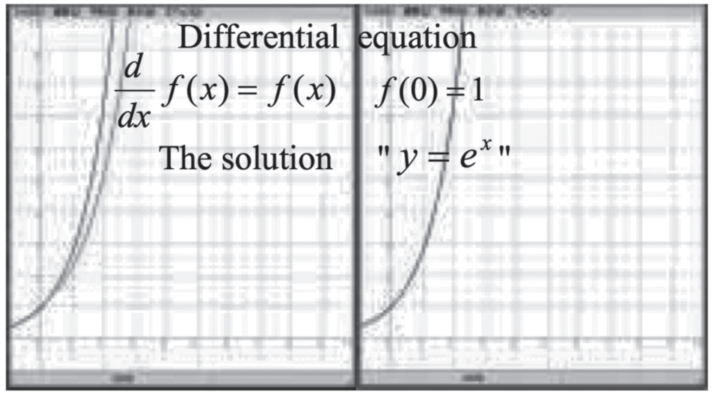


Figure 8—(left) Euler method, (right) Runge-Kutta method

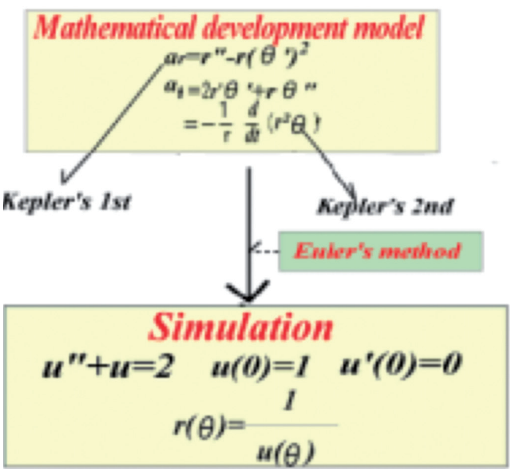


Figure 9—Mathematical development model

of support by the teachers is necessary. In addition, I tried to help the students' gain understanding, using the analysis method of numerical values by computer programming. They will be able to have full realization of Kepler's Laws and Newton's Law by drawing the solution curve using Euler's method (Fig.7).

The solution curve by Euler's method has the fault that the error grows when the input value parts from an initial value (Fig.8). When teachers give the student the base program, regard to the error is necessary.

The modeling shown in Fig.2 shows that the new mathematical development model (Fig.9) has a part that guides students to the new mathematics unit.

Fortunately, my students have already studied the formal theory as well as the calculation of differentiation and integration.

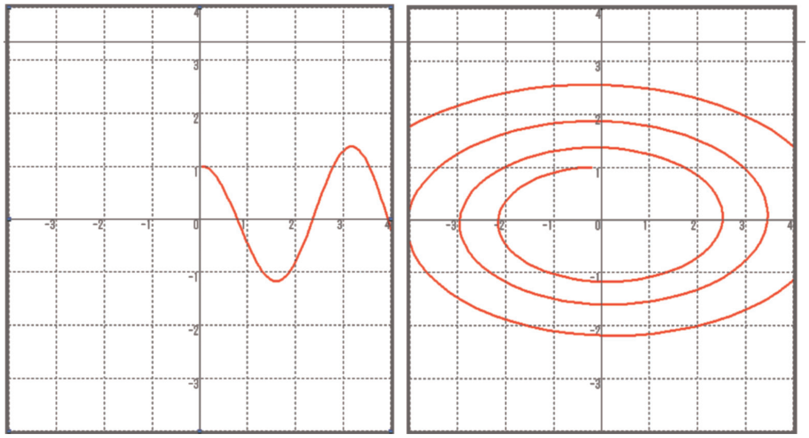


Figure 10—Simple harmonic motion by Euler's method

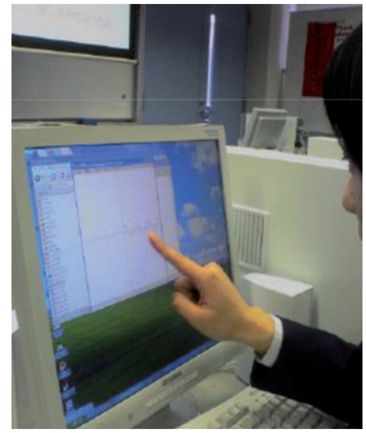


Figure 11—The work-sheet

Through these tools prepared well, students could smoothly make the mathematical development model by the work-sheet (Fig.11) and computer programming.

Finally, students verified the models of Kepler's motion which they made by themselves.

They solved

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{c^2}$$

made by Newton's Law.

In next initial condition, $u(0) = 1$, $u'(0) = 0$.

In addition, if this expression is transformed into

$$\frac{GM}{c^2} = 2, \quad r(\theta) = \frac{1}{u(\theta)}, \quad r = \frac{\frac{1}{2}}{1 - \frac{1}{2}\cos\theta}$$

is solved.

This means the polar equation of an ellipse is,

$$r = \frac{\ell}{1 - e\cos\theta} \quad (\ell > 0,\ e > 0).$$

And, they confirmed whether this solution accorded with the drawing of the solution curve.

3. Students' evaluation and impressions

These two modelling stages showed utility as an educational content with consciousness of the interrelation between science and mathematics. One student's impression was "This

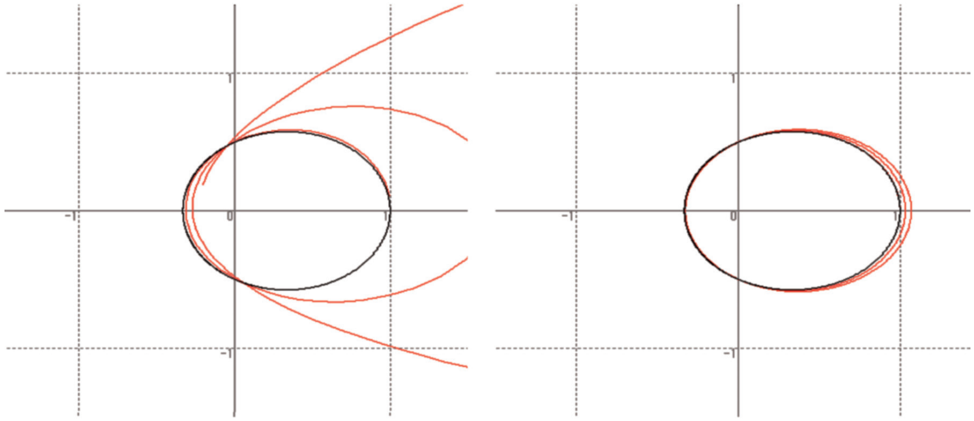


Figure 12—Simulation result by students: (left) $d\theta = 0.1$, (right) $d\theta = 0.01$

lesson was a valuable experience of learning the interest of natural science before I get into college".

But we gave a lot of content in few classes. It seems that students had difficulty understanding the content

though they were interested in the content.

Another student's opinion, "Both the content of modelling and guidance to Kepler's motion is too greedy". A little more time might have been necessary so that the students might master the knowledge. However, this modelling aims at the introduction into a new mathematics unit like differential equations by making a mathematical development model. The students would surely have experienced this step with enough practice.

In the school education of Japan, there is no custom of using information technology effectively.

The strong and weak points of the programming were caused by the students' ability.

This is especially difficult if there is no educational environment that properly treats information technology.

Moreover, there was a student who verified the situation in which the eccentricity was changed (Fig.14).

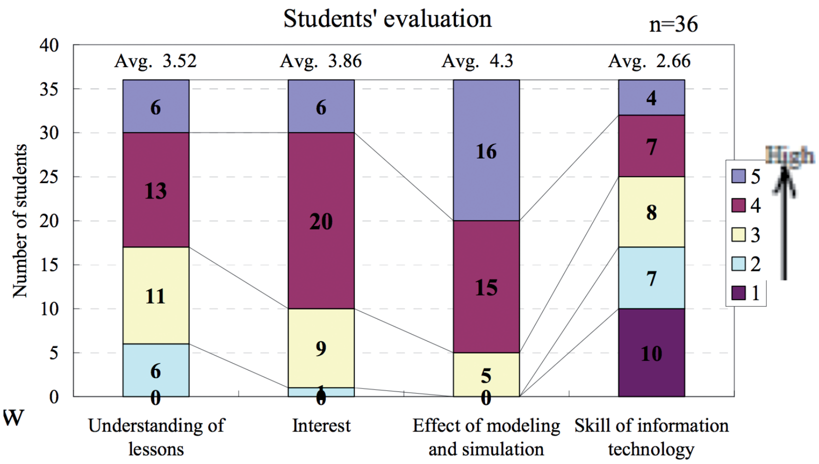


Figure 13—Students' evaluation

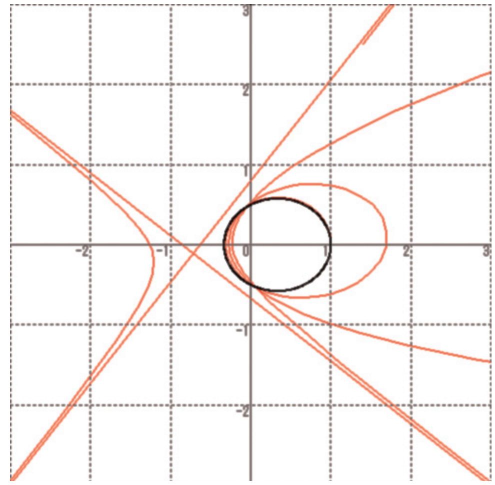


Figure 14—Exercise that student presented

4. Conclusion and future subjects

In this practice case we have gotten the result of making students understand the true concept of mathematics theory against the background of a great scientific discovery. And they could learn the relation between school mathematics and mathematics used in an information-intensive industry. Moreover, we tried to solve various worries about school education even if only a little. Students have learnt some mathematics knowledge from the mathematical modelling, elements of differential equations and computer simulation through understanding Kepler's Laws. These contents are important and necessary knowledge if students aim to be scientists, mathematicians or engineers in the future.

The high school students in Japan are not accustomed to problem solving by modelling. There are few chances for them to notice the necessity of mathematics, when students study mathematics. Even if it is at a late time when students treat the material, we think that it is necessary to improve the teacher's guidance by applying such modelling. This time, students made the mathematical model with teacher support. We do not know whether it was the minimum of support needed. But in the future, they might have to research in teams or solve problems alone. We want to make students get as much such self-help as possible at early stages of their growth.

This practice case seemed to show that mathematics was useful for science and technology. Students could catch a glimpse of scientific spirit. Though not all of the students achieved, they experienced and understood the concept of the mathematics theory. Such modelling practice is important for students who aim to become scientists and engineers and for students to become wise citizens.

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Looking at the Workplace through Mathematical Eyes – An Innovative approach

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Abstract The initiating premise of this paper is that mathematical ideas and techniques proliferate in everyday workplaces, but are dismissed as 'common sense' or 'part of the job'. This 'invisibility' presents difficulties for recruitment, adaptability, response to change, training, mobility and the Recognition of Prior Learning (RPL). Making such mathematics more visible, provides a starting point at which to address these issues. However, the mathematics that underpin the performance of work tend to be obscured from view, being encapsulated by habit and procedure, wrapped in a 'job' description, in a context of preceding and dependent jobs that comprise an organisation. This paper describes an innovative approach to penetrating the obscuring layers, to reveal the underlying mathematics content and benchmarking them with the National Framework of Qualifications in Ireland.

Introduction

The characterisation of mathematics in industry, to the extent that it exists, is contested, being described variously as techno-mathematical literacy (Hoyles, Noss, Kent, & Bakker, 2007), functional mathematics (Wake, 2005), quantitative literacy (van der Kooij & Strasser, 2004; Coben, 2009; Steen, 1997), numeracy (OECD, 2008; Maguire & O'Donoghue, 2003), and realistic mathematics (van den Heuvel-Panhizen, 1998; Treffers, 1987). The proliferation of such similar terms seeking to nuance mathematical content, tends to exacerbate the struggle to achieve shared meaning and some measure of generalisation. More than ever, problem solving, spatial awareness, estimation, interpretation and communication skills, are highly valued in the modern worker, as being essential to support change, reaction and response (Expert group on Future Skills Needs, 2009; O'Donoghue, 2000), especially given the pervasiveness of ICT and 'black boxes'. However, Mathematical Knowledge, Skill and Competence (MKSC) that underpin work, may be dismissed as 'just part of the job' (Coben & Thumpston, 1995). Skills deployed from a 'common sense' perspective may tend to conceal mathematical ability rather than expose it for development (Coben, 2009). To dismiss mathematical skills, rendering them invisible, poses significant challenges for education and training programmes for want of a starting point, i.e. the so-called 'bootstrap problem' (Klinger, 2009). This is problematic for the worker, since, in the view of the National Development Plan (NDP) in Ireland, a lack of appropriate MKSC can have a critical impact on a person's continuing employability (NDP/CSF Information Office, 2007).

Longitudinal studies, conducted in the United Kingdom, tend to confirm the correlation between low levels of numeracy and poor long term employment prospects (Bynner & Parsons, 2005; Bynner & Parsons, 1997). This seems to be further underlined by the report of the Expert Skills Group in Ireland, that persons with low levels of education are at the greatest risk of unemployment, only 5% of all vacancies being for lower skilled jobs (Expert group on Future Skills Needs, 2009).

There are many factors that may exacerbate MKSC invisibility e.g. Language / Jargon (Wake & Williams, 2007); Habitus (Wedege, 1999); Training (Brown, Collins, & Duguid, 1989) 1989); Common sense (Coben, 2000); Group Status (Gal et al., 2009) (Lave & Wenger, 1991); Artefact (Marr & Hagston, 2007; Strasser, 2003); Black-box (Hoyles, Wolf, Molyneux-Hodgson, & Kent, 2002; Wake & Williams, 2007); Culture (Colleran, O'Donoghue, & Murphy, 2003) and Anxiety (Evans, 2000; McLeod, 1992), operating to obscure or deny such skills, however unwittingly. Just as human vision perceives only a narrow spectrum of light, so too will many overlook the mathematical concepts that pervade everyday work and life. Seeing things through what might be called 'Everyday Eyes', gives prominence to what workers 'do', rather then what they 'know', and adds to the self- perception of not being a ‘maths person’.

That mathematics could become so inextricably bound up in one’s work practices as to be invisible presents different yet interrelated problems for different stakeholders. Employers, for whom a profile of the skills requirement of a particular job overlooks the MKSC elements, may struggle to recruit and train personnel. Workers that are mismatched to their jobs are less likely to realise their potential, be positively responsive to change, and to obtain job satisfaction which negatively impacts on retention and absence (McClelland, 1988). Moreover, the tendency to dismiss such expertise as mere ’common sense’ impedes the Recognition of Prior Learning (RPL). Providers of workplace learning opportunities are constrained in the design of programmes for the want of a starting point, especially for topics that are not seen to be pertinent to the job (Klinger, 2009).

In contrast, viewing the world through what might be called ‘Mathematical Eyes’, perceives problem solving, space and shape, pattern and relationship, data handling and chance, in addition to quantity and number. Opening ‘Mathematical Eyes’ heightens the awareness of what workers ‘know’ rather than what they ‘do’. Furthermore, a platform is established on which to adapt, change, innovate and create in response to an ever-evolving world. However, deriving benefits from latent mathematical knowledge, skills and competence, requires that they are first located, and then expressed in terms of a common standard in order to exploit their potential. Uncovering these MKSC, faces the workplace challenge that skills underpin tasks that form procedures and habits, that constitute a job set in a context, particular to an organisation. A number of tools have been developed by this present work, to peel away the covering layers to reveal the mathematics beneath, categorise them as to domain and benchmark them with the National Framework of Qualifications (NFQ) for visibility.

The National Qualifications Authority of Ireland (NQAI, 2003), was set up in 2001 to establish and maintain the National Framework of Qualifications (NFQ), in order to recognise and award qualifications based on standards of knowledge, skill and competence acquired by learners. Launched in 2003, the NFQ enjoys growing national and international support. Its structure offers a new metric for achievement in more breadth and depth, across all subject areas. However, since it post-dates work in progress on the development of a mathematics framework, there remain areas of dissonance between the two, adding significantly to the impact of this study.

The NFQ provides for 10 levels of achievement, ranging from the Elementary or Foundation Level 1, rising incrementally through Levels 4 and 5, equivalent to the completion of upper- secondary school leavers. This is followed by Level 6 indicating a post upper second level award often provided as a precursor to University. Levels 7 through to 10, attest

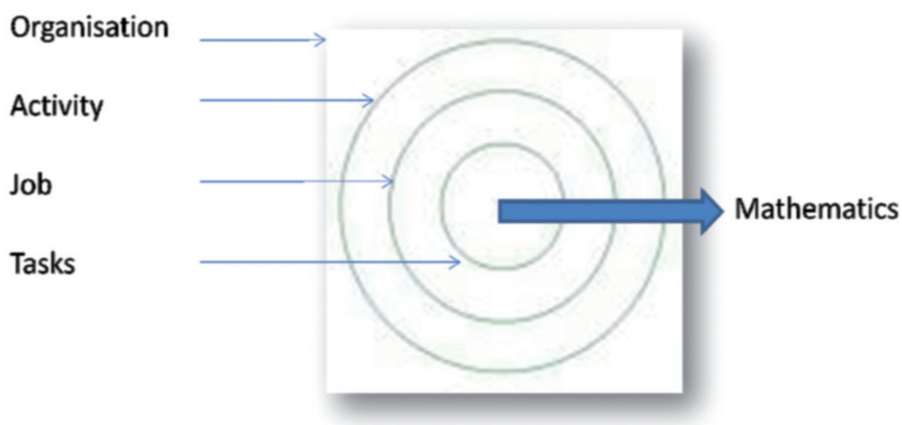

Figure 1—Workplace Mathematics Knowledge, Skills and Competence (MKSC)

to Bachelor degrees at ordinary and honours standards, Masters degrees and Doctorates respectively. Each level of achievement may be classified in 3 main strands, each refined by attendant sub- strands as follows;

Knowledge: Kind and Breadth, Skills: Range and Selectivity, and Competence: Context, Role, Learning to Learn and Insight. Differentiation between levels is guided by textual specification which may be vulnerable to idiosyncratic interpretation (NQAI, 2003).

A precise knowledge of the provisions of the National Standards, informed the selection of the methodology with the capacity to discipline the conduct of research in relatively few workplaces, seeking to link underpinning MKSC with the NFQ.

Methodology

This research is guided by the 'Building Theory from Case Study Research' (BTCSR) methodology, described by Eisenhardt (2002). In developing BTCSR, Eisenhardt draws together work previously done on 'Grounded Theory' (Glaser & Strauss, 1967) and 'Case Study Design' (Yin, 1984). The resulting platform includes construct definition, triangulation, across case analysis, and the role of pertinent literature (Eisenhardt, 2002). The richness of detail, anticipated from relatively few cases studied, rather than being a weakness of BTCSR (Huberman & Miles, 2002), supports within and across case analysis, comparison with enfolding literature and triangulation of multiple sources.

In the modern workplace, skills deployed in the performance of work lie at the kernel of a number of encapsulating layers, each exerting its own influence to hide, wholly or partially,

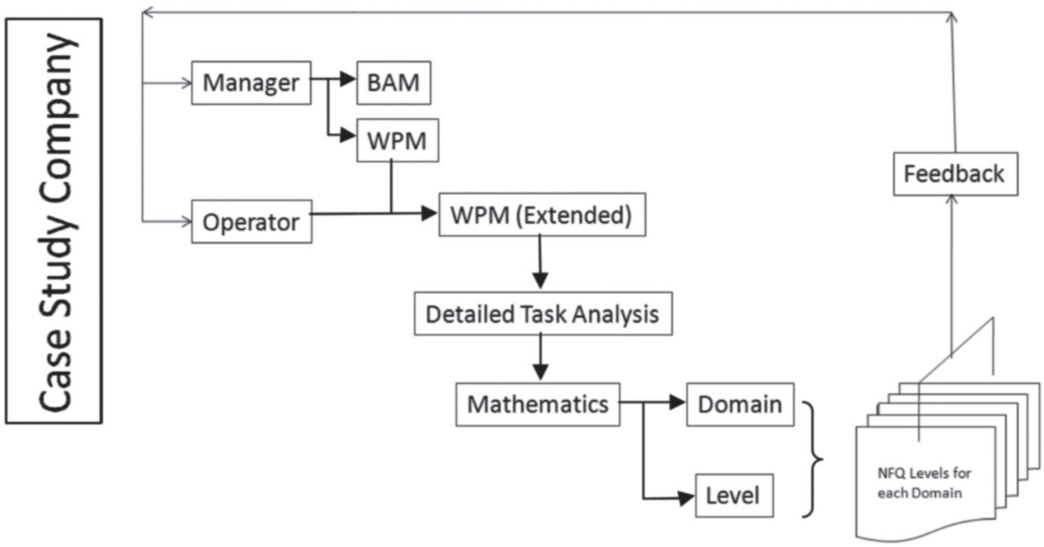

Figure 2—Overview of the Methods and Tools used.

the underpinning MKSC. Businesses comprise chains of activity that are triggered in different combinations. Activities encapsulate work practices, which themselves are a combination of tasks, normally considered to be a 'job'. Underpinning a 'job' is a collection of knowledge, skills and competence, the mathematical content of which may lie under the cover of a range of factors, whether internal, external, personal, contextual or other, (see Fig. 1).

This catalogue of circumstances and factors, aid the concealment of mathematics in the workplace. Not only are MKSC obscured from the view of Management and Operator alike, albeit in varying degrees and combinations, the performance of the tasks upon which they rely, are buried deeply in organisations. References to a job usually denote a person, working alone or in concert to maintain workflow from initiation to completion. In seeking to make MKSC visible, this research must find them in the first instance, guided by an awareness of how they might be hidden.

In a coordinated plan, designed to strip away the occluding layers, the researcher separates combinations of business activities to identify their constituent jobs. Then by setting boundaries to a specific job, makes explicit the precise component tasks. Assessing the MKSC underlying the tasks, classifies them as to mathematics domain, calibrates them for complexity and situational factors and benchmarks them against the National Framework of Qualifications in order to communicate about them. Finally, reflecting back the findings to the participants informs an assessment of overall invisibility and the extent of the influence of each factor.

Different tools have been designed for this research to untangle the challenges presented by each layer, in order to identify a 'job' and break it down into its parts (see Fig. 2).

Firstly, the Business Activity Model(BAM) tool (Ericsson, 2004), applied at the Management level, traces the individual activities that comprise a business, noting their precedents and dependents, that are triggered in various combinations by different business events.

Secondly, the Work Practice Model (WPM), tool (Sierhuis, Clancey, & de Hoog, 2009), profiles the combination of tasks that constitute a job contained in a business activity from the Management point of view initially. This is then corroborated or extended in an interview at the Operations level and sets boundaries to the selected job.

Thirdly, a Detailed Task Analysis, guided by the WPM and elaborated by researcher observation, breaks down each task element so that the mathematics can be identified and categorised as to MKSC domain.

A comprehensive Literature Review, together with extensive experience of the workplace, sensitises the researcher to MKSC and the many ways in which they may be disguised or overlooked. With the support of documents, designed to ensure completeness and consistency, the researcher identifies the mathematics and categorises them to domain. Then, guided by the provisions of the National Standards and taking situational factors into account, each instance of MKSC is calibrated for level in terms of Knowledge: Kind and Breadth, Skills: Range and Selectivity, and Competence: Context, Role, Learning to Learn and Insight. The aggregate level is then modelled onto the NFQ for each mathematics domain and reflected to the participants.

In preparation for doctoral research, this approach was recently piloted in an enterprise, selected for its compactness and labour intensiveness, in order to test tools designed to locate subtle instances of mathematical ideas and techniques.

Pilot Study

The Pilot Study was conducted in three site visits over a period of 4 weeks. The host site is a subsidiary of a larger group of companies, and is engaged in creating images of approximately 1 million documents per week on behalf of client companies. The work is completed by 10 people, both full and part time, supervised by a line manager and assistant manager. Despite its superficial simplicity, there are a number of conditional subroutines in response to customers' Service Level Agreements (SLA) that add complexity.

The BAM was drafted by the researcher and confirmed by the Management. While there were activities that involved very apparent manipulation of numbers, e.g. accounting and payroll, the researcher selected the more labour intensive scanning activities for their superficial absence of MKSC.

The researcher then produced a WPM using information provided from the management perspective. In a semi structured interview with the worker, the WPM was confirmed, modified and extended to include conditional elements that had been overlooked by the management.

A Detailed Task Analysis, based on the WPM, extended by the researcher's observations, disentangled the 'job' into 17 separate instances of underpinning mathematical ideas and techniques, albeit at 'low' levels.

Calibrating the identified MKSC, in order to benchmark them with the NFQ was challenging due to the variety of factors involved. Designed to address all learning disciplines, the NFQ does not contain mathematics specific language. The Further Education Training Awards Council (FETAC, 2009) provisions are under development and not yet aligned with the NFQ. The pilot study observed jobs containing complicated mathematical elements that had been rendered a matter of routine by repetition over time. Other jobs were characterised by an array of relatively simple mathematical activities when taken in isolation, yet combined to construct a whole job featuring a degree of complexity beyond the aggregate of its parts. Future case studies will take particular care to differentiate between complexity and complicatedness in the interests of precision.

The terms "complex" and "complicated" are commonly used interchangeably to describe the opposite of simple. It seems somewhat counter intuitive to suggest that a task may be complex but not complicated or conversely, that a task may be complicated, but not complex. However, discourse concerning contextualisations of mathematics in the workplace, requires precise use of language that is capable of differentiating between these two adjectives.

A complicated task comprises elements that are difficult to understand and analyse. It does not index volume but rather intricacy (Summers et al 2000). Degrees of complicatedness may be addressed by the national standards with references to abstractness, however, elementary concepts may be considered to be complicated in spite of being located at the lowest level of the framework.

In contrast, a task is said to be complex if it has many interconnected parts. Complexity increases in line with the number of bits of information involved, the variety of information and the number of ways in which it can be arranged (Dodge, 2009). That there are many interrelated parts of the whole, does not necessarily imply difficulty.

Typically, 'jobs' comprise multiple tasks and sub tasks, many of which are routine when viewed in isolation, yet may be combined in a wide variety of ways that increase complexity.

When some of its component parts are difficult to understand, analyse and perform, a complex 'job' becomes complicated in line with the degree of difficulty encountered.

This distinction is particularly apposite in the workplace, where complex and complicated jobs are repeatedly performed with apparent ease, belying their underlying mathematical and other characteristics. Assessing workplace 'jobs' and their constituent tasks, for the purpose of benchmarking against existing frameworks, must take account of the extent to which a 'job' component is complex and complicated, in addition to the situation within which the 'job' is performed. Moreover, workers should not 'inherit' a level of MKSC associated with complicated mathematics that are thoroughly embedded in a 'black-box' and otherwise, completely invisible to them.

The National Framework of Qualifications makes extensive provision for the calibration of knowledge, skills and competence that range from the elementary and concrete at the 'lower' end of the spectrum, to the abstract and theoretical at the 'higher' end. The degree of complexity and/or complicatedness is a qualifying rather than a determinant of the NFQ level. For example, the Knowledge Kind related to a task may be elementary and routine, but, performed in a complex situation may index breadth of knowledge. Competence Context and Role are self evident, but aspects of Learning to Learn and Insight may be collinear with situation complexity

As this research continues with an intense focus on case studies, the relationship between complex / complicated and embeddness , contrasted with the workers role, will be profiled as an aid to training and a decision support tool. For example, that the mathematical content of a job element has become complicated over time, may indicate the type of training necessary for the worker to maintain process control. Alternatively, such a job profile may justify a 'black-box' solution as a means of removing the risk of failure posed by increasing levels of complicatedness.

The extent of ' embeddedness' may be a marker of mathematics invisibility in a continuum ranging from obvious and overt at one end to completely opaque at the other. Awareness of the factors contributing to invisibility, together with the job profiled in terms of the NFQ, will inform the design of the appropriate teaching and learning response.

The standards frameworks are well developed and readily accessible by the providers of teaching and learning. This work has created and tested a range of methods and instruments that locate the MKSC that underpin workplace numeracy activity. In an iterative process over a relatively few cases, a Framework Interface Protocol will be developed and refined, such that the MKSC detected in the workplace can be consistently, reliably and objectively interpreted.

Concluding Remarks

The difficulty in researching this topic was highlighted by the multiple affectors that may contribute to invisibility, each in their own way and extent. Moreover, that the MKSC sought were to be found at the individual task level, 'buried' under a number of layers, presented particular challenges of identification. The approach to observing the workplace described above, consists of a set of tools for the exploration of mathematics in the workplace, to overcome these obstacles and meet the needs of the research.

The BAM rapidly orientates the researcher in the workplace, providing sufficient familiarity to make an informed and independent selection of candidate jobs. In this way, the possible contaminating affects of self-selectors and employer influence can be avoided.

The WPM, developed initially from the Management point of view and corroborated by that of Operations, builds internal validity and sets boundaries to the job under observation. The researcher can absorb the details of the work practices described and be better positioned to appreciate the purpose and meaning of each task that might otherwise be masked by the operator, however unintentionally.

By reflecting the job-MKSC content back to the participants, it may be possible to triangulate the extent and possible causes of mathematics invisibility in the workplace. These may provide an index of visibility, pointing to the factors to be addressed in order to open a person's 'Mathematical Eyes'.

The benefits accruing to the individual of being made aware of their facility with MKSC may include a review of their self perception of not being a 'maths person', and their being encouraged to pursue further development. Acknowledgement of their skills, may support the Recognition of Prior Learning, and empower them in their work and elsewhere. They need no longer equate what they know with what they do, and become aware of their inter-sector mobility. The employer, accurately informed by the MKSC requirement of existing, changing and planned jobs will be more likely to recruit suitable personnel. Providing learning and training opportunities, specific to the identified need could be more efficient use of time and money. Personnel, matched to the requirements of their job, whose potential to develop is leveraged by the company, provides fulfilment and satisfaction, with a consequent positive affect on retention and absence. Workplace learning and training providers will benefit from a clearer indication of the skills required by the company, and, being equipped with an audience profile based on their work, can adjust their content and delivery methods accordingly.The Framework Interface Protocol under development, together with a model to facilitate communications between stakeholders, may constitute a step towards alignment of the Mathematics Framework with the European Qualifications Framework (EQF) for the benefit of inter sector mobility both in Ireland and across the European Union.

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Educational Interfaces between Mathematics and Industry in India AND Use of Technology in Mathematics Education in India

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Abstract This paper consists of two parts. The first part deals with interaction between Mathematics and Industry in India. The interaction between the two is almost non existent. We look into some of the reasons behind this and mention some of the initiatives taken in the past to develop some interactions between the two communities. We also suggests some of the steps that must be taken in order to improve the interface between mathematics and industry in India.

The second part deals with the use of computer aided technology in mathematics education at various levels in India. We also mention a future project proposed by us that can significantly revolutionize mathematics teaching using computer aided tools.

Interfaces between Mathematics and Industry in India

Introduction

Industrial innovations are based on techniques of scientific research which most often are driven by mathematical theory. However, industry or society rarely recognize its contribution. Given the intimate connection between industrial innovations, science and mathematics, it is natural to ask whether there is any fruitful interaction between the mathematics community and industries. While in many countries, the interaction between industries and mathematics is fairly established, in India, there is hardly any interaction between the two. In this paper, we look at some of the reasons behind this non-existent interaction. We also mention some of the initiatives that have been taken in the past, like the Indian Society of Industrial and Applied Mathematics (ISIAM) and Industrial Mathematics Group (IMG), in order to develop better relations with industries. However, these initiatives have not lived up to their expectations, and seem to have not made any visible impact. We first take a brief look at the aforementioned initiatives, their aims, objectives and achievements. We then look at some of the recently initiated academic programmes in mathematics which are designed to cater to industrial needs. However, these are not sufficient for a vast country like India with its fast-growing economy. Mumbai being the financial and industrial hub of India, one could expect far better interaction between industry and the mathematics community. However, in reality, it is practically non-existent. The number of industry-supported projects in some of the very good mathematics departments and institutes is almost negligible. Lastly, we present views from the industries, which is based on interactions with people from industries and analysis of questionnaires which were sent to them.

Initiatives for Industrial Collaboration

There have been some initiatives by the mathematics community to bridge the gap between mathematics and industry. We look at two such initiatives. However, these initiatives have not lived up to their expectations. Even the people involved with these groups have hardly any industrial research projects.

ISIAM

The Indian Society of Industrial and Applied Mathematics (ISIAM) was proposed on the occasion of a symposium on “Differential Equations and Industrial Applications” in September, 1990 at the Department of Mathematics, Aligarh Muslim University, Aligarh, India, and was formally approved by its members in 1991. This society was affiliated to the International Council of Industrial and Applied Mathematics (ICIAM) in May 1999. Some of the aims of this society were: (i) creating an environment for working in emerging areas of mathematical sciences having great potential for applications (ii) efforts to create an environment to teach mathematics in the proper perspective from the very beginning (iii) creating awareness in public for the relevance of mathematics to real-world problems etc.

Several national and international conferences have been organized by ISIAM. The society also publishes its proceedings in its Journal (Indian Journal of Industrial and Applied Mathematics) and provides valuable information for teaching and research of industrial and applied mathematics. It has also constituted two awards for mathematicians with significant contributions to industrial problems.

However, the activities of the Society have of late been confined to organizing workshops and symposia, and not much emphasis is given to industry-mathematics collaborations. The society has not come out with any concrete proposal for bridging the gap and developing strong interaction between the industry and mathematics. ISIAM is going to hold a satellite conference of International Congress of Mathematicians 2010 on Mathematics in Science and Technology from August 15-17, 2010. One can hope that during this meeting the society can formulate some concrete plans to improve the mathematics-industry interaction.

Industrial Mathematics Group (IMG)

The Industrial Mathematics Group (IMG) was started in 1991 by a group of people from the Mathematics Department and the Chemical Engineering Department at the Indian Institute of Technology (IIT), Mumbai. The main aim of IMG was to provide necessary inputs for Mathematical Technology in terms of (i) Research and Development Projects (ii) Consultancy Projects (ii) In-House and In-Campus Workshops (iv) Curriculum Development Activities (v) Skill Development Activities. Initially they had discussions with a few industries and research organizations in and around Mumbai and organized a few workshops on Industrial Mathematics at the IIT Mumbai and at the MS University, Baroda. The main objective of this workshop was to provide an insight into some important mathematical and statistical techniques used in industrial applications, covering the following four modules: Systems Analysis, Optimization Techniques, Finite Element Methods and Industrial Statistics. The group also received substantial funding from the Department of Science and Technology (DST) in the late 1990s which was later discontinued.

This group does not seem to be active currently. There has not been any discernible activities recently and nothing is mentioned as to its future activities.

Industrial Sponsored Projects

We look at industrial supported projects in last ten years in some of the finest mathematics departments.

The table 1 shows that there has been no industrial sponsored research projects in these departments and only a few consultancies. These consultancies are mostly related to statistics and financial mathematics. However most of these departments do have research projects in Mathematics that are funded by the government agencies like UGC, DST, DAE, NBHM, DRDO, AICTE etc. In fact, it is easier to convince the government funding agencies for research projects, rather than Industry. This just shows that the interaction between Mathematics and Industry, even in the financial hub of the country and in some of the very good mathematics departments is almost non existent. However the fault lies with academicians as much as it with the Industry. Rarely mathematicians approach industries with research projects. Industries too do not have enough confidence in academicians for deliverables. However there are a few industries that do give financial support for organizing workshops and conferences (Table 1).

Academic Programmes

The undergraduate and postgraduate mathematics courses in India mostly include topics from pure mathematics and not much emphasis is given to practical and industrial applications. The use of computer aided tools and software in these courses is also essentially non existent. The following are some courses having industrial applications, most of which have been started recently.

1. M.Tech. in Industrial Mathematics & Scientific Computing at I.I.T. , Madras. (started in 1998)
2. M.Sc. in Industrial Mathematics and Informatics, I.I.T. , Roorkee.
3. M.Sc. Tech (Industrial Mathematics with Computer Applications) North Gujrat University, Patan, Three-Years Self-Financed Post Graduate Degree Programme (Started from Jun 2002)
4. M. Sc. Tech. Three year course in the University of Pune, started in 2007.

Department/Institute	Industry Supported Projects	
	Research Projects	**Consultancy**
Department of Mathematics, Indian Institute of Technology, Mumbai	None	12
Department of Mathematics, University of Mumbai, Mumbai	None	None
Department of Mathematics University of Pune, Mumbai	None	None
School of Mathematics, TIFR, Mumbai	None	None
Dept. of Maths and Stats Central University Hyderabad	None	only few
Department of Mathematics I.I.Sc., Bangalore	None (ongoing)	10 (ongoing)

Table 1—Industrial Projects and Consultancy in Major Mathematics Departments

5. M.Sc. (Tech.) in industrial Mathematics with Computer Application. Few colleges of the University of Pune offers this course.

6. M. Sc. in Industrial Mathematics with Computer Applications, Vikram University, Ujjain, Madhya Pradesh.

Apart from the above course some universities offer courses in applied mathematics. But it is limited in numbers. For a vast country like India with its fast growing economy, we need many more courses that cater to industrial needs. One can hope that many more such courses will be initiated (especially in Mumbai) and that it will have significant impact in future collaboration between industries and academia once these students join the industries.

Reasons behind the poor interaction between mathematics and industry

1. Mathematics graduates, post-graduates and Ph.D. students presently working in industries are very small in number.

2. There is a severe scarcity of applied mathematicians in the county.

3. Although, Mumbai is the financial capital of India, the number of courses in mathematics offered in Mumbai which are relevant to industries is negligible.

4. Minimal use of computer aided mathematical tools and software in mathematics education at all levels. At best, it is confined to handful of mathematics teachers and researchers. Many a time, people discourage students to use these tools.

5. It is difficult for mathematicians to convince industries about their work and its relevance to real world problems. In fact, most often mathematicians do not even bother about the applications of their findings.

6. Poor R & D departments in most of the industries. In fact, even in best of the industries R & D departments mostly focus on developments and cost effectiveness, the research part is very poor to say the least.

7. R & D departments in most of the industries are not willing to share their research activities.

8. Industries are not forthcoming to support mathematics projects although they see its applications and relevance.

9. In general, industries do not have enough confidence in the academic world for the deliverables of the project.

10. In general, academicians tend to approach only government agencies for research projects.

11. Indian universities have poor research culture in mathematics and the elite research institutes have enough research funds from the government agencies, therefore, they hardly bother about the industrial supports.

12. The Institute of Chemical Technology (ICT), Mumbai is having one of the strongest links with industries, so much so that it is often cited as the role model example in India. Most often the industrial projects at ICT are written by faculty members of technology branches and submitted to the industry for funding. Rarely does the industry approach them with their problems. This shows poor R & D on part of industries. Thus, if we wish to develop stronger interaction between industry and mathematics, the academia has to take a lead.

Industrial Perspective

We have tried to gather industrial industry perceptions about mathematics, by talking to various people in the industries and also by taking their views using a questionnaire. Almost all of them do acknowledge the importance of mathematics in industrial innovations, analysis, optimization, and control of industrial processes etc. Most of the industries use ready made (black-box) software for simulation, analysis and solving mathematical problems. Industry faces problems that extend well beyond the horizon of classical topics in mathematics. Many of these problems have a significant mathematical components, and intellectual challenges.

Industries acknowledge that: students having good mathematical background do much better compared to other students,however mathematics courses offered at the UG and PG level in most of the universities need to be reviewed. Mathematical background of students to handle real world problems is not adequate, visible use of mathematical software in mathematics education is the need of the hour. Hardly any industry has given support for a mathematics research project, but most of them are open to this idea provided the projects have some merits. Many industries give financial support for organizing workshops and conferences. Industries are also willing to give supports for infrastructures to the departments of mathematics that would like to offer courses that are relevant to industry. Industry do not see a need for any training programmes in Mathematics for their staffs and people working in R & D, primarily because the mathematics which they use, most often is very simple or ready made codes are available.

Most of the industries are not willing to invest money, on fundamental research as they do not see any immediate gain out of it, where as from the academician point of view fundamental research is a vital component of innovations. Most of industrial projects and consultancies at the academic institutions like Institute of Chemical Technology (ICT) and Indian Institute of Technology (IIT) are through personal contacts and reputations.

Steps to be taken for developing a closer interaction with industries

1. Curriculum must be redesigned in order to include industrially relevant topics across the board.

2. Real world problems and its solutions must be included in the curriculum. One also seriously needs to accommodate the project component especially at the final year of masters programmes in all university programs.

3. Many more masters programes specifically catering to topics of industrial impor tance must be started, especially in major universities and institutes in India.

4. Established groups and societies like IMG and ISIAM must widen their horizon and must take the lead to establish greater interactions between the Mathematicians and Industry. They in collaboration with the government and industry must form a network of experts (formal or informal, sponsored or independent) who can look into the various aspects of improving the collaboration with industries.

5. Mathematics community has to take lead and approach the industries with projects having real world applications. Especially projects dealing with social and environmental problems are the need of the hour.

6. Industry should fund postdoctoral programs at academic institutions. Such programs can offer a first-hand experience of working on an industrial issue, while preserving the option of an academic career. This can also work as a catalyst between the two communities.

7. The academia along with the government should take a leaf out of some of the established international centers like ICIAM, ICM (Poland), OCIAM (UK), SAMSI (USA), IMA(USA) just to name a few in order to bring the two communities closer for better coordination.

Conclusion

Although, mathematics is considered to be the queen of sciences, and is at the heart of any scientific and industrial innovations, the interface between the Industry and Mathematics in India is almost non existent. The onus lies with both the communities to work together to strengthen coordination and cooperation which can lead to technology development and meet the societal requirements and challenges in India. Stronger interaction between mathematics and industry will be beneficial to both to the partners and to national economies and this in turn will inspire new mathematics and enhance the competitive advantage of industries as well. Industries must show greater confidence in mathematicians to tackle their problems. They must consider it as a long term investment and also as part of their obligations to society and must not have a myopic view of immediate gain.

Use of Technology in Mathematics Education in India

Few working in mathematics education today would be unaware of the emergence of computer technologies, and related mathematical software in recent years for teaching, learning and research in mathematics. The use of Information communication Technology (ICT) in teaching mathematics can make the teaching process more effective as well as enhance the students capabilities in understanding basic concepts. Calculating technology in mathematics has evolved from four-function calculators to scientific calculators to graphing calculators and now to computers with computer algebra system (CAS). Introduction of computer algebra systems, advantages and disadvantages of using CAS have been explained in [1]. Challenges of implementing CAS based mathematics teaching and overcoming these challenges at the undergraduate and postgraduate level mathematics in India have also been explained in [1].

While use of computer aided tools in many countries in teaching and learning mathematics have made a significant impact at all levels, in India the progress and awareness of this technology has been really very slow. Mostly, it has been confined among researchers and a handful of university and college teachers in well established research institutes, IIT's and university departments. We look at the extent of the use of ICT in mathematics at various levels in India.

Use of ICT at Undergraduate and Postgraduate Level

In recent years, many people have taken the lead to create awareness about these tools among mathematics teachers, by conducting workshops and conferences and the University Grant Commission (UGC) runs refresher courses for college teachers. However, many more consolidated efforts are needed in order to create a significant impact. As a result of some of these efforts, use of some software (especially free and open source software like SciLab, GeoGebra, KASH, MAXIMA, WinPlot etc.) at the un-dergraduate level in universities like University of Mumbai, University of Pune have been initiated. However, effective use of software in mathematics teaching have to be assessed in due course of time as there are many concerns and challenges. But most college teachers in rural areas are not even aware of the existence of mathematical software and its impact in mathematics teaching. The use of ICT in mathematics teaching is mainly limited to the postgraduate levels. At the undergraduate level in most of the universities it has been almost non existent. Main reasons behind this are infrastructure constrains and unawareness of the mathematics teachers. In recent years, the government funding agencies are providing generous funding for creating computer laboratories, and related infrastructures in colleges and institutes. But what we need is to create awareness and provide teachers with innovative teaching modules. The author along with Professor S. Kumaresan of University of Hyderabad, India are planning to organize a series of workshops across India in order to create awareness. We give a brief description of this proposal later.

Even in the streams like Engineering and Technology, use of mathematical software in teaching mathematics have been very limited in most of the colleges and institutes. At best most of them are teaching some programming languages, spreadsheet programmes like excel. Use of software like MatLab, MathCAD, Mathematica, statistical software are confined to individual preferences.

Use of ICT at School Level

At school level, use of ICT is even less visible. There has been some efforts to create awareness about use of technology in mathematics teaching at school levels. One such effort is the National Conference and Workshop on Technology and Innovations in Mathematics Education (TIME) conducted at the Department of Mathematics, Indian Institute of Technology in 2005, 2007 and 2009. The main aim of this conference is to provide mathematics teachers of schools, a platform to discuss about innovative ways of mathematics teaching at the same time create an awareness of effecive use of ICT, and the kind of innovations that it can bring in mathematics teaching. One can clearly see during these workshops that many school teachers have started using tools like, GeoGebra, Geometer's Sketch Pad, Excel spreadsheets, power point Presentations etc. However, it is mostly at personal level and officially it has not been implemented in the curriculum. The Homi Bhabha Centre for Science Education (HBCSE), Mumbai, the centre devoted to research and development in science and mathematics education has also taken some initiatives recently to create awareness by conducting some workshops at schools. However, this is not enough and it must take further initiative in order to create awareness, develop innovative teaching modules for schools using various teaching tools as they have got necessary infrastructure and funding. There has been some initiatives to set up virtual GeoGebra Institutes in India which can provide on-line support system for teachers not only using GeoGebra but also using other innovative tools. There have been few other conferences also dealing with ICT use in mathematics education in India.

The above initiatives are not enough, and we need more consolidated efforts by concerned mathematics teachers at local levels to organize many more workshops and discussions to bring in technology in the mathematics curriculum and to revolutionize mathematics teaching. The different educational boards should also seriously looks into the various aspects of implementing technology in mathematics educations.

A proposal that can make a difference

The author along with Professor S. Kumaresan of the University of Hyderabad has sub mitted a project to the Department of Science and Technology, Government of India which has been approved. Under this project, we plan to conduct several workshops across the country to make the mathematics teachers aware of computer aided technologies and how to make use of such tools effectively in their teaching. We would develop teaching modules for teachers based on which they can design their own teaching modules or they can adopt modules developed by others.

We first plan to send questionnaires to university and college teachers, teaching mathematics across the country to make assessments of their knowledge in mathematical software and their opinions about whether these tools can have a significant role in improving the teaching standard of mathematics. Based on these data, we may be able to create an awareness of the importance of the use of mathematical software in mathematics teaching. We also wish to create a pool of experts who can help in creating awareness, develop teaching materials, training others and look into the various aspects of implementation of these tools in mathematics teaching at college and university levels.

We would like to give stress on the use of free open source mathematical software so that it will be easily accessible to teachers and students. We may also explore the possibilities of developing mathematical software in other Indian languages which can then be made freely available.

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A Meta-analysis of Mathematics Teachers of the GIFT Program Using Success Case Methodology

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Abstract The ICMI-ICIAM Discussion Document (DD), Educational Interfaces between Mathematics and Industry, states that "...mathematics is said to be used almost everywhere. However, these uses are not generally visible except to specialists." The GIFT program is one whose goal is to bridge this gap through substantive projects which bring practicing math teachers into industrial ("real world") projects during the summer, have them integrate their new experiences back into their classroom, and add flexibility (Section 1.1 of DD) to the thinking of their students. The point of this evaluation research is to see, using Success Case Methodology, whether the goals of the program that we see informally being achieved are supported by data. We will use any "unanticipated consequences" from this analysis to improve the GIFT program, expand the analysis to science projects, and, motivated by DD, lay the groundwork for future GIFT-related projects.

Introduction to the study

The ICMI-ICIAM Discussion Document, *Educational Interfaces between Mathematics and Industry*, talks about " ... the intimate connections between mathematics and industry" and, for example, then says that " ... mathematics is said to be used almost everywhere. However, these uses are not generally visible except to specialists." The GIFT program is one whose goal is to bridge this gap through substantive projects which bring practicing middle and high school mathematics teachers into industrial projects and then have them integrate their new experiences back into their classroom. Because of time constraints, teachers without firsthand knowledge cannot provide mathematical situations which involve "real life problems." The last sentence of 1.1 in the Discussion Document is especially in agreement with our approach; to wit, "In other words, learners should be equipped for flexibility in an ever-changing work and life environment, globally and locally." From the GIFT experience, the flexibility should change the teachers' outlook and, ultimately, the outlook of their students. The last four of the bullets of the *What are the aims of the Study?* and the last two of *Why is there a need for this Study?* fit well with the goals of the GIFT program. The issues of sections 8 (Curriculum and Syllabus issues) and 9 (teacher training) of the discussion document fits well with regard to the goals of the GIFT project.

The present work lays the groundwork for future projects motivated by the discussion document. The results of the present meta-analysis will ultimately be used as a basis for further analysis in a long term project using a variety of methods for both math and science teachers.

Introduction to the GIFT program

In a commitment to providing first hand connections between classroom activities and real world applications, the Georgia Intern-Fellowships for Teachers (GIFT) initiated in 1991 by the Georgia Institute of Technology is a collaborative effort between industry and education. GIFT provides mathematics, science, and technology teachers in grades six through twelve (students between 12 and 17 years old) "real life" experiences in the applications of those disciplines. Over time, GIFT has placed 1,515 teachers into summer internship positions of 4-7 weeklong in corporate and university research laboratory settings. From the beginning, participants in GIFT benefited from internships provided by long term industry partners such as UPS, Georgia Power, Cisco, and EMS Technologies; and more recently by partners Nordson Corporation, Solvay Pharmaceuticals, CIBA Vision, Gwinnett Hospital System, RFS Pharma, Optima Chemicals, PCC Airfoils, and Stiefel Laboratories and General Electric. We know anecdotally that these internships have contributed to teachers having increased content knowledge and enhanced teaching practices based on evidence based experiences and now move towards an in-depth analysis. Programs like GIFT are sometimes referred to as "externship" programs.

GIFT is designed with the following goals:

- Provide industry mentors an efficient method of identifying and selecting teachers interested in participating in internships,
- Quickly orient teachers to industry work environments, and mentors to K-12 workplace culture,
- Provide participants (teachers and mentors) support throughout the summer by assigning small groups of teachers to a master-teacher facilitator,
- Assist teachers with creating an "Action Plan" for implementing summer experiences into the classroom or more generally applying the GIFT experience in the classroom,
- Provide support for Action Plan implementation in the classroom through visits by GIFT staff,
- Foster the development of an extended professional community of learners, and
- Encourage extended partnerships for communication and collaboration between teachers and industry mentors and pass that approach on to the students of the GIFT teachers.

Logistics of the GIFT program

To participate in GIFT, sponsors from industry or a university submit an on-line survey which includes a position description describing the nature of the summer work, a list of the skills required of the teacher, and a letter of intent for participation. The teachers complete an on-line application that includes information about their background, courses they have taught, their technology skills and their geographical preference for work locations. GIFT uses information from the sponsor and teacher databases to coordinate the matching of skills with the preferences of both. Sponsors are given access to applications of teachers who meet their job requirements. Sponsors then interview prospective applicants and select a teacher to hire for the summer. Approximately 150 teachers from in Georgia, U.S. apply to the program each year, with a current average placement of 80 teachers per summer.

Once the sponsorships are arranged, each GIFT teacher works with two people. One is a mentor from the industry whom we call an (industry) mentor or sponsor and the second is a facilitator. A facilitator is generally an experienced teacher who has served as a GIFT intern in previous years or a college professor with significant grades sixth through twelfth experience. Facilitators provide guidance in the development of interns' Action Plans, assess the Action Plan, and make recommendation on the allocation of state granted Professional Learning Units (PLUs), a requirement for teachers in the State of Georgia.

Rationale for the study

Research suggests that the quality of the teaching workforce is the single most important factor in predicting student achievement (Darling-Hammond & Ball, 1997). "Quality" has many dimensions, however. Effective teachers must have a solid knowledge of academic content, a high mastery of different pedagogical techniques, an understanding of student developmental issues and different ways of learning, and a strong sense of professionalism. Teachers also must have a satisfactory answer to the inevitable question by students—"When am I ever going to use this"? Other than student learning or developmental issues, industrial workplace environments are in the unique position of being able to help teachers develop their strengths in most of these categories through summer internships. When teamed with facilitators, industry mentors can provide motivated teachers summer experiences that show the uses of math skills in industry, that increase the teacher's content knowledge, and that provide new teaching strategies.

These experiences also provide teachers with first-hand knowledge about how industrial scientists actually approach problems, how they design experiments, how they interpret data, how they communicate orally and in writing, and how they come to and implement workplace solutions. And, in perhaps the most powerful effect of all, the teachers' sense of professionalism from these experiences has a continuing influence on them. In that regard, the GIFT program provides teachers an opportunity to connect classroom activities to real world applications and vice versa.

The point of this evaluation research is to see, using Success Case Methodology, whether the goals of the program that we see informally being achieved are, in fact, supported by data. In addition, we will use any "unanticipated consequences" from this analysis to improve the GIFT program regardless of teacher instructional specialty as long as the insights are transportable to other disciplines.

The purpose of this study is to document the success cases of GIFT mathematics teachers in industrial workplace environment. We identified 23 mathematics teachers, each of whom worked in industrial workplace environment, to construct individual cases. This approach will help us to uncover patterns and develop themes across cases (Yin, 1994). The case studies document the prior experiences, knowledge, and beliefs these mathematics teachers brought to the program, as well as how those factors interacted with their learning from the program and from their own teaching experiences.

The Aims of the Study

It is hoped that this study will broaden the awareness of mathematics teachers and other educators with regard to industrial work place environment and needs with respect to education. In addition, we hope to explore the relationship between mathematics teachers and industrial workplace environment, and the impact of GIFT program on industrial work place environment. We anticipate having first results in March, 2010 in time to report to the ICMI/ICIAM meeting.

Study Design

The Success Case Evaluation Method (SCM)

Brinkerhoff (2003) originally developed the Success Case Method (SCM) to evaluate the impact of interventions on business industry goals. It is a simple process that combines analysis of outstanding groups with case study and story-telling. The primary goal of the model is to assess how well an organizational intervention is working by focusing on extreme (that is, both "success" and "unsuccess") groups (Coryn, Schroter & Hanssen, 2009). It is a way of exploring whether and how well an initiative is working. Furthermore, it is designed to identify the contextual factors that differentiate successful from unsuccessful cases. The stories are supported with evidence to confirm their reality. According to Brinkerhoff (2003);

> A success story is not considered valid and reportable until we are convinced that we have enough compelling evidence that the story would 'stand up in court' ... if pressed we could prove it beyond a reasonable doubt .

The core questions of the SCM approach are:

- What is really happening?
- What results are being achieved?
- What is the value of the results?
- How can it be improved?

Although SCM has been used to evaluate training initiatives and new work methods, it has also been used in educational setting to determine the reasons that influence the academic achievement of minority students (Coryn et al. 2007). More recently, SCM has been proposed as an alternative approach to reexamine causal relationships when more scientifically rigorous designs are not practical and not feasible (Brinkerhoff, 2005 ; Shriven, 2006a). SCM is a five step procedure:

1. Focus and plan the SCM
2. Create an impact model
3. Survey all program recipients to identify success and nonsuccess cases
4. Interview a random sample of success and nonsuccess cases and document their stories
5. Document findings, conclusions, and recommendations

According to Coryn (2007), in step one, the researcher needs to determine the focus of the SCM study that can be used for both formative and summative reasons. In step two, the cases are identified as high ("success cases"), moderate (average cases) and low ("unsuccess cases"). The survey method usually is used to identify cases. We are now in the process of constructing the required survey. In step four, these identified cases are used to create a sampling strata that represents both success and unsuccess cases. Therefore, SCM method is an analysis of extreme or outlier cases where independent evidence is sought to support claim of success or failure. Next, the underlying reasons for success and unsuccess cases are investigated using a semi-structured interview method that searches for an explanation from a random sample of extreme cases. In final step, the SCM findings, conclusions and explanations are put together across all cases. Furthermore, the final report is usually presented as a meta- analysis of success stories.

Why SCM method?

In order to learn from GIFT's program impact and to explore the impact of leadership improvement on industrial work place environment, an evaluation methodology was needed. SCM is a valid method to evaluate the GIFT program because it allows the researcher to assess the past efforts in a particular program (Coryn et al., 2009). It is particularly useful as an evaluation method for GIFT because it allows for an in-depth exploration of (1) what impact it is having on mathematics teachers (2) what are the specific barriers that exists in industrial workplace environment that GIFT mathematics teachers encountered. The result of using the SCM is that it can be used to understand and evaluate the effectiveness and impact of GIFT mathematics teachers. Applying SCM to historical data will provide insight for the evaluation through other methods of the GIFT program.

The Developed Impact Model

At the core of the SCM approach, the following Impact Model will be followed:

Capability—>Critical Actions—>Key Results—>Organizational Goals

CAPABILITY: What new or improved capabilities were acquired as a result of participating in GIFT?

CRITICAL ACTIONS: What the participants did with the new or improved capability?

KEY RESULTS: What did the mathematics teachers achieve?

ORGANIZATIONAL GOALS: How did the results achieved affect goals of the GIFT program and industrial work environment?

With the SCM method, we will be able to assess the GIFT program with these four categories. In other words, we will assess to what extent the GIFT intervention program developed improved capabilities, and then resulted in improved actions in industrial workplace environment, and whether those actions helped the industrial work environment. Furthermore, we will investigate how those achieved results affected the overall program goals. We will gather information to support this impact model through a combination of survey(s) and interview(s) as well as data verification(s). This research will be also a case study of “stories” as these are compelling ways to demonstrate improved leadership.

The participants for this study will consist of 23 mathematics teachers who did a total of 26 internships and were in the program at least one of the summers of 2007, 2008, and 2009. Demographically, 14 are high school teachers and 9 are middle school teachers. There were African American (16), Caucasian (6) and one Asian-American in the sample. In addition, 17 of them are female and 6 male. Their numbers represent 14 suburban, 6 urban and 3 rural school districts. To assess the program impact, we will interview the teachers about their experience in industrial working environment, and ask them to complete a survey about their experiences with the program.

This proposed study will examine 23 mathematics teacher cases (here, case refers to per internship experience) to determine what impact GIFT is having in the industrial workplace environment. Furthermore, we will identify the barriers to the work. It is possible that we may re-construct our own version of an *Impact Model* based on the information that will be found in the cases while we are analyzing each of the 23 cases. An SCM study results, typically, in two immediate ways. One is "in-depth stories of documented business impact that can be disseminated to a variety of audiences with the company." and a second "Knowledge of factors that enhance or impede the impact of training on business results." (Brinkerhoff & Dressler, 2002).

According to Brinkerhoff (2003), in some cases, the researchers view the logic differently than what had originally written, and they can change some components of the model for consistency across the models. It is important to emphasize that this method allows us to evaluate many cases across the state, and not just cases that are considered successful.

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Cultivating an Interface through Collaborative Research between Engineers in Nippon Steel and Mathematicians in University

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Abstract The steel making process requires control of a diverse range of phenomena. Nippon Steel has globally collaborated with mathematicians over a decade and resolved industrial problems by enhancing practical insights with mathematical reasoning. Engineers in Nippon Steel have learnt how to understand the phenomena in the steel-making process only by the rules of mathematics, not by a posteriori ad hoc ways. On the other hand, mathematicians in universities have learnt how to link mathematics with the physical reality of the phenomena. As a result, the collaborative research is playing a major role in mathematical innovation to broaden the diverse range of applications in mathematics and cultivation in both industry and the field of mathematics.

Introduction

Nippon Steel is the second largest producer of crude steel in the world, currently manufacturing all types of steel products at ten different steelworks and supplying economical, high-quality, innovative steel products. Our software and hardware resources amassed through steel operations offer great potential for applications to urban development such as efficient energy utilization, power generation, waste recycling, water treatment, transportation, regional developments and computer systems. We are reinforcing our manufacturing skills and continuing to refine our technologies.

The steel making process requires control of a diverse range of phenomena: combustion, chemical reactions, granulation of raw materials, fluid dynamics of molten steel, heat transfer, plastic deformation, crystallization and so forth, which involve mathematical applications for solving and modeling such problems.

Nippon Steel has collaborated internationally with various mathematicians for decades and resolved industrial problems by enhancing practical insights with mathematical reasoning. Engineers in Nippon Steel have learnt how to view the phenomena only by the rules of pure logic. On the other hand, mathematicians in universities have learnt how to link mathematics with the physical reality of the phenomena. As a result, the collaborative research is playing a major role in mathematical innovation to broaden applications of the diverse range of mathematics in both academia and industry.

Collaboration Style

Figure 1 shows our style of collaboration with engineers and mathematicians in the case of Nippon Steel and the University of Tokyo. We formed international task force teams made up of faculty members, post-doctoral fellows and doctor course students. Team members are selected flexibly to create a task force according to the characteristics of the task. Our collaboration is composed of six indispensable phases.

The first is "intuition and expertise" from industry. Intuition and expertise can be carried out exclusively by insight based on observation of phenomena in the manufacturing process. The insight should be enhanced by mathematical reasoning. The second is "communication." Communication is bilateral translations: the translation of phenomena to mathematics and the translation of mathematics to phenomena. Engineers in industry need to understand real problems on site, express them in the language of physics, and offer possible model equations to mathematicians. Mathematicians explore the underlying mathematics to the model equations. This forum for communication through the interpretation

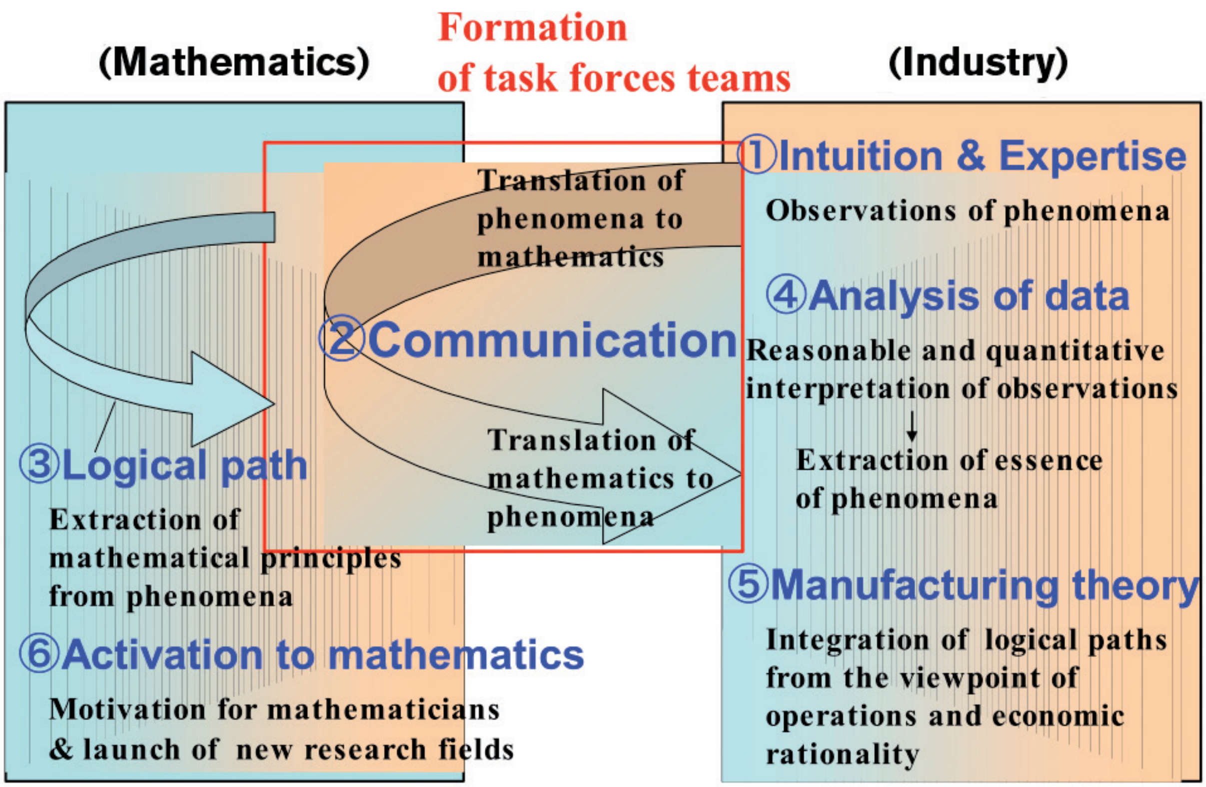

Figure 1—Collaboration with engineers and mathematicians in the case of Nippon Steel and the University of Tokyo

of phenomena is extremely important in order that engineers and mathematicians may reach a common understanding of the nature of the problem and the mathematical components. The third is "logical path." This corresponds to the extraction of mathematical principles from phenomena. Better communication can create a more logical path. The fourth is "analysis of data." This means reasonable and quantitative interpretation of observations carried out on site. This enables us to extract the essence of phenomena. The fifth is "manufacturing theory." This means the integration of logical paths from viewpoints of operation and economic rationality on site. The last is "activation to mathematics." Motivation for mathematicians has launched new mathematical research fields.

We, engineers in industry, have been eager to free ourselves from restrictions in our conventional thinking by making full use of mathematical reasoning that is free from specific industrial fields, through wider borderless collaborations. We have examined various conjectures by mathematicians and gained better practical solutions and further utilized analysis results. By repeating such phases of collaboration many times, we are able to pursue economic rationality, and mathematicians are able to find new results and describe them as theorem for future wider uses. It is important that mathematicians work not only for mathematical interests but also for the economic rationality through teamwork with engineers from a long-term point of view.

The cultivating interface between mathematics and industry has come into being as a forum for communication with the mathematicians mentioned above. Communication between the team members who are engineers in industry, and faculty members, post-doctoral fellows and doctor course students in university mathematical departments, has enhanced their communication skills day by day. As a result, several new themes have been launched.

Example of interdisciplinary collaboration

Figure 2 shows a challenge faced by Dr. Yuko Hatano. She is an associate professor affiliated with the University of Tsukuba whose major is Risk Engineering, and she had already collaborated with Nippon Steel on another subject.

The objective is to predict the progress of soil contamination. It is often the case with mass diffusion in a porous medium such as soil that numerical simulations using traditional advection diffusion equations fail to predict observation results of a real phenomenon observed in the field or laboratory tests. For instance, there are cases where actually the concentration is beyond the environmental standard as shown in Fig. 3, even when a simulation indicates that the concentration of the pollutant is below the relevant environmental standard and the danger of soil pollution is unlikely. Diffusion not following the prediction based on such a simulation is called anomalous diffusion, in contrast to the traditional diffusion equations, and is often observed in different manners with various substances in the soil or atmosphere in the real environment.

The above is the kind of problem that we encounter when numerically simulating a soil system in which voids are distributed unevenly between particles, using a grid for calculation larger than the voids. This type of problem will not occur when the grid spacing is smaller than the voids between soil particles, for instance, about 0.1 mm. However, since several kilometers or more is the normal scale for environmental studies, in view of computer load the use of such a fine grid for a three-dimensional case is extremely difficult, and is practically unsuitable for on-line field analysis. Moreover, whereas a model test covers a time scale of as short as minutes to days, the prediction of a real environmental problem must deal with a time scale as large as a few years to tens of years.

Although we have to treat widely varied sizes of data obtained through physical and numerical tests based upon different scales of space and time, the scaling law allows us to combine those data together in accordance with principles of phenomena.

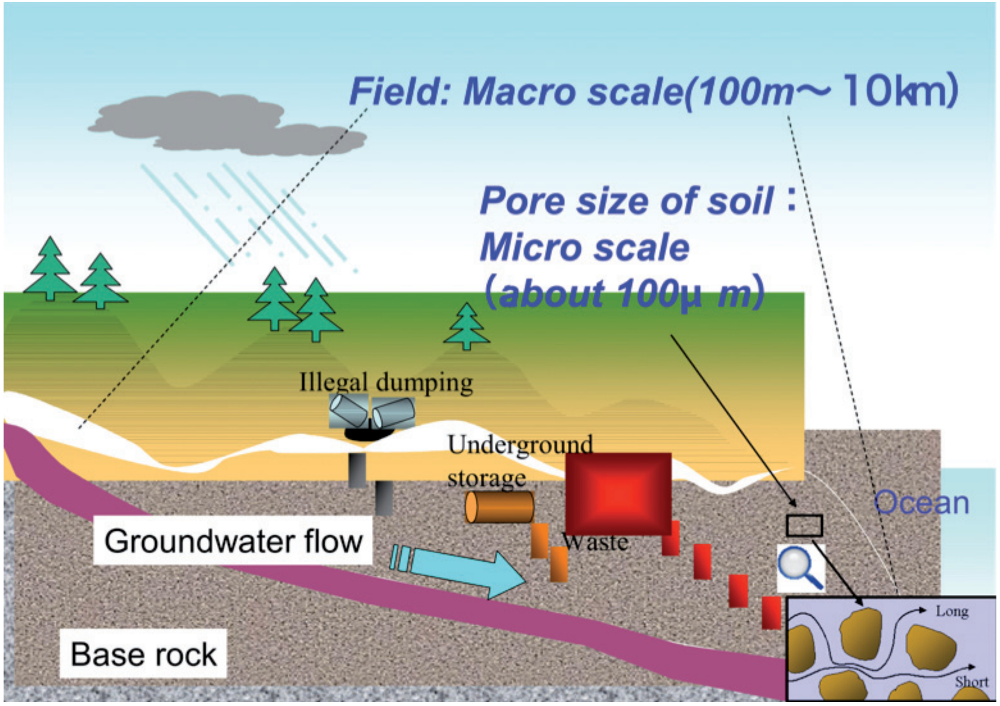

Figure 2—Prediction of soil contamination in large scale and long term

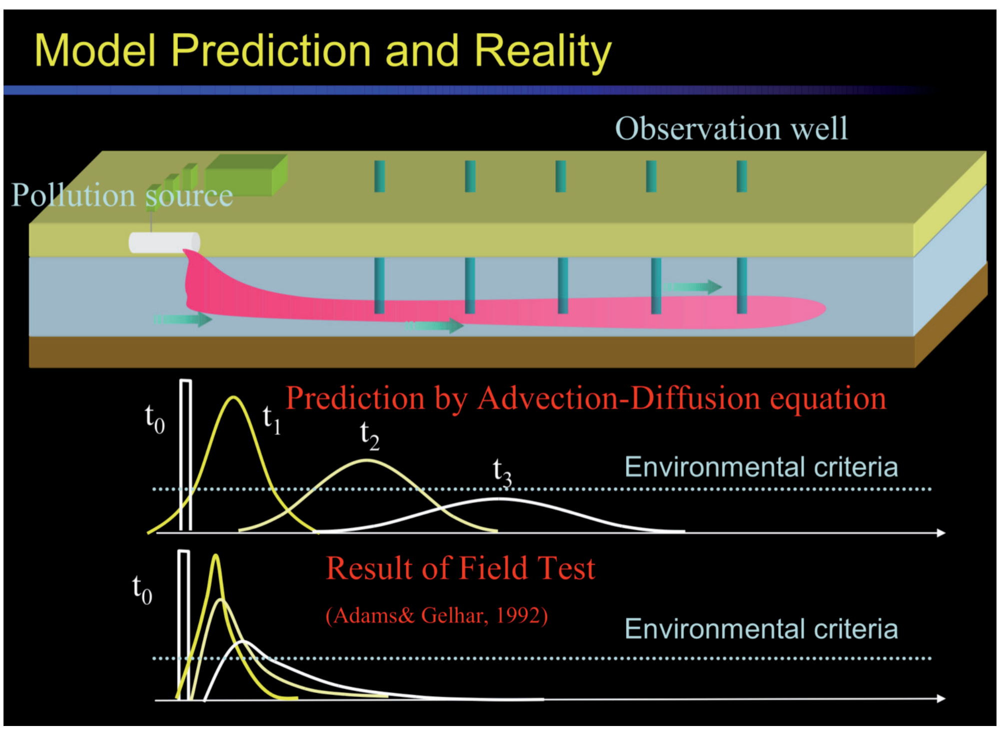

Figure 3—Comparison between model prediction and results of field tests

Large-scale numerical simulation is the principal method for the dynamic analysis of substances in any environmental medium: air, water or soil. Many detailed chemical and biochemical reactions are incorporated in the program codes for environmental simulation, and as a result, simulation programs seem to be becoming increasingly complicated these days. While a great number of numerical simulations are conducted on environmental issues, it is often difficult to tell whether each of such simulation results is valid, which fact is most serious for the problems.

Therefore the present study aims at dynamic prediction of environmental phenomena not totally depending on conventional numerical simulations but also employing mathematical methods typically such as scaling law. Toward this end, it is desirable to create a new field of environmental study involving mathematicians.

Launch of new research field in mathematics

A stochastic method employing random walk in consideration of the distribution of the waiting time of particles is used for describing mass transfer in soil. The stochastic method has been effective when applied to the small space dealt with in laboratory tests, but the limitation on the number of particles is a bottleneck due to the limit of computer capacity, and thus the method cannot respond effectively to more pragmatic requirements of calculation in a larger volume of space.

On the other hand, some fields of physics and engineering employ numerical simulation based on a diffusion equation that includes a fractional-order derivative in time. While the concept of a fractional-order derivative can be traced back to as long ago as Leibniz (see [2]), a theory of partial differential equation that is applicable to such numerical simulation has not yet been established, and the application of such a method has so far been limited to very special cases where the space has only one dimension. It is reported in the literature [3] that, according to the scaling law to the effect that the root mean square of the displacement of particles is in proportion to time raised to the kth power (t^k), the stochastic method using the random walk mentioned earlier is closely related to the Fokker-Planck equation, which leads to a fractional-order derivative:

$$(\partial/\partial t)^k u(x,t) = \nabla \cdot (\kappa \nabla u(x,t)) - \mu \cdot \nabla u(x,t),$$

where $u(x,t)$, κ and μ are the probability density function of particles, their diffusion coefficient, and mobility acting on them, respectively. It is expected that a scaling law combines stochastic methods such as the random-walk model for anomalous diffusion with the theory of partial differential equation including a fractional-order derivative to form a new field of research for mathematical concept and methodology. In [1], we discuss a related topic with such a theory.

Besides the above, Hatano et al. found that a formula empirically derived from two short-term atmospheric pollution cases (emission of inert gas Kr–85 from a nuclear plant in U.S.A. and the data of aerosol collected by an international team on global warming in the Arctic Ocean region) can describe the behavior of the pollutant of a long-term atmospheric pollution case (the accident of the Chernobyl Nuclear Power Plant) reasonably well [4], [5]. The formula is also written as a scaling law, but it is not yet been fully clarified why the formula has such a form.

Thus, through the collaboration of mathematicians and engineers from both academic and industrial fields, the present study establishes the fundamental logical structure that lies behind the scaling law observed in the behavior of pollutants in different environmental media such as soil and atmosphere, and thus clarifying the universal characteristics of scaling law.

Future Plan

In industrial practice, a reduced-scale model is constructed to analyze a phenomenon that takes place in real-size equipment, significant physical values for the phenomenon in question are described by dimensionless numbers, and the dimensionless numbers obtained from the model analysis are made to match with those of real-size equipment. This matching operation secures the similarity of the dynamic physical values between the model and real-size equipment. This similarity refers also to the scaling law. It has been found from the above viewpoint of scaling law that, in addition to the physical values such as time and length which have been conventionally used for scaling up, the fractional powers in the differentiation of time and space are essential. This means that mathematics is expected to present a new “angle of view” for the scaling law that deals with inhomogeneous media. Practically, environmental analysis deals with a scale of several kilometers or more in size. In this relation, establishment of scaling laws including an a priori choice of an exponent will make it possible to appropriately use results obtained through reduced-scale tests and clarify a real phenomenon across a large space.

By establishing scaling laws and developing mathematical methods based thereon, we can significantly reduce costs for producing high-quality products as well as energy consumption and CO_2 emission by improving production efficiency in various problems of manufacturing industries such as monitoring of sintering processes, reactions in a blast furnace, and other metallurgical reactions in steel-making processes as shown in Fig. 4.

Scaling laws and mathematical methods are applicable also to a wide variety of fields such as chemical engineering, mechanical engineering, geotechnical engineering, biotechnology, etc., and therefore, the establishment of such scaling laws is expected to be useful in remarkably accelerating the development of science and technology through the solution of important industrial problems.

Furthermore, the concept of scaling law combining micro- and macroscopic aspects is closely related to that of multi-scale modeling, the application of which is rapidly expanding in material science, chemistry, and other widely varied fields. The present study is expected to lead to proposals of new mathematical concepts and methodologies for multi-scale modeling, bringing about new problem recognition and methodology to mathematics.

Suggestions regarding future research of activities in the field of EIMI

"Mathematics for phenomena" will be the key for combining mathematics with industrial technology. Mathematical science can be understood as mathematics for phenomena; it is aimed at extracting fundamental principles behind different natural phenomena and engineering problems, and crystallizing them into mathematical structures.

Beyond the simple numerical operation of physical model equations, a methodology based on the principles and rules of mathematics makes it possible to construct mathematical models that describe the essence of a phenomenon selectively. Such mathematical models serve as important basis for understanding and controlling a phenomenon. When a mathematical model describes the essence of a phenomenon as simply and comprehensibly as possible (a minimum necessary model), it becomes easier for engineers and researchers from a variety of technical fields to study, and it becomes easier to conceive ideas that can lead to innovations.

In order to construct such a minimum necessary mathematical model that describes the essence of a phenomenon efficiently, a framework is required for the joint work of mathematicians and engineers from academic and industrial fields where they can thoroughly discuss subject phenomena and define suitable targets and milestones for different study stages. In addition, it is indispensable to mutually confirm work progress. At present, however, applied mathematics in Japan, compared with other developed countries, seems to lack such teamwork experience that helps to combine a phenomenon with mathematical methodology. In order to solve a problem as promptly as required in industry, it is too late to begin studying methodology after posing of the problem. It is necessary to contin-

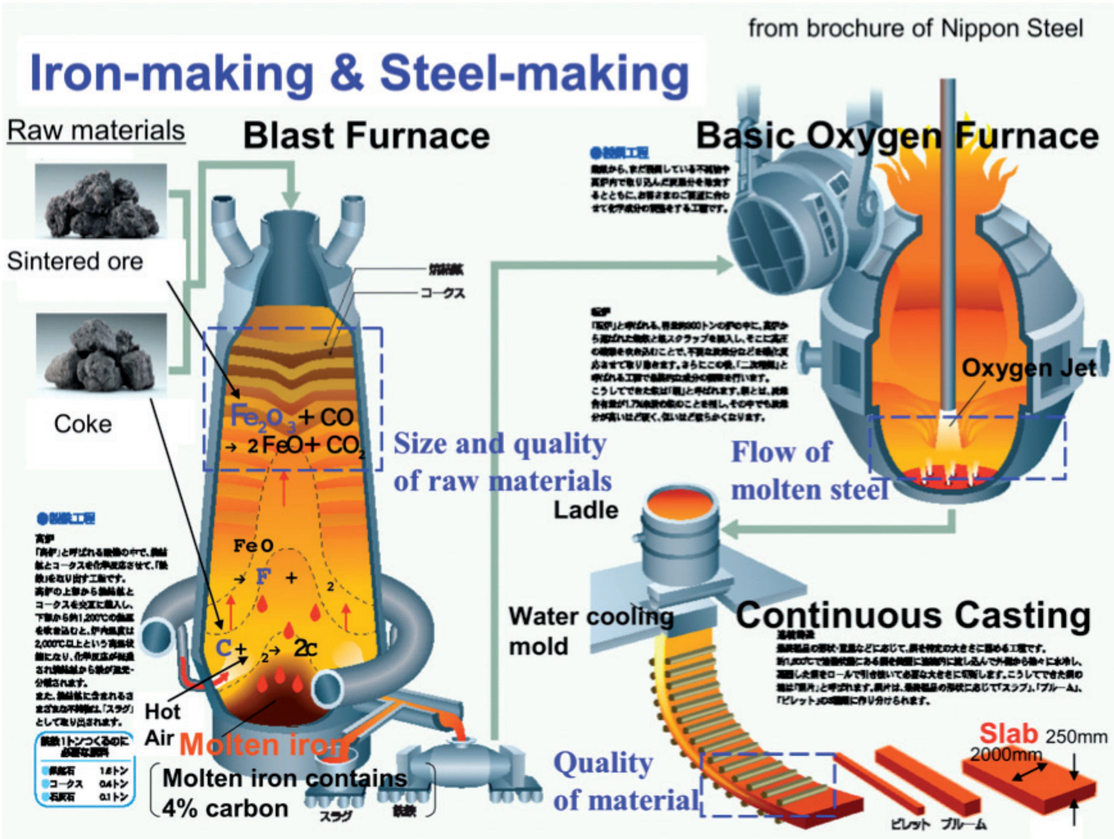

Figure 4—Example of inhomogeneous media in iron making and steel making process

ue to improve the skill to combine a phenomenon with mathematical methodology for its prompt application, and in this respect, each individual must improve their qualification to be "the right person" who can meet the above conditions and the role.

It is desirable that both mathematics and industry foster people capable of working jointly with each other from the viewpoint of "mathematics for phenomena" through academic-industrial collaboration. Towards this end, it is necessary to create a new framework independent of the structure of present industry and academic organizations. We must reinterpret and reconstruct the fundamental concept of manufacturing based on field practice, which constitutes the competitive edge in developed countries, from the standpoint of mathematical methodology while learning about interdisciplinary collaboration from abroad. By so doing, we will be able to command the most advanced industrial technology of the world.

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Computational Modelling in Science, Technology, Engineering and Mathematics Education

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Abstract Modelling is a central aspect of the research process in science, technology, engineering and mathematics (STEM) which occurs in the cognitive context of an interactive balance between theory, experiment and computation. The STEM learning processes should then also involve modelling in environments where there is a balanced interplay between theory, experiment and computation. However, an adequate integration of computational themes in STEM high school and undergraduate university curricula remains to be achieved. In this work, we present an approach to embed computational modelling activities in the STEM learning process which may be fruitfully adopted by high school and university curricula, as well as be a valuable instrument for the professional development of teachers. To illustrate, we consider the example of physics.

Introduction

Science, technology, engineering and mathematics (STEM) are evolving structures of knowledge which are symbiotically interconnected. On one hand, science is based on hypotheses and models, leading to theories, which have a strong mathematical character as scientific reasoning, concepts and laws are represented by mathematical reasoning, entities and relations. On the other hand, scientific explanations and predictions must be consistent with the results of systematic and reliable experiments, which depend on technological developments as much as these depend on the progress of science and mathematics (see, e.g., Chalmers, 1999; Crump, 2001; Feynman, 1967). The creation of STEM knowledge is a dynamical cognition process which involves a blend of individual and collective reflexions where modelling occurs with a balance between theoretical, experimental and computational elements (Blum, Galbraith, Henn & Niss, 2007; Neunzert & Siddiqi, 2000; Schwartz, 2007; Slooten, van den Berg & Ellermeijer, 2006). In this research paradigm, computational modelling plays a key role in the expansion of the STEM cognitive horizon through enhanced calculation, exploration and visualization capabilities.

Although clearly related to real world phenomena, STEM knowledge is built upon abstract and subtle conceptual and methodological frameworks which have a complex historical evolution. These cognitive features make science, technology, engineering and mathematics difficult fields to learn and to teach. To develop an approach to STEM education which aims to be effective and in phase with the rapid scientific and technological development, it is of crucial importance to promote an early integration of computational modelling in learning environments which reflect the exploratory and interactive nature of modern research (Ogborn, 1994). However, even in technologically advanced countries, computers, computational methods and software, as well as exploratory and interactive learning environments, are still not appropriately integrated in most STEM education curricula for the high school and undergraduate university levels. As a consequence, these curricula are generally outdated and most tend to transmit to students a sense of detachment from how science is currently made. These are contributing factors to the development of negative views about the education process and to an increase in student failure.

Physics education is a good example to illustrate this problem. Consider the general physics courses taken by first year university students. These are courses which usually follow a traditional lecture plus laboratory instruction approach and cover a large number of physics topics which students find particularly difficult. Due to a lack of understanding of the necessary fundamental concepts in physics and mathematics, the number of students that fail on the course examinations is usually very high. What is worse is that many of those students that do actually succeed also reveal several weaknesses in their understanding of elementary physics (Halloun & Hestenes, 1985; Hestenes, 1987; McDermott, 1991).

Research in physics education has shown that this situation can be improved when students are involved in the learning activities as scientists are involved in research (Beichner et al., 1999; Mazur, 1997; McDermott, 1997). This is not a surprising result. Scientific research in physics is an interactive and exploratory process of creation, testing and improvement of mathematical models that describe observable physical phenomena. It is this cognitive process that leads to an inspiring understanding of the rules of the physical universe. As a consequence, physics should be expected to be more successfully taught in interactive and exploratory environments where students are helped by teachers to work as scientists do. In this kind of class environment knowledge performance is better promoted and common sense beliefs as well as incorrect scientific ideas can be more effectively fought.

The scientific research process is supported by a continuously evolving set of analytical, computational and experimental techniques. The same should be true for research inspired learning environments. Consequently, another important aspect of these learning environments is the possibility to balance the role of computational modelling methods and tools. This would set the learning process in phase not only with modern scientific research where computation is as important as theory and experiment, but also with the rapid parallel development of technology.

Several attempts have already been made to introduce computational modelling in research inspired learning environments. The starting emphasis was on professional programming languages such as Fortran (Bork, 1967) and Pascal (Redish & Wilson, 1993). Although more recently this approach has evolved to Python (Chabay & Sherwood, 2008), it continues to require that students develop a working knowledge of programming, a time consuming task which hinders the process of learning physics. The same happens with scientific computation software such as Mathematica and Matlab. To avoid overloading students with such programming notions or syntax, and focus the learning process on the relevant physics and mathematics, several computer modelling systems were created, for example, Dynamical Modelling System (Ogborn, 1985), Stella (High Performance Systems, 1997), Easy Java Simulations (Christian & Esquembre, 2007) and Modellus (Teodoro, 2002).

A proper and balanced integration of computational modelling methods and tools in STEM learning environments is, thus, both a curricular and a technological development problem. In this work, we discuss how Modellus (a freely available software tool created in Java which is able to run in all operating systems, see the software webpage a http://modellus.fct.unl.pt) can be used as a central element of an approach to develop exploratory and interactive computational modelling learning activities relevant for STEM education. These activities can be adopted by high school and university curricula, and used as a valuable instrument for the professional development of teachers. To illustrate, we consider computational modelling with Modellus to teach physics, namely introductory mechanics, and discuss its impact on the student learning process.

Modellus: interactive and exploratory computational modelling for STEM education

The construction of STEM knowledge requires unambiguously clear declarative, operational and conditional specifications of abstract concepts and of the relations among such concepts. Of crucial importance in the understanding of the resulting models or theories is the interpretation process which involves operational familiarization and connection with the relevant referents in the observable universe (Reif, 2008). In education, as in research, computers, computational methods and software are cognitive artefacts (Teodoro, 2005) which may amplify the learning cognition horizon due to more powerful calculation, exploration and visualization capabilities. As a consequence, these artefacts may play a key role in enhancing the operational familiarization and the connection with the real world referents which must necessarily be involved in the STEM learning processes. To be able to fulfil such a potential key role, computational methods and tools should be used not only to display text, images or simulations but as mathematical modelling tools integrated in learning environments which reflect the exploratory and interactive nature of modern research. In addition, the modelling process should be focused on the meaning of models and avoid learning opacity factors such as too much programming and specific software knowledge.

To meet this educational challenge it is not enough to simply choose a subset of programming languages and professional computational software. It is necessary to develop computer software systems with computational modelling functionalities which contribute to nurture the progressive growth of solid STEM cognitive competencies. Among those systems which have already been created, Modellus stands out as a computational modelling tool for STEM education because of the following main advantages: 1) an easy and intuitive creation of mathematical models using standard mathematical notation; 2) the possibility to create animations with interactive objects that have mathematical properties expressed in the model; 3) the simultaneous exploration of multiple representations such as images, tables, graphs and animations; 4) the computation and display of mathematical quantities obtained from the analysis of images and graphs.

These are features that allow a deeper cognitive contact of models with the relevant real world referents and a deeper operational exploration of models as objects which are simultaneously abstract, in the sense that they represent relations between mathematical entities, and concrete, in the sense that they may be directly manipulated in the computer. In a word, Modellus allows a deeper reification of abstract mathematical objects. Because of these characteristics, computational modelling activities built with Modellus can be readily conceived as exploratory and interactive modelling experiments performed by students in collaborative groups or individually. They can also be designed with an emphasis on cognitive conflicts in the understanding of STEM concepts, on the manipulation of multiple representations of mathematical models and on the interplay between the analytical and numerical approaches applied to solve STEM problems.

As a domain general environment for modelling, Modellus can be used to conceive STEM learning activities which involve the exploration of existing models and the development of new ones (Bliss & Ogborn, 1989; Schwartz, 2007). As much as possible, such modelling activities should consider realistic problems to maximise the cognitive contact with the real world referents. This is a challenge because more realistic problems are generally associated with more complex analytic solutions which are beyond the analytic capabilities of high school or first to second year university students. With Modellus and numerical methods, which are conceptually simpler and yet powerful, the interactive exploration of models for more realistic problems can start at an earlier age, allowing students a closer contact with the model referents, an essential cognitive element to appreciate the relevancy and power of models, necessarily a partial idealized representation of their referents.

Clearly, the development of the appropriate computational modelling activities for STEM research inspired learning environments is bound to call for a richer set of modelling functionalities which are not yet available in Modellus. These events are seeds for technological evolution which should be accomplished by a Modellus enhancement program. Currently under development and set to appear in forthcoming versions of Modellus is, for example, the following set of new functionalities: spreadsheet, data logging and curve fitting capabilities, advanced animation objects like curves, waves and fields, 3D animations and graphs, creation of a physics engine for motion and collisions, video analysis and cellular automata models.

The simultaneous development of new functionalities to meet appropriate teaching goals is important because it reduces the learning opacity factor associated to an unnecessary proliferation of tools. However, there is a learning stage where it is advantageous to allow some diversity in the use of computer software tools and complement Modellus with other available tools. Indeed, in a research based STEM learning environment one of the objectives is to make a progressive introduction to professional STEM computation methods and software. For example, Excel is a general purpose spreadsheet where modelling is focused on the algorithms. In addition, it already allows data analysis from direct data logging. On the other hand, Mathematica and Matlab (or wxMaxima, a similar but freely available tool) have powerful symbolic computation capabilities. Using these different tools to implement the same algorithm is an important step to learn the meaning of the algorithm instead of the syntax of a particular tool. If more realistic simulations are needed, Modellus animations can be complemented, for example, with EJS.

Computational modelling learning activities: an illustrative example from rotational dynamics

Let us consider a computational modelling activity about rigid body rotational dynamics and angular momentum, a topic in general physics which in a course program for first year university students should be introduced after computational modelling activities covering vectors, kinematics and Newton's fundamental laws of motion, including simple numerical and analytical solutions (Neves, Silva & Teodoro, 2009 a, 2009 b).

A rigid body is a system of particles whose relative distance does not change with time. When a rigid body rotates around a fixed axis, each one of its particles has a circular motion around the axis which is characterised by a rotation angle θ, an angular velocity ω and an angular acceleration α. The kinetic rotation energy is the sum of the kinetic energies of all the particles of the body and is given by $K = I\omega^2/2$, where I is the moment of inertia of the body relative to the rotation axis. From a dynamical point of view, the rotational motion of a rigid body around a fixed axis is characterised by two vectors, the angular momentum of the rigid body and the moment of the sum of all the forces acting on the body, the latter also called the net applied torque. Newton's laws of motion imply that the instantaneous rate of change of the angular momentum is equal to the net applied torque. If the rotation axis coincides with the Oz axis, the angular momentum is given by $\vec{L} = L\vec{e}_z = I\omega\vec{e}_z$, where $\vec{e}_z$ is the unit vector of the Oz axis. The corresponding moment of the sum of all applied forces is $\vec{\tau} = \tau\vec{e}_z = I\alpha\vec{e}_z$.

A real world system which may be considered as a rotating rigid body is a wind turbine. With Modellus it is possible to model the action of the wind on the rotor blades and analyse their motion using at the same time different representations such as graphs, tables and object animations. A mathematical model example in SI units is given in figure 1.

In this model the fundamental equations of the rotational motion are written in the form of Euler iterations. Students are thus taught to apply this numerical method in a new realistic context, extending the applicability range of knowledge already acquired with a previous analysis of analogous numerical solutions of Newton's equations in translational motion settings (Neves, Silva & Teodoro, 2009 a). In this new application, students can determine

$$R = 20$$
$$Ko = 5000$$
$$wo = \frac{\pi}{6}$$
$$I = 2 \times \frac{Ko}{wo^2}$$
$$L = \text{last}(L) + \text{last}(Tau) \times \Delta t$$
$$L_0 = I \times wo$$
$$w = \frac{L}{I}$$
$$alpha = \frac{Tau}{I}$$
$$K = \frac{1}{2} \times I \times w^2$$
$$theta = \text{last}(theta) + \text{last}(w) \times \Delta t$$
$$x = R \times \cos(theta)$$
$$y = R \times \sin(theta)$$

Figure 1—Modellus mathematical model for the rotational motion of a wind turbine.

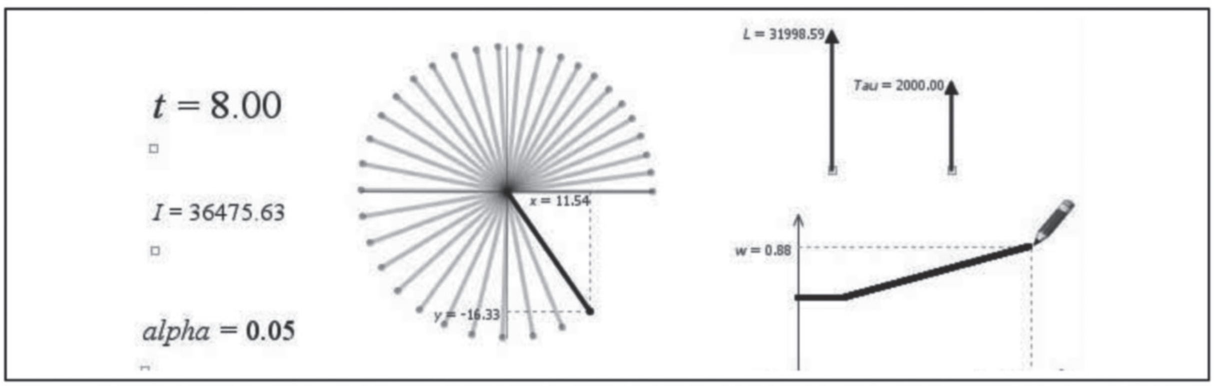

Figure 2—Modellus animation of the rotational motion of a wind turbine. The bar stroboscopic effect shows the acceleration due to the net applied torque.

the angular velocity and the rotation angle knowing the net applied torque, the moment of inertia of the system and the motion initial conditions.

The model animation is constructed with three objects: a bar representing the rotor blade, a vector representing the angular momentum and a vector representing the net wind torque (see figure 2). Because the coordinates of the net torque are independent variables and the model is iterative, students can manipulate this vector at will and in real time control the motion of the rotor blade. With this activity students can confirm that the choice of the time step is an important one to obtain a good simulation of the motion and that this is the same as determining a good numerical solution of the equations of motion. While exploring the model, students can determine, for example, what are the values of the angular momentum, the angular velocity and the rotational kinetic energy, 8 seconds after increasing the net wind torque to 2000 Nm (see figure 2). The possibility to change the mathematical model and immediately observe this action on the animation, graphs and tables is a powerful cognitive element to enhance the students learning process. Students can also change the model. Introducing a vector to represent the wind force they can explore the effect of the wind direction on the net applied torque and on the motion of the rotor blades.

Field actions, discussion and outlook

In this paper we have shown how Modellus can be used as a key element of an approach to develop exploratory and interactive computational modelling learning activities for research inspired STEM education. As an example, we have discussed the modelling of the rotational dynamics of a wind turbine. This was one of the activities which was tested on the field when we implemented computational modelling activities in the general physics course offered in 2008 and 2009 to the first year biomedical engineering students at the Faculty of Sciences and Technology of the New Lisbon University (Neves, Silva & Teodoro, 2008, 2009 a, 2009 b). For other educational applications and evaluation tests of Modellus based computational modelling activities see, e.g., Araœjo, Veit & Moreira (2008), Dorneles, Araújo & Veit (2008) and Teodoro (2002).

The 2009 edition of the general physics course for biomedical engineering involved a total of 115 students. Of these, 59 were taking the course for the first time and only they were enrolled in the computational modelling classes. To build an interactive collaborative learning environment, we organized the students in groups of two or three, one group for each computer in the classroom. During each class, the student teams worked on a computational modelling activity set conceived by us to be an interactive and exploratory learning experience with Modellus, built around a small number of problems in mechanics connected with easily observed real world phenomena. The teams were instructed to analyse and discuss the problems on their own using the physical, mathematical and computational modelling guidelines provided by the activity set documentation, a set of PDF documents with embedded video support. To ensure a good working pace with appropriate conceptual, analytical and computational understanding, the students were continuously monitored and helped during the exploration of the activities. Whenever it was felt necessary, global class discussions were conducted to clarify doubts on concepts, reasoning or calculations. Online support in class and at home was provided in the context of the Moodle platform where links to class and home work documentation was provided.

The evaluation procedures associated with the computational modelling activities with Modellus involved both group evaluation and individual evaluation. For each computational modelling class, all student groups had to complete an online test written in the Moodle platform answering the questions of the corresponding activity PDF document. The individual evaluation was based on the student solutions to two sets of homework activities and a final class test with computational modelling problems to be solved with Modellus. All students took a pre-instruction and post-instruction FCI test (Hestenes, Wells & Swackhamer, 1992) which did not count for their final course grade. At the end of the semester, the students answered a questionnaire to evaluate Modellus and the new computational modelling activities of the general physics course.

As reflected by the student answers to the 2008 and 2009 questionnaires (Neves, Silva & Teodoro, 2008, 2009b), the activities with Modellus were helpful in the learning process of mathematical models in the context of introductory physics. For students, Modellus was easy to learn and user-friendly. Working in groups of two or three was also acknowledged to be more advantageous than working individually. In addition, the PDF documents with embedded video guidance were considered to be interesting and well designed.

During class and homework communications, it was clear that the computational modelling activities with Modellus were being successful in identifying and resolving several student difficulties in key physical and mathematical concepts of the course. The possibility to explore simultaneously several different representations like graphs, tables and animations was a key success factor, in particular the possibility to have a real time visible correspondence between the animations with interactive objects and the object's mathematical properties defined in the model. During the course, Modellus and simple numerical methods, such as the Euler and Euler-Cromer methods, allowed the introduction and exploration of advanced mathematical concepts such as integration in the context of real world physics problems, prior to the introduction of the corresponding analytic techniques. However, FCI test results for the whole general physics course lead to an average FCI gain of 22%, a typical traditional instruction performance (Hake, 1998). Moreover, students manifested clearly that the new content load was too heavy and that the available time to spend working on the computational modelling activities was insufficient. Improvement actions (Neves, Silva & Teodoro, 2009 b) are now being implemented.

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Modeling Modeling: Developing Habits of Mathematical Minds

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Abstract Faculty at the Department of Mathematical Sciences of the University of Delaware have engaged in a reform effort to develop undergraduate mathematics majors specific habits of mind associated with the effective practice of mathematics in applied and industrial settings. This effort has led to the creation of a new mathematical modeling capstone course possessing many novel features. Over the past seven years, this course has developed the confidence and ability in our students to see the intimate connections between mathematics and real world processes. In this paper, we identify seven key habits of mind associated with effective mathematicians and present best practices for reinforcing these habits in our students.

"It is like gum. You chew gum and use it to freshen up your breath, but in the end, its worthless and doesnt have any nutrition or vitamins. Math is used in school to determine your intelligence, but there is no need for it later."

Introduction

Unfortunately, this students comment is typical; all too often, students dont see the connection between mathematics and real-life (Cai & Merlino, in press). Even among those students who expect to become scientists, less than 75% of those students believe that advanced mathematics or science courses are necessary for their future careers (Ma, 2006). One of unfortunate reasons for developing such beliefs is a lack of experience in seeing the intimate connections between mathematics and the real-world. Students are usually taught mathematical concepts and procedures in a vacuum. Little effort is made to connect mathematics with authentic real world applications and students gain little direct experience with how mathematics is used in the workplace (Bessot & Ridgway, 2001; Freudenthal, 1991). Over the past several years, an effort has been made to reform our major undergraduate mathematical modeling course so that students can directly experience interactions between mathematics and its applications. How can we best develop students habits of mind to see and interpret the world mathematically? This is the primary research question that has guided our reform effort. Through modeling the mathematical modeling processes in the course, we have used innovative curriculum materials and pedagogical strategies to develop seven habits of mind for students to make sense of the world mathematically. In this paper, we critically analyze and reflect on this reform effort. Our objective in this paper is to provide concrete examples of best practices to connecting mathematics and the real-world. A unique feature of this contribution is its primary focus on developing students habits of mind. Thus, the examples and ideas from our critical analysis of the reform effort can be easily adapted to other institutions and other courses.

Description of the Modeling Course

Our capstone course, Math 512: Contemporary Applications of Mathematics is the centerpiece of our undergraduate mathematics curriculum at the University of Delaware. Recently, it has become very popular for majors outside of mathematics because of the connection between mathematics and real-world. More importantly, it has been the target of our efforts to reform how we instill in students what we believe to be habits of mind that all effective mathematicians share. Our course was strongly influenced by the annual Mathematical

Problems in Industry (MPI) workshop, and Mathematical Contest in Modeling sponsored by the Consortium for Mathematics and its Application (COMAP)[1] MPI is a workshop where faculty and graduate students spend a week working on open problems contributed from industrial experts. Participants spend almost the entire meeting in breakout rooms doing new mathematics and critiquing one another in order to make progress by the end of the week. MCM is a similar experience for undergraduates except that it is a four-day contest. Student teams are given an open ended modeling problem and are allowed to use any inanimate source. Their goal is to craft a comprehensive solution which is submitted to contest judges. After completing the contest, most students comment that they accomplished more in those four days than they do in an entire semester.

Starting in 2001, we began reforming our capstone course to develop effective qualities in students that we had observed at MPI and MCM. In particular, we wanted to focus on small groups of students working on open problems arising from disciplines outside of the mathematical sciences. We designed our course around the notion that there were certain key activities associated with experiences like MPI and MCM, and that we would have to make it possible for students to engage in these activities in the context of a regular course. Since 2002, with the support of two internal educational grants at the University of Delaware, we have continuously revised and improved the course to better prepare students of all majors to see and interpret the world mathematically.

Most significantly, we added a laboratory component to Math 512 using our Modeling Experiment and Computation (MEC) Lab which is a fully instrumented wet lab housed within the Department of Mathematical Sciences.

The figure above helps helps visualize the transformation. Fundamentally, industrial mathematics involves connections between a process that must be quantified as some kind of mathematical object, mathematical techniques and methods, and increasingly computation and simulation. Our laboratory component adds a new dimension to this undertaking by connecting all these activities with direct observation and measurement.

Developing Habits of Mind

As we have developed and refined our capstone course, we have sought to identify the habits of mind that are characteristic of successful mathematical modelers (Gross, Haenschke, & Tenzler, 1988). It is these habits we attempt to instill in our students. These habits are:

1. Awareness
2. Attention

3. Resilience
4. Specificity
5. Curiosity
6. Vision
7. Communication

We first describe each of these habits in turn and explain their relationship to mathematical modeling. In the next section, we discuss the pedagogical principles to develop these habits of mind.

Awareness — Mathematical modeling begins with a problem. This first step requires the successful modeler to be aware of the world around them, to possess the listening skills to enable them to learn about the problem from other scientists, to be possess the reading skills to enable them to learn about the problem from the literature, and to possess the breadth of knowledge to enable them to understand how the problem at hand relates to the field at large. Clearly, within this broad category of awareness, several specific skills must be developed. In our courses, we aim most directly at developing the reading skills of students as this seems to be a large stumbling block on the path to their success.

Attention — Once a modeler has understood the context of a problem, it is time to focus on the phenomena before them. This requires attention. That is, they must learn to observe. This process can take many forms. It may involve gathering data, examining large sets of data, or more frequently, deciding what data to gather. In our courses, students are placed face-to-face with an experimental system and pushed to pay attention. What do they see? What do they not see? They are urged to drop pre-conceived notions of how a system behaves and to simply give their attention to the natural world.

Resilience — A successful modeler must be prepared to fail and to fail often. Assumptions often dont fit the data; there is often ambiguity about what to leave in and what to leave out, and often the natural world leads one in unexpected directions. This requires that the successful modeler develop a high level of comfort with ambiguity. This also requires that the student abandons the comfort of pre-packaged problems and develop the resilience needed to tackle the process of formulating their own problems. In our courses, we provide the students with open-ended problems to which we do not know the answer. This allows us to model the habit of flexibility as we go through the investigative process side-by-side with our students.

Specificity — A successful modeler must be prepared to fail and to fail often. Assumptions often dont fit the data; there is often ambiguity about what to leave in and what to leave out, and often the natural world leads one in unexpected directions. This requires that the suc-

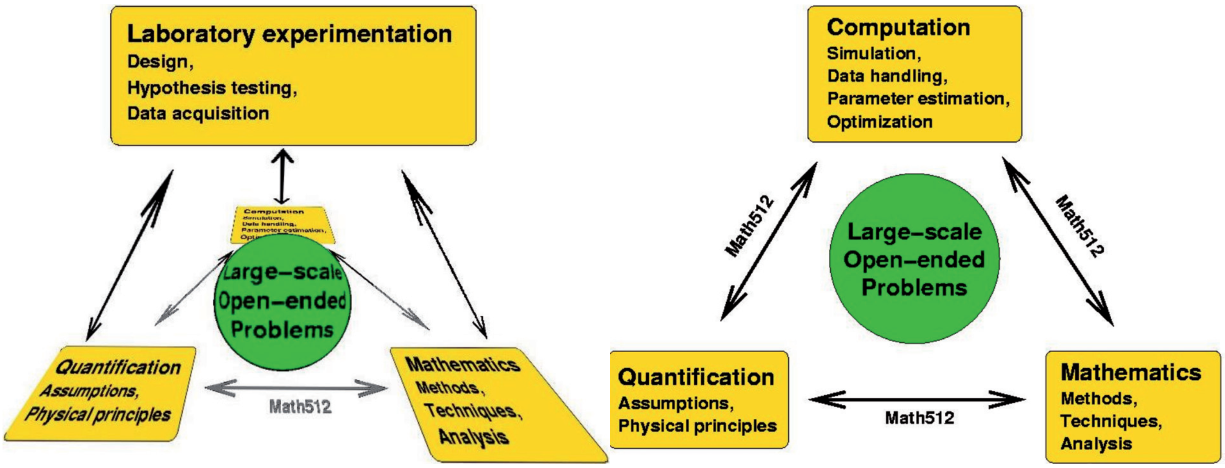

Figure 1

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Curiosity — Linked with the ability to ask quantitative questions is the desire to do so. The successful modeler must possess the habit of curiosity. They must be driven to look at the world and ask Why? This is perhaps, the most difficult of the habits to instill in students. In our courses, we grow this habit by providing them with a range of inherently interesting problems and allowing them to follow their own path in choosing which to investigate.

Vision — The modeler that knows the context of a problem, observes, and asks quantitative questions, possesses the information needed to generalize their results and synthesize others results with their own. But, they must develop a broad vision that drives them to do so. In our courses, we develop this skill by pushing the students to constantly iterate. They know context, they have asked good questions, and they have answered them. Now, they are asked how their answers compare with those of others. What are the implications of this? How does this work shed light on broader questions in mathematics and in science?

Communication — Our first habit, awareness, ultimately focused on communication. But, at the start, the focus is on input. That is, the modeler listens, reads, and digests. However, science and mathematics does not take place in a vacuum. The successful modeler is able to communicate their results, no matter how tentative, with others. In our courses, we emphasize this habit through requiring group work, asking for informal and formal presentations in class on project progress, and requiring the final product of a written report to be of high quality.

Pedagogical Principles of the Reform Effort

Four principles have guided our reform effort from the beginning. Over time, we have made adjustments and corrections, but we have found that these basic principles, and the practices that we have implemented based on them, help instill the seven habits of mind in our students.

Maps matter — The heart of Math 512 is a collection of exciting open research questions originating from outside of the mathematical sciences. The course was designed to guide them through the modeling process. The problems vary from year to year, but our goal was kept the same designing the course around effective practices for helping students develop habits of the mind, rather than teaching a rigid set of topics. Students are driven by a desire to learn, and they are keenly aware of how they will be graded. We carefully designed our course to align our expectations that students would aggressively pursue open-ended problems with a grading scheme that would reward our students' efforts. We realized that we had to design a grading scheme that formatively assess the students' progress on their research project throughout the semester. Our response to this challenge is a set milestones placed at equal intervals in the semester. A visual road map guides students through the semester, even though they may work on different modeling projects, as shown in Figure 2. We have added a column describing the habits of mind these activities instill.

On the first day of classes, students see the milestone chart as a grading scale, with their milestone grade arranged in rows. They know they will need to complete their literature review for the first milestone and that it will be worth 80% of their grade. They know that they will use the wiki to draft their report until the 4th milestone when they will start preparing a final copy in LaTeX. However, as the semester progresses, they see that milestones are markers on their way to a significant work product, and this work-in-progress is what is being graded. As instructors, we use the table to mark our own progress as we guide and lead students on their intellectual journey.

Process Emphasis — As instructors, we learned early on that we would have to teach process almost as much as we would have to teach content. Our students have all taken a semester-long composition course offered by the English Department, but we spend a significant amount of time teaching students to write collaboratively within their groups. In the writing process, students explore, understand, and learn how to pose significant questions to

Habit of mind	Milestone	1	2	3	4	5
	Feature	(wiki)	(wiki)	(wiki)	(wiki + LaTeX)	(wiki + LaTeX)
Awareness, Attention, Specificity, Curiosity	Lit. review, Assumptions, Definitions, Formulations	80%	20%	5%	0%	0%
Attention, Specificity, Resilience, Curiosity	Analysis, Model solutions, Measurements, Parameter estimation.	5%	60%	40%	40%	20%
Awareness, Attention, Curiosity, Resilience, Vision	Simulations, Comparisons Strengths & weaknesses, Synthesis.	0%	0%	25%	20%	30%
Engagement, Vision	Presentation, Style, Brevity, Clarity.	15%	20%	30%	40%	50%

Figure 2—Math 512 Milestones

research, as well as investigate the ways to solve them. The use of the advanced technology facilitated the processes. This includes mastering the latest technology for facilitating such collaborations. Students must use LaTeX, the universal typesetting language for mathematics. Over time, we have used a variety of collaborative versioning systems including Concurrent Versioning System (CVS), Subversion (SVN) and now students draft material using wikis enabled with embedded LaTeX.

Hands-on Opportunities — Students must be able to observe the processes that lie at the heart of their problems, whether it is freezing drops of water or observing an organic self-assembled network, so we have a dedicating wet lab set aside for our course. It is possible for students to acquire data from the literature or the web, but these sources of information do not respond to student interrogation. As students refine their assumptions and modify their expectations, they design an experiment and observe its outcome. Students learn that every observation comes with a cost in time and energy, and they become very adept at creating definitive experiments to resolve key issues in their projects.

Formative Assessment — If we expect students to approach problems professionally, we have to provide feedback throughout the semester tapping into the learning that we observe as instructors as well as the learning that occurs within each group outside of class. Over the last two years, we have refined a peer assessment rubric wherein students evaluate one another (including themselves). Within each group, students take responsibility for different aspects of the project. After implementing this element to our course, we find that all students are more engaged in all project activities and project groups tend to find ways to match individual skills and abilities to required tasks.

Outcomes

Our reform effort has led to multiple positive outcomes for our students and our department. Our capstone course has helped to prepare many of our students to succeed in high competitive graduate programs across the nation. Not only do we bring industrial problems into the classroom, but our students' work has attracted new industrial problems into the more professional MPI setting. Our course has been featured in the press (Pelesko, 2008) and in numerous invited talks nationally and internationally where there is interest in applying our practices at other institutions.

An example of strong connections between Math512 and industrial mathematics research activities comes from the study of air bearings. A bearing is any device that supports a load. Most students have had some exposure to ball bearings used on axles for bicycles and skateboards. An air bearing is a load bearing device that relies upon a cushion of air. Most students have experienced an air bearing when playing air hockey. Modeling air bearings has many industrial applications including the motion of the read/write head of a disk drive and control of precision, industrial robots.

At the 2001 MPI workshop, an interesting air bearing problem was posed by Ferdi Hendriks from IBM. For static air bearings, their engineering team had observed that scratching the bottom of the air bearing improved its performance in the sense that grooved surfaces could bear a greater load than smooth surfaces with the same input air flow. Optimization via computer simulation led to the same conclusion, and the optimal bearing surfaces exhibited a fractal branching structure similar to structures observed in natural systems such as the cardiopulmonary system in mammals, root systems in trees and so forth (See Figure 1). The key difference is that the air bearing surface was optimized by a computer algorithm whereas examples in the natural world are optimized by natural selection. The problem proved to be very challenging, but the workshop participants developed a model based on lubrication theory and studied axisymmetric profiles. The model predicted that minor surface variations could lead to large changes in bearing performance. However, branching structures are not axisymmetric, so the week ended without much more quantitative insight into why the observed groove patterns were optimal (Howell et. Al, 2001).

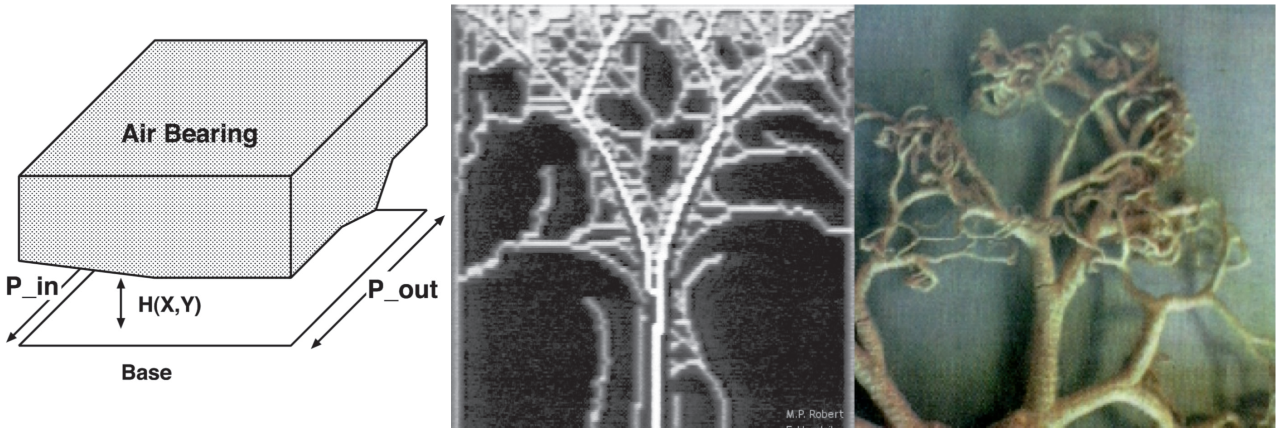

Figure 3—The geometry of an air-bearing is shown on the left. In this diagram, air flows under the bearing from left to right driven by a pressure drop. The computationally optimized surface of a static air bearing from the Mathematical Problems in Industry Workshop 2001 is displayed in the center. On the right, the circulatory system of a basket starfish, one of many examples of branching processes in the natural world.

The problem had all the ingredients of a great capstone project. The problem was compelling because the observed optimal solution was not intuitively obvious and there are could be connections to the natural world. With an interesting problem, awareness and attention are not difficult to encourage. Most students had had experiences with air bearings without really giving them much thought. We were confident that our students had sufficient mathematical background to grasp lubrication theory relatively in the first weeks of the course, and a laboratory model of an air bearing could be constructed within our model budget and facilities.

The air bearing problem was offered as one of three project topics during the first year of our reform effort in 2003. As instructors, we learned many lessons during this first offering and these experiences inspired many changes that have since because a permanent part of our capstone course.

Our experience transferring an industry workshop problem into an undergraduate classroom taught us an important lesson about preparation: The success or failure in creating these experiences for undergraduates requires instructors to lay a thorough foundation far beyond what is required for a typical classroom experience. Not only did the workshop presenter inspire an interesting problem, but the workshop report was a valuable document for the upcoming capstone class.

The workshop report was accessible to students and made it easier for us as instructors to package the problem for student consumption. However, this is not a sustainable approach for a course that will be offered once per year and will require new projects every time it is offered. Fortunately, we obtained an internal grant to hire students during the summer to explore prospective projects individually. The summer research experience lays the groundwork for the semester long project and helps identify potential hazards in the projects. For example, a laboratory model of the project might require extra equipment or may simply be too complex to use in the course at all. We have learned to use these summer experiences to sift through the literature and identify key references so that students can climb the learning curve as quickly as possible during the semester.

We note that the summer experience and the MPI workshop experience differ in important ways. The MPI workshop is focuses entirely on mathematical modeling and runs on a very energetic 5-day schedule whereas the summer experience includes access to our lab and time to pursue a variety of avenues. The resources differ as well. The MPI workshop can draw upon the skills of a dozen or more faculty and graduate students. Our summer experience will typically involve a single understand working with a faculty member. Whether we use the industry workshop or a summer student, extra preparation is required if one is to bring industrial mathematics into the classroom.

Another key lesson learned during the first offering was that we needed formal ways to instill vision in our students. Project teams worked diligently gathering data on low hanging fruit, but this often leads to an aimless project. To encourage a collective vision in the teams, we use wikis where teams identify their achievements, near term and long term goals. Every week, they revise their goals based on their goals based on their most recent experiences. To complete the process, there are weekly presentations with a lively question-and-answer period where students can critique one another.

The milestone structure has been the core of our reform effort from the beginning, but we found that we had to add features to create a successful industrial modeling experience. In an industrial setting, teams tend to be interdisciplinary and roles are well understood from the composition of the team. In a classroom setting, effective interdisciplinary teamwork is a new experience for most students who, for better or worse, are focused on grades. Therefore, we implemented a peer assessment system where students were required to assume responsibility for different portions of their project. Their evolving project report remains the basis for the grades of all the team members, but scores are adjusted up and down based on peer assessments. Peer assessment involves numerical ratings of different students on different tasks. The numbers themselves are meaningful, but it is just as important that students write why they are giving a particular score. This form of reflection helps instill all of the habits of mind identified in this paper. On the rare occasion when a student does not provide adequate reasoning for a peer assessment, the student is asked to rework the assessment and respond to provide answers to leading questions about the team's activities.

In addition to driving the development of our reform effort through its first cycle, the air bearing problem resurrected itself in a surprising way. A few months after the semester ended, New Way Precision, a Pennsylvania company specializing in porous media air bearings, contacted us. All student project reports are posted on our course page, and they had come across our students' work. New Way Precision was looking for a mathematical understanding of their air bearings, and the fluid dynamics underlying porous media bearings are sufficiently different from solid air bearings. Future discussions led to New Way Precision presenting their problem at the 2004 MPI workshop. In this particular case, industrial mathematics has come full circle, from industry to the classroom and back into the industrial setting.

Conclusions

In this report, we presented initial ideas in our reform effort to develop seven habits of mind for students to make sense of the world mathematically, through modeling the mathematical modeling processes in an undergraduate course. A mathematical modeling course has the greatest potential for students to have the direct experience about the intimate connections between mathematics and the real-world. Pedagogically, it is important to model the modeling processes in classroom so that habits of mathematical minds can be developed. In this proposal, we briefly presented the pedagogical principles for the mathematical course, and how the seven habits of mind can be addressed pedagogically. Through the example of air bearing, we provided some details about our critical analysis of using specific modeling projects and provided concrete examples of best practices to connecting mathematics and the real-world. Through these specific modeling projects and concrete examples of best modeling practices, we also discussed the lessons learned to instill the habits of mind.

Notes

1 For more information about COMAP or MCM, please visit www.comap.com.

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MITACS Accelerate: A Case Study of a Successful Industrial Research Internship Program

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Abstract The MITACS Accelerate internship program links Canadian industry with graduate students and postdoctoral fellows through applied research internships. Starting with proof of concept – 18 internships in the mathematical sciences in 2003 – the program has more than doubled in size each year, with 608 internships across all disciplines in 2008-2009 and an additional 1289 projected for 2009-2010. A key factor in the success of the program is a rigorous peer review process; internships are research projects, not co-op terms or consulting jobs, and the work often forms part of the intern's graduate thesis. Equally important is the business development team that builds connections between academia and the industrial sector, identifying opportunities and matching academic expertise to public and private sector needs. The small cost of entry of $7,500 per 4-month internship makes the program accessible to companies of all sizes, and many companies follow up with additional internships and longer term involvement. Exit surveys show a high degree of satisfaction with the program from academic and industry participants alike.

1. Introduction

Since its inception in 1999, MITACS (www.mitacs.ca) has been funding mathematical sciences research projects across Canada. MITACS has played a leadership role in linking businesses, government and not-for-profits with over 50 of Canada's universities to develop cutting-edge tools to support the growth of our knowledge-based economy.

A core objective of all MITACS programs is to support the development of the up-and-coming generation of Canadian researchers. In 2003, MITACS introduced its Graduate Research Internship Program, with 18 internships in the mathematical sciences in its first year. The program, rebranded MITACS Accelerate (www.mitacsaccelerate.ca), grew rapidly and was expanded to all disciplines nationwide in 2008, while retaining a substantial presence in the mathematical sciences. The program supported 608 internships across all disciplines in 2008-2009 and is set to deliver nearly 1300 internships in 2009–2010.

MITACS Accelerate is directly transforming the way Canada recruits, trains and retains a highly skilled workforce. In addition, by proactively matching the needs of the public and private sectors with university expertise, MITACS Accelerate is driving knowledge and technology transfer between traditionally isolated sectors.

2. Program Strategy

2.1. Objectives

Headed by MITACS as a collaborative effort of Canadian federal and provincial governments, industry, universities, and other organizations, MITACS Accelerate aims to dramatically increase Canadian research intensity through the creation of hundreds, soon to be thousands, of internships. The objectives of the program are to:

- Enhance the training experience of graduate students and postdoctoral fellows by providing opportunities to work on industrially-relevant research challenges with impact on Canada's social and economic sectors;
- Provide industry with access to state-of-the-art research;
- Promote technology transfer and knowledge exchange to improve Canada's position at the leading edge of technology innovation and commercialization;
- Introduce and build lasting connections between businesses and academic researchers;

- Increase industrial R&D spending; and
- Through government matching of these contributions, increase funding for university research.

2.2. Challenges

Envisioning and implementing the program to reach these objectives required a major effort to:

- Convince Canadian companies of the benefits of research expertise in addressing business issues;
- Encourage the academic community to consider the needs of the industrial sector in their research programs;
- Persuade federal and provincial governments to include the program in their growth strategies and budgets; and
- Recruit and place significant numbers of trainees in industrially-relevant and academically-sound research projects.

2.3. Unique Approach

MITACS was able to address the needs of all stakeholders (governments, industry, and academia) by developing a program unlike any other at Canadian universities with:

- A strong research focus (different from co-op or work terms) that results in open, peer-reviewed research (different from consulting contracts);
- Clear deliverables for the industrial partner (different from most research grants); and
- An emphasis on the expertise of the academic supervisor as well as that of the intern (different from scholarship programs which focus exclusively on the merits of the student).

3. Details of the Internship Program

A MITACS Accelerate internship is a collaborative research project that links a graduate student or postdoctoral fellow, their academic supervisor, and a non-academic partner. Internships with for-profit companies form the bulk of the program, with a small amount of funding available for internships with not-for-profit or local government partners. Internships are undertaken in a wide range of areas including manufacturing, technical innovation, business processes, IT, design and many more.

Each Accelerate internship is 4 months long. The intern gains experience outside the university environment, spending half of the internship working on site with the partner organization to understand the research problem, collect data, interact with employees, etc. The balance of the intern's time is spent at their university, collaborating with their supervisor and other researchers and accessing specialized research resources as needed.

Interns are encouraged to publish their research results, working with the industrial partner to identify aspects of the project which contribute to the research expertise in the field without compromising the company's intellectual property (IP). MITACS takes no stake in IP generated through internships; it is shared between the university and company according to their own negotiated agreements.

Internships may be combined into larger 8 or 12-month projects, which can be applied for through a single proposal. Interns gain proposal-writing experience during the application process and receive valuable feedback on their projects through the peer review process.

3.1. Funding

The industrial partner contributes $7,500 per 4-month internship, and this amount is matched by the federal Industrial Research and Development Internships (IRDI) program (www.nce.gc.ca/irdi_e.htm), provincial grants, universities, and other sources. A research grant of $15,000 is provided to the academic supervisor, of which the intern receives at least $10,000 as a stipend. The remainder of the grant can be used towards the stipend or for research-related costs such as equipment, travel, and conference attendance.

3.2. Flexibility

While the core features of an internship – a research project and time spent on site with the partner organization – are non-negotiable, there is a great deal of flexibility built into the program to enable participation by a broad cross-section of companies and interns. Options include:

- The Intern Travel Subsidy Program provides matching funds over and above the amount of the internship grant, enabling a partner organization to be matched with academic expertise from another geographic region.

- Interns may apply for multiple 4-month internships on one proposal.
- Multiple interns may collaborate on a group of internships related to a larger research project.
- While we recommend that the intern write their own proposal as part of the learning experience, supervisors may apply for internships in order to gain funding to attract a student.
- There are no application deadlines. Proposals are reviewed as they are received, and proposals which pass peer review are approved (subject to availability of funds).
- Internships for postdoctoral fellows are eligible for double-funding ($30,000 per 4-month internship) to provide more competitive salaries.

3.3. Benefits to Stakeholders

MITACS Accelerate provides unique benefits to all three participant groups:

INTERNS: Internships provide exposure to industrially-relevant applications of graduate-level research. They are an opportunity to connect with potential employers and to gain business skills in a non-academic setting, as well as to gain additional funding for graduate school.

FACULTY: MITACS Accelerate provides opportunities to apply research expertise to real-world business problems and to build long-term collaborations with companies.

PARTNER ORGANIZATIONS: The program is a low-investment way for companies to tap into research expertise at Canadian universities. Successful internships can lead to long-term collaborations with academia and further access to early-stage research results. Internships may also be a valuable recruitment tool for future employees.

The low investment required for a single internship ensures that the program is accessible to start-up companies and small and medium enterprises, providing them with access to research expertise they may not have in-house, and increasing their competitiveness in the global marketplace.

4. Success Factors

MITACS has identified two primary factors that have led to the ongoing success of the Accelerate program: a business development team and rigorous peer review.

4.1. Building Connections

The success of MITACS Accelerate hinges on building connections between academia and industry. Professors are often expected to take on this task, but in a large and sparsely populated country such as Canada, a company with a research challenge may be thousands of kilometres away from applicable university research expertise. Making these connections is an overwhelming task for most faculty members, resulting in a perceived lack of interest in collaborating with industry.

It is our experience that the small number of joint research projects is less indicative of interest and more of frustration on both sides; organizations often have difficulty finding appropriate academic talent, and professors often cannot identify partners who would benefit from their research expertise.

The solution to this problem lies at the core of the success of MITACS Accelerate: its business development (BD) team. The BD team consists of highly talented individuals with a mix of academic and business experience, who are familiar with the talent available in Canadian universities. This team is skilled at matching industrial opportunities and academic expertise, facilitating initial interaction between parties, and providing feedback on proposal development. BD staff manage expectations on both sides and help applicants frame proposals for mutually beneficial research projects.

Each member of the BD team has responsibility for a geographic region, an industry section, and a number of academic disciplines, ensuring coverage of the entire country. The team approach allows opportunities to be shared and matches made across sectors. To maximize the program's reach, the BD team seeks out companies with little or no history of university collaboration and academic disciplines traditionally distant from industrial interaction.

In addition, the BD team actively seeks internship opportunities in areas of high priority to our federal and provincial government funders. For example, Ontario provincial priorities include clean and sustainable technologies, health science and advanced health technologies, digital media, information and communication technologies, finance, advanced manufacturing, and the automotive sector. First Nations and northern communities are also priority areas across the country.

MITACS also relies on the Accelerate Consortium, which consists of 17 leading Canadian research networks that partner with MITACS to identify and deliver internship opportunities.

4.2. Peer Review

The objective of the process is to maintain a high standard of research quality for internships funded by the program and to provide valuable feedback to students and their supervisors from expert reviewers. Peer review is an essential part of maintaining the academic relevance of the internship program and ensuring that research faculty view internships as part of their student's research training.

Internal reviews of large proposals are conducted by the Accelerate Research Review Committee (RRC). The RRC is a committee of the MITACS Board of Directors mandated to ensure the research excellence of the internship program through a fair and transparent evaluation process.

Arms-length external reviews of all proposals are provided by the MITACS College of Reviewers (COR), a body of over 350 faculty from Canada and around the world. Reviews may be solicited from other sources if appropriate expertise is not available on the COR.

5. Outcomes

The continued growth of the program and positive feedback from participants are indicators of its success. Exit surveys[I] completed since 2007 show the following outcomes for MITACS Accelerate:

- 87% of interns were of the opinion that there might be future employment opportunities with their internship partner company
- 98% of interns felt the internship experience would assist them in securing future career opportunities
- 99% of interns would recommend the MITACS Accelerate Program to other potential interns
- 92% of professors anticipated future collaborative projects with the industry partner
- 98% of professors felt the internship was beneficial to the intern's research program
- 99% of professors would recommend the MITACS Accelerate Program to other university professors
- 96% of industry respondents indicated that they will utilize the techniques, tools and research advances developed during the internship

- 97% of industry partners would recommend the MITACS Accelerate Program to other organizations
- 95% of industry respondents indicated that the opportunity to collaborate with university researchers was beneficial to their organizations

6. Future Directions

Building on the success of the standard internship model described above, MITACS is exploring several new options to expand the program. One possibility under discussion at several universities is the integration of internships with graduate programs, especially in master's degree programs in fields with a strong applied focus, such as health informatics. ``Development internships'', focusing on the commercialization of research results, are also under consideration for a pilot in 2010.

"Internship clusters" consist of six or more related internships that form a coherent research program. The cluster model was developed in early 2009 in response to growing interest from both academia and industry for larger collaborative research projects. A single research proposal covering the whole project streamlines the application and review process. Clusters provide a higher matching ratio, allowing more student stipends per industry dollar, and provide flexibility in budgeting research costs across all internships in the cluster.

Finally, the MITACS international program connects Canadian mathematicians with scientists in similar organizations around the world, and discussions are underway about the possibility of international internships for graduate students.

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Notes

1 These results comprise responses from 133 interns, 86 professors, and 86 industry partners.

Applied Mathematics in Secondary Modern School

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Abstract During the last years the situation in German Secondary Modern School degraded. Secondary Modern School students often are not able to meet their job aspirations especially in the so called MINT-field (vocations characterized by mathematics, computer, nature and technical sciences). Otherwise, the German economy shows a lack of qualified personnel in the MINT-field — the so called skill shortage.

The research project "Applied Mathematics in Secondary Modern School", developed by research group "Mathematics Education" at Freie Universität Berlin, trains Secondary Modern School students in Mathematics needed in different vocational trainings in the MINT-field. The goal must be to better students chances on vocational training market and to counter the skill shortage.

Introduction

During the last years, Germany's policy and economy complain the skill shortage especially in the so-called MINT-professions, which are effected by mathematics, computer, nature and technical sciences, which show a high level demand of qualified personnel. This causes a demand of skilled labor from other countries, a great expenditure - as not even native young skilled people do have the accession to MINT-professions.

In Germany, there are different ways of vocational training. One possibility is to pass it in the dual system. Besides training in company, the vocational training takes place at special schools. In theory, each young adult who obtains a leaving certificate from school in one of the four possible German school systems, is allowed to start such a vocational training. In reality, the vocational training market looks quite different. Not every graduate has the possibility to choose his most requested one. Especially graduates from Secondary Modern School (in German called "Hauptschule", the lower school level in German eduaction system, abbr. SMS) are compelled to start a different training. Particularly, in the MINT-field there is a great imbalance between SMS-graduates' aspiration and the actual started vocational training.

The basic concept of the research project "Applied Mathematics in Secondary Modern School" is to develop mathematical training programs for SMS in order to arrange mathematical competences, which are needed in MINT-vocational trainings. The development of the training program is carried out in close cooperation whith SMS's, vocational training schools, companies, the chamber of commerce and industry and the research group "Mathematics Education" at Freie Universität Berlin, head of the research group: Prof. Dr. Brigitte Lutz-Westphal.

The following article illustrates the actual situation in the German vocational training market in times of skill shortage and the SMS-students' whishes on this market. Therefore, it is important to give an introduction of the German education an vocational training system. Additionally, there will be a short presentation of the research project "Applied Mathematics in Secondary Modern School".

The skill shortage in Germany

For several years, German economy system, based on supply and demand, has lost its stability. Among the actual financial crises, demographical aspects are playing a decisive role. The demographical change is caused by a higher mortality rate than the birth rate, so that the natural population development is negative. The consequence is a decreasing population (cf. Kuhn & Ochsen (2009), p. 122ff). Therefore, not only less people are living in Germany, but also there is an increasing loss of qualified labors avaiable for commercial development. To have a successful German economy, constant technology progress and economic growth has to be taken place. The success can only be carried out, if new technological production facilities are getting geared with highly qualified personnel (cf. Koppel & Plünnecke} (2009), p. 5). If there is a lack of qualified personnel, we call it "skill shortage". Skill shortage might have enormous consequences for an economic system like the German one. It can lead to vast deficits in the added value within the economy. Analysis of the Institute of German Economy to the Federal Ministry of Economics and Technology are showing, that in 2006 there have been damages in double-digit billions (cf. IW} (2007)). Especially in the MINT-field skill shortage is noticed.

Highly qualified personnel often have an academic degree or further qualifications, which can be acquired by the German further education system. In the MINT-field, this personnel is mainly found in one of the four MINT-professions: engineers, technicians (including foremen), other scientists and mathematicians and data handling experts (cf. Koppel & Plünnecke} (2009), p. 16ff.).

The skill shortage in the MINT-field can be calculated eminently by kind and number of disengaged jobs. In contrary to the trends in all other activity fields, in high qualified MINT-jobs until July 2008 there were more disengaged jobs than registered jobless people; there was a lack of 144.000 qualified skilled employees.

In order to sustainable controvert of the skill shortage in the MINT-field, the attention of professional relevant competences should be concentrated to the scholar and vocational training. Furthermore, young people have to be enthused and motivated for jobs in the MINT-field, so that the German economic system will be primed for future times. A long-term alternative to decreasing skill shortage might be the infantile education in full-time schools and/or to advance and upgrade the general conditions (ibid.).

The German Educational and Vocational Training System

As yet, we only mentioned highly qualified personnel, generally academics resp. employees having had an advanced vocational training. But as well less qualified professions have to be focused, because German educational and vocational training system offers possibilities for a SMS-student to get a training qualification in the dual system, which is the basis for further qualification steps.

Germany is divided in different states, which have their own educational system. Each of these systems has to meet nationwide requirements according to the principle of federalism. In general, the educational system is divided into six parts beginning with pre-school education up to tertiary level (cf. KMK (2003)).

The part focused in this article is within the secondary level I. Secondary level I is divided in four different school types: “Gymnasium”, “Realschule”, “Hauptschule” (SMS) and “Gesamtschule”, whereas the last mentioned one means a kind of combination of all former school types. SMS-qualification can be obtained either at SMS or at “Gesamtschule” after nine classes. “Realschul”-graduation can be obtained in all different school types and takes ten classes. Empirically, students’ lowest activity and education level is at the SMS and the highest one can be found at the “Gymnasium” (ibid.).

The focused SMS-students regularly have the possibility to start a vocational education in dual system or to get a “Realschul”-qualification by extending attendance for one year - to ten classes. With this qualification the ancient SMS-students will be able to get on to the general qualification at “Gymnasium”. The SMS-qualification after the ninth grade means the first general school qualification, a middle school-qualification is seen as “Realschul”-qualification after tenth grade.

The German eduactional system seems to be transparent to give students the chance to get general qualifications for university entrance after “Realschul”-qualification and SMS-qualification. In reality, it is not as simple as it seems.

There are multiple methods to start a vocational training in Germany, usually on the basis on the dual system. Besides a practical training in companies of economy, trade, administration etc., the training takes place in vocational training schools.

Each student is able to start a training organized in the dual system with each of the German possible school-qualifications. Besides the dual system there are generally two different methods of vocational training. On the one hand there are courses of education, which are guiding students on an academic path to a vocational qualification. At the other hand courses of studies offer students vocational qualifications (cf. BMBF (2009), p. 19).

At present, there are 349 different formally accepted vocational occupations to be passed by the dual system (actual level: 01.08.2009, cf. www2.bibb.de). In general, the high number of occupations can be divided into different categories, especially in the MINT-field. The Federal Institute for Vocational Education and Training (BIBB) choose a category system for the different vocational trainings, which is for statistical analysis divides between technical and non technical occupations. Usually, professions of the categories production and service are listed, which require for a high rate of technical qualifications (cf. Uhly (2007), p. 7f., 48ff.). Actually, there are listed 91 vocational trainings in MINT-field.

The Problem of Secondary Modern School

During the last years, the discussion about the consistency of SMS in Germany has been aroused. The state Rhineland-Palatinate is going to abolish SMS until school year 2013/2014. In a so called "Realschule plus", future generations of students will be able to acquire different qualifications from SMS with the aim to get an advanced technical college entrance. The main ambition of closing SMS is to confront the skill shortage concertedly. Furthermore, by this campaign politicians in Rhineland-Palatinate want to confront the bad image of SMS-qualification, which can be measured by the drastic sinking numbers of SMS-students. An abolishment of SMS is getting labored in the states Hamburg and Schleswig-Holstein as well. In Bremen and in several eastern states there have not been any SMS for years. Other states like Hesse want to abide by SMS, because - as they say - an abolishment of SMS cannot solve young people problems with SMS-degree at the vocational training market (cf. www.faz.net).

Taking into consideration the historical development of the different qualifications in Germany, in contrast of age groups, the first thing that is noticed is that in the actual generation less people try to get a SMS-qualification. The percentage of 60- to 65-year old people with SMS-qualification as highest qualification is within the range of 60% (cf. AG BBE (2008), p. 39 f.). The tendency of younger age groups is falling. In school year 2005 there were only 24% of the young people, who attended the eighth class at SMS (cf. BMBF (2007/2008), p. 25). This means the amount of young people who are trying to absolve SMS-qualification as the highest school-qualification is permanently decreasing. The aspired qualifications are getting relocated more and more to "Realschul"- and general qualification for university entrance (cf. AG BBE (2008), p. 39 f.).

The sinking quality of SMS-qualifications is getting clearly by view to the ways of education and vocational training of SMS-students after qualification. Just 29 % of young people with SMS-qualification in school year 2007 started a vocational training up to November 2007. The largest group (44%) in that school year attended a further educational school. All other students either have an employment without vocational training (12%), or attending to pre-vocational-trainings (20%), have no employment or are in other situations like military or national duty (cf. BMBF (2008a), p. 24 f.).

Compared to the realized ways of education and vocational training by SMS-students in school year 2007, there is a big discrepancy to their wishes before the qualification. In March 2007, just before qualification, 68% of the SMS-students have been looking forward to start a vocational training in dual system. Just 13% of those young people wanted to attend to further educational schools. The percentage of those who wanted to take part at pre-vocational training was within the range of 9% (ibid.).

Nor the realized ways after SMS-qualification, neither the vocations meet the wishes of SMS-students. A comparison of the male and female SMS students` wishes of vocational training vs. the started vocational trainings of SMS-students shows that there are big differences. In a longitudinal section study about the intersection panel of the first final year 2004, information of the educational and vocational-training attitude of SMS-students have been collocated by the German Youth Institute (DJI) on behalf of Federal Ministry of Education and Research (BMBF). In summary, about 3.900 SMS-students are getting scientifically focused concerning to their further educational and vocational training attitude up to 2009 (cf. BMBF (2008a)). BMBF is editing an annual report of vocational education in which the educational and vocational training attitude is described (cf. BMBF (2008b)).

The aspirations of vocational training by female SMS-students mainly go directly to the service sector, mainly to social vocations like medical secretary or other service vocations like specialist in the hotel business (cf. BMBF (2008a), p. 18). The aspir-ations of vocational training by male SMS-students are quite different. Here especially MINT-vocations like mechatronical engineer, electronican in automation technology or motor vehicle mechatronics technician are aimed (ibid.).

The reality looks quite different. In 2006, most of SMS-students took vocational trainings like salesman, salesclerk in retail or salesman in food trade (cf. BMBF (2008b), p. 133 ff.). In contrast to the wanted vocations the realized ones were those in the sector handcraft, which are slightly embossed by technology and in service sector without social influences. SMS-students with aspirations for jobs in MINT-field do not find an apprenticeship training position.

Hence, there are considerable differences between the expected outlooks of SMS-students concerning a vocational training, further education and other employments and their real situation both in vocational training and in educational field as in the actual vocational choice. Especially noticeable is that much less SMS-students than imagined start a vocational training in the dual system. Regarding the career aspir-ations, there is a rearrangement from highly qualified technological and social affected apprenticeships to the less qualified technological, manual and trading vocation fields. Highly qualified technological vocations like mechatronic engineer, trade vocations like industrial clerk or social vocations like those in nursing service and medical domain are taken by “Realschul”- or “Gymnasium”-students (ibid.).

With regard to PISA and similar comparative studies, the Confederation of German Trade Unions (DGB) demands the basic right of vocational training for each adolescent like a guarantee (ibid., p. 34). For implementation, all public schools have to impart as much competences and knowledge to their students as possible to guarantee an ample qualification for vocational training. Basically, there is a barrier of language, which has to be overcome, because after all there are 15% of all youth, who have large deficits in language and therefore limited competences in communications as well as personal and social deficits. General education schools for this reason have to assume responsibility for adjusting these deficiencies of competences (ibid.).

In comparative studies, “German” and “Mathematics” play a decisive role. In average, 40% of the queried SMS-students state their mark in these subjects of a grade below the average or inferior (ibid.). How does SMS confront the deficits and how does SMS prepare its students for a successful start in vocational training?

In SMS, the preparation for the professional world is taking place in different situations. On the one hand there are taken training measures to correct deficits of competences, and on the other hand measures for targeted development of students individual potency have to be taken. Training education in small groups (age of the students between school and vocational training) of at most twelve students is to guarantee targeted training of competences (cf. BMBF (2008a), p. 11). This training offer is accepted by 25% of all students in this age group in at least one school subject. Especially the subjects which are very important for a successful SMS-qualification are getting asked for; the subjects “Mathematics” with a percentage of 13%, “German” with 9% and “English” with 8% of all students are basically required. The attendance to this training education increases school achievement in 65% of all cases. SMS-teachers here are important vocational advisors and instructors. 62% of SMS-students state, that their teachers play a decisive role in bringing out a decision and that they are important for pupils support. The far the most important construction in SMS for bringing out decision are professional practical trainings (ibid.).

Which are the most important occasions SMS-students should be prepared for? To start with the subject “Mathematics” there are the questions: Why to mathematics play such a big role in SMS-qualification? Why is it important to specially teach math-ematics? The results of studies are clearly: many students have deficits in the subject `Mathematics`. The great percentage of mathematics in training education shows that a part of the students attach importance to get better in “Mathematics”. It is known, that grades, especially the grade in “Mathematics”, play a decisive role in interviews for a job (cf. BMBF (2008b), p. 34). As seems as if personnel offices prefer an adequate mathematical education.

Project Description

The research project “Applied Mathematics in Secondary Modern Schoo”` is getting developed by the research group “Mathematics Education” at Freie Universität Berlin, head of the research group: Prof. Dr. Brigitte Lutz-Westphal. The project has been carried out predominantly in SMS, vocational training schools, companies and the chamber of commerce and industry in the Oldenburger Münsterland and in the Dümmer region in the State Lower Saxony, known for its increased social and economic position.

The research hypothesis on the project directly concerns to the given situation on Germanys economy and school situation. Thesis is that mathematical training programs, being integrated in the context of all-day SMS out of the general mathematical education, which trains special mathematical skills for students to better their possibilities realizing their most wanted vocational trainings.

The questions, which have to be answered to proof the thesis and to arrange the training program, are:

- Which mathematical skills are required for different vocational trainings?
- Which mathematical competences should be trained more intensively in SMS to achieve a higher number of trainees in MINT-field?
- Which topics and mathematical methods should be integrated in a training program?
- Is a targeted mathematical training program, given beyond the general mathemat-ical education, able to work off the deficits?
- Which effects and references have to be fixed for general mathematical education in SMS?

The main aim of the research project is to develop an additional mathematical training program for SMS-students to get extensive mathematical contents, skills and competences to them, required to start special vocational trainings in the MINT-field.

The project is departed in three main phases. The first phase is the so called preparation phase, to build up contacts with partners out of companies, SMS, vocational training schools and the chamber of commerce and industry. Furthermore, basic information about the configuration of a training program are collected and analyzed. First of all, the information are getting collected by interviews and questionnaires, which are evaluated in the partnership. The interviews and questionnaires should show how instructors appraise and estimate the discrepancy between the SMS-students vocational aspirations and the realized ways of vocational training. Furthermore interviews and questionnaires should give information about the mathematical skills, which are required for special vocational trainings and which have to be trained more intensively in SMS. In addition the information should state, which role SMS and teacher in vocational training schools are playing in training of vocational relevant mathematical competences. Besides the interviews and questionnaires the current curricula in mathematical education in SMS and the curricula for different vocational trainings are getting compared to concern similarities and differences in field of required mathematical skills.

Based on the knowledge by first phase, the training program can be developed and carried out. For this purpose there must be a decision, which topics of mathematics have to be handled in the training program. These topics have to be boxed into a modular structure, which can be carried out in SMS. In return, the question has to be answered, which mathematical educational method has to be preferred. If the educational, meth-odical and content questions are answered, the training program can be planned and carried out. For this purpose, it is important to cooperate with the called partners. It is planned to give successful participating students a certificate, so that they are able to demonstrate their mathematical interests, knowledge, skills and competences for vocational training application in MINT-field.

The third phase contents the evaluation of the training program. Besides the carrying out of the training program there must be an investigation, how SMS-students train their mathematical skills and competences and how this training is appropriate to a successful application for a vocational training in MINT-field. In addition, effects and references can be given for a further development of the training program to judge the success of the program. As the success of the project may be, there might be references about the positioning of that kind of mathematical training program in general all-day life in SMS.

Actually, the project is arranged in the phases I and II. Within the phase I, partners are getting acquired. For this purpose, contacts to SMS, vocational training schools, companies and the chamber of commerce and industry are built up. The demand in the partner is to carry out interviews and questionnaires to get information how to construct the training program. Contemporaneous, curricula for mathematical education in SMS, especially those who are placed in the state Lower Saxony, and the curricula for different vocational trainings in the MINT-field are getting sighted and analyzed for common and different distinct of mathematical skills and competences.

By appraisements out of the analysis of the curricula of SMS and vocational training, first offers for mathematical topics, which can be contented in the training program, are put out. First analyses show, that there are a lot of overlaps in vocational training and SMS curricula. For example in vocational training of motor vehicle mechatronics technician, a student should be able to work confidently in the field of different display formats, especially in case of circuit diagrams, electrical and electronics circuits, basic parameters and signals. For this purpose, it is important to deal with various strategies of troubleshooting, which are named in competence fields of mathematical modeling and problem solving. In handling with troubles, data and structures, the fields of mathematical communication and argumentation have to be named.

Some offers of modules, which can be trained in the training program, can be:

Data:

Collecting and converting data is a focus in some vocational trainings, like motor ve-hicle mechanics technician. In this module, operations like collecting data by measuring or using information systems can be trained. Therefore, technical instruments by some vocations can be used. In addition, methods to calculate areas, volumes and dimensions can be trained. Furthermore, it is important to analyze data, so that different methods in data handling, like interpretation and calculus of data, can be processed.

Technical drawing:

Technical drawing is important in many MINT-vocations like draftsman, architectural draftsman, surveyor etc. Therefore, some mathematical drawing methods must be handled. Not only drawing by hand but also drawing programs should be discussed. Some topics of geometry and methods of geometry can be centered.

Optimization:

In many vocations, it is important to optimize processes, data, procedures, accounts etc. Optimization is also fundamental in project scheduling. In this case, some linear and combinatorial methods of optimization can be trained. Algorithms for different problems of optimization in application by computer can be handled. Methods like Simplex or Dijkstra algorithm are possible ways for an introduction into algorithms and optimization problems.

Financial and business mathematics:

Some vocations in the MINT-field are affected by financial and business problems. Therefore, some topics in financial and business mathematics can be trained. Topics like calculation of percentage, of interests or charges etc. can be handled.

Mathematical physics:

In MINT-field, there are many vocations, in which to deal with physical dimensions and formula. In the training program some topics in the physics can be integrated like calculus with units, calculus with physical values etc.

Mathematical modeling and problem solving:

Mathematical modeling is an important mathematical competence. In some vocations, it is useful to know some modeling structures to solve problems and requirements. In case of troubleshooting, mathematical modeling can help students to make an efficient way of problem solving. Therefore, some mathematical models out of concrete vocational situations can be handled by students. Mathematical modeling might be integrated in other modules, but it is important to know, how to build up models and how to deal with them in mathematical and vocational context.

The named modules are offers to handle topics, competences, skills and applications of mathematics in the training program. The topics have to be prepared by experiences of instructors and teacher in SMS and vocational training schools.

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Minnesota Programs and Activities in Industrial Mathematics

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Abstract The Institute for Mathematics and its Applications (IMA) and the Minnesota Center for Industrial Mathematics (MCIM) offer a number of research and training programs in industrial mathematics. We will describe the activities within these programs and highlight some of the accomplishments. We will also review some lessons learned over the course of time, and identify key factors that make such programs work.

Introduction

The Institute for Mathematics and its Applications (IMA) and the Minnesota Center for Industrial Mathematics (MCIM) offer a number of research and training programs in industrial mathematics. The IMA is a US National Science Foundation mathematical sciences research institute founded in 1982 and located on the campus of the University of Minnesota. MCIM is a center within the School of Mathematics at the University of Minnesota, and was founded in 1994. They each have very distinct missions and scopes. What they share is a common goal of providing training in industrial research and engaging companies in research involving mathematics.

MCIM serves the School of Mathematics by administering a degree program as well as facilitating research collaborations with industries. The IMA offers a variety of programs serving graduate students and postdoctoral fellows, whereas the programs at MCIM benefit Minnesota mathematics students and faculty. The cooperation between these centers creates synergy.

In the following, we will go over each of the activities of MCIM and IMA in industrial mathematics.

IMA Industrial programs

One of IMA's mission is to bring mathematics to bear on pressing problems arising in industry. Since its establishment, companies may join the IMA as Participating Corporations. At present, the IMA has 12 active corporate members, each paying an annual fee. There are several clear benefits to being a member, the most valuable of which is the possibility to jointly hire a postdoctoral fellow with the IMA. Another benefit is that members send representatives to IMA Industrial Advisory Board meeting where they learn about IMA's program development plans and provide input to the program development. Members therefore know much sooner than the public about upcoming programs and can make plans to take advantage of them. The access to the broad mathematical community is another benefit.

IMA Industrial Postdoctoral Fellowships. The IMA industrial postdoctoral fellowship program was started in 1990 by its then director Avner Friedman. In this program, the IMA collaborates with an industry partner to create a project. A postdoc is jointly selected by the IMA and the sponsoring company based on his/her potential contribution to the project. The appointment is for two years with the agreement that the postdoc will put 50% effort on the company's project and 50% effort on his/her own research. Funding for the positions is also split between the IMA and the sponsoring company.

The nature of the projects have been quite diverse. Mathematical areas involved have included Numerical modeling, optimization, medical imaging, materials modeling, finance, communications network, computer graphics, and filtering theory.

Since the inception of the program, there have been 49 IMA Industrial Postdocs, involving 22 companies. Of the 45 that have finished their fellowships, 28 are in academic positions. The rest are in industry or government laboratories. Four of the postdocs were hired by the companies with whom they worked immediately after finishing their fellowships. One postdoc, David Dobson, received the 2000 Felix Klein Prize from the European Mathematical Society for his work with Honeywell.

There are many success stories from this program. An example is a project former postdoc Jay Gopalakrishnan did with Medtronic. He modeled the physical process of ablation therapy for atrial fibrillation. His work with David Franciscelli led to new products and procedures. We will highlight several projects in the presentation.

MCIM Programs

MS and PhD in Industrial and Applied Mathematics. MCIM administers degree programs leading to MS and PhD degrees in Mathematics with Emphasis in Industrial and Applied Mathematics. The Masters degree requirement includes an internship with a company. For students pursuing an MS degree, the internship is carefully arranged not only to ensure a successful outcome but also with the goal of turning the project on which the student work into a thesis topic. Industry mentors are often kept engaged by serving in the student's thesis committee.

The MS degree program takes 2 years to complete and involves course work in mathematics as well as courses in other fields. A Master's thesis and an oral defense of the thesis are required. Since 1994, there have been over 30 MS students who finished the program. Many have gone on to PhD programs while others are in the industrial workforce.

The written exam requirements for students pursuing Mathematics PhD with emphasis in Industrial and Applied Mathematics differ from the standard PhD requirements. Students are required to pass either Real or Complex Analysis, and Algebra or Topology. They must also take an advanced course sequence outside of Mathematics.

Many of our current students choose to pursue a PhD without the MS thesis. These students also request internships at the end of their first or second year of studies. With solid course work and the written qualifying exam behind them, these students go on internships during the summers. Many of these internship projects have led to PhD theses. Some of these summer internships have grown into multi-semester projects which provide funding to the students (and faculty).

INDUSTRIAL INTERNSHIPS. The MCIM makes every effort in ensuring that the internship is productive, both for the student and for the company. It is important that the project engages the student in mathematical research, and that there is a responsible mentor assigned to the project. Since 1994, MCIM has arranged over 100 internships. Students have been place in over 40 companies. It is estimated that about ¥¥¥ of all graduate students have had an internship experience. Over 15 PhD theses can be attributed to internships.

Since 2007, the School of Mathematics at the University of Minnesota has been offering a Masters degree program in Financial Mathematics. Students in this program are also encouraged to take internships. MCIM has taken the responsibility of arranging internships for students in the Financial Mathematics program.

Placing students in internships continues to be a major activity of MCIM. Even with the contacts MCIM has developed over the years and the reputation of its students, it takes a major effort to arrange each one of these internships. On the other hand, this a worthwhile activity. The value of these internships to the department, to the students, and to the companies involved has been tremendous.

Joint Programs

The IMA and MCIM also collaborates on two other activities.

IMA-MCIM INDUSTRIAL PROBLEMS SEMINAR. Industrial Problems Seminar, which has been running continuously at the University of Minnesota since 1988, is now a joint effort between these two entities. The seminar is a forum for an industry scientist to present his/her research work and to highlight how mathematics play an important role in it. The seminar speaker usually spends a day on campus, meeting with students, postdocs, faculty, and IMA visitors. These visits have the desired effect of raising industry's awareness of IMA and MCIM's resources and often lead to new collaborations. In a typical year there are 12 to 15 seminars.

IMA MATHEMATICAL MODELING IN INDUSTRY WORKSHOPS. The IMA Mathematical Modeling in Industry workshop for graduate students is offered yearly. MCIM affiliated faculty serve as faculty advisors during the workshop. This program is open to all interested graduate students from all over the world. The IMA recruits 6 industry scientists to come to the IMA to work with 6 teams of 6 students. The scientists are responsible for choosing a problem to pose to the students and for serving as a project supervisor in the duration of the workshop. The students, working closely in teams, have 10 days in which to produce the best solutions to the problems. A written and oral final reports are required of each team at the end of the period.

Since 1998 there have been 9 workshops run in this format. A total of 29 companies and laboratories have sent mentors to pose problems. Project topics have been quite varied, from optimal design of airplanes to financial mathematics. Several of the final reports have appeared as journal articles or in conference proceedings.

Faculty serve mostly as "consultants" and to make sure that the students have all the resources they need. Students take ownership of the projects and are very proud of what they can accomplish in the limited time. Many find the experience exhilarating and important in their careers. Industry scientist who act as mentors in this program find the work produced by the students to be of high value. The experience also provides the mentors with important new ideas which they take back to their companies.

Discussions

Each of the programs and activities described above will be discussed in detail in the presentation. In particular, the presentation will highlight lessons learned and identify key factors that make such programs work.

Teaching Non-Traditional Applications to Engineering Students

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Abstract This paper presents results of two studies on using an innovative pedagogical strategy in teaching mathematical modelling and applications to engineering students. Both studies analyse engineering students' attitudes towards non-traditional for them contexts in teaching/learning of mathematical modelling and applications: environment and business. On the one hand, the contexts are not directly related to engineering. On the other hand, chances are that most of the graduates in engineering will be dealing with mathematical modelling of environmental and business systems in one way or another in their future work. This is because nearly every engineering activity has an impact on the environment and has commercial implications.

Introduction

There are many papers devoted to investigating undergraduate students' competency in the mathematical modelling process. We mention some recent research. A measure of attainment for stages of modelling has been developed in (Haines & Crouch, 2001). The authors expanded their study in (Crouch & Haines, 2004) where they compared undergraduates (novices) and engineering research students (experts). They suggested a three level classification of the developmental processes which the learner passes in moving from novice behaviour to that of an expert. One of the conclusions of that research was that

> students are weak in linking mathematical world and the real world, thus supporting a view that students need much stronger experiences in building real world mathematical world connections (Crouch & Haines, 2004).

It echoes with the findings from a study of 500 students from 14 universities in Australia, Finland, France, New Zealand, Russia, South Africa, Spain, Ukraine and the UK (Klymchuk & Zverkova, 2001). The study indicated that the students felt difficult to move from the real world to the mathematical world because of the lack of practice in application tasks. Many researchers and practitioners consider skills in mathematical modelling to be different from skills in mathematics.

> Model building is an activity which students often find difficult and sometimes rather puzzling. The process of model building requires skills other than simply knowing the appropriate mathematics (George, 1988).

Modelling "can be learnt but not taught in a usual way" (Neunzert & Siddiqi, 2000). Some relationships between students' mathematical competencies and their skills in modelling were considered in (Galbraith & Haines, 1998) and in (Gruenwald & Schott, 2000). Engineering students' attitudes towards ecological modelling were investigated in (Klymchuk et al, 2008) and towards epidemic modelling in (Narayanan et al, 2009). The role of entrepreneurship in engineering education was studied in (Gruenwald & Krause, 2006) and (Chisholm & Blair, 2006). Kadijevich pointed out at an important aspect of doing even simple mathematical modelling activity by new coming undergraduate students:

> Although through solving such ... [simple modelling] ... tasks students will not realise the examined nature of modelling, it is certain that mathematical knowledge will become alive for them and that they will begin to perceive mathematics as a human enterprise, which improves our lives (Kadijevich, 1999).

In this paper we consider engineering students' feedback on introducing non-traditional for them contexts in teaching/learning of mathematical modelling and applications: environment and business. Practice was selected as the basis for the research framework and, it was decided "to follow conventional wisdom as understood by the people who are stakeholders in the practice" (Zevenbergen & Begg, 1999). The students' mathematical and modelling activities in the class as well as their attitudes were the research objects.

The First Study

The first study was conducted with 2 groups of students. The first group consisted of the first-year engineering students studying mathematics courses in the Auckland University of Technology, New Zealand and the Wismar University of Technology, Business and Design, Germany. The models used with the students in a project have the following features:

- Each model is environmental. Environmental issues are getting more and more important for many human activities worldwide. This, on the first glance nontraditional area of application for engineering students, will help them to broaden their vision and prepare them to take ethical responsibility in future because nearly every engineering activity has an impact on the environment.
- Each model is adjusted to the region where the students study by selecting a nearby lake or an island, putting its name in the title of the model and entering the corresponding values of the parameters into the model. We assume that this psychological strategy will help students to relate to the models in a *personal* and an *emotional* way and increase their motivation and enthusiasm.
- Each model is developed by professional mathematicians working in industry and is based on a real practical problem.
- Each model is adapted and presented in a way understood by engineering students.
- Each model is a little bit beyond the scope of the mathematics course the students study. So they need to learn on their own some new concepts. For example, a model can be based on separable differential equations that students study but further investigation of equilibrium solutions and stability that are not covered in the course is required. We assume that this *discovery learning strategy* can help the students enhance their investigation and research skills.
- Each model is mathematized to a large extend. So the students do not go through the first few steps of mathematical modelling process (collecting the data, making as-

sumptions, formulating the mathematical model, etc.). But, apart from solving the given mathematical models, they do practice in other important steps of mathematical modelling process including interpreting the solutions, discussing the limitations of the models and communicating the findings through a written report.

One of the models is given below.

Model of Water Quality Control in Taupo Lake

Polluted water enters Taupo lake from a recently built factory at a constant rate N.A mathematical model of concentration of pollution has been developed under certain assumptions including the following:

- the upper levels of water are mixed in all directions
- change in mass of pollution is equal to the difference between the mass of the entering pollution and the mass of the pollution which is being decomposed
- the rate of decomposition of pollution is constant
- decomposition of pollution takes place due to biological, chemical and physical processes and/or exchange with the deeper levels of water.

Under the assumptions the equation of the balance of mass of pollution on any interval of time Δt can be written in the form:

$$V\Delta C_N = N\Delta t - QC_N\Delta t - KVC_N\Delta t$$

where V–volume of the upper levels (constant), Q–water consumption rate (constant) $C_N = C_N(t)$–concentration of pollution at time t, K–decomposition rate (constant).

Questions:

1. Set up a differential equation for the concentration of pollution from equation (1) by dividing both sides of equation (1) by Δt and taking a limit when $\Delta t \to 0$.
2. Solve that differential equation to find the concentration of pollution as a function of time provided that the initial concentration of pollution was zero.
3. Determine the equilibrium concentration of pollution C_{Ne} (that is the concentration when $t \to \infty$ or $dC_N/dt = 0$).
4. Determine time needed to reach p portion ($p = C_N(t)/C_{Ne}$) of the equilibrium concentration. Will it take more time to reach p portion of the equilibrium concentration in case when there is no decomposition of pollution?

5. Set up a differential equation for the concentration of pollution from the differential equation in question 1 in case when the initial amount of pollution entered the lake was C_0 and after that pollution is not coming to the lake anymore (that is $N = 0$).

6. Solve the differential equation from question 5.

7. Determine time needed to reach the $(1 - p)$ portion $(C(t)/C_0 = 1 - p)$ of the initial concentration C_0.

(1)

The total number of students who completed the project was 147 in both universities. The number of students who answered the anonymous questionnaire was 63 so the response rate was 43%. Participation in the study was voluntary. After completing the project the students were asked to answer the questions below.

Question 1.—Do you find the project to be practical?

a) Yes Please give the reasons:

b) No Please give the reasons:

Question 2.—Do you find the project to be relevant and useful for your future career?

a) Yes Please give the reasons:

b) No Please give the reasons:

A brief statistics and common students' comments are presented below.

Question 1.—Practical? Yes — 48%. Selected students' comments were: "The models describe the real world", "A good way of increasing students interest in the subject", "It was so helpful for my other subjects", "I didn't realize modelling is used for fishing quotas. It also helped me realize the effects of sneaky illegal fishing (which most of us have done)".

Question 1.—Practical? No — 52%. Selected students' comments were: "It is not possible to calculate the nature", "It did give a practical situation but you bearly think about that at all when doing the assignment".

Question 2.—Relevant for your career? Yes —35%. Selected students' comments were: "Mathematics is the base needed to go into the Engineering World, so it will help a lot", "In engineering, we will be dealing with these kind of situations", "We are more motivated to solve such real problems than working with dry examples", "Everything you learn is bound to be beneficial at some point".

Question 2.—Relevant for your career? No — 65%. Selected students' comments were: "I don't see how it relates to mechanical or electrical engineering" (most common comment), "I don't compute formulas, I have to calculate beams".

The second group consisted of a mixture of year 2–5 engineering students studying the experimental course Mathematical Modelling of Survival and Sustainability at Wismar University, Germany. The course was taught in English by a visiting lecturer from New Zealand. The course has a multidisciplinary character and is very practical. Matlab is used throughout the course. The students solved a number of ecological and environmental models in their individual and group projects and also on the exam. They could have their group project based on an appropriate scientific journal paper selected from the course website. They could also approach a local business/industry or contact a relevant department of the local government to choose a real practical project based on the local data.

One of the models is given below.

Populations of Birds and Insects on Poel Island in Mecklenburg-West Pomerania

Populations of birds and insects on Poel Island are modelled by the following system of the equations:

$$\begin{aligned} \frac{dx}{dt} &= 0.4x - 0.002xy \\ \frac{dy}{dt} &= -0.2y + 0.000008xy \end{aligned} \tag{2}$$

QUESTIONS:

1. Which of the variables, x or y, represents the bird population and which represents the insect population? Explain.
2. Find the equilibrium solutions and explain their significance.
3. Find an expression for dy/dx and solve this separable differential equation by hand.
4. Use Matlab (pplane) to draw the phase trajectory corresponding to initial population of 100 birds and 40,000 insects. Use the phase trajectory to describe how both populations change with respect to time.
5. Use part 4 to make rough sketches by hand of the bird and insect populations as functions of time. How are these graphs related to each other? Create those graphs with Matlab (pplane).

Suppose equations (2) are replaced by the following equations:

$$\begin{aligned} \frac{dx}{dt} &= 0.4x(1 - 0.000001x) - 0.002xy \\ \frac{dy}{dt} &= -0.2y + 0.000008xy \end{aligned} \tag{3}$$

6. According to equations (3), what happens to the insect population in the absence of birds?

7. Use Matlab (pplane) to draw the phase trajectory for equations (3) corresponding to initial population of 100 birds and 40,000 insects. Use the phase trajectory to describe how both populations change with respect to time.

After completing the course the students were asked to answer the question "Do you think this course is suitable for engineering students and if so, why?" There were 25 students in the course. The response rate was 100%. Participation in the study was voluntary. All 25 students answered "Yes" to the above question. The main two reasons were:

- Improving knowledge in mathematics, Matlab and mathematical modelling that is useful for engineering —23 (92%). Typical students' comments were: "You consolidate your mathematical knowledge", "Raise knowledge about differential equations and especially how to build them", "Increasing skills in Matlab", "In my opinion many problems or predictions in the 'engineering world' could be handled/solved with the techniques that you can learn here", "Because you learned how to put some problems into a mathematical system", "To see new ways (models)", "In the course you can better make a statement for normal problems about the life", "Because I could improve my understanding for differential equations", "The mathematical models all around us and the true way for an engineer is to understand how a model from the nature reacts if you change one parameter".

- Practical and interesting — 10 (40%). Typical students' comments were: "To get practical problems", "It is very important to use practical part in the course as it is done here to help students to understand what are they going to do in their future jobs", "Of course it deals not with typically engineering problems but after all it was an interesting subject", "Engineering students can apply their knowledge and broaden their horizon", "It is nice to see we can use differential equations in other areas", "I think that every subject which has a lot of practical things is very useful. This mathematical course was very useful for me and I think, that in our university everyone must study mathematics in this way".

Conclusions from the First Study

There is a big difference between the students' responses in the two groups about the relevance of the suggested context of applications. Only 35% of the students in group 1 (first-year students) indicated that the environment/ecology context is relevant for their future career whereas 100% of the students in group 2 (year 2–5 students) commented that the course was suitable for engineering students. One of the reasons for such a difference might be the difference in maturity. Another reason might be the difference in students' mathematics background. It was reflected in the exam performance. The pass rate of the first-year students in their maths courses was around 50%. The pass rate of the year 2–5 students in the modelling course was 100%. Moreover, all 25 students from the second group received excellent or very good final grades. From informal talks to the students of the second group we received a strong indication that their enthusiasm and positive attitudes towards the course significantly contributed to their high performance in the course and very positive attitudes towards the unusual contexts. It was a risk to offer such non-traditional course to engineering students. In spite of concerns of some engineering staff and to their surprise the students were very positive about the course. They were mature enough to value the new knowledge in mathematics and modelling they received from the course that can be applied in engineering (92%). They also enjoyed the practicality of the course that enhanced their problem solving skills (40%). The main lesson for us as teachers was: students' feedback should be taken into account when designing curricula for their study.

The Second Study

The second study was conducted with 2 groups of engineering students — Bachelor and Masters — studying Operations Research course at Wismar University, Germany. The course was taught in English by a visiting lecturer from New Zealand. The students studied a variety of linear programming models dealing with maximizing profit and minimizing expenses. On the one hand, the (business) context of the course is not directly related to engineering. On the other hand, chances are that most of the graduates in engineering will be dealing with mathematical modelling of optimization problems in one way or another in their future work because nearly every engineering activity has commercial implications. The course did not require any special knowledge from economics or business studies. It was based on a spreadsheet modelling and had many real practical applications from the business world. Excel was used as the main program being a standard for small businesses and the chosen program of the course textbook. Apart from using the Excel Solver tool the students learnt many useful Excel functions thus mastering their spreadsheet modelling skills. There was much attention put into enhancing students' generic mathematical modelling skills while doing sensitivity analysis and discussing assumptions and limitations of each model.

After completing the course the students were asked to answer the question "Do you think this course is suitable for engineering students and if so, why?"

The First Group (Bachelor Students)

There were 16 students in the first group. All 16 students answered "Yes" to the question on the suitability of the course to engineering students giving the following reasons:

Benefit for the future work — 15 (94%). The typical students comments were: "Because in all jobs in engineering you must maximize the profit and minimize the cost", "Companies want engineers who are trained in financial questions of small companies", "It is also useful for independent engineers to optimize their own profit", "In small firms engineers are responsible for calculating the costs and make sure the company makes profit. And also in bigger firms engineers are used for management positions thus a basic idea of business and optimization is very useful", "To know how to maximize the profit is very important for me. This could be a help for us if we have our own firm later on".

Enhancing the skills in Excel — 7 (44%). The typical students comments were: "It is good to know how Excel works", "It is very good because the skill to handle Excel is very useful. We use Excel very often in our mechanical engineering", "To learn to work with Excel and to use its 'hidden' functions".

Improving the English language — 5 (31%). The typical students comments were: "Good for the improvement of English", "Practice the speech".

The Second Group (Masters Students)

There were 17 students in the second group. All 17 students answered "Yes" to the question on the suitability of the course to engineering students giving the following reasons:

Benefit for the future work — 8 (47%). The typical students comments were: "Students learn to fix problems in a way that is quite often used in 'real life' business", "It is good for me to know how to optimize a problem and it will make my chances bigger on employment market", "It is very suitable because it teaches to implement the problem which is similar with the problem in a company".

Enhancing the skills in Excel — 10 (59%). The typical students comments were: "This course shows me new ways to work with Excel and solve mathematical problems", "Excel is software that almost every company owns. For that reason it is quite useful to know this program very well", "Learning basics and extended applications in Excel".

Improving the English language — 8 (47%). The typical students comments were: "To get in contact with more specific English integrated in a technical course", "Students are getting more common with lectures, scripts and even speech skills in the English language".

Conclusions from the Second Study

All 33 students in both groups indicated that the course was suitable to engineering students. They appreciated the practical nature of the models, the opportunity to enhance their problem solving, modelling and computer skills and the chance to improve their English language. Bachelor students saw more benefits from the course for their future job than Masters students: 94% versus 47%. One of the reasons for this difference could be that some Masters students are planning to do PhD study so the relevance of the course to their employment is not a priority at this stage. Both groups gave very similar responses concerning improvement of their computer and English language skills from taking the course. As with the First Study we think that students' feedback should be taken into account when designing curricula for their study.

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Mathematics and Industry – a complex relationship

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Abstract The relationship between mathematics and industry is a very complex theme. This complexity is mirrored in the various aspects of this discussion document. First of all mathematics in industry leads to questions about the education of mathematicians at universities. Furthermore the collaboration of university and industry is a very vivid research field with many new research results and new mathematical methods. Transferring these research results and methods from the university to the companies is economically necessary and lead to educational questions in the adult and further education. An important problem is the general public view of mathematics. This view is strongly influenced by the experiences with school mathematics. Opening the mind of the general public to the new meaning of mathematics for our industrial society leads to educational challenges in teacher education as well as in school mathematics. If we see mathematics as a technology we have to analyze the consequences for the society.

This paper is a report about more than thirty years of research and activities in industry mathematics at the University of Linz, the Austrian centre of industry mathematics. Many of these activities had an educational background.

Introduction

This paper describes more than thirty years of activities and research in the field of mathematics and industry at the University of Linz.

Starting point of this process was an educational reason in the seventies of the last century. The University of Linz educated Technical Mathematicians in a diploma study program (master level) and the university was convinced that our graduates were needed in industry. Therefore it was really surprising for us that industry did not think that there was a need for mathematicians in their companies. As a consequence of this situation members of the Institute of Mathematics started to collaborate with companies in different fields (energy, steel production, agriculture, ...). The aim of this initiative was to convince important persons in the companies that mathematics could be important for the economical and technological success and that it would be useful to have mathematicians working in their company (Schlöglmann and Wacker, 1988, 1991; Maaß and Schlöglmann, 1991a).

Today, after more than thirty years work in this field, our graduates have no problems finding jobs in different parts of industrial companies anymore.

This paper is dedicated to the initiator and motor of these activities O. Univ. Prof. Dr. Hansjörg Wacker who died much too early in 1991. But his work was continued by others and today the University of Linz is one of the European centres in this field.

The paper will, in a way, be chronological so as to make the development more apparent.

Preparing mathematicians for a job in industry – a first concept

To be well educated for a job as a mathematician in industrial companies an only content orientated education is not sufficient. While the basic mathematical contents (analysis, algebra, stochastics, differential equations, function theory, functional analysis, approximation theory, numerical mathematics, ...) are of course important, a career in industry requires more.

Practical problems are usually not formulated as a mathematical problem, they are problems within a content, often described by a non-mathematician in his or her language. That means that mathematicians have to discuss problems with non-mathematicians in order to understand their language and recognize the mathematical kernel of the problem. This step of problem formulation is a very important step in the problem solution circle and takes time. A mistake in this step can lead to an insufficient problem formulation and as a consequence to a solution which does not solve the problem of the problem poser. If the mathematical problem is formulated the mathematician has to work on the problem with mathematical means. In most cases it is necessary to use numerical methods and write a computer program. If the problem is mathematically solved the solution has been tested in the reality.

The educational concept was gradually increasing the complexity of problem solving tasks. After a basic education in mathematics and programming the students (mostly in groups of two students) had to work on a bigger problem with a programming part. The second step in this education was the so-called „project seminar". In this seminar the companies posed real problems and the students had to discuss the problems with the collaborators of the companies in order to solve them supervised by the university (Lindner, Schlöglmann and Wacker, 1987). For the students the last step of the program was often working on an industrial problem as their master thesis. Once the students had finished their education at university they often started working for the company that had originally posed the problem for their master thesis.

International activities in the field of mathematics and industry

The University of Linz was of course not the only university working in this field, which naturally led to collaboration with other European universities. First with the University of Kaiserslautern (Prof. Dr. Helmut Neunzert) and followed by more universities in the years to come.

At that point the most important question was: Who did mathematics in industry? On the one hand we knew that there was a demand for mathematicians in industry. On the other hand only few mathematicians actually worked in industry. Research done at the University of Kaiserslautern and our own, the University of Linz, showed that the mathematical work in industry was done by engineers (Neunzert and Schulz-Reese, 1984; Maaß and Schlöglmann, 1989c). Another problem was that the education for mathematicians at many universities imparts all the qualifications for successful career in industry. In order to counteract the European centres for mathematics in industry founded the European Consortium for Mathematics in Industry (ECMI). This consortium created a two years course for the education of mathematicians (Schlöglmann, 1991). Part of the curriculum was an exchange program for all participants, which means that each participant had to study one semester at another university. Furthermore the participants had to work on an industrial project. In the following years the collaboration between the ECMI-centres was extended through various activities. The last consequence was a double-degree study program, which means that the students had to study at two universities and get a degree from both.

Further education in mathematics for industry for engineers

If engineers did the mathematical work in industry the engineers needed a qualification in modern mathematical methods. A group of mathematicians at the University of Linz developed a two years university course “Mathematical methods for users” (Maaß and Schlöglmann, 1989a, 1989b, 1991b, 1991c, 1992a, 1992b, 1992c). The concept of this course contained the following contents: Differential equations, Stochastics (2 semesters), Programming in Fortran, Numerical Mathematics (2 semesters), Optimization (2 semesters) and Control Theory. As a first step study materials for all courses were developed. Our idea was that the participants would work through the materials at home and in presence phases at the university there would be discussions about ideas and problems. As we translated our course concept into action we were confronted with two significant problems. First and most importantly the engineers did not understand the mathematical language. They had been educated in a time before the “new math” had been developed and secondly the participants of the course were hard working people and did not have enough time to work on the materials at home. Therefore we extended the program. We added a course to give an introduction to the mathematical basics and extended the number of presence phases at the university. In these phases the contents were presented and the participants worked on extended tasks at home.

The university course “Mathematical methods for users” was implemented twice in Linz and once in Bregenz.

Mathematics in Industry – a theoretical analysis

In the course of the activities to mathematics in industry emerge also more theoretical questions: Why is mathematics used in industry? In which form is mathematics used? What has changed about the use of mathematics over the last years? In what ways does this influence society?

To study these questions we investigated the consequences of the projects, interviewed managers, engineers, participants of the university courses and initiated a research project (funded by the Austrian Foundation for Scientific Research) called “To the Acceptance of Mathematics as Technology”.

One of the first questions was: Can we call mathematics a technology? Without doubt we can say that the influence of mathematics on industry and society has changed during the last decades. This is mostly seen as a consequence of the “new technologies”, but mathematics is part of this new technologies. So wrote Edward E. David, President of Exxon Research And Engineering, in a report of the National Research Council of the USA: "Apparently too few people recognize that the high technology that is so celebrated today is essentially mathematical technology" (David, 1984). The European Consortium for Mathematics in Industry, that "in modern industry formulated analogously mathematical methods and ideas play an increasingly important role in research and development, production, distribution and management" (Wacker, 1988).

If mathematics is a technology, what are the consequences of this fact especially for the society? Let us explore in the problem of technology further.

To characterize a modern industrial society the philosopher Heinz Hülsmann developed the concept of "technological formation" (Hülsmann, 1985). According to Hülsmann this term means that a modern industrial society is formed by the technologies that are used by it. Technology as a term "grasps" precisely the structure and situation of this societal formation, supplies so to speak the "differentia specifica" (Hülsmann, 1985; 9).

Technology is for Hülsmann the consequence that natural science turns into technology (Hülsmann, 1985; 9).

If natural science turns into technology then this is "at the same time the change in sociality. Conversely this says that technology is social reality and realizes real sociality" (Hülsmann, 198; 9). Furthermore we have to consider the simultaneity of the formative power of the technology. "We don't have first a certain technique that is later social effective and structuring. Modern technology as large scale research isn't even later societal fomating but is as such already formativ" (Hülsmann, 1985; 10).

Hülsmann emphasizes the significance of the natural science for the development of technologies, but it is important to say that the technological formation of our society is also a consequence of the results of other sciences especially the social sciences and the economics (Schlöglmann, 1992). Today social organisations in our society are organized by abstract concepts that come from the social and human sciences, and the educational field works with results of the pedagogical science. Also large-scale research is no longer only a domain of natural science. Big research institutions are built up to give government advice.

Peter Heintel (Heintel, 1992) pointed out the close relationship between technological formation and societal organisation. Especially the form of communication within a society is important. Whereas in small group societies the direct communication from individual to individual is possible and for its stability necessary, big societal units require technological formation. In societies with many members direct communication cannot be the dominating form of communication, because it is not possible to reach all members in a short period of time. Therefore such societies need an organisation form mediated by indirect communication where information reaches the members by special mediums like written documents. That means technological formation is only possible in big societies. These societies have a demand on intermediate levels, like writing, tools but also mathematics to communicate and to produce a common thinking and dealing. Such societies also have a highly developed system of division of labour and therefore a demand on intermediate mediums for a "loss free" transport of contents (mathematics is such a medium) (Schlöglmann, 2002).

Sciences are driving forces in a society that is formed by technologies and is organized by indirect communication. Therefore specifics of sciences are also important for the conditions within the society and the structure of development. If the state of research, as the form in which sciences produce new results, is important for the competiveness of the economy because new products and the alteration of production conditions need new results of research, we get a growing of research. But more research leads to a dynamic in the changing process of the society, especially in the conditions of labour.

Furthermore research and sciences are based on the "logos", the reasons. This has a tendency to general validity and a more general result has influence on more parts in economy, production and society. Another consequence is also an increasing "abstraction of the technologies". Which means that the scientific basis of these technologies is more abstract and formal. This growing formalisation is seen in the case of computer technology. On the one hand this increasing formalisation requires a formal basis to formulate results (this basis is mathematics) while on the other hand it is responsible for the penetration of more and more parts of our society by new technologies.

For a long time mathematics was used for a long time to organize economic processes and even democracy needs mathematics. In many occupations is used, what we can see in investigations to work place mathematics. But what has changed the position of mathematics in the industrial use? If knowledge is used in a production process this knowledge must be usable as a "black box". That means the user of the black box does not need deeper knowledge to use it. On the one side the computer gives the opportunity to use mathematics as a black box and on the other side makes it possible to use complex mathematical methods to automate, for instance, production processes. But the use of mathematics as black box is not really new. Already in the beginning, as mathematics helped to organize and control economic processes, tables were used for calculations (for more about this problem see: Maaß and Schlöglmann, 1988a, 1988b). A consequence of this "black box use" of mathematics is the "disappearance of the use of mathematics out of the awareness of the people" (Schlöglmann, 1998). But in the meaning of general public the tool, the computer, controls processes and not the mathematical program. This challenge, education, to correct this view of the public.

Another interesting question is: “What is a mathematical product?”. Usually mathematical research products are theorems and their proofs. But in collaboration with industrial companies theorems and their proofs are not products that can be sold. Mathematical products in this field are usually programs, often in form of a black box, that solve an industrial problem.

The conference “Mathematics as Technology? Interaction between Mathematics, New Technologies, Education and Further Education”.

The purpose of this conference (held 1988 in Strobl, Austria) was to discuss the situation of mathematics and industry. Therefore we invited mathematicians from university and industry, mathematics educators, mathematics teachers and philosophers to give presentations and discuss in working groups the consequences for education of mathematicians, further education, school mathematics and mathematics and society. The results were published in a book (Maaß and Schlöglmann, 1989c).

I will now summarize the results if this conference with some additions of newer activities.

During the late seventies and the early eighties some universities in Germany and Austria started application oriented education programs (Techno-Mathematics or Industry Mathematics) with project work and elements of collaboration with companies within the study program. An important question in this connection was which kind of problems should be addressed in this education. Mathematicians at the University of Linz decided that no routine problems should be solved through a university. Universities have research as their main duty and therefore universities should only accept problems with a research part. The collaboration led to more difficulties like financing the project worker, secrecy of results, publication of the results, and so on.

If mathematics is a technology and technology is the consequence of research than we must also discuss the problem of technology transfer. A university has different ways to transfer research results to other systems of the society. The first way is the education of students. During their education students learn new theoretical results and apply these new results if they work in companies after their graduation. A second way is further education for persons who use mathematics in their practice. The mathematical knowledge of for instance engineers is based on the knowledge learned in education. To get access to new research results universities can organize programs for persons who need this new knowledge. A third way of technology transfer is collaboration with companies and a fourth way is the developing of a black box at the university.

An important point to consider when discussing the problem of mathematics and industry is also the picture of mathematics in the general public. Many people see school mathematics as mathematics and don't realize that mathematics is the foundation of the organization of the economic and societal life. But especially the new applications of mathematics are unknown to the public. This intensifies the view of the tool, the computer, in the foreground and the mathematical program in the background. To change this view is only possible if the new applications of mathematics are also a theme in school mathematics. As a first step insight into the new situation must be given to mathematics teachers. In the mathematics teacher education curriculum at the University of Linz there is a course "applications of mathematics". In this course groups of teacher students work on simplified problems of industrial mathematics. The idea of this course is that teacher students learn to trust their own ability to handle such problems. A second activity is directed to in-service teachers. The university organize one-day courses for teachers to give them a view to the new projects in the field of mathematical applications and organize working groups for the simplification of real problems.

If we take a look at school mathematics our aim is to give students a realistic picture of mathematics. In this sense we propose to work on different levels: information about mathematical applications – examples of industrial mathematics – problem of black box. Teachers should have more information to the new applications of mathematics and we hope that our teacher activities will form the basis for this aim. Examples of industrial mathematics should start with a problem description and in a first step with a simplified model. A second possibility is to use simulation programs. Those help to see consequences of the mathematical realization (see Maaß and Schlöglmann, 1994a). For teaching black boxes we proposed 1994 (see Maaß and Schlöglmann, 1994b) using computer games and analyze the effect of intervention in the game. The basic idea of this education concept is that computer games are based in on mathematical functions in many cases and by systematic testing students can find these functions.

During the last years the University organized together with the foundation "Talente" project weeks for students who are interested in mathematics. During such a week high school students work in groups on simplified mathematical problems.

If mathematics is a technology it is necessary to discuss the consequences of the industrial application of mathematics for a society. That such a discussion is necessary we could see from the problems with the finance system during the last year.

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A View on Mathematical Discourse in Research and Development

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Abstract This paper argues that analyzing the discourse in industrial mathematics is key to understand how mathematics innovates, where obstacles occur, and how innovation can be organized more systematically in the future. Ethnographic research with focus on language use may reveal best practices and help to develop consulting and training offerings for mathematicians who work in R&D units or study mathematics.

Introduction

Awareness is rising that mathematics plays a crucial role for innovation in many industries, including logistics, finance, electronics, and the chemical and pharmaceutical industry. The growing demand of mathematical expertise in industry has led to a series of initiatives from mathematical communities in many countries. Aiming at a more systematic co-operation, new university programs for applied mathematics have been defined and research centers for industrial mathematics, special interest groups, and faculty positions focussing on industrial mathematics have been founded within the last decade.

These initiatives have had a positive effect, in some cases boosting the knowledge transfer into R&D departments. Nevertheless, a systematic exploitation of mathematical knowledge in industrial settings does not happen yet, at least not on a large scale. Reasons for this gap are seen in a different terminology and language use in mathematics and the application domain, deficits in the education of engineers, as well as in practical and organizational conditions (Grötschel/Lucas/Mehlmann 2009:16)

In addition, there are educational issues within mathematics that go beyond the sheer knowledge of mathematical theories and application domains. Many graduates pursuing a career in industry feel that they are not able to apply their mathematical knowledge in the industrial setting except their general ability of logical thinking. One reason might be the discourse that takes place in R&D units. There, mathematicians must carry through their ideas in a setting of conflicting views and different levels of mathematical knowledge, with constraints in time an budget and hierarchies to take into account.

This paper argues that analyzing this discourse in industrial mathematics is key to understand how mathematics innovates, where obstacles occur, and how innovation can be organized more systematically in the future. Analyzing the use of language – whether in oral or written form – may show how mathematicians use their mathematical knowledge in practice. It can even help to reveal a specific mathematical way of thinking.

A new form of participatory research will be introduced. It is based on the role of an "useful linguist" who takes an active role in research and development teams as he helps with the creation and review of documents such as technical documentation and PR articles. In addition to interviews and observations, he uses the work on documents to make visible and document the discourse that is taking place in the research and development teams.

This Linguistic research may reveal best practices and help to manage a fruitful application of mathematics, including consulting and training offerings for mathematicians who work in R&D departments or study mathematics.

Role of Language

Focussing on language has been a promising approach for analyzing the nature of mathematics from begin on. Mathematics is a science that examines abstract objects and methods. It therefore relies on language when it comes to defining and communicating the objects under investigation and proving mathematical findings. This is why the reflection on mathematics must always take into account the language of mathematics and the language use of mathematicians.

There is a significant difference between both terms. The first one stands for an internal view on mathematics whereas the second term allows for an external view on mathematics.

Mathematics is often seen as an exclusive domain that is only accessible for those who can understand the formalisms that are used by mathematicians. This view is mirrored in numerous popular math books written by mathematicians. Best example is the classic book "What is mathematics?" written by Richard Courant and Herbert Robbins. When reading the book title one could get the impression that the book is an essay about mathematics. But it is not. It is rather a text book that invites readers to learn mathematics by doing it. The message is that learning the language of mathematics is a prerequisite for talking about mathematics.

This view has been challenged. Another classic mathematics book, "Experience Mathematics", written by Philip Davies and Reuben Hersh, paved the way to external accounts on mathematics. It catches the essence of mathematics in an every day language without explaining mathematics in the traditional sense. It focuses on explaining how the mathematicians' practice looks like.

This approach is the basis of the following argumentation that argues for a meta-research on mathematics using linguistic and ethnographic methods for analyzing discourse at the work place of mathematicians in industrial research and development. The idea is not to focus on the mathematical core of innovations, but to investigate how these innovations evolved. It takes a closer look on the conversations that precede and follow mathematical innovations. This view on industrial mathematics can be based on philosophical and sociological ground work.

Philosophy of Mathematics

The early investigations on mathematics were conducted by mathematicians themselves. They portrayed the ideal use of mathematical language with emphasize on mathematical formalism. This was mirrored in the opinion that mathematics is a "hard science" that is error prone and produces findings of eternal truth. The body of mathematical knowledge is seen as cumulative, consensual and historical invariant.

This view was challenged in the second half of the 20-th century by philosophers that followed the ideas of social constructivisms (and others). They stated that even mathematics is socially construed and therefore not free from human influence, for example power, taste and will. This thesis made possible a sociological and linguistic investigation of the practices of mathematicians (for example Ernest 1998, Heintz 2002). This research could not transform mathematics into a "soft science", not even from a theoretical point of view. But it opened the view to the basic characteristics of mathematics and how they are influenced by people and historic circumstances.

Imre Lakatos' essay "Proofs and Refutations" can be seen as a milestone. Lakatos constructed a fictive dialogue of students with a teacher who moderates the conversation. The group talks about the Euler formula E-K+F=2. They exchange ideas, claims, proofs of their statements, and counterexamples. The dialogue is fictive, but it is not created from scratch. All contributions from the students are in fact historical statements from mathematicians who worked on the Euler formula. The statements are composed in the form of a conversation. The dialogue shows that the invention of a mathematical proposition is not a linear process. It involves detours, errors, and controversies. The mathematical form and content of the proposition was developed under heavy influence of opinions, feelings and taste of the involved persons. The main arguments are written in the natural language, not in the language of mathematics.

Analyzing mathematical discourse in industrial contexts, as proposed in this paper, uses the idea of analyzing a mathematical discourse, but without creating it from historical sources. Similar conversations occur when mathematicians work in interdisciplinary teams with a common goal but team members who have different views on how to reach this goal. Analyzing discourse in industrial mathematics takes Lakatos' approach one step further since it is focussed on the discussion in our time and analyzes the language use not within a discipline (as did Heintz 2002), but investigates the interface between mathematics and other disciplines. In doing so, assumptions become visible that underlie the common understanding of the application of mathematics.

Mathematics as Lingua Franca

One claim is that mathematical formulas are the lingua franca of science and technology — an opinion that is closely linked to the famous quote from Galilei saying that mathematics is the language of the book of nature. Two more sayings gained fame within the mathematical community. Eugene Wigner saw an “unreasonable effectiveness” of mathematics in science and technology, and the Davis Report concluded that high technology is always mathematical technology.

Observing the daily work of interdisciplinary R&D teams lead to the impression that mathematics is indeed useful in industrial settings and can serve as lingua franca. But this not the complete picture. When mathematicians talk at their work place, mathematical formulas are always embedded in natural language. Mathematicians use a mixture of formulas and words which makes the natural language as crucial for industrial mathematics as formulas are. They use metaphors, examples, and stories to explain mathematical ideas to colleagues and to convince them that these ideas are the right ones. Natural language serves also as a means for searching the right abstraction of phenomena within the application domain. The goal of this mathematical discourse might be a mathematical formalism, but formulas are only reached through a discussion with extensive use of natural language.

Transfer of mathematical knowledge

The transfer of mathematical knowledge is often seen as a mechanical process which covers the packaging of mathematical ideas and methods and their working into a technical product. This might include stimulation of mathematical research through the interdisciplinary work with engineers. Nevertheless, knowledge transfer is mainly seen in the opposite direction using preexisting mathematical knowledge in an application domain. A first view on mathematical discourse in an industrial setting shows that knowledge is not transferred in this sense. It rather changes while being applied, since it must be verbally constructed anew in discussions with engineers and managers of the application domain. Even more: Mathematicians must see to get their perspective and ideas applied. Engineers and managers are supplied with different knowledge and different views on the technical product in development. Technical products can be construed with less (or no) mathematics although more mathematics promises to make them better. To carry through the mathematical ideas is a central challenge for mathematicians working in R&D units. This challenge is taken up verbally in the interdisciplinary dialogue. As field work shows, this dialogue includes rhetoric strategies for hiding mathematical content, showing its usefulness and proving its cost-efficiency. These characteristics of the mathematical discourse in R&D shed a light on the actual behavior of mathematicians, how they integrate themselves in interdisciplinary teams and which rules and strategies they use for positioning mathematics.

Methodology

Analyzing mathematical discourse requests methods that are stringent and provide general insights. They must go beyond examples that mathematicians tell from their individual experience and point of view. This is why we will conduct a participatory observation, including the work with documents that are created within development projects such as technical documentation and marketing collaterals. A linguist takes part in the project work as a "useful linguist"; aside from observations he prepares for example documentation and other writings and manages review cycles that allow mathematicians, engineers and managers to articulate their views in written form (Schmidt 2009).

Field work should include observation of the daily work of R&D groups in mathematical industries like finance or optimization in logistics and transportation, as well as the application of simulation and control theory across the industries. Collaboration is planned with university institutes as well as research centers designed for knowledge transfer and R&D departments of private companies.

Ground Work in Linguistics

Much work has been conducted in the area of analyzing public discourse on scientific results, showing that a funneling process takes place which shapes the presented knowledge (Liebert 2002) and which adds – by using natural language – specific views on this knowledge resulting in a semantic battle (Felder 2006).

Semantic battles usually take place in the public arena, when a group of individuals want to dominate the discourse on a topic, but others with an opposed view try the same. The semantic battle can concern topics that are per se controversial and belong to the sphere of politics, such as taxes or school education. Linguists have observed a specific language use in discourse about those topics. Each party tries to set their views dominant by using terms that support their views and criticize the view of others. Even if the used terms are neutral, they normally set one aspect dominant which is used to influence the direction of the conversation. Research had been conducted for analyzing semantic battles in several domains, including public debates on biology, especially genetics (see Felder 2009).

When analyzing discourse in R&D units, the scope is of course different. Not public debates are analyzed, but discourse within an organization. One assumption is that the linguistic tools for observing public discourse can be applied to organizational communication. As already mentioned, this discourse contains also different views on a subject, and each team member in an interdisciplinary team brings in his specific knowledge, ideas, and goals which lead to semantic battles in a similar way.

When it comes to analyzing documents with respect to their mathematical content, there is also linguistic work available. Text linguistics focuses on text structures, language use in texts, but there is also research conducted on mathematics and its popularization in different texts types (Schmidt 2003).

Ground Work in Sociology

The investigation of scientific knowledge and its creation has also a tradition in sociology. Ground work for the proposed approach are studies that have challenged the opinion, natural sciences are sciences that are clean from human influence such as battles on power or pressure from outside the research teams (Knorr Cetina 1984). The proposed work applies studies that identify different scientific cultures across disciplines (Knorr Cetina 2002) and that define a research program for a sociology of knowledge (Keller 2005). It expands existing studies on (pure) mathematics that were conducted in this tradition (Heintz 2002).

The mentioned sociologists used the participatory observation to analyze the behavior of individuals and groups, organizational set-ups and power structures. Knorr Cetina spent time at the CERN in Geneva regularly to talk to scientists, conduct interviews, watch them and take notes. Heintz joint the Max Planck institute for mathematics in Bonn for several weeks and gained her insights also by watching and talking to the mathematicians there. In addition, both sociologists analyzed documents that had been written by the scientists and how they were reviewed.

The method of participatory observation is also at the heart of the study we will conduct. However, it will be adapted to the domain of industrial mathematics and to the purpose of analyzing innovations in this domain.

The Useful Linguist

The work on documents is an important part of the scientific work, since results must be published, and scientists must apply for grants. Knorr Cetina showed that all insights she got from participatory observation were mirrored in the joint work of the scientists on a scientific paper, including the text revisions, comments from reviewers and the kind of document cycling during the writing and review process.

In industrial research and development, documents have a similar importance as in natural sciences. Nevertheless, they are of another kind and variety.

In software development, there are for example internal documents like specifications and design documents that are used to prepare the development of technical artifacts, such as algorithms or interfaces. In addition, there is project documentation including project charters, minutes of team meetings and status reports. Other documents are prepared for the external audience. They include product documentation that is shipped with the technical product, for example installation guides or operating instructions. Companies prepare also marketing documents such as White Papers, Solution Briefs or Leaflets about products.

Usually all those documents are written and reviewed by project members and other experts, which normally leads to a number of revisions and several document versions. The revisions, especially in this variety of document types, make visible technical problems, discussions, and solution proposals as well as different views on how to position the later product in the market. That is why the work on texts serves as a tool for gaining a closer look on semantic battles that come with the application of mathematics.

The useful linguist joins research and development projects in order to draft and edit documents, and to organize the cycling of documents for reviews. He uses his role as technical writer to get to know the inner world of the project. As a project member he is at the core of the innovation and can observe how mathematics comes into play. He joins the project on a long-term basis so that he is able to dive deeply into the topics and to communicate at eyes level with the engineers and mathematicians. This helps to reveal what is happening in the project and to draw the right conclusions.

As a technical writer he does not belong to the inner group of colleagues in the project, since he is a co-worker with focus on language. Therefore he has an internal, but distant view on the product development. He is not concerned with the product itself and also not with the mathematics in use. He focuses on the communication about the product, its features and how the mathematicians were involved during development. As a linguist he can use the creation and review of document to control the document cycling and to enrich the participatory observation.

Lines of Investigation

Industrial mathematics is a diverse field. It takes place at university departments, mostly as project-based collaboration with companies. There are spin-offs that often productize one specific mathematical invention. Innovation in small and medium enterprises may come from local or regional collaboration with universities or public research institutes. Larger companies can afford an own research and development department, some companies even have units that focus on mathematical consulting. The different industries have their own culture and tradition, also from a mathematical point of view. For example, insurance companies build their business on statistics.

This diversity must be taken into account when conducting research about industrial mathematics. Since a full coverage of all possibilities of mathematical innovation is not possible, only exemplary studies are realistic. However, they need a central theme and guiding research questions.

The following questions may lead to a clearer picture of mathematical innovation in research and development:

- How do mathematicians argue for the use of mathematics? What barriers are conceived by the mathematicians that hinder mathematical innovation? How do they position their mathematical ideas in this context?
- How is mathematics sold? Do mathematicians use arguments from an economic point of view such as addressing costs and benefits of the use of mathematics? Which role have patents?
- How do mathematicians find a mathematical model of the central objects of the application domain? What strategies are used for developing a common language? How are objects of the application domain changed or redefined to make them fit to the mathematical model? What issues influence the mathematical model? Are only aspects from the application domain relevant or also organizational issues like time constraints and the availability of budget?
- Which mathematical theories and tools are in use? Are they developed anew or reused, for example from a software library? Which level of proficiency do the project members have? Do the team members judge the level of sophistication of the used and proposed mathematical models?
- Which strategies are used to make the mathematical tool set visible or to hide the mathematical content? How is the mathematical content documented in the product documentation? What is explained and what is left out? Are mathematical artifacts visible on technical interfaces or user interfaces?
- Which role have proofs in industrial mathematics? Which standards from research mathematics are applied? Do mathematicians refer to truth, beauty, or similar concepts?
- Is there a mathematical way of thinking that goes behind the application of mathematical models and methods? How do mathematicians bring in their implicit knowledge and their experience with abstract mathematical structures?

Outcomes

When addressing the interface of mathematics and industry, the organizational development of industrial mathematics and education are without doubt the main issues.

The proposed analysis of mathematical discourse is meta-research that can support organizational concerns. It may contribute to both mentioned areas of activity and help to leverage the use of mathematics in industrial settings and to leverage communication skills in R&D teams. Linguistic and Sociology help to find best practices for the transfer of mathematical knowledge which may lead to a better management of organizations for industrial mathematics and a better integration of mathematicians in R&D units.

Field work may lead to the documentation of best practices, it can reveal success stories and can help to detail out shortcomings of today. In addition, it can be a means for specifying needs for mathematical research and education, addressing them to the mathematical research community. Outcome of the linguistic field work can include the specification of technical tools, platforms for community building across mathematicians in academia and industry.

Last but not least, training can be developed that focuses on soft skills that are needed by mathematicians who work in R&D units. This can lead to a higher impact of mathematics in industry through people at their work place. Furthermore, industrial mathematics will be promoted as a whole which helps to close gaps in the interface of mathematics and industry.

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Graduate Student Training and Development in Mathematical Modeling: The GSMM Camp and MPI Workshop

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Abstract This paper describes a program of graduate student training and development in mathematical modeling consisting of the Graduate Student Mathematical Modeling Camp and the Mathematical Problems in Industry Workshop. The program, run annually over a two-week period in June, begins with the Camp, a meeting designed to promote problem-solving and scientific communication skills. The Workshop is held during the week following the Camp and the students attending the Camp also participate in the Workshop. The Camp/Workshop program has been run successfully for the past six years, and an objective of the present paper is to describe the program so that similar programs may be considered elsewhere.

Introduction

Research in the mathematical sciences has become increasingly interdisciplinary requiring both depth in mathematics and the ability to understand and communicate across a variety of areas outside of mathematics. While students in mathematics are often trained well in mathematical analysis and methods, they often have less experience with (or are not exposed to) mathematical modeling and problem solving in areas of application outside of mathematics. The Graduate Student Mathematical Modeling (GSMM) Camp at Rensselaer Polytechnic Institute (RPI) is an educational tool that has been developed during the past six years, and run primarily to promote problem-solving and scientific communication skills. At the Camp, graduate students work together in teams, with the guidance of a faculty mentor, on highly interdisciplinary problems usually inspired by industrial applications. The problems are brought by faculty mentors who are experts in mathematical modeling, and who have extensive experience covering a wide range of areas of application. The problems are carefully selected to promote a broad range of problem-solving skills including mathematical modeling and analysis, scientific computation, and critical assessment of solutions. As a result, the GSMM Camp exposes graduate students to real-world problems of current scientific interest, and provides them with a valuable educational and career-enhancing experience outside of the traditional academic setting.

An important aspect of the GSMM Camp is its link with the Mathematical Problems in Industry (MPI) Workshop. This problem-solving workshop, run annually for the past 25 years, attracts leading applied mathematicians and scientists from industry, universities and national laboratories. The focus of the workshop is a set of problems brought by participants from industry. The problems span a wide range of application areas, including fluid and solid mechanics, mathematical biology, data analysis, and mathematical finance, among others, and require a strong set of problem-solving skills to handle them. As with the Camp, the work on the problems is done in teams, but at the Workshop each team has a broader membership which includes graduate students, postdocs, faculty and attending scientists, and the industrial participant who brought the problem. The MPI Workshop runs during the week immediately following the Camp, and students who attend the week-long Camp also attend the Workshop.

Experience over the years with the MPI Workshop has shown that the problems considered there are very difficult, and to make significant progress on the problems requires a strong background in a wide range of areas of application together with strong problem-solving skills. Graduate students often lack the experience and breadth to participate significantly to the problem-solving effort of the team. They need additional practice and confidence in their abilities, and the GSMM Camp is designed to give them this. So, the Camp and Workshop compliment each other. The Camp is primarily an educational tool for graduate students, whose purpose is to build strength in problem-solving and scientific communication. The problems brought to the Camp are intended to challenge, but not overwhelm, the students, and to provide a warm up for the difficult problems considered at the Workshop. There, the graduate students who attended the Camp are better prepared and participate more effectively in the team effort, and thus gain even more from the Workshop experience. It has been found that the graduate students now play a much more significant role in the Workshop as a direct result of the earlier participation in the Camp.

Other industrial mathematics and/or problem-solving workshops exist already in the U.S. and elsewhere. The closest program to the one discussed here is a two-week program organized by the Pacific Institute for Mathematical Sciences (PIMS) and run for several years at universities in Canada primarily. Their program involves a one-week workshop for graduate students which precedes a one-week workshop for industry, and this was the original model for the GSMM Camp and its link to the existing MPI Workshop. Other mathematical problem-solving workshops for graduate students exist in the U.S., notably at the Institute for Mathematics and its Applications, North Carolina State University and Claremont Graduate University, but these individual workshops lack the comprehensive program provided by the GSMM Camp and the MPI Workshop.

The purpose of this paper is to describe the format and organization of the Camp and Workshop, and to discuss some of the interdisciplinary problems considered at these meetings. The goal is to pass along my experience as an organizer of the Camp and organizer and co-organizer of the Workshop with the idea that similar programs may be initiated and run elsewhere.

GSMM Camp

As noted previously, the focus of the GSMM Camp is graduate student education and development directed towards scientific communication and problem solving. The activities of the week-long Camp, run annually in June, are centered around the participating graduate students who work in teams with the support and guidance of a faculty or industry mentor. Approximately 28 students attend the Camp each year, and this group is divided in four teams of approximately 7 students each. Experience over the past six years has shown that groups of this size are optimal. (Students can get "lost" or can "hide" in larger groups.) Each team works on an interdisciplinary problem as a prelude to those to be considered at the MPI Workshop to be held in the following week. In this way, the graduate students are given the opportunity to develop problem-solving skills prior to exercising them at the Workshop.

Also, the team approach to mathematical modeling and analysis promotes scientific communication. In this environment students learn from one another and derive strength from the combined talents of the group, very much in the spirit of true collaborative research.

The day-to-day flow of the Camp follows a well-established path, similar to that of the long-running MPI Workshop discussed below. In the morning of the first day, the invited mentors present the problems to the whole group. Following the presentations, each graduate student is asked to rank the problems according to his or her interests, and the Camp organizer uses this information to distribute the students into teams with approximately equal numbers. In addition to student preference, the teams are balanced in terms of male and female students, and in terms of university affiliation. After lunch on the first day, and continuing through the second and third days, the students work on the problems with the guidance of the faculty mentors. Often the first afternoon is spent, in part, with the mentor giving "mini lectures" to provide additional background on the problem beyond that discussed in the initial problem presentation. Students digest and then use this information to begin the mathematical modeling process. At a later stage, once the basic models have been established, the student teams often split up into subgroups to work on various aspects of the problem. For example, one subgroup might pursue an asymptotic analysis while another subgroup works on a numerical approach. In the end, the subgroups report back to share their results, and in the morning of the last day the students prepare summary presentations. The subsequent afternoon is devoted to presentations by the student teams of the work carried out during the week to the whole group.

It should be stressed that while the general flow of the Camp is similar to that of the Workshop the emphasis is different. This difference is centered essentially around the choice of problems, a key element to both activities. At the Workshop, the problems brought are often quite difficult having been considered already by scientists and/or engineers at the participating industry. These problems require experienced applied mathematicians with strong problem-solving skills to play a leading role in the group effort in order to make progress. In this environment there is a tendency for students to *observe* rather than *do*. The situation at the Camp is reversed. At the Camp the emphasis is on building the problem-solving and communication skills of the graduate students by *doing*. In order to accomplish this, the problems are carefully chosen to challenge but not overwhelm the graduate students so that the graduate students are the 'doers' while the mentors are the 'observers.' The mentors and the organizer work through many aspects of the problems in advance and know available modeling paths so they may act as effective "camp counselors" providing encouragement and guidance to the graduate students, but remaining on the side to let the students ask the right questions, formulate the models, work out the solutions (both analytical and numerical), and interpret the results. The work of the team is allowed to flow freely and while modeling paths are known in advance, these are not always the paths that are followed. The students are in charge and ultimately choose their own path and obtain their own solutions.

The mentors play a very important role in the success of the Camp, and it is essential that they be chosen carefully. The mentors select the problems brought to the Camp, often based on problems considered in their research or at previous MPI Workshops (or other Industry-Math meetings), and the choice of problems and their presentation is crucial to the success of the Camp. Of course a positive attitude and energy of the mentors is important to the activities of the student teams. Typically, the mentors are invited from faculty who attend and enthusiastically support the Workshop, and show a willingness to work with and help students. Generally it has not been difficult to attract highly qualified mentors to the program.

The university facilities needed to run the Camp are modest. A lecture room is required for the initial problem presentations on the morning of the first day and for the summary presentations on the afternoon of the last day. The discussions of the student teams takes place in traditional classrooms fitted with plenty of blackboard and/or whiteboard space. It is important that the classrooms are close to one another and have internet access, and that there is a central meeting space nearby for refreshments and social interaction. Students attending from universities outside of RPI stay in campus housing (dorms), and meals are provided each day at the meeting site on campus. (Students may also wander off campus for dinner in small groups.) Library and computing facilities on campus are also available during the week of the Camp.

In order to run and maintain a successful program it is important to advertise so that a strong group of students register and participate in the Camp each year. The main vehicle for advertising is the Camp web pages. These provide general information about the Camp as well as information about past and present modeling problems and online registration. The program is also advertised using the Camp/Workshop email list, and by placing announcements in SIAM News, the SIAM web site, and elsewhere. Further advertising is done via word of mouth through contacts made by the organizer and by personal communication of past participating students and their faculty advisors.

Finally, the program requires financial support and this has been provided primarily by the U.S. National Science Foundation (NSF). The NSF is eager to support programs that promote training for graduate students in the mathematical sciences, and the Camp/Workshop combination fits well within their goals. Additional support for the program has come from the Department of Mathematical Sciences at RPI which provides secretarial services leading up to and during the Camp.

MPI Workshop

The MPI Workshop is a week-long meeting that has run annually starting in the summer of 1985. It was modeled after the long-running and successful Study Group with Industry held at Oxford University at the time, but now called the ESGI (European Study Group with Industry) held at participating universities throughout Europe and Great Britain. Originally, the Workshop was organized and run locally at RPI, but now it is run at participating universities including the University of Delaware (MPI 1999, 2000, 2004 and 2007), Worcester Polytechnic Institute (MPI 2003 and 2005) and Franklin W. Olin College of Engineering (MPI 2006). All other MPI Workshops were held at RPI except MPI 1995 which was organized jointly with the Center for Nonlinear Studies at Los Alamos and held at the University of New Mexico. The Sloan Foundation provided initial support for the meeting, and from 1990 support came primarily from the participating industries and the hosting universities. Starting with MPI 2008, additional support has been provided by the National Science Foundation.

Over the past 25 years, the Workshop has attracted participants from a wide range of companies, both big and small, including Alcoa, Biogen, BOC Gases, Chevron, Citibank, Corning, Genentech, General Electric, Hitachi, IBM, Kodak, and Merril Lynch, among others. Throughout the years, the main objectives of the Workshop have been

- to promote links between research mathematicians and industry,
- to provide fresh input for real-world problems from industry,
- to stimulate new ideas and interdisciplinary research, and
- to provide a learning environment for graduate students and postdocs.

With the link to the Camp and with new support from NSF, the Workshop has experienced a significant increase in participation from both academics and industry in recent years. This has strengthened the Workshop and helped fulfill the objectives listed above.

The Workshop takes place during the second or third week of June immediately following the Camp. The primary focus on the first day is the problem presentations by the representatives from industry. Four problems are usually presented, and each problem typically has multiple parts. Following the problem presentations, the Workshop participants break up into smaller working groups consisting of faculty, postdocs and graduate students, each devoted to work on a particular problem directed by a representative from industry. The division of labor occurs naturally according to each participant's background and taste, but there is an attempt by the organizers to maintain approximately equal numbers amongst the groups. Progress on the problems is made by the combined efforts and talents of the Workshop participants within each team. The work is informal and highly interactive, requiring strong problem-solving and communication skills. By the end of the week, full solutions are found in many cases and significant progress is made in all cases. On the morning of the last day, summary reports detailing the progress made on the problems are given by an academic participant from each group.

Additional work is done following the Workshop including preparation of a report on each problem which is submitted to a Workshop Proceedings. The Proceedings are unpublished documents, distributed to all participants, and used mainly to record the work done at the meeting and to help promote the meeting to prospective industry participants. In many cases, the Workshop problems have led to further research and published articles, including, for example, work on the behavior of catalytic converters (Please, Hagan & Schwendeman, 1994), on ohmic heating of multiphase food materials (Please, Schwendeman & Hagan, 1995), on the behavior of shear-thinning flows (Brewster, Chapman, Fitt & Please, 1995; Chapman, Fitt & Please, 1997; Fitt & Please, 2001; and Sun, et al., 2004), on filter design and fouling (King & Please, 1996), on lubricating flows in computer hard drives (Witelski, 1998; Witelski & Hendirks, 1999a and 1999b; and Hendriks, Tilley, Billingham, Dellar & Hinch, 2005), on modeling multiphase flow in a paper manufacturing device (Fitt, Howell, King, Please & Schwendeman, 2002), on the numerical and asymptotic analysis of crystal growth (Khenner, Braun & Mauk, 2002a and 2002b; and Khenner & Braun, 2005), on modeling pad conditioning in chemical-mechanical polishing (Borucki, Please, Witelski, Kramer & Schwendeman, 2004), and on epidemiological models for alcohol problems (Braun, Wilson, Buchanan, Pelesko & Gleeson, 2006), among others. Many industrial problems, inspired by problems presented at MPI Workshops and similar workshops, appear in the collection edited by Cumberbatch and Fitt (2001).

Recent Modeling Problems

A wide range of problems have been considered at the Camp and Workshop. For example, at the Camp in June 2009, the following problems were discussed:

1. *Design of Fuel Tanks for High-Speed Vehicles*, presented by Tom Witelski (Duke)
2. *Bistable Nematic Liquid Crystal Display Device Design*, presented by Linda Cummings (NJIT)
3. *An Ecological Modeling Problem*, presented by Mark Holmes (RPI)
4. *Decompression During Scuba Diving*, presented by Colin Please (Southampton)

Students at the Camp then worked on the following problems given at the Workshop:

1. *Homogenization of Self-Acting Air Bearing Problems for Patterned Magnetic Recording Media*, presented by Ferdinand Hendriks (Hitachi)
2. *Characterization of Porous Media Using Network Models for Filtration Applications*, presented by Uwe Beuscher (W.L. Gore & Associates)
3. *Development and Persistence of 'Static' or 'Dead' Zones in Flows of Certain Materials*, presented by John Abbott (Corning)
4. *The Dynamics of a Model of a Computer Protocol*, presented by Fern Y. Hunt (NIST)
5. *Vulnerable Plaque: Can We Predict Which Plaque Will Lead to the Next Adverse Cardiac Event?*, presented by Nowwar Mustafa (Christiana Care)

Further information on these problems may be found at the website for the Camp:

http://www.math.rpi.edu/GSMMCamp/

and for the Workshop:

http://www.math.udel.edu/MPI/

A full list of problems brought to the MPI Workshop since 1995 may be found at

http://eaton.math.rpi.edu/Faculty/Schwendeman/Workshop/MPI.html

Concluding Remarks

Research in the mathematical sciences requires both disciplinary depth in mathematics as well as the ability to model and solve problems that arise in applications. Graduate education in mathematics is typically strong in the former while often being weak in the latter. The Graduate Student Mathematical Modeling Camp at RPI has been developed to give students a rich experience in mathematical modeling and to expose students to a wide range of applications. This is done using a team approach which has the additional benefit of promoting skills in scientific communication. The Camp is organized in conjunction with the long-running MPI Workshop which provides a further educational experience for the participating graduate students by exposing them to interesting mathematical problems and potentially new areas for collaborative research. The combined Camp/Workshop program has been run successfully for the past six years, and provides an excellent model for graduate student education and career development.

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Consuming Alcohol – A Topic for Teaching Mathematics?

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Abstract In Mathematics education students can be motivated by discussing examples, which may be found in the students' life-world. Thinking about such examples is a very challenging but complex task. Especially by looking for useful examples a lot of knowledge is necessary, so that well designed examples can be found for education. In text-books challenging examples, which are good modelling tasks can hardly be found. But if one has a look on different text-books for Mathematics education it will be possible to find different examples for different purposes. A lot of these examples want to show the possibilities of Mathematics in real-life-situations. Often they are used as introduction examples to motivate students to think about situations which can be found in their living environment. In my paper I show the discussion about "absorption & degradation of alcohol" and its implementation in class by technology.

Introduction

The development of Mathematics has been influenced by the development of technology from the beginning. In the age of information technology and 'New Media', electronic additives are broadening the horizon in both the education and teaching of Mathematics.

Electronic devices support cognition. The support is given by the possibility of transferring complex operations to technology and the use of different computer models. The efficient use of methods involving intensive computation and/or models is guaranteed.

By using technology in Mathematics education an enormous shift from the accomplishment to the planning of problem-solving can be done. Therefore a useful shift of emphasis from mathematical operations to the use of mathematical knowledge and reflections can be realized. If you use technology, some inner-mathematical reflections have to be done, because the solution created by technology has to be kept in mind.

Looking back in history the use of technology in Mathematics education started in the 1970's (Siller, 2008). At that time numerical calculators were state of the art. Today the development in technology has been very successful. Numerical calculators still exist, but other electronic devices such as CAS-calculators (e.g., CASIO ClassPad) are more powerful instruments. Such additives should not only be used as 'number crunchers', they can be used as a method for communication in Mathematics education. Abstract mathematical objects can be visualized very easily using these instruments by using numerical, graphical or symbolic manipulation.

As the role of technology can strongly influence Mathematics education, in particular mathematical modelling, it is necessary to include the role of technology-additives in the modelling circle. An approach has been conceptualised by Siller and Greefrath (2009).

Taking this graphical illustration as a basis it is possible to discuss processes in the real world including the use of technology in terms of modelling in a more detailed way. Cross-curricular aspects can be discussed with the help of mathematical knowledge, so that students can recognize the helpfulness of mathematics.

Alcohol in Blood – a prospective example for modelling with technology

The starting point of a lot of modelling examples is a real-life situation, at best procedures, scenes or incidents in real life, shown in Siller and Maaß (2009) or the ISTRON-series (http://istron.ph-freiburg.de — last access: 24.08.2009). But it is possible to find 'simple' examples in school-books, too. The only condition a teacher has to provide is that students

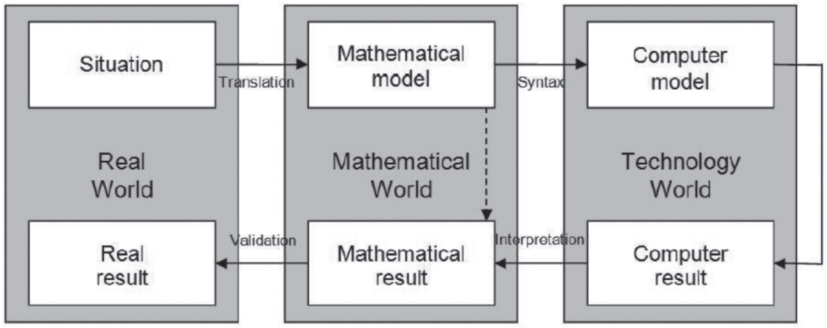
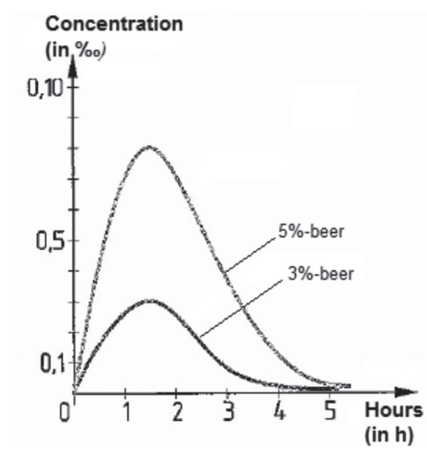


Figure 1—Modelling circle concerning technology

are allowed to discuss such examples in an extra-ordinary way so that they are thinking about several opportunities that could happen. By wading through Austrian school-books in Mathematics an interesting example on the topic of alcohol can be found. Students are confronted with it, nearly their whole life because in the press it is possible to find different announcements on this topic very often. Links to teaching Mathematics can easily be found, and a starting point for cross-curricular teaching could be met. In Malle et al. (2006, p. 113) such an example is stated:

> *"In the adjoining graph you can see the chronological sequence of the concentration of alcohol in blood (in ‰) after the consumption of a particular abundance of beer with 5% or alternatively 3%. Describe the progress in your own words. Can you find similarities; in what way are they different?"*



Modelling a theoretical concentration

By discussing this example with students several different questions may arise. They could be taken as a starting point for modelling activities in class. Some questions which could be interesting for students are:

- We have learned that the reduction of alcohol in blood takes place at a constant rate: In every hour the same amount of alcohol is reduced! Is that right?
- In the case of the same amount of alcohol in blood the reduction of the 5%-beer is done faster than with the 3%-beer. Why?

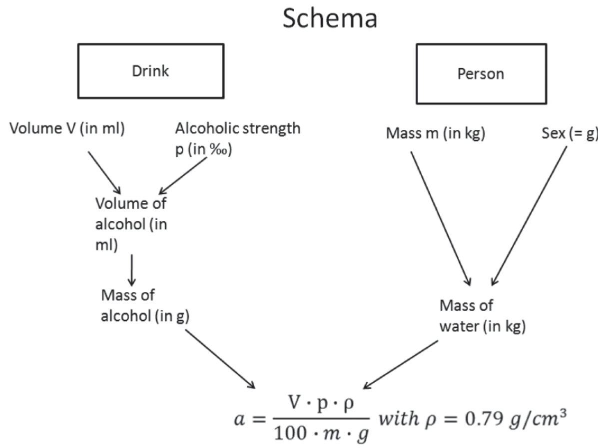


Figure 2—Schema of the Widmark-formula

- In the graph it is shown, if you consume alcohol once, it will never disappear in blood. Is that right?

This leads to the question: How does the degradation and absorption of alcohol take place in real life? By searching for an answer a lot of different and inconsistent results can be found — especially by using the world-wide-web. A definite answer cannot be given. Some more questions, which are of mathematical relevance, may help to find a possible answer and are the starting point for this modelling approach:

- Somebody is drinking ½ l of beer. How much alcohol will remain in this person's blood circle after 2 hours?
- Somebody has caused an accident. Two hours later the driver has to deliver a blood-sample. The alcohol test produced a blood alcohol level of 0.7. What was the alcohol-concentration in blood when the accident happened?

A first and very rough calculation can be done by the Widmark-formula (Widmark, 1932). The concentration of alcohol only depends on the person (drinking it) and the alcoholic drink itself. The parameters which are included are the amount of alcoholic drinks consumed and the alcoholic strength, as well as the mass of the person drinking alcohol and the sex. The age or the height is not considered, as in the Watson-formula (Watson et al., 1980). By following the schema (cf. Fig. 2) first considerations are possible. Think about a man ($g = 0.7$) with a mass of 63 kg, drinking half a litre of beer ($p = 4.5$ %). Taking this formula you will find that this person has a theoretical amount of $a = 0.4$ alcohol level in blood. Now it is possible to think about the reduction of alcohol. Some assumptions that are made are

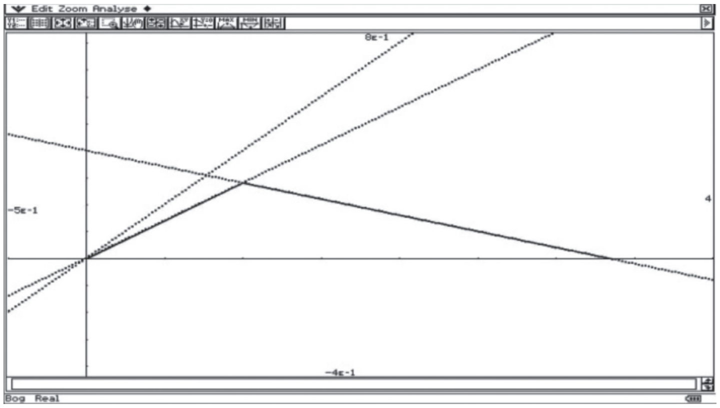


Figure 3—Linear processes

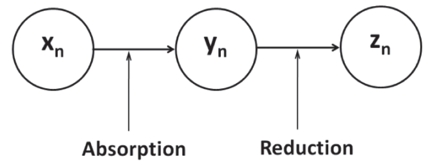


Figure 4—Absorption and Reduction

- the constant reduction of alcohol per hour (is an empirical value of 0.1 alcohol level/h $\leq d \leq$ 0.2 alcohol level/h depending on sex, constitution etc.),
- $b(t)$ as a linear model for the concentration of alcohol after t hours ($b(t) = a - d \cdot t \Rightarrow b(t) = 0.4 - 0.12 \cdot t$).

However, by using these assumptions it can be seen that those ideas may be easy but are not very realistic. In reality the process is more complex because consumed alcohol will not fade into the blood suddenly; it is mostly absorbed by the gastrointestinal tract and the absorption and reduction of alcohol are two overlapping procedures. So we have to think about using another approach.

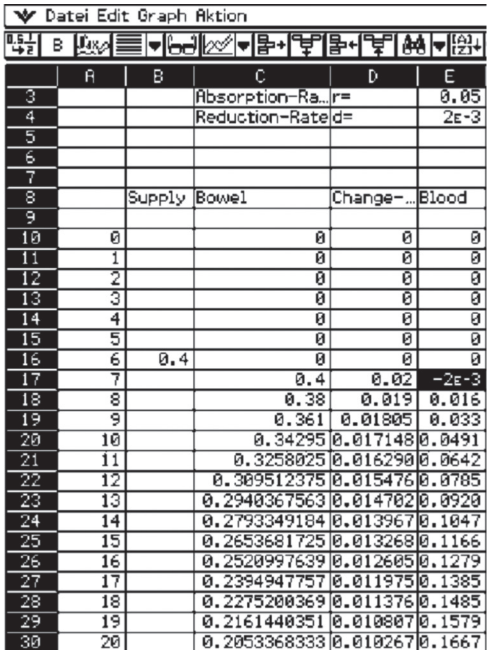
Absorption and reduction shown as a mathematical process

Considering how to show these two procedures leads to two different models. The first one describes the situation as a possible starting point to the problem; the second one leads to some interesting solutions.

First model – Linear approach

The time until the whole alcohol is faded into the blood is a well known empirical value – it lasts about 60 minutes. Knowing this value it is possible to make an assumption, which should help to find an appropriate linear model. It should be that in the same unit of time the same amount of alcohol is absorbed. Therefore the linear model described in Fig. 3 is constructed.

For the reduction of alcohol the linear function $b(t) = 0.4 - 0.12 \cdot t$ can be found. The absorption of alcohol can be shown through $a^*(t) = 0.4 \cdot t$. But alcohol is already reduced af-



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	A	B	C	D	E
3			Absorption-Ra…	r=	0.05
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8		Supply	Bowel	Change-…	Blood
9					
10	0		0	0	0
11	1		0	0	0
12	2		0	0	0
13	3		0	0	0
14	4		0	0	0
15	5		0	0	0
16	6	0.4	0	0	0
17	7		0.4	0.02	-2E-3
18	8		0.38	0.019	0.016
19	9		0.361	0.01805	0.033
20	10		0.34295	0.017148	0.0491
21	11		0.3258025	0.016290	0.0642
22	12		0.309512375	0.015476	0.0785
23	13		0.2940367563	0.014702	0.0920
24	14		0.2793349184	0.013967	0.1047
25	15		0.2653681725	0.013268	0.1166
26	16		0.2520997639	0.012605	0.1279
27	17		0.2394947757	0.011975	0.1385
28	18		0.2275200369	0.011376	0.1485
29	19		0.2161440351	0.010807	0.1579
30	20		0.2053368333	0.010267	0.1667

Figure 5—Spreadsheet solution

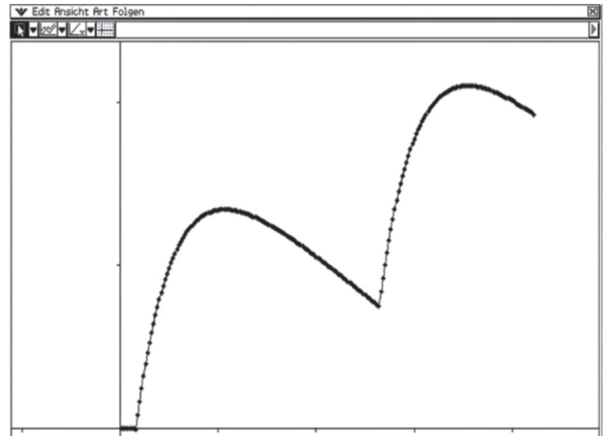

Figure 6—Graphical output of the solution

ter one hour, so we get the following (linear) function for the real absorption $a(t) = 0.28 \cdot t$. By looking at the graphs of these functions (Fig. 4) it is possible to see, that although the progress is similar to the one shown in the school-book example, the curve does not fit at all. So another model has to be found.

Second model — Semi-Linear approach

The absorption of alcohol in blood is realized by a process of diffusion. That means that a fixed portion r, which can be found in the gastrointestinal tract, is absorbed by a body. Not the whole amount of alcohol is absorbed in one hour, but nearly all of it, let us say about 95% (because $0.95 \approx 1-0.95^{60}$, that is about 5 % per min). Therefore the factor r is equal to 0.05. Now a discrete model with the parameters x_n (amount of alcohol in the gastrointestinal tract after n minutes), y_n (amount of alcohol in blood after n minutes) and z_n (reduced alcohol after n minutes) with the initial values $x_0 = a$ and $y_0 = z_0 = 0$ can be constructed. By thinking about the process the figure of absorption and reduction could be mentioned. The following equations show the results:

$$x_{n+1} = x_n - 0.05 \cdot x_n = 0.95 \cdot x_n$$

$$y_{n+1} = y_n + 0.05 \cdot x_n - 0.002$$

$$z_{n+1} = z_n + 0.002$$

And it is valid that $x_{n+1} + y_{n+1} + z_{n+1} = x_n + y_n + z_n = \cdots = a$, for every $n \in \mathbb{N}$.

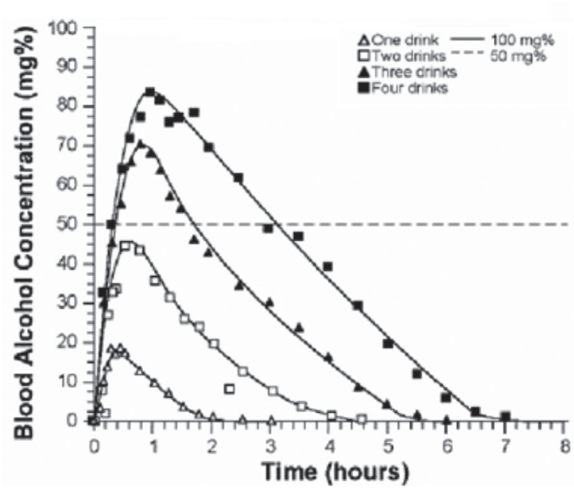

Figure 7—Alcohol concentration in Wilkinson et al. (1977, Fig. 6, p. 221)

By recognizing x_n as geometrical sequence and z_n as arithmetical sequence the possible solution $y_n = a \cdot (1 - 0.95^n) - 0.002 \cdot n$ can be found. By using technology this model can be simulated in many different ways. One is shown in Fig. 5 with a graphical representation in Fig. 6.

By thinking about such natural processes a lot of other models (e.g., continuous model or a model by differential equations) can be found, too. Of course a validation is necessary. It can be done easily with Wilkinson et al (cf. Fig. 7).

Third model — Differential equation(s)

By substituting the discrete parameter $n \in \mathbb{N}$ by $t \in \mathbb{R}$, in the found formula in the semi-linear approach — $y_n = a \cdot (1 - 0.95^n) - 0.002 \cdot n$ — a simple function which describes this process for every moment is found — $y(t) = a \cdot (1 - 0.95^t) - 0.002 \cdot t$.

This function, describing the process of absorption and diffusion of alcohol in blood can also be found by thinking about differential equations describing a mathematical model for this situation:

$$x'(t) = -c \cdot x(t)$$
$$y'(t) = c \cdot x(t) - d$$

Integrating the first differential-equation returns $x(t) = a \cdot e^{-ct}$. With this solution the second equation can be written as $y'(t) = c \cdot a \cdot e^{-ct} - d$. Now it is possible to integrate the second equation. Its solution can be found as $y(t) = -a \cdot e^{-ct} - dt + C$, with $C = a$ because $y(0) = 0$.

So the solution of this mathematical model can be written as

$$y(t) = a \cdot (1 - e^{-ct} - d \cdot t.$$

By taking appropriate values for c and d into account the function above can be found in another way.

Perspective — Bateman functions

As a perspective to similar processes students should discuss absorption- and diffusion-processes of medicaments in blood. If such processes are seen as diffusion-processes similar models can be created. One result, which is returned by solving a system of two differential equations, is the Bateman-function. The necessity of mathematics in different fields of interest can be shown evidently through such aspects. Therefore two assumptions have to be done, so that $x(t)$ shows the concentration of medicament in the gastro-intestinal-tract and $y(t)$ describes the concentration of a medicament in blood. The equations describing such processes have to consider the mass which was eaten a, the absorption-velocity b and the reduction-velocity c in blood. So the equations for such a diffusion-process can be written like the following.

$$x' =- c \cdot x \qquad x(0) = a$$

$$y' = c \cdot x - b \cdot y \qquad y(0) = 0$$

Such a system of two differential-equations of first order can be solved easily by transforming it to a differential-equation of second order. Solving such an equation is complex for students in the beginning, especially when technology is not allowed to be used. But a little artifice can help. The first equation of the system has a simple and unique solution $x(t) = a \cdot e^{-ct}$. Therefore the other equation can be written as $y'(t) = -b \cdot y(t) + ca \cdot e^{-ct}$, which is an inhomogeneous linear differential-equation. By multiplying this equation with the factor e^{bt} another equation $(y(t) \cdot e^{bt})' = c \cdot a \cdot e^{(b-c)t}$ is returned. By integrating this equation and differentiating between $c \neq b$ and $c = b$ a solution can be found.

1. $c \neq b$

$$y(t) \cdot e^{bt} = \frac{a \cdot c}{b - c} e^{(b-c)t} + C$$

$$y(0) = 0 \Rightarrow C = -\frac{a \cdot c}{b - c}$$

$$\Rightarrow y(t) = \frac{a \cdot c}{b - c}(e^{-ct} + e^{-bt})$$

2. $c = b$

$$y(t) \cdot e^{ct} = c \cdot a \cdot t + C$$

$$y(0) = 0 \Rightarrow C = 0$$

$$\Rightarrow y(t) = a \cdot c \cdot t \cdot e^{-ct}$$

The model of the Bateman-function (cf. Garrett, 1994) is a very simplified model for the absorption and reduction of a certain substance in blood depending on the time. But for a first approach to modelling activities in classes including cross-curricular aspects of biology it can be introduced very impressively.

Summary

The discussed example(s) show(s) that technology can be helpful at any step of the modelling circle. Using technology does not only create an appendix to the modelling circle (see Fig. 1), but influences each part of the circle.

Discussing such a process-oriented task in education should be enforced; especially cross-curricular aspects should be encouraged, so that students learn to cross-link their thinking. The motivation for mathematics can be enhanced, the necessity for mathematics in education is becoming evident and even a political dimension can be mentioned in mathematics education, when modelling activities are considered in classes.

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The Other Side of the Coin-Attempts to Embed Authentic Real World Tasks in the Secondary Curriculum

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Co-authors Dawn Ng
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Abstract Research studies involving implementations of two different models of curriculum embedding intended to bring authentic real world applications into secondary school curricula are discussed The first has a system wide focus emphasising using an applications and modelling approach to teaching and assessing all mathematics subjects in the last two years of pre-tertiary schooling This relatively well established implementation resulted from a response to influences prevalent in society in the late 1980's The second is through interdisciplinary project work from upper primary through secondary where the anchor subject could be mathematics This newer implementation is yet to realise its full potential In both assessment, gestation time of new ideas and rapid technological changes are drivers and enablers of curriculum change.

According to the EIMI-Study discussion document “at all educational levels, students typically have been taught the tools of mathematics with little or no mention of authentic real world applications, and with little or no contact with what is done in the workplace” (Damlamian & Sträßer, 2009, p. 525). There are, however, instances of secondary school systems (e.g., the Queensland senior secondary mathematics curriculum in Australia and the Ontario secondary mathematics curriculum in Canada) where this is not the case and more recently another move by educational systems in many parts of the world such as Singapore (Ng, 2009; Wong & Lee, 2009), Germany (Höfer & Beckmann, 2009), and Denmark (Andresen & Lindenskov, 2009) to expect that their teachers will include real world tasks in mathematics and in interdisciplinary and multidisciplinary activities which involve mathematics. In fact, the secondary mathematical landscape seems more like a system of beautiful inland lakes and back waters behind an ebbing and rising tide on the more visible coastline. Since the 1960s there has been a continual ebb and flow in secondary school curricula with real world applications and modelling gaining in prominence and then retreating in some systems (e.g., Victoria see Stillman, 2007) at the same time as the opposite is happening in other educational systems (e.g., Queensland see Stillman & Galbraith, 2009). As a backdrop to this there are the deep backwaters where no change has happened for many years where mathematics is indeed taught as “a dead science and a finished product” (Damlamian & Sträßer, 2009, p. 530) but there are also, but much less frequently, beautiful lakes where innovative curricula have been promoting realistic applications and modelling as an integral part of the curriculum for many years and teachers are expected to do this and actually do. In this paper we address the issue of curriculum change where the purpose is (a) to include real world applications and modelling in the mainstream mathematical course or (b) through interdisciplinary activities with mathematics as the anchor subject. The state of play with respect to industrial applications in these settings will also be explored.

Background

During the 1960s and early 1970s marked changes occurred in the social fabric of many countries all over the Western world. Youth and students in many countries revolted against what they saw as the unnecessarily restrictive shackles of the values of those in authority. A long lasting ideological legacy of this time has been that:

> Young people are no longer prepared to rely blindly on authorities or to follow directions, let alone orders, without asking critical questions. They demand arguments and good reasons to be motivated (convinced, stimulated, persuaded) to accept and adopt activities. (Niss, 1987, p. 491)

At both the secondary and tertiary levels of education students began to actively, but sometimes passively, question the relevance of the content and form of the mathematics they were studying; "and right from the beginning relevance was interpreted by students, teachers and educationalists as *applicability*" (Niss, 1987, p. 491). The nature of this applicability varied greatly ranging from general societal applicability or applicability in other disciplines to applicability in the everyday life of the students or their expected future societal roles but from the students' perspective this applicability was what was meaningful to them.

At the same time there were other influences such as employer dissatisfaction with graduates of mathematics departments of universities (see Gaskell & Klamkin, 1974; McLone, 1973) with comments such as "the request most often made is for mathematics graduates with an appreciation of the applicability of their subject in other fields and an ability to express problems, initially stated in non-mathematical terms, in a form amenable to mathematical treatment with the subsequent re-expression in a readily understandable form to non-mathematical colleagues ... this is not often found in those currently graduating" (McLone, 1973). Educational reform movements were another influence.

In their *Everybody Counts: A Report to the Nation on the Future of Mathematics Education*, the National Research Council (1989) in the U.S. flagged that in order to change mathematics education dramatically before 2000 it would have to go through several transitions. One of these transitions was a shift from an " emphasis on tools for future courses to greater emphasis on topics that are relevant to students' present and future needs" (p. 83). Modelling was identified as one of the " distinctive modes of thought" (p. 31) offered by mathematics and mathematics was said to play a special role in education because of " its universal applicability" (p. 31). However, " Model-building, which facilitates systematic, structured understanding of complex situations" (p. 83) was identified as an additional topic area rather than a framework for teaching the application of mathematics.

Systemic Focuses on Modeling and Applications in Mathematics Curricula

From this milieu of influences, arose genuine attempts in educational systems in several countries to include real world applications of mathematics and mathematical modelling more centrally within the secondary mathematics education curriculum in the late 1980s and early 1990s. Some of these innovative curricula enjoyed spectacular success for a short time and then were lost for various reasons (see Stillman, 2007, for an example). Others have persisted to this day given new impetus as digital technologies became to be seen as a means of enabling modelling (Stillman & Galbraith, 2009; Suurtamm & Roulet, 2007).

In this paper we provide details of one substantial initiative covering 20 years — mathematical modelling in Queensland senior secondary mathematics courses. Applications and mathematical modelling were first introduced in 1990-1 within senior mathematics curricula in Queensland in a limited number of schools; however, by 1995 all secondary schools across the state were using such syllabuses. The development of the initiative from the perspective of implementing teachers and key curriculum figures responsible for its introduction and evolution has been the subject of a longitudinal research study, *Curriculum Change in SecondaryMathematics* (CCiSM) which is on-going.

The objectives of the 1989 Trial/Pilot Syllabuses included the indentification of the assumptions and variables of a mathematical model, formulation of a model, derivation of results from a model and interpretation of these in terms of the given situation. Mathematics C, the most challenging Mathematics subject, also required the modification and validation of the model. While syllabus refinements have taken place since and some changes of emphasis have occurred, the essence of the mathematical modelling component has remained essentially stable for over a decade although different emphases in wording and in placement of modelling and application aspects in organising categories for the general objectives has brought to the fore different aspects in implementation over this time frame.

The Queensland system of assessment at the senior secondary level is entirely school based which means that the production of assessment tasks and the awarding of levels of achievement is in the hands of teachers in individual schools, with panels at district and state levels performing critical reviewing roles to assure comparability of outcomes across schools and regions. In keeping with the school-based nature of the Queensland context, individual schools and teachers design individual work programs (including assessment tasks) under the syllabus umbrella.

The tasks vary from applications of mathematics to open modelling tasks incorporating real situations such as sagging power lines, burgularies, cars overtaking and accidents from building implosions. Applications require students to carry out specified mathematical calculations but in a real world context without any assessment of whether the proposed models make sense in this real problem context. In these application examples essential activities, central to modelling, are absent. Their purpose is purely illustrative of where mathematics is used. Modelling tasks, on the other hand, are much more open, with the making of assumptions, choice of mathematics, and interpretation in context being aspects which students carry out. The contexts used vary greatly depending on the experiences of the teachers or their network of sources. Queensland is a large state with geographically isolated schools and with many schools where there is only one qualified mathematics teacher at this level so some teachers do not have this network to draw on.

When applications and modelling were first introduced into the Queensland system it required " a major shift in thinking that it was maths as it is used and get out there and use it" (QKTG1, November, 2005 CCiSM key teacher interview). The focus on modelling was generally not evident in implementation in the initial years although applications were very much present particularly in assessment (Fuller, 2001; Stillman, 1998) which now included a substantial weighting towards successful solution of application or modelling tasks in examination and alternative assessment contexts. This was attributed by many teachers and curriculum figures interviewed for the CCiSM project to an initial lack of understanding by teachers as to what was the difference between these or a reluctance to take up modelling as it was considered too time consuming or too removed from their current practice. However, for others the intention to include both was there and modelling was seen as developing over the senior years as this teacher points out:

> I think there was a sense that both were important but a larger scale modelling of a situation is obviously going to be far more complex and demanding but at the end of the day these are school students. We are still just giving them the building blocks where they could perhaps apply the Maths in a trivialised or smaller problem that was a stepping stone towards modelling with the intention, I suppose, that by Year 12 we would be able to put a lot of that together and do some genuine, or more genuine sorts of modelling ... like that where they have built up the skills in a number of areas. (QKTG1, November, 2005 CCiSM teacher interview)

Initially, many teachers had to be convinced of the efficacy of the change from purely abstract mathematics with a major focus on concepts, skills and processes with the question being raised: " Is it really going to make any difference to the end results?" However, teachers quickly came to realise there were many benefits and the major ones, as seen from one teachers's perspective, were:

> I think it gives students opportunities to (the word " apply" comes to mind) use the skills in something other than contrived or abstract situations so they can actually have experiences of their maths being used so there is some relevance either to the real world or where it might take them or some glimpse of modelling and how maths is used. But also to encourage, I suppose you would call it, the transfer of knowledge so that familiarity is a word that became quite bandied around a lot. It was all very fine to use your mathematical techniques, procedures or whatever in a familiar scenario but many students really struggle when it is slightly unfamiliar or they are not sure what mathematical tools to use and obviously they need practice at doing that. (QKTG1, November, 2005 CCiSM teacher interview)

Over the years modelling has gained a toe-hold in classroom practice through a " slow evolutionary process" (QKTG4) where genuine modelling is carried out " in some schools more than others" (QKTG1). This evolution has been strongly supported by the assessment monitoring panel system and insistence on use of alternative assessment. Technology was also another driver rapidily enabling progress in what was achievable with using real world contexts in the classroom:

> I have worked with some people who are quite passionate about it and they have actually taken modelling to quite an advanced level in that their students are really focussed on technology. ... they have really looked at doing all sorts of things using graphics calculators and computers, for example, to do a whole range of different things with a given scenario. (QKTG1, CCiSM interview)

Thus, it would be fair to say that teachers in the senior secondary area in Queensland do use applications and modelling of real situations in their teaching and assessment. The scope of the real situations used in these is limited to the experiences of the teaching staff at a school and their network of colleagues (Stillman & Galbraith, accepted). It would thus appear to be important that if these teachers' awareness of industrial applications and modelling situations is to be raised that case studies from industry be made available to teachers at an accessible level as has happened in the past (e.g., Project CAM — Careers and Mathematics) and that teachers and teacher educators be involved in programs where they experience industrial applications first hand. In addition special enrichment programs for secondary students (e.g., AB Paterson Modelling Challenge in Queensland) need to be fostered with industrial mathematicians contributing to these.

An Interdisciplinary Project Approach to Making Connection with the Real World

A second approach to involving the tackling of tasks with real world elements in the mathematics classroom has been to include interdisciplinary activities which involve mathematics. In 1999, the Singapore Government accepted the recommendation of the Committee on University Admission that project work be included as part of university admission. The purpose of such project work according to this committee is to "inculcate and measure qualities including curiosity, creativity and enterprise. Projects also nurture critical skills for the information age and cultivate habits of self-directed inquiry" (Ministry of Education, 1999). Interdisciplinary project work was then introduced as an educational initiative in primary, secondary, and pre-university institutions in 2000 in order to prepare students for contemporary workplace demands (Quek et al., 2006). The promotion of flexible and adaptive application of mathematical knowledge is thus an objective of interdisciplinary projects involving mathematics. A second objective is to ensure explicit links between different subject knowledge are made so students learn to " appreciate the inter-connectedness of disciplines and see the relevance of classroom learning to their current or future interests" (Chan, 2001, p. 1). Following a trial in junior colleges, it was decided to include performance on project work as an admission criterion to local universities from 2005.

Ng (2009) researched the implementation of a design-based interdisciplinary project she designed involving mathematics, science, and geography. Sixteeen classes of students (N = 617) from grades 7 and 8 (aged 13–14) in two educational streams (high and average) across three Singapore government secondary schools were involved in this study.

Ten student groups from these classes were the focus of intensive observation and data collection. The project was conducted through weekly meeting sessions facilitated by teachers in normal curriculum time over a 15-week period.

The aim of the project was to enhance students' environmental consciousness and explore methods of environmental conservation in the way Singaporeans live. It required student-groups to decide on the type, purpose, location, and facilities of a building of their own design where they explored environmentally friendly features to include before making physical scale models of their buildings from recycled materials, based on scale drawings. The context of designing and building scale models of buildings was considered to be within the life experiences of students. Mathematical tasks in the project included: (a) *decision making* about the various aspects of the building (i.e., size, dimensions, location, purpose, environmentally friendly features, and design), (b) *cost of furnishing* and fitting out a selected area in the building (i.e., budgeting including flooring, painting, choice of electrical appliances, and furniture), and (c) hand-drawn *scale drawings* of the actual building with the number and types of drawings decided by students.

Some students in the focus groups did not perceive the usefulness of school subjects during the project activities. Certain students did not perceive the project task as mathematical; thus, they did not consider using mathematics for the project with the consequence they reported during interviews that mathematics was not useful for the project. These students did not participate when mathematical knowledge was being applied by others in their group or did not draw on their knowledge and skills for the project in a connected manner so some expected mathematical knowledge and skills were not applied. Several student-group members did not make conscious and consistent efforts to monitor the accuracy and reasonableness of their application of mathematical knowledge and processes. There were also students who had difficulty identifying connections between mathematical concepts such as choice of scale and area estimations during the application of mathematical knowledge for real world purposes.

To a large extent, coverage of mathematics by each group during the tasks depended on sensitivity to task features, engagement with the task, task scaffolding, and a shared repertoire of mathematical concepts, skills, and domain knowledge such as environmentally friendly features of building design, among other factors. It was assumed that students working in groups would complement each other on the aspects mentioned and promote higher quality mathematical outcomes but this did not happen. Such complementarity does not happen automatically but depends on close monitoring by facilitating teachers. At times, students struggled with whether and how to engage mathematically with the task. Yet, too regimented scaffolding might not do justice to the open-ended nature of such tasks for creative problem solving and inter-connected meaningful learning. It is thus an open question as to how teachers achieve a " balance" in scaffolding during such tasks in order to retain the mathematical rigour of the tasks in their eyes.

In addition, there was limited activation of real-world knowledge by the focus groups despite the presentation of the project within a real world setting. Only three groups, all from high stream, applied real world knowledge during all three mathematical tasks. On the positive side, there was at least one member within each focus group who attempted to remind other group members of real-world constraints during mathematical decision making. In addition, all but one student group used some form of inter-subject connections during the project in contrast to the findings of Chua (2004) who investigated project work outcomes in a primary setting.

It would appear that at this stage the weighting of project work for tertiary entrance (i.e., one-ninth of admission score) or its remoteness from grades 7 and 8 is not sufficient incentive for many students and teachers to view interdisciplinary projects as an integral part of teaching, learning and assessment (Quek & Fan, 2009) and of equal importance to school based end-of-term and end-of-semester tests which are used within schools to determine progress, selections or placement (Quek & Fan, 2009). Perhaps in the future when students have more experience of interdisciplinary project work and more recent changes have had time to be established, there is a chance for a breaking away by students and teachers from a focus on being exam smart and showing little interest in applications. With the new emphasis on application and modelling advocated at the pre-university level and the introduction of technology into primary and secondary school, it is likely "students will be able to solve real-world problems with messy data and to undertake mathematical modelling" (Wong & Lee, 2009, p. 36) in the future. As has been found in other implementations that have successfully incorporated an applications and mathematical modelling approach to teaching and learning mathematics, there needs to be an "extensive period of exposure and gestation during which teachers" develop and share new ideas and are allowed to build the momentum required to experiment and change practice (Suurtamm & Roulet, 2007, p. 495). For teachers to be able to incorporate the modelling of industrial problems into project work, however, they need support in broadening their awareness of industrial practices and industrial mathematics related to the production of goods and services in society of relevance to the curriculum they are expected to teach.

Concluding Remarks

The two examples we have presented give a glimpse of the complexity of the vexed question of why something that seems so natural to others outside of schooling to be embedded in the mathematics curriculum in a meaningful manner, namely, "authentic real world applications" (Damlamian & Sträßer, 2009, p. 525), has been ignored in many systems, given lipservice by teachers when it is embedded in curriculum documents, or had a rather chequered career in many parts of the world. For curriculum change in this area to involve industrial mathematics examples it appears that both support from industrial mathematicians and the allowance of time for change to take hold in the mindset of the implementers and consumers of the curriculum, namely teachers and students, are necessary before the potentials expressed in curriculum documents are realised. Only then can it result in " renewal of the profession as a whole" (QKTG1, CCiSM teacher interview).

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Using Popular Science in a Mathematical Modeling Course

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Abstract A student-driven approach to a mathematical modeling course is presented. Students choose their topic based on an article from a popular science medium, spend time researching the background information of the article, constructing a mathematical model, solving it, and either writing a report or giving a brief presentation before a critical audience. Through this experience, students learn the underlying techniques of formulating relevant mathematical models. One example is presented, which focuses on the treatment of post-operative pain using patient controlled analgesia.

Introduction

How do you know whether a story in the media is true? How do you evaluate quantitative evidence that is presented? How do you judge the conclusions drawn from this quantitative evidence? What skills does a person need in order to approach these tasks independently and confidently? The ability to mathematically model a problem, formulate a solution, and articulate this solution in the problem's original context effectively answers these questions.

The skill to translate a contextual problem into a quantitative framework is fundamental for citizens to perform their role in a technology-based democratic society. Quantitative courses in different disciplines focus on the mathematical methods that elucidates the underlying concept, such as in economics, engineering, or the sciences. However, the translation from context to mathematics has already taken place. Students see the mathematics involved as a tool for solving a problem in a particular topic and may not understand the connection of the mathematics to a significantly different context.

Ever since the first "study group" at Oxford University in 1968, applied mathematicians have been developing mathematical models to solve industrial and scientific problems. More recently, these groups have expanded to several countries, including the Mathematical Problems in Industry Workshops (MPI) in the US and the PIMS Industrial Workshop in Canada. The Consortium for Mathematics and its Applications (COMAP) has been running the undergraduate Mathematical Contests in Modeling (MCM) for nearly twenty years. The common format for all of these programs is that a new, original problem is presented in a scientific or engineering context, and the contestants have a finite time (within five days) to formulate a mathematical model, solve it, and present the work in the original context of the problem. This format is an engaging (and sometimes enraging) experience to any who have participated. If communicated to undergraduate students in a cogent fashion, such an open-ended experience describes the discipline of applied mathematics well.

> QUESTION: Can a mathematical modeling course be run in this format, and if so, how does the student benefit?

Integrative experiences as a pedagogical technique have been of interest in engineering education (see Somerville et al. (2005) for a review). Our Practicum to Industrial Mathematics course begins with students opening that day's the Science Times section of the New York Times on a Tuesday morning and read over the articles. From this selection of articles, groups of students discuss the different aspects of the topic that are interesting to them. By the end of the same class, and with the instructor's input, the student team formulates a mathematical question that, if solved, would provide some insight into the topic itself or some related problem. This course has been implemented at Olin College, Wellesley College and WPI.

This process has three benefits for the students. The first is that the question is necessarily current: the matter is a current topic in the popular media. The second is that they provide their own motivation for the problem and its solution, and they are responsible for its success. This is *their* problem, and not one posed by the instructor. The third is that the students understand how to formulate a tractable question based on a practical set of criteria.

The class time between the assignment of the problem and the report due date is spent with teams working on their projects. *There are no formal lectures.* Each team is guaranteed to have a specific amount of time with the instructor without interruption, typically around 15–20 minutes. The students are advised to come prepared with a small list of the questions, prioritized from most to least important in their view.

Our example below describes a model of a technique in pain management called patient-controlled analgesia (PCA). The final model is a simple three-dimensional system of first-order nonlinear equations. However, the choice of the nonlinearity leads to a stable fixed point that is not present in the linear model (all previous modeling attempts from a subsequent literature search at representing pain in this situation appear to be linear). The qualitative results from this model agree with what can be found from clinical studies of this pain management technique. The fact that the student has multiple opportunities to discuss the impact of modeling assumptions in their quantitative description is a significant benefit to this approach.

Example: Patient Controlled Analgesia

There are two principal methods of administering intravenous pain medication. The first involves the administration of the drug according to a regular medication schedule called continuous analgesia (CA). The second gives control to the patient through a process called Patient-Controlled Analgesia, or PCA. The advantages of CA include the ability of the caregiver to limit the concentration of the pain medication given to the patient in addition to better pain management overall (Sucato *et al.* (2005)). However, when the patient is susceptible to side effects, the response time by staff predicts the renewal of the drug supply. With PCA, the patient is allowed to self-administer a dose by simply pushing a button whenever the pain is sufficiently great. The overall dose is limited by the total amount of drug that can be administered over a typical period: if this dose is administered before the period is complete, the button ceases to function. Since the dose is limited, the degree of pain management is suboptimal by design, but complications due to side effects are greatly reduced.

PCA is also used as a pain management treatment for chronic conditions with the use of opiates (such as morphine) as the medication. These conditions are notable since the physical source of the pain may either not have a definite cause or there is no other treatment option to eliminate the pain source. Further, morphine effectiveness varies greatly from patient to patient, which makes it difficult to prescribe a regular dosage schedule that will treat a particular patient's pain without causing undesirable side effects.

Pain measurement is still primarily performed by asking the patient to rate their pain level on a scale from 1 (no pain) to 10 (as much pain imaginable) with no agreement on a uniform metric. One scale that is used for children is the Wong-Baker faces scale shown in Figure 1 with a 0–5 scale, but a 0–10 scale is also used, such as in the visual analog scale (VAS). These metrics are by definition qualitative: they are based on the patient's perception of the pain level they are experiencing, and not a measure of the strength of the pain source (which is called discomfort in the literature). Of interest in models of PCA is the choice of a pain measure 'pang', defined as the amount of pain a PCA patient experiences such that they would push the button, when the medication is locked, once per second (Jacobs *et al.* (1985)). Again, this unit relies on the patient's interpretation of the pain being experienced, which varies from patient to patient. If professionals must design a medication regimen keyed to patient feedback, perhaps eliminating the one degree of separation and giving control of the medication to the patient could reduce the complexity of the system and improve the overall treatment of the patient's pain.

The modeling of PCA has centered on using linear control systems with a stochastic pain source (Jacobs *et al.* (1985,1995), Liu et al. (1990), Chase *et al.* (2004), Shieh *et al.* (2004, 2007)). Central to these studies is the connection of how this source of pain is perceived by the patient. Liu and Northrop (1990) make the following distinctions of pain. Discomfort arises from the actual source of pain, which is transmitted from the pain location along nerves to the brain. This level of discomfort is called neuro*logical pain*, Q. However, the medication acts to prevent this pain signal from being perceived by the patient, and the patient's pain response is a measure of perceived pain, P. They modeled the perceived pain as a linear combination of the neurological pain and its rate of increase over time. In our study, we want to understand the implications of these two modeling assumptions: the metric used in measuring pain levels and how pain is mitigated through PCA. One case of interest is that of *chronic pain*, during which the patient reports perceiving pain but no source of discomfort can be identified.

In the following, we make the assumptions that the medication has the same effect on pain regardless of food intake, sleep, time of day or mood. Further, we assume the patient will initiate PCA at exactly the same pain level every time. The time for the medication to take ef-

Figure 1—Wong/Baker Faces scale. This graphic is used by doctors to determine pain levels in children (see the MD Anderson Cancer Center website http://www.mdanderson.org/pdf/pted painscale faces.pdf for more information). Note that the numerical values of these faces can range from either 0–5 or from 1–10.

fect once it is administered is assumed to be instantaneous, and the experiencing of intense pain over time will increase a patient's sensitivity to pain for some period of time thereafter.

Pain Model

We model pain as a system of differential equations. The principal quantities are the amount of perceived pain P and the amount of neurological pain Q. We assume that the perceived pain is a linear combination of the neurological pain and its rate of change

$$P = b_1 Q + b_2 \frac{dQ}{dt} + D,$$

where b_1, b_2 are phenomenological constants. In Liu and Northrop (1990), this equation is forced by the level of discomfort D, but for this section we are concerned with the modeling of how a patient responds to an initial pain state ($D = 0$). The rate of change of perceived pain decays based on healing and a second term from the sensitivity to neurological pain

$$\frac{dP}{dt} = -a_1 P + a_2 Q.$$

Note that $a_2 > 0$ implies that the patient is more sensitive to pain if pain has been experienced in the past, while $a_2 < 0$ implies that the patient develops a tolerance to the previous pain levels.

Note that both of these equations are effectively compartmental models of the underlying biological phenomena which senses pain and transmits the information to the patient. Mathematically, they are appropriate provided that the level of pain is not considered "too large", or deviations of pain around some known constant level remains small. However, no deterministic representation of these processes exists at this time. There is no measure of what level of pain is "too large", nor an understanding of when these models break down.

To simplify the number of constants, we scale

$$P = p\,,\ Q = qQ_0\,,\ t^* = tT_0$$

where p, q, t are dimensionless, and results in the equations

$$\frac{dp}{dt} = -\alpha p + \beta q; \qquad \frac{dq}{dt} = p - q,$$

where $\alpha = a_1/b_2$, $\beta = a_2 Q_o T_o / P_o$, and $T_o = 1/b_2$, $P_o/Q_o = b_2/b_1$. A linear stability analysis of this system yields growth rates λ about the steady state $p = q = 0$ as

$$\lambda = -\frac{\alpha + 1}{2} \pm \sqrt{(\alpha + 1)^2 + 4\beta}\,.$$

Hence, if $\beta > 0$, $(p, q) = (0, 0)$ is a saddle point and the state is unstable to linear disturbances. If $\beta < 0$, then $(p, q) = (0, 0)$ is stable. This is consistent since a patient that would become more sensitive to pain the more pain that is experienced results in a positive feedback loop. One model of chronic pain one can introduce a source term in (5) as a measure of discomfort and find a local stable solution. However, this model assumes that a physical source of pain is present.

However, the measurement of pain on a linear scale suggests that the patient senses relative changes in pain on a linear scale, or

$$\frac{\Delta P}{P} \approx k.$$

However, scales that relate to human sensation such as touch (Richter), sound (decibel), sight (brightness) and smell (European Odour Unit) are proportional to the logarithm of a known quantity. All of these units are logarithmically based on the amplitude of the original stimulus. This suggests that relative changes in pain can be modeled with a logarithmic rule

$$\frac{\Delta P}{P} \approx k \log\left(\frac{P}{P_0}\right),$$

where $P_o \neq 0$ is a characteristic pain level reported by the patient. The system (4),(5) then becomes

$$\frac{dp}{dt} = -\log p + \beta q; \qquad \frac{dq}{dt} = p - q.$$

However with pain measurement, we note that there is a filter between the source of pain and what the patient perceives. Note that this representation is a different compartmental model for the the interaction between neurological and perceived pain. It is not the unique representation of how these two quantities are related, but one in which we feel incorporates how human perception is related to changes in external stimuli. The results that follow could be simulated with different nonlinear models, such as a quadratic function of p in place of the $\log p$ term.

The system (6),(7) exhibits a *stable* solution for $p = q = e^{\beta/\alpha}$ for all values of α and β. This result is convenient in the study of PCA management, since this pain level is independent of the strength of the discomfort. A sample phase plane of the system (6),(7) is shown in Figure 2 for $\alpha = 1$, $\beta = 2$. The arrows denote the phase field, the dashed blue curve corresponds to the nullcline for (6) and the solid red curve corresponds to the nullcline for (7). The dot corresponds to the equilibrium pain level.

PCA Model

To model the management of chronic (perceived) pain, we look at the diffusion of a medication concentration C within the bloodstream, that is introduced into the patient through the nonlinear source Φ. In dimensional form

$$\frac{dQ}{dt} = b_1 P - b_2 Q; \qquad \frac{dC}{dt} = -b_3 C + \Phi(t, P_t, d, Td, T_l)$$

where Φ is a piecewise expression representing the drug added by the PCA machine. When P becomes greater than the pain threshold P_t, the patient initiates a dosage cycle wherein a fixed amount of medication (usually an opioid such as morphine) at a constant rate is introduced into the bloodstream followed by a set lockout period. This behavior can be modeled as

$$\Phi = \begin{cases} \dfrac{d}{T_d} & 0 < t < T_d, P = P_t \\ 0 & T_d < t < T_d + T_l \text{ or } P < P_t \end{cases}$$

where Φ is the pain threshold at which the patient will initiate PCA, d is the dosage of the drug, T_d is the time over which the dose is delivered to the patient, and T_l is the lockout period during which the patient cannot administer more of the mediation.[1] Using the scaling found in (3) and setting the concentration scale to be $C_0 = T_0 d / T_d$, we arrive at the dimensionless system

$$\frac{dp}{dt} = -\alpha p \log p + \beta q - \gamma c; \quad \frac{dq}{dt} = p - q; \quad \frac{dc}{dt} = -\sigma c + \phi(t)$$

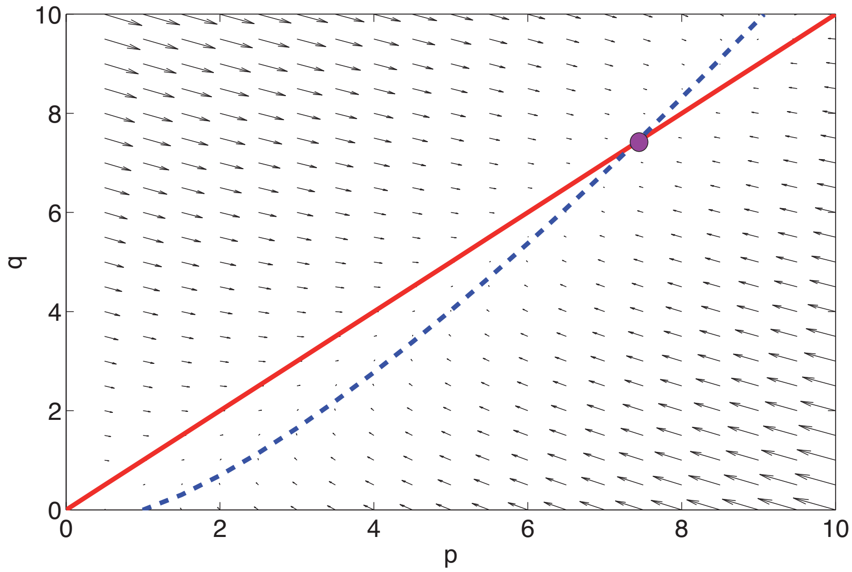

Figure 2—Sample phase plane diagram of (p, q) defined by (6),(7) with $\alpha = 1$ and $\beta = 2$. The blue dashed curve corresponds to the p nullcline, the solid red curve corresponds to the q nullcline, and the solid purple dot is the equilibrium point. This model represents a chronic pain level independent of a prescribed discomfort level.

where $\gamma = T_0^2 d/(T_d P_0) a_3$ measures the efficacy of the drug for pain relief, $\sigma = T_0 b_3$ is the diffusion rate of the drug in the bloodstream, and

$$\phi = \begin{cases} 1 & 0 < t < \tau_d, p = p_t \\ 0 & \tau_d < t < \tau_d + \tau_l \text{ or } p < p_t \end{cases}$$

Note, however, that the treatment requires a criterion for the maximum concentration in the patient's bloodstream. With this choice of scaling, and from (13), this restriction fixes $\sigma = 1$ and the maximum nondimensional concentration is given by $c = 1$.

We note that in the case $\gamma = 0$, the drug has no effect on pain management. However, we can show that the concentration levels are always bounded below the critical level $c = 1$. Consider the case when $c(0) = A_0$, then the solution of (13) over $0 < t < \tau_d$ gives

$$c(t) = \{1 - (1 - A_0)e^{-t}\}.$$

At the time $t = \tau_d$, the source ϕ turns off, and the resulting initial-value problem becomes

$$c(t) = -c(t), \; \tau_d < t < \tau_d + \tau_l, \; c(\tau_d) = B_0,$$

with $B_0 = [(A_0 - 1)e^{-\tau_d} - 1]$. The concentration A_1 at the end of this period becomes

$$A_1 = B_0 e^{-(t-\tau_d)}\big|_{t=\tau_l} = [(A_0 - 1)e^{-\tau_d} - 1]e^{-\tau_l}$$

However, this gives a recursion relation for A_n after n lockout periods as

$$A_{n+1} = [(A_n - 1)e^{-\tau_d} - 1]e^{-\tau_l}$$

which in the limit as $n \to \infty,\ A_n \to A,\ B_n \to B$, where

$$\overline{A} = \frac{e^{-\tau_l}(1 - e^{-\tau_d})}{1 - e^{-\tau_d + \tau_l}},\ \overline{B} = \frac{1 - e^{-\tau_d}}{1 - e^{-(\tau_d + \tau_l)}}.$$

Note that both $\overline{A} < 1,\ \overline{B} < 1$, implying that the drug concentration level of the patient never exceeds the maximum value $c = 1$. This factor implies that the patient never receives the equivalent dose for a continuous analgesic treatment, and has heightened discomfort. However, the patient will be less likely to experience side effects from the drug compared to a continuous treatment. This result is found in clinical findings comparing these two pain management schemes for postoperative pain (Sucato *et al.* (2005)).

We close our module by investigating (11)-(13). Note that there is a significant difference in the time scales involved in healing and memory of neurological pain affecting perceived pain compared to that of the drug efficacy, and the diffusion of the drug in the bloodstream. We anticipate that healing has a time-scale of a day or two compared to minutes for that found in common opioid. We assume the typical pain level shown in Figure 2 as a reference, but assume that $\alpha = 0.01$, $\beta = 0.02$. The times $\tau_d = 1$ and $\tau_l = 15$ were chosen based on typical operation settings, and an arbitrary pain threshold $p_t = 6$ was chosen to instigate the PCA treatment. The final free parameter is the efficacy of the drug γ.

We simulate the model using Matlab's standard numerical integration package ode45. We plot four examples of this simulation in Figure 3 for $\gamma = 0.1$, $\gamma = 1$, $\gamma = 2$, $\gamma = 5$ respectively. The top figure corresponds to the (p, q) phase-plane plot of the transient (red solid curve) compared to the q nullcline $p = q$ (dashed blue curve). The bottom figure shows the concentration level over time of the simulation. In all the cases, the transient approaches a periodic-solution where the pain is reduced significantly during the drug infusion and then gradually increases along the nullcline as the concentration of the drug decays in the bloodstream. In the case of $\gamma = 0.1$, the final periodic state has not been achieved.

In summary, we have developed a nonlinear model for patient-controlled analgesia. This model incorporates a pain model where the perceived pain depends on the impact of neurological pain linearly, but its rate of change depends on the perceived pain nonlinearly. This pain model can then exhibit equilibrium pain solutions that may correspond to chronic conditions where no physical cause of the pain is found. The analgesic scheme is found to be suboptimal in that the desired concentration for treatment is never achieved. This agrees qualitatively with clinical results for postoperative pain management as well as other mod-

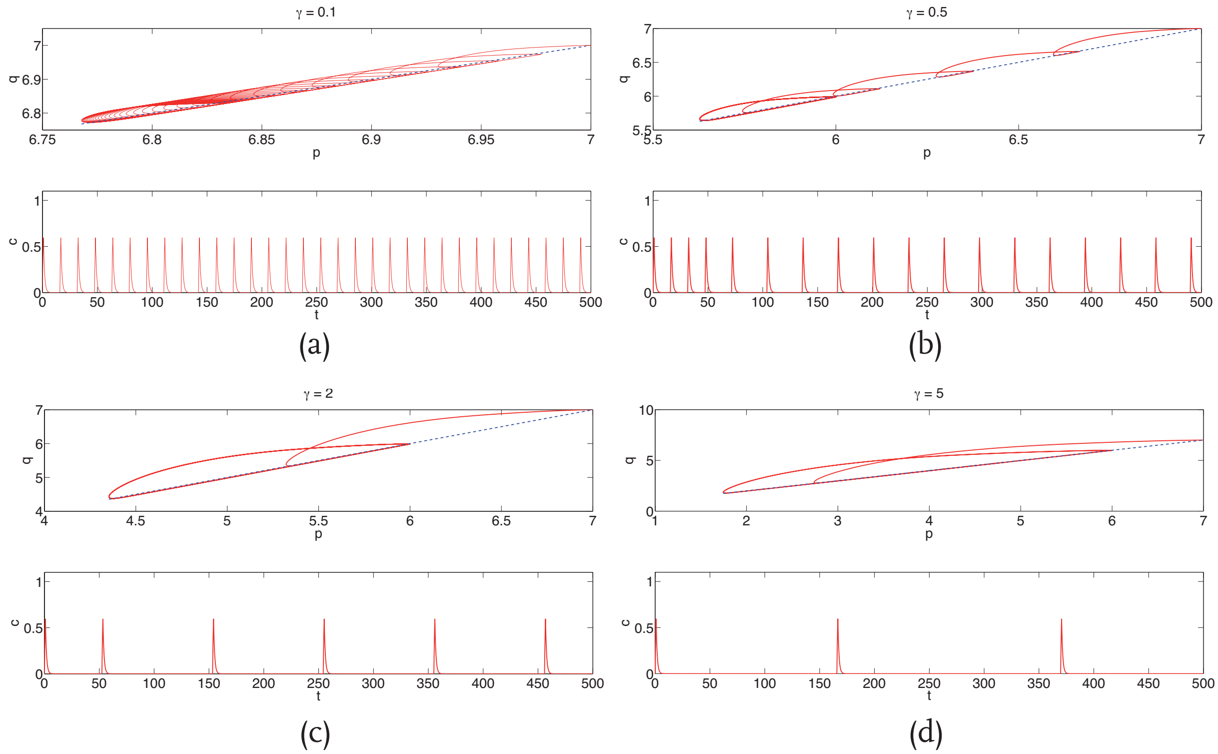

Figure 3—Results of simulations of (11)-(13) using $\alpha = 0.01$, $\beta = 0.02$, $\tau_d = 1$, $\tau_l = 15$, $p_t = 6$ for (a) $\gamma = 0.1$, (b) $\gamma = 0.5$, (c) $\gamma = 2$, (d) $\gamma = 5$. The top figure corresponds to the (p, q) transients (red solid curve) plotted in the (p, q) phase plane with the nullcline $p = q$ (dashed blue curve). The bottom figure shows the drug concentration c as a function of time during the same transient. Note that a period oscillation develops for sufficiently large drug efficiencies γ. The peak concentrations $c \approx 0.632$ agree with the analysis of $\overline{B}$ described above.

els of PCA. Extensions to this model would include allowing the healing coefficient α to vary in time along with a comparison of the pain response to patient movement or stimuli.

Acknowledgments

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Notes

1 See http://www.fpnotebook.com/PHA27.htm for the current age-based protocols based on morphine.

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The Threefold Dilemma of Missing Coherence – Bridging the Artificial Reef between the Mainland and some Isolated Islands*

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Abstract This report shall be referred to as an 'inventory' of mathematical deficiencies of fresh apprentices in industry. This report can also be regarded as a first attempt at establishing a cooperation with steel industry, focussing on educational aspects for the apprentices and simultanously for their instructors.

Where we identify deficiencies on one side, we are astonished to discover an interesting spectrum of mathematical problems in which apprentices have to develop competencies. The dilemmas which we meet are *threefold*:

The steel companies, looking for qualified apprentices yearly, are constantly confronted with the fact that students who are leaving school possess unsatisfactory mathematical knowledge. Unfortunately, in industrial education, mathematics is no longer an independent subject as it has been in school. Now, the subject matter knowledge is interwoven with a technical curriculum, so that the individual and mathematical follow-up work is difficult with respect to both the education at vocational school as well as in the company itself. Moreover, the education of apprentices in Germany is organized as follows: The instructors bear also a responsibility inside the company, and these persons — and that is a *third* dilemma — are in most cases no trained educators in mathematics. At the moment we are still in a discussion phase on which interventions might improve the situation.

*** The title is using a metaphorical language while speaking of missing connections or bridges and relations. Thus, it stands verbally in the tradition of others papers reffering to analogous settings (see [E 98]: Drawbridge up and [TH 04]: Bridges over troubled waters ...), which try to describe some complex deficient educational problematic issues by also using metaphors.**

1 Introduction

1.1 The 'DAMPF' activities of the mathematicians as a hotbed for the educational interface

The reader should know that the Duisburg region is traditionally an industrial area where the last (very large) German steel factories are located, e.g., the globally operating ThyssenKrupp Steel[1] (TKS) and Hüttenwerke Krupp Mannesmann[2] (HKM). Currently, more than 36.000 persons are working in this industry branch and 15 million tons of steel are produced and processed at Duisburg every year.

This explains that for more than two decades the mathematical department of the University of Duisburg-Essen has been organizing half-yearly a one-day conference where engineers and scientists working in steel production and university research mathematicians meet. Each conference is structured in the same way: A talk of an industrial mathematician is followed by a talk given by a university professor on actual developments in mathematics. Finally, there is a lunch with intensive personal conversations. These meetings are organized by the so-called DAMPF-group[3] where the German acronym DAMPF means 'steam, power and energy'.

Meanwhile, cooperative projects have also been initiated and first masterand PhD-theses dealing with scientific problem settings were supervised by university researchers. And it was ThyssenKrupp who decided to participate actively in the nationwide Year of Mathematics in 2008 in Germany. That is, meetings with teachers, courses for students and exhibitions were successfully arranged and financed, emphasizing the importance which the company is attributing to mathematics as a school subject and beyond, last not least, also to the competencies of the company's apprecentices.

1.1.1 Educational interfaces – an issue in deficit

The success of these events served as a hotbed so that the first author was invited to comment on further educational problems since TKS is suffering daily from the low mathematical competencies of their apprentices. The company has becoming aware of these deficiencies when entrance examinations were set up. Further, the problems have been encountered when these young people (16-18 years) enter the firm and attend a 3-year course. Every year, the company is looking for 250 apprentices for technical professions such as electronics technician for industrial engineering, industrial mechanics, mechatronics, and so on. The basic mathematical knowledge needed for these professions is not inconsiderable. This is true for the location of Duisburg as well as for the five other company locations. Thus, the problem which we are faced locally is just the tip of a larger iceberg.

1.2 The three dilemmas

1.2.1 Dilemma 1

To become more precise, the first dilemma is constituted by the fact that the ThyssenKrupp company, which is looking for qualified apprentices, is nearly never satisfied with the competencies ([NEtAl 02]) of school leavers who are now applying for a position in the company. This deficiency is obvious when we consider that in the year 2009 more than 53% of all appliers passed the initial test in mathematics with less than 50%!

The exercises in the initial test have elementary character: changing units of measurement, converting elementary formulae, calculating surfaces (rectangles, trapezoids and sectors of circles), averaging volumes. Additionally, basic skills in algebra and geometry are tested. Over several years the company has kept statistics in view of deficiencies, which would be worth analyzing, but this is not the subject of our work that we present here.

For various reasons it is not possible to reject all applicants who show a lack of mathematical skills. In this respect, compensating for deficiencies – in view of mathematics – is an integral part of the so-called dual education system in Germany. 'Dual' means, that the apprentices go to 'Berufsschule' (vocational school) twice a week and work three days within the company, where there is theoretical instruction aiming also at minimizing the mathematical deficiencies.

1.2.2 Dilemma 2

It should be remarked — and this is the second dilemma — that mathematics at vocational school (and within the instructional lessons in the company itself) is no longer an independent subject, but part of metallic engineering, electrical engineering, etc. For the most part, our subject is understood as an instruction of 'technical mathematics' (e.g. [MEtAl 07], [FEtAl 05]). Thus, mathematics looses its central role and it is not possible to catch up the subject-specific deficiencies. Teachers experience apprentices' weaknesses regarding the content, without having time to strengthen the mathematical foundations. This makes it difficult to identify deficits in terms of didactical categories. It is well known that learning takes place first of all within a contextual process. The transfer of situative heuristical solutions and rudiments has then to be carried into a different, however mathematically equivalent context. Nevertheless, such a transfer is not trivial because of many reasons.

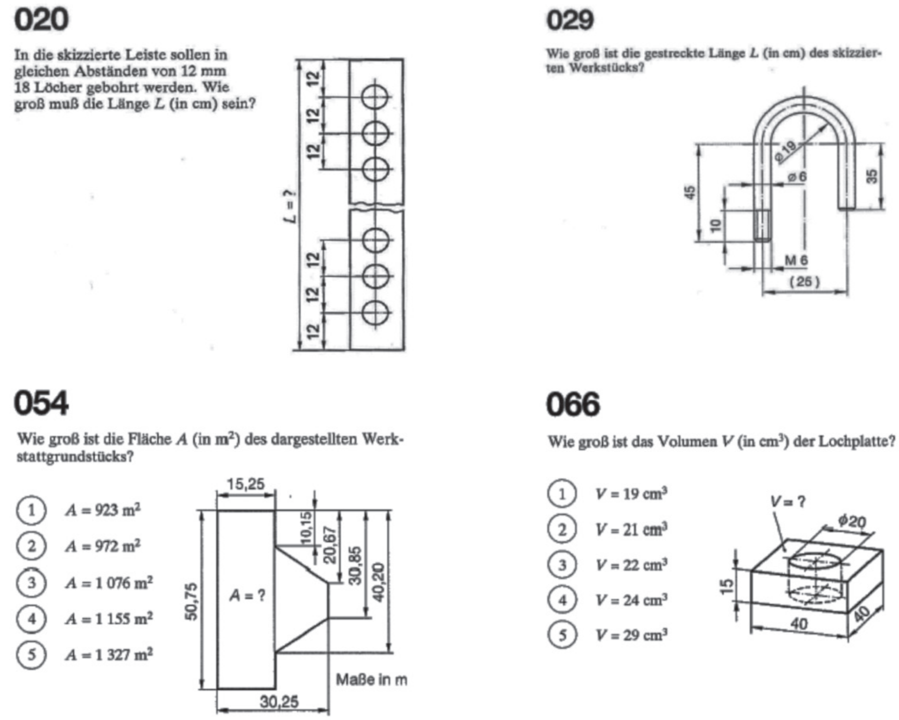

Figure 1

1.2.3 Dilemma 3

On the other hand, there is some training at the company's side, since the company has clear concepts and expectations what should be handled mathematically and 'mastered' by these employees. This training is done by the instructors, however, these trainers are seldom qualified (in this area) and mathematically trained experts. This leads us to the *third dilemma*. Although they try to give their best, the instructors advise their students to manipulate mathematics as they personally think it should be done — in most cases they are not aware of alternative approaches.

2 The mathematical strands

This is not the right place to present the corresponding mathematical contents of the intraplant apprenticeship training in detail, but in the following, we provide some information on typical tasks that apprentices encounter. We will focus on presenting four geometrically based operating tasks (see Fig 1).

We possess numerous student documents waiting for being analyzed carefully. What in fact seems to be very interesting is the type of exercises (see page 5), which one generally cannot find in standard secondary school books. It reveals that the complexity of the exercises is not mainly based on managing arithmetical calculus. It is much rather the amount of information and geometrical decoding that has to be evaluated and is difficult to handle (see the comments in [Str 03] for the school context). The exercises tranport a sense of realism since they are connected to technical contexts and they do not have the nimbus of classical school exercises.

3 The struggle of the instructors and their dilemmas

3.1 Missing continuity and coherence

But still, how can we solve the appearing difficulties? It is a moot point to complain about deficiencies and that minimum standards have not been reached at school (see the discussion in chapter 4 of [HG 09]). Rather, the problems should be seen in *non-existing continuity* and *coherence* (see again [HG 09]) for the different fields of school and vocational education. What is missing are communication and cooperation among those partners. However, it is not our issue to discuss and benchmark the approach of the vocational school, nor to suggest isolated solutions (see [Str 82], [Str 84]).

As aforementioned the companies' instructors try:

- to work independently and to supplement vocational school, and
- to solve the deficits inside the company.

The instructors are indeed certified and experienced technicians with long lasting experiences with apprentices, but in turn have limited global didactical and mathematical potential to diagnose the deficiencies in the learning of their pupils. Last year, the first author held a one day workshop for this clientele. In the following, we refer to that event with a few remarks.

3.2 The 'folklore estimation' of the instructors

It is a matter of common knowledge that sustainable advanced training without acknowledging the underlying beliefs of the persons involved is blind (see [Rö 09]). Hence, before designing the in-service training course, the first author tried to get in contact via email with the instructors who intended to attend the course. We obtained five replies, far beyond being representative, however, those provided us with some kind of snapshot showing at least some typical convictions. Unfortunately, the statements focus only on the learners' side and the instructors have not reflected their role in the learning process. We summarize some of these quotations and comment them from our point of view:

(a) School is responsible for the deficits that our apprecentices are showing.

It would be necessary to identify the deficiencies and to suggest solutions by a third party, because the instructors of the company may just cover operating skills.

(b) Students are not able to set up and manipulate formulae. They try to look them up in books.

Rearranging terms in the elementary algebra is in many cases a necessary skill. Converting formulae is on the other hand every-day business, because it is mastered by 'memorizing'. In this case, it would be helpful to discuss the approaches of Barnard/Tall ([BT 01]), who are using the term of 'cognitive units' to describe such phenomena.

(c) Students are unable to qualitatively understand the functional relations in formulae.

With the overhasty change of representation mode a new problem comes up.

(d) Pocket calculators are used without thinking.

(e) The general attitude towards mathematics can be characterized by sentences like *boldness towards knowledge gaps* and *I am not talented for mathematics*

4 Possible research topics

The classic teacher-student relationship is primarily based on teacher' s competency in specialized knowledge, and some elaborated pedagogicalcontent knowledge. In front of such a background, the main task of the teacher is to initiate understanding and insight. Although this is the standard philosophy in school, in our context, such an aim almost seems to be very ambiguous from both the teachers' and the pupils' position.

So at first, we are missing:

(1) An in itself coherent, intellectual and simple philosophy that points out how mathematics can be thought on a minimum standard level with a lasting effect.

(2) Such a philosophy has to be positively experienced and put in to action at the same time by teachers as well as by pupils.

Sträßer ([SBEW 89]) made clear that skill training versus insight is an incorrect dichotomy. Still today, persuasive approaches that balance these two aspects have not been noticed by third parties and the authors have no information regarding the following important aspect:

(3) Coping with mathematical deficiencies in a specific context of application, e.g., metal engineering. We do not know how to transfer insights into a quite different context, e.g., electrical engineering and vice versa.

Finally we cannot offer any straightforward convincing concepts for further education to the instructors at the moment, but we have some visions …

5 Conclusions and possible consequences

It seems remarkable to us, that our first steps to strengthen the educational interfaces are based on a fruitful cooperation with mathematics in industry over many years which enforced the belief that university partners can also increase the didactical needs of industry. This estimation results from the cooperation between the university and the company. Without such a cooperation we — responsible for mathemat- ical education at university -— would never have been asked!

Besides, it is important to build a trusting atmosphere between the instructor and the didactician. Research on professional development of teachers — and the situation of instructors is quite similar — can only be sustainable and successful if the teachers are willing to accept any kind of support and empowerment. Therefore the offers for instructors should not create the impression of compensating for deficiencies on the teachers' side. Thus, we need some specific strategy to address the individual needs of educators that is based on positive messages.

Since the company under discussion has at least five locations spread over a large area, it is not easy to design *one* measure. With respect to designing a training for the apprentices, one should notice that there are more than 1.000 people starting their job every year and one cannot gather them together to just one course. These organizational difficulties have to be solved. Up to now, the first author has not had any direct contact to the apprentices, and that is why we have to rely on information from the instructors. From our research in the field of beliefs, we know that negative emotions are often linked to the subject *math*. Because of this, every supporting measure has to start right (here) with the apprentices and has to spark interest, self-confidence and self-efficiency. So, we conclude with the following question:

How is it possible to establish an intellectual philosophy for mathematics teachers at school that will cover a basic level of competencies and enable a vision of mathematics education for a better performance of students starting their job training.

Notes

1 http://www.thyssenkrupp-steel.de/de/

2 http://www.hkm.de/english/the-enterprise/the-ironworks.php

3 http://www.uni-due.de/mathematik/DAMPF.shtml

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A Holistic Approach to Applied Mathematics Education for Middle and High Schools

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Introduction

In this paper we describe an integrated collection of programs whose overall objective is to enhance mathematical education, and especially applied mathematics education in middle and high school grades. The overall approach includes elements of professional development, student-based projects and programs of different lengths, levels and intensities, and the use of student contests to help motivate students' interest in applied mathematics. The term applied mathematics is used here in a broad sense and its relevance to real life problems and industrial mathematics is implicit in all our activities.

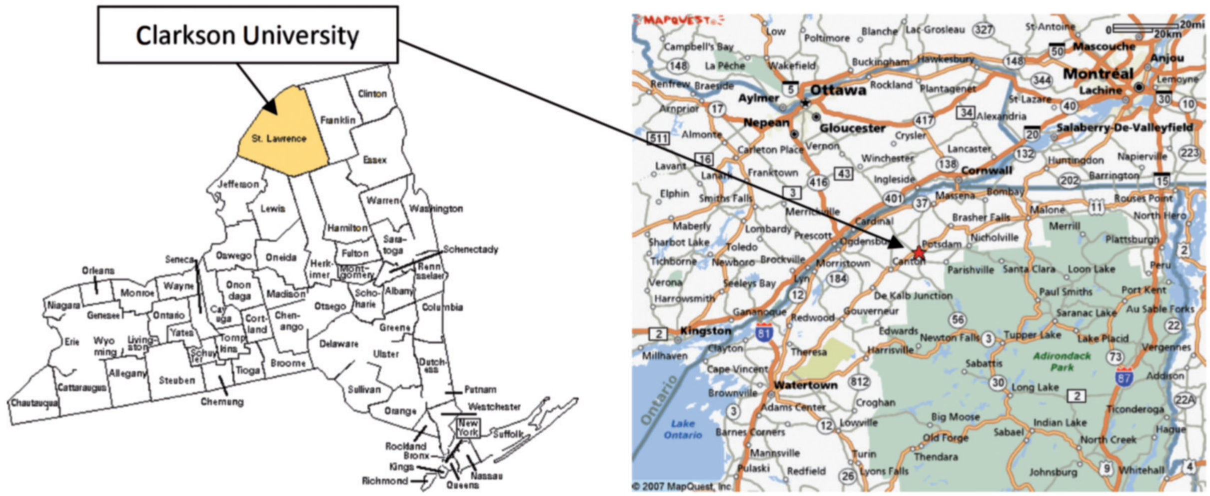
The range of topics used is also very broad, encompassing both traditional engineering-oriented problems and newer areas of applied mathematics such as may be encountered in computer graphics.

The programs described involve a large team of colleagues from the spectrum of STEM disciplines and departments at Clarkson and partners from many local school districts. The partnerships are coordinated through the Offices of Educational Partnerships at the University and at the local Board of Cooperative Educational Services (BOCES). We are operating in a very rural, low income area of Northern New York State, which makes the need for efficient collaboration among all partners vital.

A high proportion of our student population is drawn from poor families where a student going on to college education is likely to be the first from his/her family to attend college, occasionally even the first from their particular village. The need for, and the rewards to be gained from, an enhanced educational experience in areas that students see as relevant to their lives are therefore great. The median household income rate in this region is less than 75% of the state average and poverty rates are among the highest in the state with nearly 1 out of 5 children living below poverty level, according to the U.S. Census Bureau and The A.E. Casey Foundation's KIDS COUNT Data Center [1-2].

In area, St. Lawrence County (SLC) is the largest county in the state, located in the northernmost section along the Canadian border. SLC contains 17 rural school districts, ranging in size from very small districts of approximately 350 students (K-12), to one larger district of over 2,000 students. SLC unemployment level was nearly 11% in the first six months of 2009, well above the state average of 8.0% for that same time period, according to the NY State Department of Labor [3].

In the subsequent sections of the paper, we describe some of the components of our integrated program in more detail and conclude with assessment data and results that demonstrate significant progress. Several sources of external funding have contributed to this success, most notably the National Science Foundation, the US Department of Education,



and New York State Education Department (NYSED). Although much of what is discussed is local in nature, connections to broader programs are included where appropriate, in particular drawing on connections through the work of the SIAM Education Committee.

Roller Coaster Project

The *IMPETUS for CareerSuccess* program is funded by NYSED as part of its STEP, or Science and Technology Entry Program. This overall program is replicated in most States as a way of encouraging and preparing students from under-represented groups for college and careers in the STEM disciplines. The target audience for our program is students in Grades 7 – 12 from families who meet criteria as either minority populations or being economically disadvantaged.

The theme of the program is the Mathematics and Physics of roller coaster design. Different aspects are presented at the different grade levels of the students. The focal activity is a week-long summer Roller Coaster Camp. Examples of some of the specific activities at the camp are included below. There are several other components to the IMPETUS program which run throughout the academic year. These include some of our Clarkson students going into the middle and high schools to lead after school activities related to both the school curricula and the roller coaster theme. The school year program also provides opportunities for the school students to visit the University and some its labs for activities related both to their immediate mathematics and science education and some of the college and career options that education may be open to them.

The Roller Coaster Camps cover six days of activities that address many of the key aspects of a successful industrial mathematics education. Students use technology for data collection and analysis, apply basic mathematical and physical principles to issues of motion and safety for roller coasters, run simulations and physical experiments to deepen their understanding.

They also learn the importance of communication and teamwork. Students are assigned particular roles within a "company" structure, each contributing specific aspects to the design. The companies are then also expected to produce posters and other deliverables which are showcased for parents and teachers. Students get to ride their design on a physical roller coaster simulator which has full 360° motion about two different axes. This makes their experience real to them in a direct way.

In the next few paragraphs we highlight some of the specific mathematics content and its relation to roller coaster design in terms of understanding simple dynamics, safety, the thrill factor, etc.

Automatic data collection using accelerometers is used to gain understanding of the "shape" of various component motions of a roller coaster. Students also learn to interpret graphs of that data, and identify hills, turns, loops, and rolls from the related graphs. Understanding the roles of velocity and acceleration in producing a thrilling ride requires an understanding of the basic mathematics and physics of motion. Labs and classes using motion detectors and data collection devices help students develop a practical understanding to go with the theory that is presented.

Extending students' knowledge of accelerations to circular, or more general curvilinear motion is used in the development of elementary safety analysis. The role of curvature in determining the magnitude of centripetal accelerations is critical in safety considerations. At the top of a hill, that centripetal acceleration must be dominated by gravity in order to maintain contact with the track. While on a loop it must exceed gravity. Both are simple to explain, and to test. One exercise in this context has hills, valleys and loops being approximated by circular arcs connecting straight line tangents. Students are then able to use the equations derived to test track segments against basic safety requirements.

One immediate practical design lesson that students learn is the important role of the various additional features such as upstop wheels that prevent the cars from leaving the track when the fundamental dynamics show that it would. Initially the analysis is based on simple conservation of energy principles, and on an idealized two-dimensional roller coaster track. The biological effects of high accelerations such as black out and red out are explained so that the limitations on accelerations experienced in safe roller coaster designs are again made realistic.

Students in the camp are divided into three groups based on current grade levels in school. They are also divided vertically into different companies. Within each company the different levels are given different titles and are responsible for different aspects of their overall roller coaster design. The coasters comprise four different sections including hills, valleys, loops and perhaps a corkscrew or other more complicated component.

The more junior students are responsible for the initial design, which is mostly conceptual and graphical. The next group has to examine the ride from the point of view of conservation of energy to make sure that a rider could get all the way through. At this stage the analysis is, which is still two-dimensional, is not concerned with safety issues. Those are the province of the most senior students who have been exposed to a more advanced analysis based on smooth linking of straight line and curved segments and the examination of speed and curvature to determine whether the cars would stay on the track and the riders would not be subjected to unsafe accelerations.

Before riding their ride on the Virtual Roller Coaster, the senior students also use the *No-Limits* software to create a computer simulation. This software simulation applies a more complete analysis of the dynamics of the ride than the students are yet capable of and provides feedback on the total acceleration, and it three components, indicating areas where accelerations exceed safety limits.

The target audience for this program consists of students who are traditionally not likely to go on to a college education. The program begun in 2007 and so only a small number of students have completed high school yet. Last year, there were seven graduating High School seniors in the program. Five of those have started a college education. Three others who would have been on that list had already entered college early. Some are STEM majors at Clarkson and now work with our team as student assistants to help pass on the benefit to future classes.

Integrated STEMProfessional Development

Professional development for local middle and high school teachers has been an important aspect of our programs. The specific programs began in 2004 with a different BOCES district than we have partnered with recently. The successful partnerships with the St. Lawrence-Lewis BOCES began with the St. Lawrence Mathematics Partnership, which was succeeded by the St. Lawrence STEM Partnership. These programs were both funded by NYSED, spanning the years 2005 – 2010. Many details of the programs can be found at the STEM Partnership website http://stlawcostempartnership.org/

Throughout our activities there has been a strong emphasis on applications and making math classes at all grade levels relevant to their students' lives. Many of the teachers, especially in the lower grades, are not mathematics or science specialists. Frequently they lack confidence in their own understanding or knowledge, or even lack some of the fundamental content knowledge themselves.

A range of (typically) weeklong Summer Institutes led by University faculty have been used to address these issues in ways that improve both the teachers' content knowledge and their understanding of why it is important for their students. As will be seen in the assessment summary, there has been a measurable effect on the performance of students in standardized tests in mathematics. The planned next phase is to focus on ways in which a greater proportion of students can advance from basic competency to high levels of achievement.

Part of the requirement for teachers attending these institutes is that they prepare some lesson plans or, preferably, project-based learning experiences that can be used in their classrooms in support of the regular curriculum. These are in turn subjected to peer review and can be published on the partnership web site, which in turn is linked from the NYSED Mathematics and Science Partnerships page. Several examples can be found on the website mentioned above.

We finish this section with a brief description of three of the institutes. These illustrate the variety of topics and grade levels targeted, and the fact that not all industrial mathematics has its roots in traditional engineering applications.

One of the first such instates that was developed had the theme of using technology and applications to enhance the teaching of mathematics. This has undergone evolutionary changes over the years, but has retained the same basic philosophy of using applied projects to help students (and their teachers) to understand both the mathematics and the reasons for studying it. The applications are drawn from (simplified) real world problems, and typically require computation, too.

The message that not all mathematical problems are amenable to simple paper and pencil algorithmic solutions is an important lesson in itself. Among the examples used was a pre-calculus version of Euler' s method to investigate the motion of a projectile subject to air resistance. This can be posed within a sports context. Problems of scaling and drawing to scale also provided opportunities to reinforce geometric reasoning as well as algebra and computations. New York State' s mathematics standards are expressed in terms of content and process strands. Most problems and projects that were used were designed to draw on several of them.

At a practical level using hand measurements of radii and circumferences of circular or spherical objects, provides a valuable introduction to the concept of π and its estimation. A discussion of experimental and computational errors is then easy to motivate.

The primary goal was improving the technical content knowledge of the teachers. The examples were however also chosen with the intention that they could be adapted top many grade levels so that teachers could expand upon them in class.

This summer, for the first time, we designed a summer institute specifically targeting the use of mathematical modeling in the middle school mathematics classroom. The challenge in focusing solely on middle school mathematics teachers is to break through the need to "teach to a test" and convincing them that by solving real-world, open-ended problems that the same goal is accomplished.

Daily curriculum included working in teams on modeling problems relevant to middle school students such as improving the flow of the lunch line to maximize time to eat, analyzing the feasibility of starting a new extra-curricular activity, and studying how rumors spread. Part of the day was spent diving deeper into defining a problem statement, making assumptions, identifying variables and relationships, solving equations, and analyzing results. To help make connections to the classroom and help justify the use of modeling, each activity included specific links to the NY State Standards.

Afternoons were designated as developmental times that teachers could spend creating a learning experience to implement in the fall. These projects included an interdisciplinary project with an art teacher to create a ceramic cup and analyze mathematically various properties such as volume and surface area. Another posed the question, "Does time spent on text-messaging affect a student' s grades." By the end of the week, participants not only understood the difference between a word problem and a real mathematical modeling problem, but also gained confidence and experience in both solving and creating these activities.

A recently added institute was devoted to an introduction to the mathematics of Computer Graphics at a very elementary level. Much of the material was focused on motion and collisions in two dimensions, although extension to three dimensions was encouraged for teachers with a stronger background. We began with rectilinear motion and the effect of perfect reflections of the edges of the screen (or some other box).

The teachers developed their understanding in a simple visual game programming package, *Scratch* (http://scratch.mit.edu/). The programming environment was simple enough that the institute could concentrate on extending this basic idea to situations such as imperfect bounces, motion under a fixed acceleration force such as gravity, and collisions with slanted "mirrors" . Circular motion, the geometry and algebra of rotations, the effect of spin on a collision were all added to individual teachers' projects depending on their own understanding and the level for which they were preparing to teach. Questions of collision of two moving objects, requires the concept of parametric descriptions of curves. Relatively simple problems generated significant gains in teachers' understanding of fundamental concepts of mathematics.

An important piece of all these professional development institutes (including the many that have not been mentioned explicitly here) is that teachers could take ownership of their own project development. While some were pleased to be able to achieve the basic tasks, others took the ideas to levels that we had not envisioned. That sense of challenge and achievement is one which those teachers are likely to generate in their classes.

Contest to Classroom

Student contests can be a strong motivator for continued interest and enthusiasm for mathematical education. Within our program we have encouraged participation in a wide range of such contests. We will focus here on the more mathematically oriented competitions but the overall program has also included Science Olympiad, FIRST Robotics and the JETS/ TEAMS pre-engineering competitions. Inevitably the mathematical content of these is a significant component but not the focus.

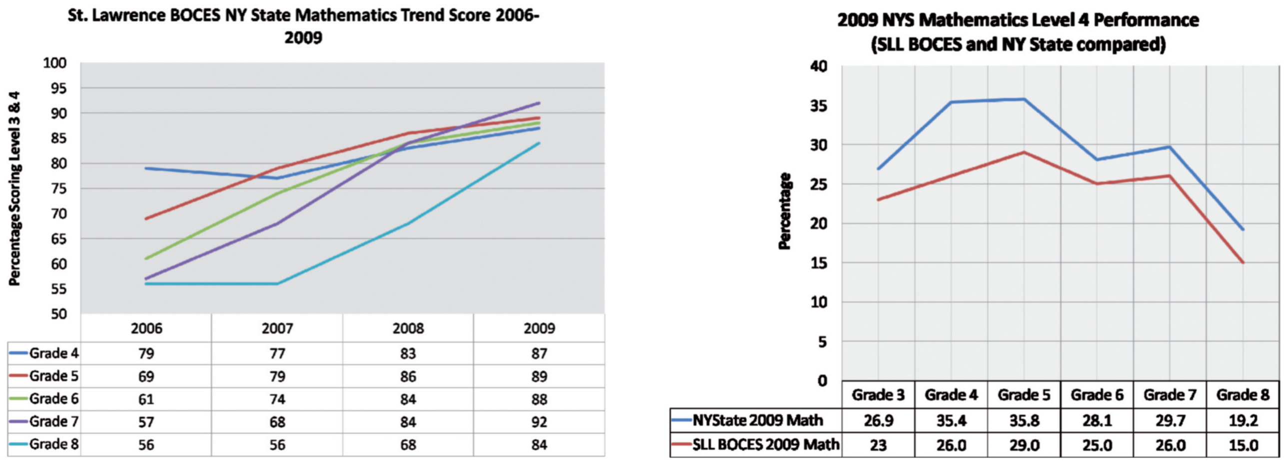
We do not go into detail on the structure and content of any of the contests. They all have excellent web presence which we cannot possibly summarize adequately in the space available.

There are two primary mathematical contests that we have used. In terms of grade level the first is MATHCOUNTS (www.mathcounts.org) which targets students in US grades 6 – 8. While the problems used are all short and tend to emphasize basic skills and computation, they are frequently posed in ways which require students to interpret and think before answering. In this sense they provide a good introduction to the world of applied and industrial mathematics.

The authors took over the coordination of the local MATHCOUNTS competition with help from some of the grants several years ago. During that time the numbers of participants has grown substantially. We also added less formal local event *ClarksonMathletics* earlier in the school year. The program gives valuable experience to some of our students, too, by having them act as coaches for the middle school students.

The High School version of the Mathematical Contest in Modeling, HiMCM (http://www.comap.com/highschool/contests/) is operated by the Consortium for Mathematics and it Applications (COMAP). The Consortium provides a wealth of resources at both high school and college levels.

This is a team competition for senior students which we have only used in the last two years. The effect on those team members' interest in pursuing mathematics and science



	2006	2007	2008	2009
Grade 4	79	77	83	87
Grade 5	69	79	86	89
Grade 6	61	74	84	88
Grade 7	57	68	84	92
Grade 8	56	56	68	84

	Grade 3	Grade 4	Grade 5	Grade 6	Grade 7	Grade 8
NYState 2009 Math	26.9	35.4	35.8	28.1	29.7	19.2
SLL BOCES 2009 Math	23	26.0	29.0	25.0	26.0	15.0

Figure 1—(a; left)Percentage of students in Grades 4 – 8 achieving "proficiency" on th NY State Mathematics tests, (b; right) Overall percentage achieving the top level in our district compared to the State as a whole.

is impressive. Teams work under time constraints over a 36 hour period. The problems are open ended and the judging is based on the best papers submitted. Teamwork, modeling, applications and communication are all emphasized – real world applied or industrial mathematics in microcosm.

This year we will be adding the Moody' s Mega Math Challenge sponsored by Moody' s and organized for them by SIAM. This new contest is just starting to spread inland from the Atlantic coast of the US. It is expected to be nationwide in another 5 years.

Our intention in involving students in these contests is twofold. First is the motivation and interest that the students gain directly. Second but similarly important is that by involving teachers, they too begin to deepen their own understanding and gradually to bring some of the ideas and materials into their own classrooms to enhance the applied mathematical experiences of all.

This transition from Contest to Classroom is being encouraged but is a slow process. Regular workshops try to guide teachers in ways this can be achieved.

Results

While it is always difficult to attribute gains to any one cause, the strength of the overall improvement suggests that our holistic approach is working.

Figure 1 shows that we are seeing significant upward trends in the proportions of students who are reaching proficiency (Levels 3 or 4 out of 4) on their annual mathematics tests. Indeed at that level our students are catching up to the rest of the State. However, Figure 1(b)

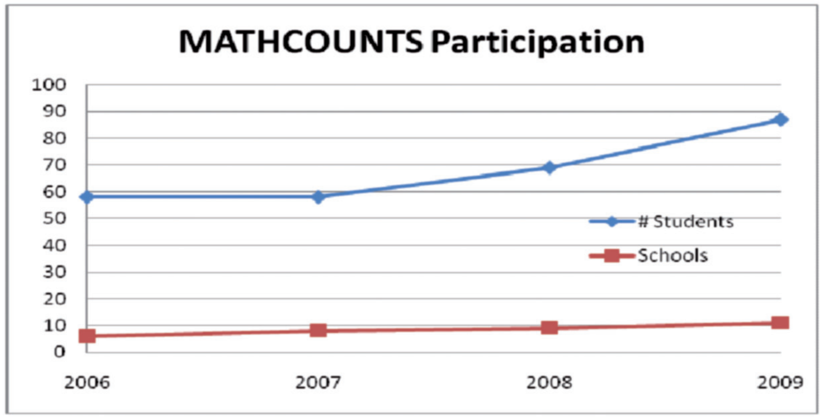


Figure 2—Growth in participation in MATHCOUNTS over academic years 2006–09

shows that we continue to lag behind the State in the proportion of students that achieve Level 4.

Continuing emphasis on applications and relevance should help address this issue and encourage a sustained interest in mathematics and science. Professional development through our applied and industrial mathematics institutes plays a critical role. Improving teachers' content knowledge and awareness of the power of mathematics in practical settings will give them the confidence to address these issues in their classes.

Table 1 provides the evidence for the gains in teachers' content knowledge as a result of the more mathematically oriented susummer institutes. The overall effectiveness is tested at the 0.001 level, and the overall effectiveness of all institutes is indicated in the final row of the table. Table 2 shows teacher responses to three fundamental questions about their experiences with the STEM Partnership and its institutes. These responses are overwhelmingly positive and give confidence that the benefits can be sustained.

The impact of contests is harder to assess directly. However the growth in terms of participating schools and student numbers in Figure 2 implies significant progress in terms of students' and teachers' interests in mathematics and its applications. The overall story that these results tell is one of definite progress that has been firmly rooted in emphasizing the role of applications in mathematics education. It is perhaps too early to draw strong inference but the signs are positive.

Acknowledgements

We are pleased to acknowledge the roles of many of our colleagues who contribute to different aspects of the work described. In particular, David Wick and Michael Ramsdell (Clark-

Summer Institute	Number tested	No. with Gains
Computer Graphics	N=7	7*
CSI Potsdam	N=16	16*
BioInformatics	N=5	3*
Integrating Conserv. Sci. & Math.	N=9	8*
Promethean Integrated Math & Sci.	N=42	38*
*Sig. at.001 level P≤1.46428E-08 Total for all Institutes	224	188 (84%)

Table 1—Teacher content knowledge. Statistically significant gains based on pre- and post-testing

	Rewarding experience	Gain Math knowledge	Students benefited
Disagree (all shades)	2	8.4	2.1
Slightly agree	12	16.7	18.8
Agree	36	50	31.3
Strongly agree	50	25	47.9
N	119	48	119

Table 2—Teacher responses to STEM Partnership(percentages of responses)

son, Physics Department) were co-designers of the Roller Coaster project, and contribute in several other areas, too. Susan Powers (Clarkson University, School of Engineering) was a driving force behind Clarkson' s involvement in Educational Outreach programs and has played an important role throughout. Bruce Brydges (SUNY, Potsdam) leads the external assessment and evaluation efforts for several programs. Gail Gotham, Ellen Glasgow and Michael Montgomery head the BOCES team in our partnership and work closely with Diane Brouwer and Mary Margaret Small in Clarkson' s Office of Educational Partnerships. Funding from several agencies has been important. NSF funded the GK-12 Project-based learning program, NYSED funds supported the St. Lawrence Mathematics Partnership, the *IMPETUS* STEP program, and the St. Lawrence STEM Partnership, and the US Department of Education funded the Safe Schools Healthy Students program which in turn helped with many of the student contests. Finally, colleagues on the SIAM Education Committee have been a valuable source of ideas and collaborations that have helped improve our own programs as well as making their own contributions to applied mathematics education at all levels and in many geographical areas.

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Mathematics in the training of engineers: an approach from two different perspectives

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Abstract Our proposition is based on the PhD carried out by A. Romo Vázquez (2009), supervised by M.Artigue and C.Castela. This research addresses the issue of the mathematical preparation of engineers. It started by a historical inquiry of both the Ecole Polytechnique (Belhoste 1994) and the work of the ICMI since the beginning of 20th century. The study revealed disjunctions between theory and practice that underlied the initial models for training and the related historical debates. It also analyzed prior research on the training and professional practice of engineers (Bissell & Dillon (2000) ; Bissell (2002, 2004) Noss, Hoyles, Pozzi (2000); Noss et Kent (2002); Kent (2007); Howson et. al (1988); Prudhomme (1999)) in order to situate the issues within a modern context. The central part of the research was twofold, consisting of an analysis of both engineering projects (practical activity) and Automatics and Mathematics courses, offering thus two complementary perspectives on the issue at stake. In the following we briefly present the two of these, linking them with questions raised in the Discussion Document.

Mathematical needs within engineering projects: a case study

In the frame of this research work, engineering projects, carried out as part of the training of engineers at the Vocational Institute (IUP) at the University of Evry in France, were monitored over the course of two years. We analyzed three of these projects in depth to better articulate the participating students' mathematical needs and the ways they cope with them. Examining the fundamental role of technology in the students' work offers some basis from which to address the question:

Does the existence of special types of technology hiding mathematics from the view of the user imply a change in the mathematical demands on the user? How?

Engineering projects

Within the Evry IUP, the projects observed are carried out by teams of students in their fourth year of tertiary education, following a didactical organization which attempts to reflect the actual organization in workplaces. A University teacher plays the role of a client who requests an engineering work following some precise negotiated terms and conditions. A group of students is supposed to work on their own to come up with an answer to the client's request. At the same time, some aspects of the research world are included in the project conception. The subject of every project is open and the client-tutor has generally no prior complete sense of how to get through the project. The final production and the route towards it have to be built together in the same process. Therefore, the students have to organize and plan their work and to look for solutions; this requires that they adapt or develop knowledge acquired in the academic world for the professional realm.

The project is assessed from two perspectives, combining workplace and engineering school requirements. On the one hand, the client must be convinced that the technological solution the students have come up with is an adequate solution. On the other hand, the evaluation is also academic: the students present their work to a jury composed of IUP teachers. The jury evaluates the students' ingenuity in regard to using knowledge taught in IUP courses. Moreover, the students are often asked to justify their choices, and engage academic knowledge in this justification.

The projects are realized over five weeks in two phases. After the first phase the students write an intermediary report in which they describe the pre-project, as supported by a study of the subject. They then present the technological solution they chose from among those they envisaged during their exploratory work. In the second phase, the pre-project is realized, which in most cases leads to a concrete production.

Research methodology

In the frame of her PhD, A. Romo conducted two observations of the projects over two years, using Dumping methodology. During the first phase of project (two weeks), interviews and questionnaires were carried out with the students and the clients-tutors. After this phase, the institutional data, specifications, intermediary reports and documents used for the development of projects were collected. This allowed A. Romo to gain, from her mathematician culture, some familiarity with the projects. However such an insight was not sufficient to go on with the study.

For the second phase three projects were selected, in order to obtain a deeper and more precise understanding and ensuing analysis. Selection was based on the interviews, questionnaires and analysis of intermediate reports, following two criteria: 1) the presence of explicit mathematical knowledge and 2) the project domain such as aeronautics, mechanics, electronics, etc. For the three projects, the students' second phase work was followed, up to the final stage, in which students presented their work in front of the jury. Interviews were conducted with the teams and with each student individually. These series of observations produced substantial data to analyse.

Praxeological analysis of projects

In coherence with the theoretical framework underlying this research work, the Anthropological Didactic Theory developed by Chevallard, the chosen option was to identify the mathematical praxeologies students had used in their work, i.e. the type of tasks and relative techniques involved as well as the discourse developed to describe, explain and justify these techniques [1]. The analysis of the students' techniques was very difficult due to gaps in their descriptions and documents (intermediary report and others). In particular, while mathematical tasks made up a substantial portion of the selected projects, yet the students' priority was not to present the corresponding techniques, and still less to justify them. Furthermore, software was broadly used for the mathematical work. These two factors made it necessary to reconstruct the techniques and the associated discourse using two sources: 1) analysis of intermediary disciplines courses and 2) the view of expert engineers. This work was a substitute to the usual a priori analysis, which is so important within the French didactical research. Mathematical praxeologies involved in the project being thus reconstructed, it was then possible to compare the students' production with the experts' views, regarding both the chosen techniques and the technological discourse.

Two kinds of mathematical needs

One result is the necessary distinction between two kinds of mathematical needs: elementary and advanced.

Elementary needs. These rarely exceeded high school mathematics: working with formulas involving several variables anticipating the respective effect of change in values of these variables; analyzing and using functional relationships; making estimations; evaluating intervals of possible values for given magnitudes; calculating simple integrals; solving simple differential equations and using elementary trigonometry. Nevertheless the students observed met evident difficulties at coping with these elementary needs, as if such elementary mathematics could not be mobilized outside the precise mathematical contexts where they had been taught. One of the projects illustrates these difficulties quite well, and how they affect the way students use technology, in that case spreadsheets, for optimizing the dimensions of a particular object needed for the realization of their project, in a situation where some form of proportional reasoning could be used for anticipating the respective influence of the different variables.

Advanced needs. In the three projects observed, the notions involved were Laplace transform, dimensional analysis and finite element methods. The corresponding mathematics was encapsulated in professional software that could be used as 'black boxes'. This situation obviously facilitated the students' mathematical work and limited their needs in terms of technical mathematical knowledge. Our observations showed nevertheless that the mathematical work did not disappear at all, even if it was substantially modified: software allowed easy exploration, and trial and error strategies, but as well the selection of the parameters for exploration as the interpretation of results became fundamental in this work, and required some conceptual understanding of the mathematics encapsulated in the software. Students' work on the ANSYS and Matlabs software for instance demonstrated this, as we propose to show in our detailed contribution.

Beyond this distinction between two kinds of mathematical needs, our observations and analysis also showed that what was required could not be reduced to mere mathematical knowledge. An efficient use of this knowledge within the projects required at least its productive composition with knowledge in automatics, mechanics, hydrodynamics or other engineering sciences. This led us to pay specific attention to these intermediary disciplines and the role of interface they played between mathematics and practice.

Due to the central role played by the Laplace transform in one of the projects, we decided to focus on this notion and compare the way it was taught in courses in Automatic Control and Mathematics, in different institutions. In our opinion, this part of our research could also contribute to the fifth topic of the discussion document i.e. the teaching and learning of industrial mathematics issue. Three of the questions submitted to discussion refer to the idea of 'explanation':

How much is it appropriate to explain? How to decide the level of *explanation* for various groups? Who decides what will be *explained* and to whom?

We think that for answering such questions, a reflection on the idea of explanation itself is needed, and we propose to elaborate on it, at the light of the analysis we have carried out, and of the conceptual tools we have developed for it.

Teaching mathematical techniques in a vocational context: A tool to investigate the textbook explanatory discourse and unearth possible options. First results relative to the Laplace Transform

Romo 2009 studied three tertiary vocational education (2nd and 3rd university years) textbooks written by automatics lecturers and one written by a mathematics lecturer. The investigation focused on teaching the Laplace transformation. To do so, we developed a general (non specific to the studied mathematical subject) analytic tool which considers different dimensions of the explanatory texts. Using this tool, we have unearthed several options for effective teaching from which a lecturer may choose the most appropriate.

The analytical tool was based on the following considerations: an interest in the teaching of mathematical techniques (ways of doing) in a vocational context, and an account of the knowledge produced about these techniques both in the mathematical world and in the applied world (engineering sciences and professional fields). In addition, this knowledge satisfies many diverse needs. We focus on six of them: describing the technique, validating it i.e. proving that this technique produces what is expected from it, explaining the reasons why this technique is efficient (knowing about causes), motivating the different gestures of the technique (knowing about purposes), making it easier to use the technique and appraising it (with regard to the field of efficiency, to the using comfort, relatively to other available techniques).

Distance to Mathematics: the Validation issue

The different course texts have been studied with this analytical table. We began with an examination of the issue of validation. From the mathematical text we have drawn a precise idea

- of the theorems involved in the uses of Laplace transformation in automatics, especially concerning their hypotheses;
- of the theoretical knowledge that should be taught to give a complete validation (according to the mathematical standards) of these theorems.

Using theoretical knowledge as a reference, it appears that the Automatics textbooks vary in the level of mathematics included. Results may be given without hypotheses, with an incomplete list or, in a few cases, with all the hypotheses required. It seems that the authors of the textbooks implicitly assume "good behaviour" of the functions involved in automatics. As for the validation process itself, based on our study we created a four-level scale of proximity to the mathematical standards:

- LEVEL 3: mathematical proofs are provided, with more or less attention to detail;
- LEVEL 2: the existence of a mathematical proof within a given theory is explicitly referred to, so that students are aware of the results' origin;
- LEVEL 1: the mathematical nature of the results is inferred from the general environment;
- LEVEL 0: the mathematical necessity to prove the results and justify what is given as true is hidden.

This scale was interpreted in terms of proximity to mathematics, considering the question of proof as a crucial dimension of a result's mathematical nature.

Distance to engineering sciences and professional world

To study the presence of the applied world within the texts, we have at first furthered the study of the validation process with an analysis of other forms of explanations (referencing the explaining function considered above). We have looked for arguments which we consider to be convincing evidence of the result's validity in the non-mathematical applied world. An especially relevant example of such explanation concerned the Dirac impulsion: after referring explicitly to the Theory of Distributions (Level 2), the lecturer added further explanation, referring at first to the classical but not fully satisfying approach in terms of limits of functions and then moving to the physical phenomenon context.

We have also investigated within the explanatory texts components which are more tightly oriented towards practical use of the techniques. One of the automatics textbooks provided a rich example. The motivating, appraising and facilitating functions were intertwined to teach a technique especially well adapted to the necessities of system control. In this case, the emphasis was not on the certainty that technique produces what is expected, but rather on the ways in which it could be used more conveniently and effectively.

Finally, we completed the research into explanatory texts with an analysis of the examples the textbooks use to teach mathematical techniques. We have distinguished three levels of tasks:

- Level 0: the task refers only to a mathematical context, without any specification in any applied field;
- Level 1: the task refers to a general automatics context, without specification of the domains involved in the system functioning;
- Level 2: the task refers to a specified control problem (electricity, mechanics, hydraulics, thermodynamics, etc.), parameters and equations are contextualised.

The mathematics text only gives Level 0 tasks; while the automatics texts obviously propose Level 1 tasks, they vary as to their inclusion of Level 2 tasks. This scale is interpreted in terms of proximity to the professional world: level 0 corresponding to a maximal distance and level 2 to the minimal one among the tasks we encountered in the automatics textbooks we studied. The whole corpus is composed of classical scholarly exercises in which the students are supposed to use only the taught techniques. We have introduced a 4th level (Level 3), consisting of completely contextualised tasks in which the students are in charge of every step, including modelling. We have not yet met this level in the data we have studied; but it is possible that such tasks could be presented in the tutorials. However, the distance between Level 3 and the professional context is still large, especially given that in the analysed texts, the teaching purpose is focused on automatics as an engineering science.

The teaching of 'industrial mathematics' suffers from limitations due to inherent constraints: the lecturer should try ideally to reduce the distance from the mathematics world, while at the same time remaining close to the engineering sciences and professional world. Time, however, is limited and, most importantly, the students are not necessarily interested in the mathematical aspects. The question is how to find a reasonable balance. From our investigation, several possibilities emerge, which, as we have pointed out, may be interpreted in different ways. The list of possible options is far from complete. The analysis tool presented here could be further used to investigate other effective teaching methods. The limits and advantages of given options should be assessed with regard to the students involved, and to the types of technicians or engineers addressed by the training institution. So we propose to present the analysis tool we have elaborated, giving examples based on the specific study we have carried out.

Notes

1 The anthropological theory of didactics uses the word “technology” to label the descriptive, explanatory and justifying discourse used for communicating about techniques.

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The Vertical Integration of Industrial Mathematics – the WPI Experience

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Abstract The project oriented education curriculum, introduced almost 40 years ago at WPI, has facilitated a major change in the mathematics education. Since the mid 90's the WPI faculty have developed a successful model that introduces real-world, industrial, projects in the mathematics training, at all levels from middle school to the Ph.D. program and faculty research. The faculty and students affiliated with the Center for Industrial Mathematics and Statistics have developed project collaborations with over 35 companies, businesses and government labs. These projects serve to motivate students to study mathematics and prepare them for interdisciplinary work in their careers. With funding from NSF, SIAM, the GE Foundation, the Alfred P. Sloan Foundation and Intel, several vertically integrated educational programs have also been developed.

Introduction

The need for modeling and simulating emerging technologies, financial products, biological or social phenomema has dramatically increased in the recent years and thus has substantially expanded the need for mathematically trained professionals. As the questions become more complex, the need for modeling, analysis and computing becomes more sophisticated and mathematicians in collaboration with other professionals can provide technical and economical advantages, important for a company's competitive edge.

At the same time, reports concerning four-year college and university undergraduate mathematics programs (e.g. Lutzer, Maxwell & Bodi, 2002) describe a drop in the number of mathematics bachelor's degrees in the US. Thus the need to find new ways to attract students to mathematics is a very real challenge for educators. Applications can provide motivation; while some students are attracted by the intrinsic beauty of mathematics, with logic and proofs, others find additional appeal in the relevance of mathematics to societal needs. Several authors (Friedman & Littman 1994, MacCluer 2000 etc.) show how industrial problems can be used to introduce new topics in mathematics.

The industrial projects can provide at the same time an essential training for industrial carers, and can also have an additional impact and motivation that cannot be obtained in any standard course or academic experience. Many students are motivated by working with a company on an industrial problem: the problem is real and the company needs a solution; having a mathematical solution impact a corporate decision is both challenging and rewarding for the students.

At the foundation of its educational philosophy, Worcester Polytechnic Institute has had, from its very beginnings in 1865, the balance between theory and practice. The vision of WPI's Founders to emphasize the mutual reinforcement between theory and applications, reflected in the university's motto "Lehr und Kunst", is present in all campus activities.

Starting from its successful experience of project-based curriculum, WPI has been at the forefront of the national movement to develop opportunities for mathematics students and faculty to gain experience with the applications of mathematics in real world settings, in particular in industrial problems typical of those in which scientists and engineers would depend upon mathematics for solutions (Davis 1998, 1999). In 1997 the Center for Industrial Mathematics and Statistics was established, as the interface between the Mathematical Sciences Department and business, industry and government; guided by an Advisory Board formed by industry executives, the faculty members affiliated with CIMS made a concerted effort to integrate industrial applications vertically at all levels in the mathematics curriculum (Berkey & Vernescu, 2007). CIMS has provided a wide range of opportunities from industrial projects for mathematics seniors, to industrial projects and internships for Master's and PhD. Students, to the development of K-12 outreach programs in industrial mathematics.

Undergraduate Projects

Senior-year projects are part of the degree requirements of all WPI seniors. Most usually a team endeavor, these projects are a substantial piece of work, equivalent to three courses in terms of credits and spanning over 3/4–4/4 of the academic year. The projects provide a capstone experience in the student's chosen major that develops creativity, instills self-confidence and enhances the ability to communicate ideas and synthesize fundamental concepts. By completing the project, the students are expected to be able to formulate a problem, develop a solution and implement it competently and professionally; be exposed to interaction with the outside world before starting their careers; be able to work in teams and communicate well orally and in writing. A cornerstone of the WPI education, the senior-year projects have been highly successful at involving undergraduates in significant research.

The first senior-year projects in industrial mathematics were developed in the early '90s and were sponsored by PresMet, a manufacturer of pressed-metal parts for the automotive industry and Morgan Construction (currently part of Siemens), at that time the largest US manufacturer of steel rolling mills. Since then, faculty affiliated with the CIMS have developed over70 senior-year projects, in collaboration with over 45 sponsors from industry, business and government such as: Bose, Compaq, Deka Research, GE, IBM, Procter & Gamble, Travelers, United Technologies, Veeder-Root, etc. Together with the sponsoring company, the advisors provide the initial formulation of the problem, in such a way that the mathematical and/or computational modeling are the essential part of the project, at the same time maintaining its industrial relevance. The students start by understanding the problem, which requires most often understanding the engineering language and learning possibly new mathematical theory. Students come to appreciate the difference between the textbook exercises and the industrial problems; in textbooks, problems are always well formulated and are always based on the material in the preceding chapters, while the industrial problems may require them to ask more questions for a complete formulation and always requires them to figure out what mathematics tools are needed. Students also understand that re-formulating the problems is an essential part of the industrial experience; better understanding the problem can lead to a different formulation. Students realize that in the corporate world time is of essence: an answer in the timeframe given by the economic cycle is at times more valuable than the best answer given a few years later. In most of these projects the need and benefits of teamwork is reinforced; students with different backgrounds bring different perspectives to the problems. And finally the students are forced to improve their communication skills as they need to present the results in a language accessible to their corporate audience, most often a different language than the one needed for the presentations to their mathematics faculty advisors. The entire experience is different from any type of coursework or even other types of project work.

The following are a few examples of past projects.

Mathematical Modeling in Metal Processing (1996 and 1997). 6 students; Advisor: Prof. Bogdan Vernescu; Sponsor: Morgan Construction Co.

These projects developed several methods to simulate the wear and optimal geometry of the Laying Pipe mechanism, used for coiling steel rods in rolling mills. The wear models included inertial effects and Coulomb-type friction. Strong correlation between the computed wear and the measured data was obtained. To improve the wear distribution an optimized shape was obtained using optimal control theory and calculus of variations techniques.

Mathematical Models of Damage Spread in Networks (2002). 1 student; Advisor: Prof. Arthur Heinricher; Sponsor: Lehman Brothers

This project described mathematical models for how damage, measured in lost capacity, can spread through an organization. The model allows a company to simulate damage spread and compare different strategies for allocating repair resources after the initial damage has occurred. This project was completed in collaboration with Lehman Brothers investment firm in the aftermath of the terrorist attacks in September 11, 2001.

Modeling of Torque for Screw Insertion Processes (2006). 3 students; Advisor: Prof. Suzanne Weekes; Sponsor: Bose Corporation.

A self-tapping screw is a high-strength one-piece fastner that is driven into preformed holes. Students analyzed and improved a mathematical model of the self-tapping screw insertion process so that it can be used in manufacturing processes at the BOSE Corporation. They built a Graphical User Interface in MATHLAB which allows users to enter fundamental data and produces the corresponding torque curve. The accuracy and robustness of the model was tested by comparing predictions to empirical data collected at BOSE.

Research Experience for Undergraduates

Since 1998 we have provided research opportunities for undergraduates from other universities through an NSF sponsored Research Experience for Undergraduates (REU) in

Industrial Mathematics and Statistics, the first of its kind, to our knowledge (Vernescu & Heinricher (2000), Heinricher & Weekes (2007)). In this eight-week summer program, we have replicated for students from other universities the experience our own students have in their senior-year project. Teams of, usually four, students have a faculty advisor and an industrial advisor and work on an industrial mathematics project of interest for the sponsor. In addition to project work, we invite industrial mathematicians to share how mathematics is used in the real world. Mathematicians from companies such as Microsoft, John Hancock, Fidelity Investments and United Technologies are invited to talk with the REU students, and we take the students to visit companies like Bose, GE Plastics, IBM T.J. Watson Research Center, The Mathworks, United Technologies Research Center, and DEKA Research and Development.

Since 1998, over 130 students from 101 universities across the United States, Purto Rico, England and France have participated in the program. REU students have worked on over 35 different projects for over 15 different companies with 15 different faculty advisors. It is a measure of success that several companies, including Bose, John Hancock Insurance, DEKA Research and Development, Premier Insurance, State Street and Veeder-Root have sponsored several projects.

Here are some sample REU projects from past research summers (more are available on the CIMS web at www.wpi.edu/+CIMS):

Optimal Cession Strategies. Sponsor: Premier Insurance Co.; Faculty advisor: Arthur Heinricher; Industrial advisors: Richard Welch, CEO, and Martin Couture.

In the state of Massachusetts, the automobile insurance industry is highly regulated. Not only are insurance rates fixed by the state, but no company can refuse insurance to anyone who requests coverage. Companies do have the option to "cede" high-risk policies to a state agency. For every new policy, the company must make a decision: keep or cede the policy? This problem contains many of the interesting/difficult aspects of real decision problems. One project team analyzed the "risk" associated with cession strategies. A second project team developed an efficient algorithm for identifying "good" and "bad" agents, and showed that the agent who sold the insurance was one of the better predictors of loss.

Modeling Fluid Flow in a Positive Displacement Pump. Sponsor: DEKA Research and Development; Faculty advisor: Suzanne Weekes; Industrial advisor: Dr. Derek Kane. This was one of four projects completed for DEKA Research and Development. Dean Kamen, the founder and owner of the company, said in a recent interview, "I don't work on a problem unless it is going to make life significantly better for a lot of people." This philosophy has impressed and inspired four different REU teams at WPI. One particular project focused on modeling flow and phase transition in a simple type of pump. The company was able to use the models to optimize design parameters for the pump and improve efficiency of this component for a highly efficient water purification system for use in third-world countries.

Statistical Procedures for Failure-Mode Testing of Diagnostic Systems (2001 and 2002). Sponsor: Veeder-Root; Advisor: Arthur Heinricher; Industrial advisor: Robert Hart.

In most states, gas stations are required to have equipment that will collect and contain harmful vapors emitted during the refueling process. Stations must also have in place diagnostic systems that continuously monitor key aspects of the vapor recovery systems and issues a warning if any of the components are not operating within required limits. The project goal was to develop statistical tests that would certify that the monitoring systems met the standards set by the California Air Resources Board (CARB). The project team developed failure-mode tests in which artificial failures are created for the system to identify. Since 2007 these tests are part of the certification procedure used by CARB and EPA.

Estimation and Optimization for Constructing Hedge Portfolios. Sponsor: Deutsche Bank; Faculty advisors: Arthur Heinricher and Carlos Morales; Industrial advisor: Dr. Ara Pehlivanian.

Finance has been one of the fastest growing areas for applications of mathematics for several years. In the summer of 2003, a team of six REU students worked on a portfolio analysis problem for Deutsche Bank in New York. The goal of this project was to develop and test a new method for building a short term hedge portfolio. The team was divided into two groups, one to focus on statistical analysis to build good estimates for the covariance matrix and another to focus on optimization algorithms for identifying (quickly) good hedge pairs and constructing efficient portfolios using the hedge pairs. The students learned about the difficulties of working with “dirty data”. They also learned to trade optimality for efficiency and found that an approximate solution on time is better than an optimal solution too late.

Professional Science Master’s

At the graduate level, with support from the Alfred P. Sloan Foundation, we have introduced in 2000 a Professional Science Master’s degree in Financial Mathematics and one in Industrial Mathematics (http://www.wpi.edu/academics/Depts/Math/Grad/profms.html).

The relevant industry was involved from the very beginning in the design of these programs’ curricula, to provide the necessary training that would make graduates successful in industry (see Tobias et al. 2001, and Sims, 2006). Specifically designed to provide the training for the workforce, these programs do not preclude students from continuing in a

Ph.D. program. The programs preserve the core component of the traditional curriculum, and also require exposure to applications of mathematics in sciences and in engineering, experience in formulating and solving open-ended problems, and computational, communication, and teamwork skills. The Professional Science Master Programs require students to complete courses from other departments, introduce them to professional skills through a special seminar, and provide them with internship opportunities. In both programs, industrial experience is gained through an industrial project sponsored by local industry. Industrial summer internships are encouraged and facilitated by CIMS through its industrial partners.

The Financial Mathematics graduate program at WPI has been designed to lead students to the frontlines of the financial revolution of the new century. The program offers an efficient, practice-oriented track to prepare students for quantitative careers in the financial industry including banks, insurance companies, investment and securities firms; it features coursework on mathematics and finance, computational laboratories, industrial internships, and project work. The goal is to provide the knowledge, skills, and experience necessary for the quantitative positions in investment banks, securities houses, insurance companies, and money management firms. A strong mathematical background is build for developing mathematical models for risk in relationship to returns, trading strategies, structured contracts, and derivative securities. A strong collaboration has been built between our faculty and the financial services industry concentrated in the Boston-Hartford corridor. Our graduates have started their careers in jobs involving financial product development and pricing, risk measurement and control, and investment decision support or portfolio management.

The Industrial Mathematics Program is aimed at training students for professional careers in industrial environments. In addition to a strong mathematical background with depth in one area, the program emphasizes the breadth required by the industrial environments through elective coordinated modules of mathematics and engineering/science courses (e.g. physics, computer science, mechanical engineering, electrical and computer engineering, bioengineering), tailored to individual students' interests and needs. By also developing the students' communication and business skills, the program aims to creating successful professionals for the corporate world. Our graduates have started, or continued, their jobs in the manufacturing, software and environmental industry.

Ph.D. Program

A multidisciplinary experience, rooted in the real-world is also part of the Ph.D. Program in Mathematical Sciences, which has as degree requirement a 1–9 credit project outside the Department. Taking students out of their comfort zone and challenging them with real-world problems has a significant impact on their mathematics understanding and appreciation and at the same time can open new career options, so valuable in times of economic downturn. Ph.D. students worked on projects for the Kodak, Air Force Labs, IBM, Sandia and LawrenceLivermore National Laboratories and United Technologies.

Workshops for Faculty

WPI was chosen to be the site for the first SIAM Mathematics in Industry Regional Workshop (Davis, 1998), held in May 18-19, 1998. The purpose of the workshop was to help mathematics faculty study and implement some of changes recommend in the SIAM Report on Mathematics in Industry. The keynote speakers from IBM, Kodak, Lucent and Lehman Brothers discussed the need for sophisticated mathematics modeling in industry, and participants presented successful university-industry project collaborations. This workshop was the first in a series of six regional workshops that brought together mathematicians from industry and academia to showcase their scientific and educational collaborations. It was followed by five other regional workshops at the University of Illinois in 1998 (Midwest), Claremont Colleges in 1999 (Western), North Carolina State in 1999 (Southeast), University of Washington in 2000 (Northwest) and University of Huston in 2001 (Southwest). These workshops have emphasized the need for better connecting the mathematics world with the industrial world and are now followed by the biannual SIAM Mathematics for Industry Conferences.

The Mathematics Problems in Industry Workshops (MPI) were hosted in 2003 and 2005 at WPI, as part of a joint effort between faculty members at RPI, U. Delaware and WPI. The workshops attract leading academic mathematicians and graduate students who work for the week on problems posed by engineers and scientists from industry. In the past, these problems have included, but were not limited to engineering and product design, process design and control, environmental remediation, scheduling and optimization, financial modeling. As outcomes, these workshops provide a better understanding of existing models and methods, access to advanced computing solutions, and open a dialog with mathematicians in academia and government labs that provides a new look and fresh ideas.

The problems presented in 2003 and 2005 at WPI were: Safe Fuel/Air Slow Compression (Gilbarco/Veeder-Root, Simsbury, CT); Optimal Wear for a Laying Pipe (Morgan Construction, Worcester, MA); Need a Lift?: An Elevator Queuing Problem (UTRC, East Hartford, CT); Enhanced Leak Detection in Fuel Tanks (Gilbarco/Veeder-Root, Simsbury, CT); Stability of the Oil-Air Boundary in Fluid Dynamic Bearings of Hard Disk Drives (Hitachi GST, San Jose, CA); Lubrication Layer Perturbations in Chemical-Mechanical Polishing (Araca, Inc., Mesa, AZ).

Middle and High School Outreach Program

Business and industrial applications of mathematics are clearly a valuable but often underutilized means of both teaching and motivating middle and high school students in mathematics. For too many students mathematics seems irrelevant, disconnected from their career interests. The main cause is that mathematics is often taught disconnected from applications and the real-world problems it is so well equipped to solve and thus mathematics becomes a barrier to overcome. Thus, it should come as no surprise that most students, particularly women and minority students, lose interest in mathematics. Research reveals (Fennema & Leder, 1990, Moses, 2001) that young women and underrepresented minority children do not persist in mathematics because they appreciate neither its connection to careers nor its utility in helping to solve society's problems. One-time school visits by a working scientist or engineer able to explain how mathematics can be used is not always the best way to encourage students to continue their studies (Campbell et al. 2002). It has also been shown that student performance is greatly improved when teaching concepts and applying them to concrete problems as opposed to only drill and practice (Wenglinsky, 2000). Therefore professional development for teachers that relates to the content and materials used in the classroom has been shown to be more beneficial for teaching and learning than professional development on abstract concepts.

In June 2000, with support from the NSF through the SIAM, we organized our first Mathematics in Industry Institute (MII) for teachers; thirty-six teachers from New England worked for five days on industrial projects. With support form the GE Foundation this pilot work was expanded into the Mathematics in Industry Institutes for Teachers, for four consecutive years: 2001-2005. The Institutes consisted of: workshops actively involving teachers in developing industrial projects suitable for use in the classroom; development of dissemination activities to bring these idea and resources to entire school districts; and incorporation of these projects into their curriculum. A follow-up workshop in the spring allowed teachers to share "best practices". With the completion of the 2005 MII, more than 240 teachers from 18 states and Canada have come to WPI to work on industrial mathematics projects. There are now 23 industrial mathematics projects for classroom use on the CIMS web site (http://www.wpi.edu/Academics/Depts/Math/CIMS).

The feedback on the impact of the institutes showed that teachers have changed the way that they think about mathematics, talk about applications for mathematics, and discuss quantitative careers in the classroom. For example, we saw an increase from 25% to 58% in the percentage of teachers who regularly use engineering and quantitative homeworks and examples. Approximately 50% of teachers reported an increase in the use of industrial math examples in the classroom.

The broader impact of these activities was reflected in the development of new courses, programs and student competitions in schools. In addition, versions of the projects were been included in the Benchmarks for High School Mathematics Education developed at the American Diploma Project (an affiliate of Achieve) and were used in the NSF-sponsored Mathematics and Sciences Partnership project connecting WPI, Boston University, University of Massachusetts Lowell, the Educational Development Center, and five Boston-area school districts.

Conclusions

The model developed at WPI is one of vertical integration of innovative industrial projects in the mathematics curriculum at all levels, from middle and high school, through undergraduate programs and up to graduate programs and faculty research. At each level the industrial problems have a different impact, and the different components reinforce each other. For students, the educational impact cannot be replicated in the regular classroom setting; the industrial projects provide a different type of challenge and motivation and at the same time they provide the training needed for a more diverse set of career options. For faculty these offer an important connection to industry, business and government, and a challenge to use these in the educational program. For industry, business and government these projects have a significant impact: they provide solutions and insight on important problems and they provide insight in and access to the educational system. By seamlessly integrating the industrial projects at all levels an economy of scale has been created and the program can respond to both educational and scientific needs.

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Mathematics in transition from classroom to workplace: lessons for curriculum design

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Abstract This paper explores how curriculum specification and consequently classroom activity in mathematics might be informed by findings from research into the use of mathematics in workplaces. In doing so we consider the problem of 'transfer' by drawing on Beach's construct of collateral transition to conceptualise the transformation of mathematics required as it crosses boundaries. In considering how we might better prepare students to make use of mathematics in different settings we therefore recognise the need to understand how mathematics is constituted differently in college and workplaces and suggest some ways in which the 'academic practice' of mathematics might be developed and enriched in ways suggested by its use by workers.

Theoretical perspectives

This paper attempts to identify ways in which we might inform curriculum design, and consequently the mathematical experiences of college students so that they are better prepared to eventually apply mathematics in workplace settings. In particular we draw on findings from a research project that investigated the activity of workers with researchers, college teachers and their students interacting with workers to develop a dozen case studies across a range of workplace settings. In our analysis of these case studies from a socio-cultural perspective we drew in particular on Cultural Historical Activity Theory (CHAT) to understand the different practices we observed and to explore the relationship between workers, mathematics and their socially constituted workplace practices. In summary we noted the following important features relating to the practices of these workers as they went about those of their day-to-day activities that involved in some way use of mathematics:

- Knowledge is often crystallised (eg Hutchins, 1995) in artefacts, including tools and signs, often as a result of reification by workplace communities (Wenger, 1998).
- Use of mathematics is often "black-boxed" (Williams and Wake, 2007) and engagement with mathematics often only occurs at 'breakdown' moments (Pozzi et al, 1998).
- The fusion (Meira, 1998) of mathematical signs (in the sense of Pierce) with the reality they represent reduces cognitive effort.

These findings point to the problems "outsiders", such as researchers and college students, face when attempting to make sense of mathematics of workplaces and highlight that there are in fact different genres of mathematics that have culturally and historically evolved in the different settings of colleges and workplaces. Indeed, because of the situated nature of workplace activity, it is perhaps easier to detect different mathematical genres across different workplaces than to immediately see similarities. It seems essential to tackle this particular issue, however, if we are to be able to draw some inferences for curriculum development that might better prepare young people mathematically for future workplace activity. Before doing so we draw further attention, albeit briefly, to further theoretical conceptualisations that might inform our deliberations.

In an attempt to identify and understand the "space" in which students/workers struggle to "transform" their knowledge of the academic genre of mathematics to assist with mathematics for the workplace we turn to the CHAT constructs of boundary crossing and boundary objects (Star, 1989). It is in such "spaces· where people and objects cross boundaries that contradictions and dissonances arise potentially leading to conflicts for individuals, resolution of which may lead to the type of learning that we commonly understand as "transfer". Our research methodology of developing case studies with researcher, college students

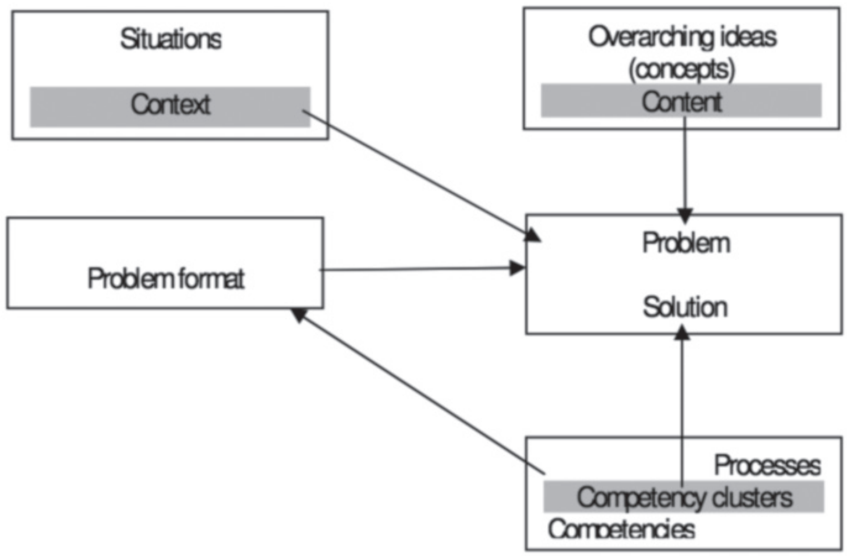


Figure 1—OECD / PISA framework of the mathematics domain.

and workers, therefore, deliberately prompted boundary crossing and invoked "breakdown" moments that we found particularly productive in provoking engagement with, and the unpacking of, mathematics in its widest sense. To explore the transformation of mathematics between "academic" and workplace genres we have found Beach's (1999) construct of consequential transition particularly helpful. In his critique of the usual understanding of "transfer" Beach considers how we might better conceptualise the (re-) construction of knowledge, skills and identity in relation to different communities of practice (Wenger, 1998) as involving a horizontal developmental process. In doing so he draws on Engestrom (2001) who in discussion of expansive learning argues for greater attention to be paid to development that might be considered horizontal, as opposed to the usual notion of developmental progress being vertical, such as when moving form everyday to scientific understanding (Vygotsky, 1986). In the case of transition from college to workplace, therefore, we suggest it is such horizontal development that we seek, as individuals as boundary crossers ideally deconstruct and reconstruct mathematical understandings in a mutually recursive way. In doing so, as we shall argue in conclusion, we wish to move away from the usual conception of developmental progress as constantly being "upwards and onwards", but rather, in relation to preparation for work, we seek horizontal development that results in enriched mathematical understanding and application.

Mathematical Practice

In turning to consider essential factors that impact on mathematical activity, potentially in any setting, we turn to a framework used to describe the domain by the PISA (OECD, 2003) series of international comparative studies as schematised in Figure 1.

Although this was developed for use in educational settings we suggest it may have some validity in both college and workplace settings drawing, as it does, attention to the key factors of (i) content, (ii) competencies and (iii) context that contribute to the mathematical *practices* in which students and workers engage. Despite its limitation in not recognising other important socio-cultural factors such as power, identity, and agency as contributing to how mathematical practices evolve, it does signal the important contribution of *each* of these key factors. We find this helpful in case we are tempted to become sidetracked and consider any one of them in isolation. It is our experience, in England, for example, that much discussion focuses on content whereas the domains of context and competencies are given far less attention; possibly because culturally and historically these are less well evolved and we consequently have less of a common understanding of them. Before synthesising our thinking with reference to this framework, however, we take an illustrative diversion by describing a little of one of our case studies that might serve to illustrate some of our overall findings. We do so with the health warning that each workplace we visited, and therefore case study we developed, cannot be considered in any way to be 'typical'. It is the richness and uniqueness of each case study that perhaps highlights how transformation of mathematics from college classroom to "shop-floor" seems particularly challenging. However, it is also this diversity of practice that points to the potential of mathematics to empower us to quickly come to understand a wide range of workplace practices in complex settings.

Illustrative case study

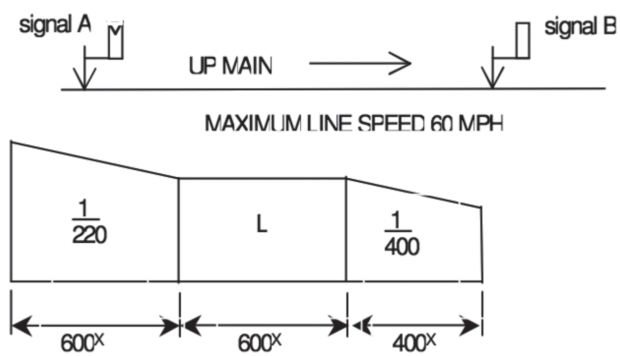
One case study in which there appeared to be immediate and obvious evidence of mathematical activity was that of the railway signal engineer. Figures 2 and 3, show the artefacts at the heart of the work of Alan, a railway signal engineer, who had worked his way up from a junior position to his role as "design checker".

As part of his job Alan checked the work of colleagues who calculated where to place signals alongside the railway track to give advance warning to train drivers that ahead there is a signal at which they should bring their train to a halt.

Having explained the context of his work, Alan went on to explain how he uses the chart in Figure 2 to determine the distance between signals for a section of track for which he has calculated the average gradient and for its maximum speed limit. He explained that, for example, for a gradient of 1 in 400 rising he notes that this lies between gradients of 1 in 200 rising and the level (gradient = 0), and for safety he would therefore take the value associated with a level gradient. He went on to further explain that the steeper the rising track the shorter braking distance that is required due to gravity acting to slow the train down.

INITIAL SPEED (Mph)	GRADIENT Rising 1 in 50 2.0%	1 in 67 1.5%	1 in 100 1.0%	1 in 200 0.5%	Level Level	Falling 1 in 200 0.5%	1 in 100 1.0%	1 in 67 1.5%	1 in 50 2.0%
20	175	180	195	215	240	275	320	395	520
25	240	255	280	315	355	410	485	625	840
30	320	340	380	425	485	575	700	895	1425
35	405	440	485	550	635	780	1010	1380	2237
40	495	550	620	720	865	1080	1420	1903	2237
45	630	710	805	935	1130	1435	1660	1903	2237
50	688	748	816	935	1130	1435	1660	1903	2237
55	770	831	901	984	1130	1435	1660	1903	2237
60	849	911	980	1061	1165	1435	1660	1903	2237

Figure 2—Table used to determine the distance required between signals for different speeds and gradients.



SECTION	DISTANCE IN YARDS	GRADIENT	RISE (YDS) (DIST * GRAD)	FALL (YDS) (DIST * GRAD)
X - A	600	1/220	-	2.73
A-B	600	L	-	-
B-Y	400	1/400	-	1.00
TOTALS	1600			3.73

1600/3.73 = 428.95 THERFORE AVERAGE GRADIENT= 1/429 FALLING

Figure 3.—Example average gradient calculation from training handbook.

It was clear that Alan doesn't need to think about this, the required understanding has become operationalised in his practice as he knows to always use the value from the table "to the right". Follow-up discussions with the student group who visited Alan pointed to a lack of understanding of the nature of a gradient of 1 in 400 in relation to other gradients such as 1 in 200 rising and level. Equally problematic was their ability to understand the training calculations of Figure 3 that shows how to find the average gradient over a number of sections of track. Alan explained that such averaging needs to be performed prior to using the chart of Figure 2. Crucially the students were unable to bring together understandings of gradient and average to make sense of the process illustrated. This brief extract from the transcript of the discussions between researcher and one student from a group of three illustrates this.

Researcher: Yes... So can you just explain what's going on in there [indicating the Table in Figure 3]?

Student: ... used different gradients for each slope and he's averaged it out...

Researcher: yes can you sort of explain the detail ...

Student: you started adding them together --- adding the gradients to gether and divide by two.

Researcher: Perhaps if we describe what each column is doing

Here the student appears to associate finding an average with the school mathematics procedure of "adding the values together and dividing by the number of values" – presumably in this case discarding the level gradient. The ensuing discussion was lengthy requiring the researcher to explain the basic concept of gradient, by drawing diagrams that illustrated, "for every 220 it goes along, it comes down 1, so when you've gone along [the track] 220 it's come down 1". Of course the worker's familiarity with the procedure causes him no problems with his calculations, but coming to understand this for the first time was clearly demanding for the students.

The nature of workplace mathematics

The above glimpse into the detail of one of our workplace case studies serves to illustrate very briefly just some of the challenges that we face when considering how school / college mathematics often does not prepare students well in making a transition from college to workplace activity. The comments below structured using three of the sub-domains of the mathematics domain framework of Figure 1 synthesise our analysis across case studies and draw on the detail of this particular illustrative example where possible. We have chosen here not to comment on the fourth sub-domain of problem format as this would lead into extensive discussion of workplace and college organisational structures.

(i) Context

We find that workplaces provide what appear complex contexts in which relatively straightforward mathematics is used in ways that are unfamiliar to college students. In the case study we illustrate above, the expectations in terms of complexity of the calculations to be performed are relatively low, but before being able to identify their exact nature, a great deal of background information is required. Indeed much of this is omitted here due to lack of space and it may be having read our brief account and inspected the figures you, as reader, are left with a number of questions about the exact workplace procedures. In many cases we found that workers had difficulty in identifying what was mathematics in their day-to-day activity. This is perhaps inevitable due to issues of crystallisation, but on other occasions we found that activity that was in effect mathematical looked somewhat different due to the workplace specific notation and tools that were used. For example, we often saw algebraic thinking and activity, somewhat disguised, perhaps in developing or using spreadsheets. On another occasion we saw a worker programming using a machine-specific language using mathematical thinking that included understanding of coordinates and vectors.

(ii) Content

As we have hinted above the mathematics content we saw workers relying on, or in a number of cases, (re-) constructing to solve specific problems, is often, if not always, in some form in the school/college curriculum. Take, for example, the calculation of average gradient illustrated in our railway signalling case study. Here, Alan brought together two pieces of mathematical content we typically find in the secondary school curriculum. However, as we point out our students had great difficulty in using their knowledge of these two concepts: we are left wondering why this is the case. In considering this we are particularly exercised by the nature of the incident that uncovered this lack of facility: the students were not asked to carry out the problem from scratch, but rather understand the mathematical working of someone else. Here, we asked the students to deconstruct and reconstruct an 'industry-standard' method, and even in this case they were not able to understand the application of two familiar concepts.

(iii) Competencies

It seems that workers use a range of competencies that embrace those of typical curricula (for example, representing situations using mathematics and interpreting mathematical models of situations) but also develop other competencies in relation to their use of mathematics. For example, above, we suggest that workers, do not always develop mathematical methods for themselves from scratch. "Making sense of the mathematics of others" is a competency that we observed in many situations. For example, a worker may be asked to use spreadsheets that have been developed by others to do a specific task. In such cases it may be that he or she has to explore the structure of the sheet and the algebraic thinking that underpins its operation.

On only a few occasions did we find workers setting out to model a situation from scratch However, we did find more instances where workers applied models that are standard to their work domain, or using artefacts that reified such models. These often bring mathematics from different content domains together in common ways. We have identified some of these as "general mathematical competences" (Wake & Williams, 2002) and note that other researchers seek to develop other constructs such as "techno-mathematical literacies" (Hoyles et al, 2007) to describe such competencies. Underpinning successful use of such competencies workers appear to need to have a deep understanding of the context in which they are being applied and an awareness that they are indeed working with a modified model and the assumptions on which it is based. Indeed, this awareness is often crucial in their interpretation of the results of the mathematics with which they engage.

Seeking to inform curriculum development

We return now to consider how our understanding of the use of mathematics across a range of workplaces might helpfully inform curriculum development to better support students in using mathematics in their transition from college classroom to workplace.

However, before focussing on specifics in relation to mathematics we return to consider how we might heed Beach's plea that we pay attention to horizontal developmental progress to inform our discussion. This provides a challenge as it is vertical development, with implicit upward motion through a hierarchy of knowledge, skills and understanding, that become more and more generalised, abstract and distant from the specifics of human activity that provides the dominant notion of human progress. Here in transition from college to work we ask students becoming workers to transform or create new relations between knowledge and social activities. We take the view alongside Beach (1999), Engestrom (2001) and others that this needs to be considered as progress and as such when proving problematic for an individual is not seen as a deficiency but rather something to be valued and supported.

Informed by this perspective, and in relation to mathematics we now draw attention to a number of important issues that we suggest should inform future curriculum development.

i. School/college mathematics is just one genre of mathematics and should be recognised as such with attention being drawn to the diversity of ways in which mathematics might appear elsewhere. This suggests that it is important to focus clearly on key mathematical concepts and principles and experience how these can be applied in a variety of different situations using a range of different notations, inscriptions and so on.

ii. Mathematics is used in a rich variety of contexts both in workplaces and more generally in communicating information in all walks of life; these contexts are often complex and detailed, although often simplified to allow mathematical analysis. Mathematics curricula should allow time and space for students to experience using their developing mathematical knowledge, skills and understanding in increasingly complex situations.

iii. Mathematical content, often due to curriculum specification, is seen as compartmentalised and atomised, with resulting pedagogies often being transmissionist and concerned with the development of instrumental rather than relational understanding (Skemp, 1976). Consideration should be given to how connections can be made across mathematical content areas in ways that may be commonly found in workplaces and other areas of application.

iv. Students appear armed with competencies in relation to mathematics that sees them particularly inadequately prepared to engage in using mathematics in workplaces. Particularly important in this regard is their lack of skill in making sense of the "mathematics of others". This is something that many workers have to do, given that they often take over parts of the work process that have previously been established. We note that our research pointed to a number of strategies useful in this regard (Wake, 2007) and suggest that these could be highlighted in curriculum specification.

v. Mathematics is used in workplaces to model complex situations and workers are often so immersed in their practice that the mathematics becomes 'fused' with their reality. Assumptions that underpin the mathematics are not made explicit but workers fully understand how these underpin the mathematics involved and can interpret changes in mathematical models in terms of workplace processes and outcomes. Curricula should provide students with experiences of working with mathematics in complex situations that mirror such scenarios with particular attention being paid to interpretation, variation and adaptation of models.

vi. Our research identified seven general mathematical competences (for example, interpreting large data sets, costing a project) (Wake & Williams, 2001) each of which we saw in use across a number of different workplaces. Fundamental to these is the expectation that technology is integral as a tool when mathematics is being applied. This is mirrored in other recent research that identifies and organises mathematics around techno-mathematical literacies. Curricula should recognise and emphasise such competencies.

Conclusion

Mathematics plays an important role in many workplaces, but this mathematics is not necessarily the same mathematics as that we see practised by students in college classrooms. There are many different genres of mathematics with college mathematics being just one, albeit privileged, genre. Coming to understand and operate with workplace mathematics requires additional knowledge, skills and understanding in relation to mathematics to those developed through current formulations of mathematics curricula. This requires developmental progress that might be considered horizontal (Beach, 1999) in that it does not involve students in the usual sense of making progress to higher levels of generalisation and abstraction, but rather in developing new relations between mathematical knowledge and social activities. In some instances such developmental progress may be supported by students taking part concurrently in both college learning and workplace activity, but for the many occasions in which this is not the case, mathematics curricula can be informed by a better understanding of issues in relation to mathematical activity considered in its widest sense. We make a number of specific suggestions regarding this here but consider that these are but a starting salvo in a battle to effect major changes to mathematics curricula in ways that might empower students in transition from classroom to workplace. We also draw attention to the need for further research to identify, for example, the content, competencies and contexts that should inform curriculum design and classroom pedagogies and the need for shared and common understanding and terminology relating to these so that future curricula might be designed to support transition and transformation of mathematics.

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Researching workers' mathematics at work

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Abstract School versus workplace knowledge is a fundamental issue in mathematics education. Mathematics is integrated within workplace activities and often hidden in technology. The so-called "transfer" of mathematics between school and workplace is not straightforward. However, lifelong learning assumes that learning takes place in all spheres of life. This paper discusses terminological and methodological issues related to reversing the one-way assumption from school knowledge to workplace knowledge and to learn from workplace activity what might be appropriate for adult vocational education. It is argued that any working model for researching the dynamics of workers' mathematics has to combine a general approach with a subjective approach.

A broad perspective on education, knowledge and technology

The perspective in the idea of lifelong learning, which structures today's educational system, demands a rupture with the limited conception of education, learning and knowledge. Individual and collective learning processes do not only take place as schooling within formal education, and the focus has now shifted from teaching to informal learning in the workplace and everyday life (Salling Olesen, 2008). This is also the case with mathematics and workers develop to a great extent their mathematics-containing competences through participation in the workplace communities of practice (FitzSimons & Wedege, 2007). In "World education report 2000", UNESCO (2000, p. 41) presents a terminology where this idea is set out explicitly with the distinction between informal, formal and non-formal education. *Informal education* means "the truly lifelong process whereby every individual acquires attitudes, values, skills and knowledge from daily experience and the educative influences and resources in his or her environment – from family and neighbours, from work and play, from the marketplace, the library and the mass media." For the most part, this process is relatively unorganized and unsystematic. *Formal education* refers to "the highly hierarchically structured, chronologically graded 'educational system', running from primary school through the university and including, in addition to general academic studies, a variety of specialized programmes and institutions for full-time technical and professional training." *Non-formal education* is defined as "any educational activity organized outside the established formal system – whether operating separately or as an important feature of some broader activity – that is intended to serve identifiable learning clienteles and learning objectives". However, this broad perspective on education is not reflected in the Discussion Document *Educational Interfaces between Mathematics and Industry* (2009) where education is recognised as formal education within the educational system or with non-formal education in the workplace (p. 3).

School knowledge versus everyday knowledge – such as workplace knowledge – is one of the fundamental issues in educational sciences in general and in mathematics education research specifically. In the educational discourse, *everyday knowledge* has the double meaning of (1) knowledge acquired or developed by an individual in her/his everyday life, and of (2) knowledge required in the individuals' everyday life as citizens, workers, students, etc. By *workers' mathematics at work* I mean mathematics acquired and developed by the individuals in their working life on the basis of previous experiences with mathematics in everyday life and formal education. Previous research mapped workplace mathematics onto school mathematics curricula where simplistic interpretations of mathematics used in the workplace were implemented (FitzSimons, 2002). During the last 15 years more sophisticated interpretations have taken into account social and cultural, even political contexts, but in the EIMI Discussion Document (2009) there is an implicit assumption of a one-way development from school knowledge to workplace knowledge. The document identifies mathematics with academic mathematics and does not acknowledge any importance of the workers' mathematics. Just after the definition of mathematics as "any activity in the mathematical sciences", it is stated that "Workers at all levels utilize mathematical ideas and techniques, consciously or unconsciously, in the process of achieving the desired workplace outcome" (DD, 2009, p. 2). Moreover, in the document, the learning of mathematics is always mentioned side by side with the teaching of mathematics. Thus, the only form of workplace knowledge occurring in the document is mathematics required by the workers.

In order to investigate the relationship between mathematics education and technology in the workplace, it is necessary to have a broad conception of mathematical knowledge and of technology as well. In the Discussion Document (2009, p. 4), it is stated that "'Technology' is understood in the broadest sense, including traditional machinery, modern information technology, and workplace organisation." However, my understanding of technology is even broader and more dynamic. I see *technology* on the labour market as consisting of three elements: technique, work organization, human competences and vocational qualifications – and of their dynamic interrelations. *Technique* is used in the broader sense to include not only tools, machines and technical equipment, but also cultural techniques (such as language and time management), and techniques for deliberate structuring of the working process (as for instance in Taylor's 'scientific management' and ISO 9000 quality management system). *Work organization* is used to designate the way in which tasks, functions, responsibility, and competence are structured in the workplace. *Human competences* are worker's capacities (cognitive, affective, and social) for acting effectively, critically and constructively in the workplace. *Vocational qualifications* are knowledge, skills and personal qualities required to handle technique and work organization in a work function (Wedege, 2000, 2004). Thus, in the third dimension of technology, one finds the two types of workplace knowledge mentioned above and a possible tension between them: knowledge developed in the individual's working life, in human competences, and knowledge required by the labour market, in vocational qualifications.

With a broad definition *artefacts* is "anything, which human beings create by the transformation of nature and of themselves: thus also language, forms of social organisation and interaction, techniques of production, skills" (Wartofsky, 1979, cited in Strässer, 2003, p. 34). The understanding of technology presented above involves three types of mathematics-containing artefacts. This is an important statement for me as it expresses my overall view on technology and on mathematics as both created by human beings. Thus, I welcome the EIMI study with its focus on educational interfaces between mathematics and industry because it reflects the need of and opens for an updated view on the relation between education, knowledge, mathematics, humans and technology.

In the international research project *Adults' mathematics: In work and for school*, we seek to explore the development of school knowledge and of workplace knowledge as a two-way process. The objective is to describe, analyze and understand adults' mathematics-containing work competences – including social and affective aspects – complementing studies of mathematical qualifications in formal vocational education in ways that will inform vocational mathematics training and education. This will be done through empirical investigations – quantitative as well as qualitative – in interplay with theoretical constructions[1].

The question to be discussed in the last part of this paper is the following: How is it possible to study semi-skilled workers' mathematics at work in a way that enables the researcher to learn from workplace activity what might be appropriate for vocational education and training? Thus, I discuss methodological issues related to reversing the one-way assumption from school knowledge to workplace knowledge. But first, I present a brief summary of previous research on adults' mathematics in the workplace.

Adults' mathematics in the workplace

In education research, the overall interest in studying adults' mathematics *in* the workplace is mathematics education *for* the workplace. Inter-disciplinarity is a significant feature in the field and researchers are drawing on research on mathematics education, adult education and vocational education (FitzSimons, 2002, Wedege, 200, 2004). Within mathematics education, the research field of vocational education and training has been cultivated internationally since the mid 1980s (Bessot & Ridgway, 2000; Strässer & Zevenbergen, 1996), and the research field of adults learning mathematics since the mid 1990s (FitzSimons et al., 1996). Concepts which recognise people's social competences, like ethnomathematics and folk mathematics, as well as concepts of adult numeracy, mathematical literacy and of mathemacy, have expanded the problem field of mathematics education research (Jablonka, 2003). Today it is scientifically legitimate to ask questions concerning people's everyday mathematics and about the power relations involved in mathematics education, and anthropological studies such as those of Scribner (1984), Lave (1988), and Nunes, Schliemann, & Carraher (1993) are paradigmatic when studying adults, mathematics, school and work.

In the international literature on mathematics in and for work, there are two problems that researchers agree upon, of which the Discussion Document (2009) has only acknowledged the first problem:

- Mathematics is integrated in the workplace activities and often hidden in technology (Bessot & Ridgway, 2000; Hoyles et al. 2002; FitzSimons, 2002; Wedege, 2000; Strässer, 2003; Williams & Wake, 2007).

- The so-called "transfer" of mathematics between school and workplace – and vice versa – is not a straightforward affair (Alexandersson, 1985; Evans, 2000; FitzSimons & Wedege, 2007; Hoyles et al. 2001; Wedege, 1999).

In summary, these studies with their focus on differences between mathematics in school and mathematics in the workplace show that mathematical elements in workplace settings are highly context-dependent. They are subsumed into routines, structured by mediating artefacts (e.g., texts, tools). It is the working task and function, in a given technological context, that control and structure the problem solving process. Some of these problems look like school tasks (the procedure is given in the work instruction) but the experienced workers have their own routines, methods of measurement and calculation. Thus, mathematics is intertwined with professional competence and expertise at all occupational levels, and judgments are based on qualitative as well as quantitative aspects. Circumstances in the production might cause deviations from the instruction or might for example raise or reduce the number of random samples in a quality control process in industry. Unlike students in the majority of school mathematics classrooms, workers are generally able to exercise a certain amount of control over how they address the problem solving process, albeit within the parameters of the expected outcome of the task at hand, regulatory procedures, and available artefacts. In the workplace, solving problems is a joint matter: you have to collaborate, not compete; and the activity of solving problems always has practical consequences: a product, a working plan, distribution of products, a price etc. Finally, because the focus is on task completion within certain constraints (e.g., time, money), mathematical correctness or precision may be somewhat negotiable, according to the situation at hand (FitzSimons & Coben, 2009, Wedege, 2002). One of the consequences of these differences between mathematics in work and mathematics in school is that the workers do not recognise the mathematics in their daily practice. Mathematics is invisible in technology but this is not the only reason. Workers do not connect the everyday activity – and their own mathematics – in the workplace with mathematics which most of them associate to the school subject or the discipline (Wedege, 2002).

Methodology

The worldview guiding thinking and action in a study of semi-skilled workers' mathematics at work – in a way that enables to learn from workplace activity what might be appropriate for vocational education and training – can be described as the *transformative paradigm*. Within this paradigm, research places central importance on the lives and experiences of groups (in this case semi-skilled workers) that traditionally have been marginalised (in this case in relation to mathematics education) and on issues of power relationships (in this case between academic mathematics and workers' mathematics). According to Mertens (2005), within the transformative paradigm, multiple realities shaped by social, political, cultural, economic and gender values are recognised (*ontology*). The relationship between the researcher and participants is viewed as interactive and knowledge is seen as socially and historically situated (*epistemology*). Finally, the approach to systematic inquiry includes qualitative, dialogical methods but quantitative and mixed methods can also be used (*methodology*).

In a study of workers' mathematics in the workplace two different lines of approach are possible and intertwined in the research: a *subjective approach* starting with people's competences and subjective needs in their working lives, and a *general approach* starting either with societal and labour market demands to qualifications and/or with the academic discipline mathematics (transformed into "school mathematics") (Wedege, 2004). The subjective approach is to be found in studies like the one that I did in 1997-98 where the focus was competent semi-skilled workers mathematical activities in different work functions within four lines of industry: building and construction, commercial/clerical, metal industry and transport/logistics (Wedege, 2000). The methods used in this study were inspired by a project initiated by the Australian Association of Mathematics Teachers in 1995-1997, but the approach was different. The starting point in the Australian project was that workers used mathematical ideas and techniques (Hogan & Morony, 2000). The aim was to generate 40 stories with rich interpretations of workers use of mathematics, and the people shadowing and interviewing workers in a long series of different workplaces were mathematics teachers. Thus, their lens was school mathematics and the approach was general. Two of the workplace studies in mathematics education research illustrate the conflict between the general and the subjective approaches. In a study on proportional reasoning in expert nurses' calculation of drug dosages Hoyles et al. (2001) compared formal activities involving ratio and proportion (general mathematical approach) with nurses' strategies tied to individual drugs, specific quantities and volumes of drugs, the way drugs are packaged, and the organization of clinical work (subjective approach). In their large project involving 22 case studies, Hoyles et al.'s, (2002) research questions were about employers' demands for mathematical qualifications, competencies and skills (general societal approach) and about what skills and competencies the employees felt were needed for the job, and what they currently possessed (subjective approach). However, to understand the cognitive, affective and social conditions for adults' knowing mathematics one has to take both dimensions into account (see Fitzsimons, 2002; Wedege, 1999; Wedege & Evans, 2006).

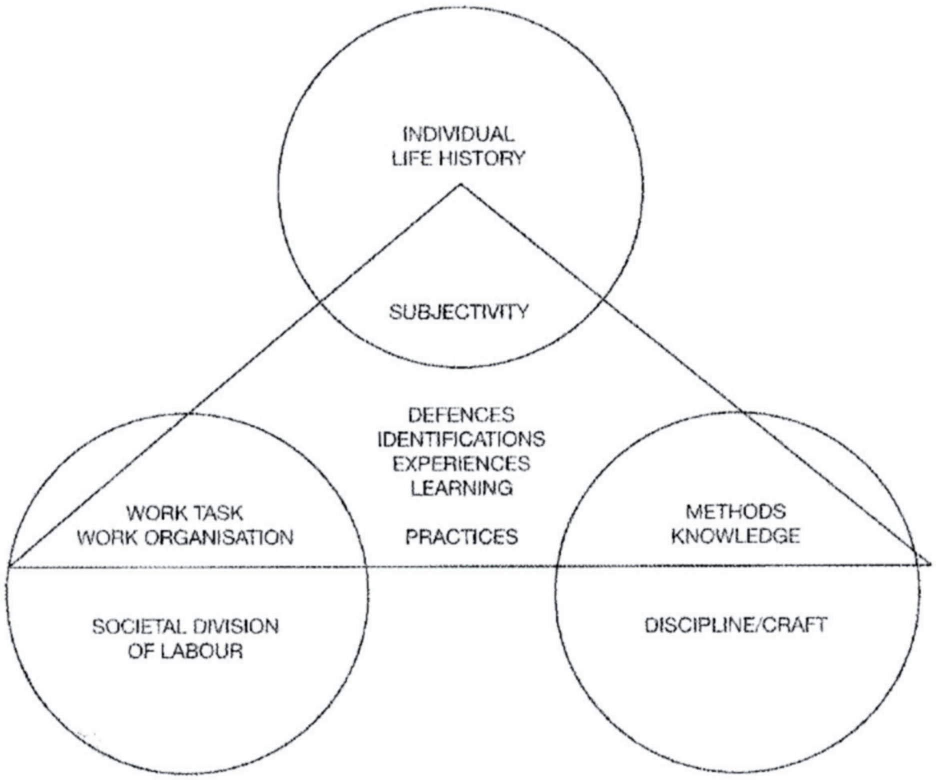

Figure 1—Workplace learning (Olesen, 2008, p. 119).

In Salling Olesen (2008) we find a working model for researching the dynamics of workplace learning in general which includes and combines the general and the subjective approaches. "It mediates the specific relation between three relatively independent dynamics: the societal work process, the knowledge available and subjective experiences of the worker(s)" (p.118) — see Figure 1.

When researching adults' mathematics in work, this model focuses on the cultural and societal nature of the knowledge and skills (competences) with which a worker approaches and handles a mathematics-containing work task, whether they come from the discipline (mathematics), a craft (vocational mathematics), or just as the established knowledge in the field (ethnomathematics).

Qualitative research is necessary to capture the complexity of the worker's cognitive, social and affective relationship with mathematics. We will use observations in the workplaces in combination with semi-structured and narrative interviews. We will also combine theoretical perspectives to capture different aspects of mathematical competence and take the importance of the institutional frame seriously into account for recognizing what are important mathematical qualifications in the particular context (FitzSimons, 2002; Wedege, 1999).

Quantitative research in the shape of a survey can provide an overarching picture of the workers' social and affective relationships with mathematics. Supporting this, the voices of the adults have already been heard in a series of qualitative studies (e.g. Evans, 2000; Hoyles et al. 2001; Wedege, 1999). With this information it is possible to create test batteries based on prior knowledge to find dimensions in the worker's conceptions about mathematics. Based on our assumption that the importance of knowing mathematics is experienced differently by men and women, gender will be an explicit dimension throughout the whole study (Henningsen, 2007).

Perspectives

As stated above, one of the fundamental issues in mathematics education is school mathematics versus out-of-school mathematics. In the context of the study *Educational interfaces between mathematics and industry*, this topic should also be talked about in terms of power in the labour market. In his book "The Politics of Mathematics Education", Mellin-Olsen (1987) stated that it is a political question whether folk mathematics, like workers' mathematics, is recognized as mathematics or not. Similarly, FitzSimons (2002) claims that the distribution of knowledge in society defines the distribution of power and that, in this context, people's everyday competences do not count as mathematics. The research project *Adults' mathematics: In work for school* is innovative in that it seeks to reverse the one-way assumption from school knowledge to workplace knowledge and to learn from workplace activity what might be appropriate for vocational education and training.

Salling Olesen (2008) asserts that "every work situation has elements of subjective engagement, cognitive construction and social interaction" (p. 126). In the context of their individual life histories and social experiences, workers can decide to develop new qualifications while resisting or neglecting others. This is critical in the case of mathematics learning, because many workers at all levels have experienced the institutional culture of formal schooling as alienating and have made little or no identification with the teaching and the texts. They tend to lack confidence in using formal mathematics because of the traditional focus on the discipline as absolute and infallible, which is in marked contrast to the negotiability (when reasonable) and commonsense being valued attributes in the workplace. Moreover, there is an apparent contradiction between many adults' problematical relationships with mathematics in formal settings and their noteworthy mathematics-containing competences in working life. These are some of the phenomena causing resistance to learning mathematics (Wedege & Evans, 2006). Our research project has the potential to help adult workers to overcome some of the cognitive and affective obstacles if they can recognise their own realities reflected in the official mathematics instruction in general schooling and vocational education and training.

Conclusion

In summary, I believe that the theoretical foundations of the Swedish research project discussed above, together with the evolving literature review and preliminary findings, have much to contribute to the EIMI study. The perspective of focusing on the workers' mathematics rather than the discipline has the potential to open up the discussion and offer new insights to mathematics educators and mathematicians alike.

Notes

1 The research project "Adults' mathematics: In work and for school" will involve 12 academics from 11 universities in 6 countries. Tine Wedege (Sweden) is the research leader of the project. Gail E. FitzSimons (Australia) will be involved in all phases of the project. Inge Henningsen (Denmark) will assist in designing the survey and in handling and analyzing of the quantitative data. Lisa Björklund Boistrup (Sweden) will assist in developing new methods for investigating adults' mathematics in work and participate in the qualitative study. An international reference group consists of researchers from adult and lifelong learning, mathematics education, learning in the workplace, mathematics in and for work, and vocational education & training. Among the members are: Corinne Hahn (France), Eva Jablonka (Sweden), Henning Salling Olesen (Denmark), Rudolf Strässer (Germany), and Geoff Wake (UK).

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Industrial Mathematics & Statistics Research for Undergraduates at WPI

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Abstract At Worcester Polytechnic Institute, we provide a unique educational experience for students of mathematical sciences by introducing them to the ways in which mathematics and statistics are used in industry and finance. Students work in teams on research problems that come directly from industry and which are of immediate interest to the companies. This opportunity to work on "real-world" projects is afforded to WPI students through the WPI educational plan, as well to students across the U.S. through the Research Experience for Undergraduates Program in Industrial Mathematics and Statistics which is funded by the National Science Foundation.These research experiences provide an excellent experience for students interested in following both nonacademic and academic career paths.

For the past twelve years, the National Science Foundation has funded the REU program in Industrial Mathematics and Statistics at Worcester Polytechnic Institute (WPI) in Worcester, Massachusetts. During the summers of 1998–2009, the program has worked with 130 students who have come from 99 different U.S. colleges in 31 states and Puerto Rico. Exactly 59 of the 130 students were female. Of the 99 universities that have been represented at the REU program over the last 12 years, 50 of them do not offer a PhD program in mathematics.

The goal of the WPI REU program is to provide a unique educational experience by introducing students to the ways that advanced mathematics and statistics are used in the *real world* to analyze and solve complex problems. The students work in teams on problems provided by local business and industry. They work with a company representative to define the problem and to develop solutions of immediate importance to the company; they work closely with a faculty advisor to maintain a clear focus on the mathematics and statistics at the core of the project. When students work with a company on an industrial problem, the problem is real and the company needs a solution. This is usually the first time that students are placed in a situation where someone is going to make a decision, perhaps an expensive decision, based on their mathematical work.

The WPI REU program provides a glimpse of the many career possibilities which are open to students with a strong mathematical background. The hope is that by the end of the summer, the students will have better answers to the questions:

i. *What is the role of a mathematician in business and industry?*

ii. *What is it like to work with technical experts on a problem that requires significant mathematics but also must satisfy real-world constraints?*

iii. *What kind of mathematical and statistical tools are used to solve problems in business and industry?*

The program provides challenges not faced in standard undergraduate programs and strengthens skills not always developed in traditional educational programs. The *SIAM Report on Mathematics in Industry* provides a comprehensive study of these special skills, and they include:

(a) *communication at several levels, including reading, writing, speaking, and listening;*

(b) *problem formulation as an interactive, evolutionary process;*

(c) *the ability to work with a diverse team.*

The REU program at WPI provides an excellent experience for advanced undergraduate students going on to graduate school, whether they choose to specialize in applied mathematics or not. The experience is certainly valuable for students interested in following nonacademic career paths, but it is just as valuable for students who enter "traditional" graduate programs and go into academic careers.

WPI Project-Based Undergraduate Program

WPI has a special infrastructure and a long history that supports intense project activities like those in REU's. Project-based learning has been central to the WPI educational program for more than 40 years. In 1971, the *WPI Plan* marked a departure from the conventional approaches to undergraduate education. It introduced as degree requirements three types of projects: the Humanities Sufficiency, the Interactive Qualifying Project (IQP) and the Major Qualifying Project (MQP). The last is a senior-year project completed in the major field of study. It is often the work of a team and spans over 3/4 – 4/4 of the academic year. The purpose of the MQP is to provide a capstone experience in the student's chosen major that will develop creativity, instill self-confidence and enhance the student's ability to communicate ideas and synthesize fundamental concepts. In completing the MQP, students are expected to:

- formulate a problem, develop a solution and implement it competently and professionally,
- interact with the outside world before starting their careers,
- work in teams and communicate well both orally and in writing.

This project activity has been highly successful at involving WPI undergraduates in significant research with faculty. In 1987, NSF began funding Mathematics REU programs. In 1988, the WPI Mathematical Sciences department hosted its first REU program and was then able to involve non-WPI students in undergraduate research as well.

In order to enhance the industrial project experience for our students and to help make new contacts with business and industry, the Center for Industrial Mathematics and Statistics (CIMS) was established at WPI in 1997. Members of the Center work to establish contacts with industry, businesses and government labs and to develop industrial projects at both the graduate and undergraduate levels for our majors. The industrial mathematics program at WPI has been extremely successful. More than 200 hundred students have completed industrial projects with 30 different companies. We work throughout the year to make new company contacts, maintain existing contacts, and to develop new project opportunities for our WPI and REU students.

REU Recruitment

Each summer, we recruit between 10 and 12 undergraduates to take part in our intense, residential REU program which lasts nine weeks. Interested students apply online at

http://www.wpi.edu/+CIMS/REU

They fill out a standard application form with personal data, education history, courses taken, and are asked to rank a list of preferred project areas. Recommendation letters from faculty are also required. Very importantly, students must write a one-page essay describing his or her interest in participating in the WPI program. After review by the Principal Investigators, promising applications are selected and these students receive a phone call from the program coordinators in a joint information exchanging and interview process. Following NSF guidelines, consideration is given to only U.S. citizens or permanent residents. However, exceptional applicants who do not satisfy these requirements are also considered, provided the availability of additional funding from our industrial partners; we have had one British student from Oxford University and a French student from the Université de Savoie. The final group of students gathered are all interested in applied and industrial mathematics and statistics, and have a diverse range of course backgrounds and interests since the industrial projects usually require a mix of probability, statistics, differential equations, numerical analysis, and optimization.

REU Program Structure

The process of meeting a real-world problem, learning to ask good questions and doing the research needed to identify the key mathematical structure, and then refining and redefining the problem, is a crucial part of the industrial mathematics experience. Also, the project is not finished when the problem is "solved." It is important that the students communicate their solution to the company in a form that the company can understand and use. Communication skills, written and oral as well as listening skills, are crucial for a successful industrial mathematician. This is developed via

i. *daily meetings* with faculty advisors,

ii. *weekly presentations* to fellow students and faculty, and

iii. *regular meetings with industrial sponsors*.

The REU students work in teams of 2–4 and each group has at least one faculty advisor plus an industrial advisor. Each team is given an office with 2 or 3 networked computers. (In 2003, while the Deutsche Bank research team was testing their portfolio model, the students had a total of 7 computers working in parallel.) Teamwork is one of the skills required for a mathematician working in industry and one responsibility of the faculty advisor is to observe and guide the team-building process. The students meet with the faculty advisor(s) every day over the course of the nine week project.

Each team makes periodic progress reports, in the form of written reports and oral presentations, for their fellow students, the faculty and industrial advisors. We also invite faculty from other departments and representatives from local companies to attend these *weekly presentations*. The students receive extensive feedback as the projects evolve through the summer. The students gain valuable practice in presenting their work; the improvement in quality during the summer is quite impressive. At the end of the two months, a *Presentation Day* is organized for the students to formally present their final results for the faculty, invited university administrators, and industrial advisors. This is followed by a special, celebratory lunch.

Each team of participants must complete a *written report* based upon the research they have completed during the summer program. The purpose of this report is to describe the problem considered, the background literature read, the approach(es) taken, the results that have been obtained, and the questions motivated by the research. Participants begin writing parts of this report as early as the first week of the program so that the faculty advisor has an opportunity to assist the students in developing a proper style for writing mathematics.

We have also invited mathematicians and statisticians working in industry to meet the students and to discuss with them what sort of work they do, what their career paths have been, what their work environment is like, and other issues of interest to the students. Our special guests have included Dr. Keith Hartt, Founding Partner and Director of Research at Bogle Investment Management; Dr. Bill Browning of Applied Mathematics Incorporated; Dr. Robert LaBarre, Principal Mathematician at United Technologies Research Center; Mr. Bruce Kearnan, Senior Associate Actuary and General Director, John Hancock Life Insurance Company. Additional speakers have been "alumni" of our REU program who have gone on to graduate school or are currently working in industry.

There are also at least three special events scheduled during the program. The following are a some of the special events held in the past programs:

i. a tour of The MathWorks, with a presentation by technical staff on engineering applications for Matlab as well as career paths for mathematical sciences majors;

ii. a visit to the Research and Development labs at BOSE in Framingham, Massachusetts, with a tour of the acoustics and destructive testing labs;

iii. participation in the Math Problems in Industry one-week workshop which was held at WPI in June of 2003, 2005, and 2008;

iv. attendance at the SIAM Annual Meeting in 2006 which was held in Boston.

In order that student-faculty interaction is not limited to the academic dimensions, a group recreational activity is planned for most weeks. Activities have included a lobster dinner in Mystic, Connecticut, Boston Red Sox games (tickets provided by John Hancock), trips to view the Fourth of July celebration in Boston Harbor, as well as barbecues at the faculty advisors' homes.

The industrial projects have become the foundation for several successful outreach programs; this vertical integration of our work is described in a separate article.

REU Projects Completed

Below, we list just a few of the projects completed in our summer REU program along with the industrial sponsor.

(a) *Modeling Fluid Flow in a Positive Displacement Pump*
DEKA Research and Development Corporation, Machester, NH

(b) *Mathematical Model for an Electro-Pneumatic Pulsed Actuator*
Applied Mathematics, Inc., Gales Ferry, CT

(c) *A Continuum Model for the Growth of Brain Tumors*
IBM Corporation, Boston, MA

(d) *Statistics Procedures for Failure-mode Testing of Diagnostic Equipment*
Veeder-Root, Simsbury, CT

(e) *Adaptive Risk Score Assignment Model for Underwriting Long-Term Care Insurance*
John Hancock Life Insurance Company, Boston, MA

(f) *FEMLAB Electromagnetic-Thermal Model of Microwave Thermal Processing*
Ferrite Corporation, Nashua, NH

(g) *Portfolio Optimization with Non-Smooth Constraints*
Goldman-Sachs, New York, NY

(h) *Mathematical Model of the Self-Tapping Screw Insertion Process*
BOSE Corporation, Framingham, MA

(i) *Quantifying Uncertainty in Predictions of Hepatic Clearance*
Pfizer, Cambridge, MA

(j) *Passive Currency Hedging Analysis*
State Street Global Advisors, Boston, MA

REU Alum

Our WPI Industrial Mathematics and Statistics alumni can be found in academia, as graduate students and faculty all over the U.S., while others have gone on to successful careers in business and industry. From the feedback and correspondence that we get from our students, the program has made a positive impact on their lives.

From a WPI REU alum via email:

> **Subject:** I got a job!
>
> I was offered a job as an "engineering assistant" with the company Applied Research Associates at their lab on Tyndall Air Force Base near Panama City, FL. My position will mostly be mathematical modeling using MATLAB and FEMLAB (and some other random tasks, occasional lab assistancy).
>
> ...
>
> Clearly, I have the REU to thank. The company actually discovered me through Monster com because I had FEMLAB in my resume, and that was one of the main reasons they were interested in me. But beyond the "marketable skill" of knowing FEMLAB, the REU is what got me really interested in mathematical modeling--and working in a research environment – in the first place. So to my colleagues and our advisers and everyone else who helped us during those transformative two months last year, thank you!

Comments from Industry

Below we provide quotes from letters from two sponsors, State Streel Global Advisors, and Bose, that give their view of sponsoring research with the undergraduates in the WPI Industrial Mathematics and Statistics Research Program, and the impact that the research has on their work.

From State Street Global Advisors:

> Because the REU program provides faculty support and enables students to access the university IT infrastructure we are able to specify much more complex projects

to challenge the students. ... We were extremely impressed with the skill that the students displayed in the areas of financial mathematics and computing. Not only did the students meet the high expectations required by the complexity of the project, they showed genuine insight and at times even ìstumped the experts...

Overall, the REU program opens the door to a realistic and complex learning experience that is not easily attainable in either a pure classroom or pure internship setting. Thank you for the opportunity to participate in the program and we look forward to future project partnerships.

From Bose:

I very much enjoy working with the students since they bring a level of passion to the projects that is not often seen in industry. They bring a "can do" attitude to every study and that is always refreshing. ... The torque model that WPI has helped us develop over the last 3–4 years has helped us make progress in fastener joint design for the projects we design. The torque model has direct applicability in supporting and improving our designs for assembly. ... I believe that these types of efforts help the student and our engineers. The students get an idea of what happends in industry. They can see how their discipline can be applied to various manufacturing problems. It also allows them to see and understand the compromises that sometimes need to be made given the constraints of the problem. ... I have enjoyed working with the WPI "family" over several years and I plan to keep WPI in mind for future student projects.

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A framework for mathematical literacy in competence – based secondary vocational education

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Abstract In competence-based vocational education there is a risk that mathematics becomes invisible and unassessed, with deteriorating mathematical skills as a consequence. This is what happened in the Netherlands. To reverse this trend we developed a framework of reference for mathematical literacy in secondary vocational education (MBO) with which domains and levels of mathematical literacy required in all MBO occupations could be identified. This chapter focuses on the lessons that we learned from the process of developing the framework with all relevant stakeholders, and from how it was used by the Centres of Expertise. In order to generalize our experiences we formulate criteria that such a framework should fulfil in order to be a useful interface between representatives of school mathematics and of services and industry.

Introduction

The position of mathematics in vocational curricula is the topic of a long-standing debate, especially in countries that move towards more competence-based vocational education (CBVE). In CBVE the structure with vocational and general subjects with attainment targets (including targets for mathematics) is substituted by a qualification structure based on general and vocational competencies. Despite potential advantages of CBVE such as higher student motivation (Van den Berg & De Bruijn, 2009), there can be major consequences regarding the visibility and accountability of mathematics in vocational curricula. Visibility here refers to how explicitly mathematics is mentioned in qualification files and therefore how visibly it ends up in curricular materials. Accountability refers to the responsibility stakeholders feel to assess students' mathematical knowledge and to pay attention to mathematical literacy as part of competence-based projects or apprenticeships.

In the Netherlands, the introduction of CBVE has led to a situation in which mathematical knowledge and skills are still required as part of occupational core tasks and work processes, but in the qualification files that describe the competencies a person needs to fulfill a job or function, hardly any explicit reference is made to the mathematics (or any other discipline) involved. This had a major impact on the visibility and accountability of the mathematics required in the 241 occupations that the Dutch senior secondary vocational education system prepares for. For some programmes (e.g. educational and nursing assistants) this has led to worrying situations and political uproar (Bronneman-Helmers, 2006).

To improve the visibility and accountability of mathematics in vocational education, several Dutch institutions concerned with mathematics education and vocational education collaborated in developing a national framework for mathematical literacy in secondary vocational education (in the following we use the Dutch abbreviation MBO). The framework is modelled after the Common European Framework of Reference for Languages (CEFR, 2004) because it is widely used in Europe to identify and prescribe the standards of language fluency for different vocational qualifications, because it supported the visibility and accountability of language requirements in Dutch vocational education, and because many Dutch teachers know this framework.

The main challenge was to develop a framework that could be used in a variety of communities such as schools, companies and national Centres of Expertise for Vocational Education, Training and the Labour Market (in the following just 'Centres of Expertise') and that would allow mathematics teachers, teachers of vocational subjects and professionals to communicate about the domains and levels of mathematical literacy required. Mathematical literacy is defined in the OECD Programme for International Student Assessment PISA as: "an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen." (OECD, 2003, p.15). The aim of this chapter is to draw lessons from our experiences in such a way that readers from other settings can learn from them. In particular the central question addressed in this chapter is: *What criteria should a framework of reference for mathematical literacy meet, in order to make mathematics in competence-based vocational education visible and accountable?*

Answering this question requires some background on the specific features of the Dutch educational system. Next we describe criteria for the framework for mathematical literacy in MBO and discuss to what extent the framework has indeed contributed to the visibility and accountability of mathematics in MBO. Finally we formulate more general lessons learned from the design and implementation of the framework.

Background information about the Dutch vocational system

About 40% of the Dutch 12-year-old pupils attend general secondary education (pre- university track or general education track). The remaining 60% attend VMBO — pre- vocational secondary education; such early pre-selection is unique in the world. A large percentage of the VMBO students, when 16 years old, move up to MBO. A minority of students move from general secondary education into MBO (see Figure 1). A small percentage of MBO students — only from level 4 — continue their studies in higher professional education (HBO). With 480,000 students in regular MBO and 26,300 in Agriculture/Green, MBO is the largest and most diverse sector of Dutch senior secondary education. It provides both theoretical instruction and practical training in preparation for the practice of a wide range of occupations for which a vocational qualification is necessary or useful. Its main target group is young people from the age of 16 (average age 18.5). There are four sectors:

- economics and business (e.g. sales agents, salary administrators, secretaries);
- engineering and technology (e.g., bricklayers, car mechanics, electricians);
- agriculture and food technology (e.g., florists);
- health care, social care, welfare and sports (e.g. hairdressers, nursing assistants).

MBO is provided at four qualification levels:

level 1: assistant training;

level 2: basic vocational training;

level 3: professional training;

level 4: middle-management training and specialist training.

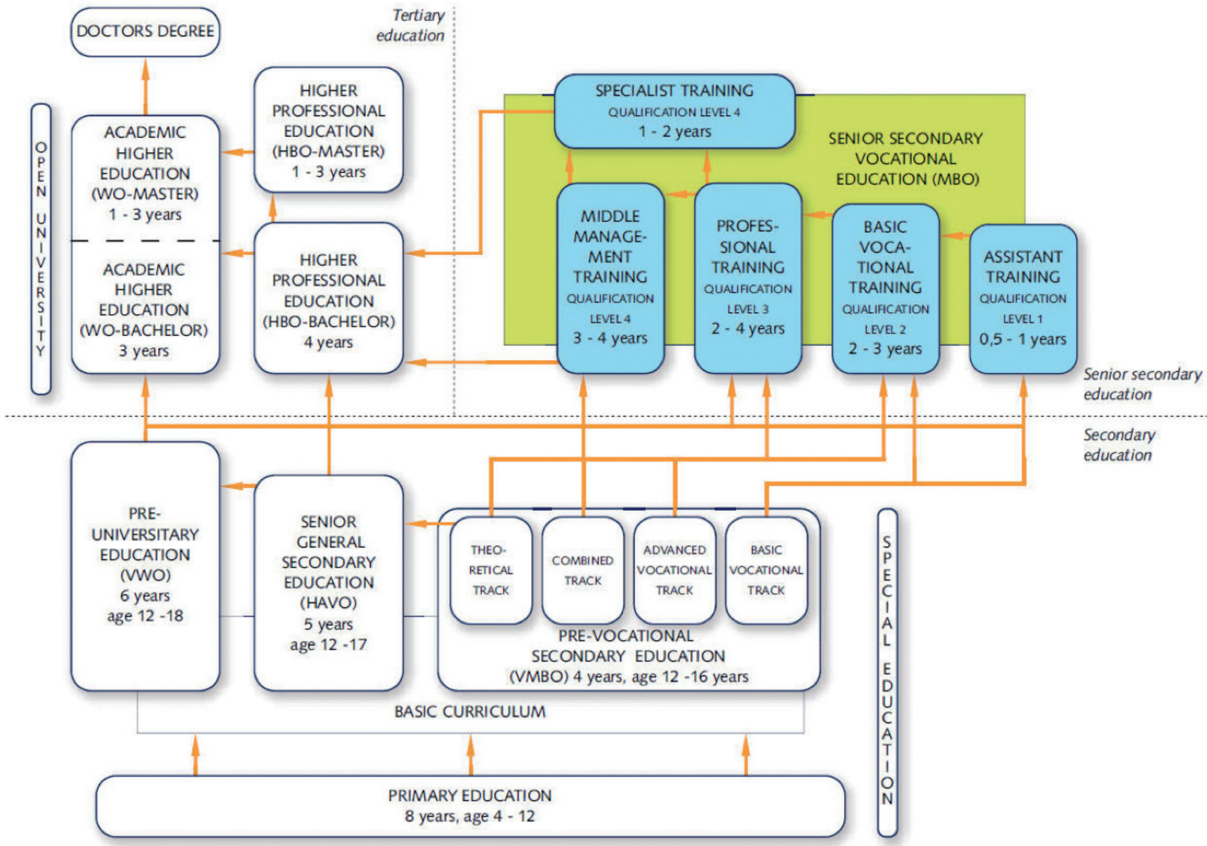

Figure 1—Dutch educational system (MBO is highlighted)

There are two learning pathways: vocational training (BOL) where practical training takes up between 20% and 60% of the course; and block or day release (BBL) where practical training takes up more than 60% of the course (formerly the apprenticeship system).

The competencies required for starting professionals are laid down in qualification files, which are produced by the 18 national Centres of Expertise. A qualification file can include several occupational profiles on the same or different levels; these profiles are variants of the main occupation (e.g., service engineers who concentrate on washing machines, central heating, air conditioning etc.). Once approved by the Ministry of Education, these qualification files have a legal status. Based on the qualification files, each school is expected to design appropriate educational programmes. Thus put simply, Expertise Centres determine the ‘what’ and schools the ‘how’. Apart from the requirements for the starting professional as described in the qualification files and in the source document for ‘learning, career and citizenship’, there are no other curricular requirements. School are free to make their own decisions, as long as they can account for them.

The previous qualification system provided lists of attainment targets for all subjects including mathematics (if required). In technical programmes this included a list of about

50 mathematical concepts and skills, but many students and teachers often did not see the relevance of many of these targets. The current qualification files summarise the vocational core tasks and work processes but pay little attention to the underlying knowledge and skills: In the qualification files it is very common to read just very general requirements such as 'basic mathematics' in the column of knowledge required for specific work processes. The effect in many schools was that the teaching of mathematics was reduced considerably – the idea being that students would learn the mathematics needed in competence-based projects. However, companies, teachers and students increasingly complained that students learned too little about mathematics (and the languages). Stakeholders therefore agreed that the domains and levels of mathematical literacy required for core tasks and work processes should be formulated more explicitly in the qualification files. For this purpose we developed a framework that we characterise in the next section in terms of the criteria it had to meet.

Criteria for a framework for mathematical literacy in CBVE

The framework for mathematical literacy had to provide the relevant actors in the MBO community with an instrument that met the following criteria:

Criterion 1: it has to facilitate communication about role and place of mathematics in vocational education in general.

Criterion 2: it has to allow stakeholders who are not educated as mathematics teachers to identify the domains and levels of mathematical literacy required for each specific occupation.

In order to meet these criteria it seemed wise to focus on two additional criteria:

Criterion 3: create common ground and support for mutual understanding.

Criterion 4: align the framework with existing instruments that MBO stakeholders know.

In the remainder of this section we describe how we tried to meet these criteria.

In terms of criterion 3, we (Freudenthal Institute of Science and Mathematics education, Fi) decided to involve all relevant stakeholders: the association of VET Colleges (MBO-raad), the sector organisation for AOCs (AOC-raad), the association of centres of expertise (Colo), the process management of CBVET (MBO 2010), the national organisation for curriculum development (SLO), the research and consultant agency for VET (Cinop), the teacher education department of the university of applied sciences Utrecht (HU). The actual development of the framework was led by the Freudenthal Institute and carried out with help of Cinop,

	Number, quantity, measure	Space and shape	Data handling and uncertainty	Relations, change and formulas
Z2	s capable of mathematically modeling, at a professional level, a practical or theoretical problem situation in the area of numbers, amounts and measures, of judging the validity of the model and analyzing the problem within that model, of generating solutions and reflecting critically on them	Has an understanding of advanced mathematical methods in geometry, for instance from analytical geometry and linear algebra, can apply these at a professional level for modeling a geometrical problem situation and can use them to analyze the situation and reflect critically on the whole of model an	Can independently set up a statistical study at a professional level and analyze data using advanced techniques and draw sound conclusions from that analysis.	Is capable of using, at a professional level, advanced mathematical instruments in the area of relations and changes to independently model and solve complex problem in the personal/public domain and in the workplace.
Z1	Uses numbers, amounts and measures in complex, non-standard situations, can work with a mathematical model of the situation and adapt it if necessary, is capable of developing procedures to reach a solution to a problem.	Interprets and analyses complex situations in 2D and 3D using geometrical concepts, properties and techniques. Can set up a mathematical (geometrical) model of the situation and calculate, construct and reason within that model to solve a complex problem.	Collects, combines, interprets and analyses data, including in very complex situations, while utilizing statistical methods and models. Can formulate a (mathematical) of the situation and calculate and reason within that model to solve a complex problem from daily life, the workplace or education.	Is capable of typifying, analyzing and describing connections and changes in complex, non-standard situation, using mathematical symbols, notations and concepts.
Y2	Uses numbers, amounts, measures and efficient procedures in somewhat complex and new situations, and can, if necessary, let go the relation to the situation and use a mathematical model of the situation.	Reasons and calculates with the aid of geometrical concepts, properties and techniques in 2D and 3D, and can, if necessary, let go the relation to the context and work with a mathematical model of the situation at a more abstract level.	Collects and processes data, also in new and unique situations, through using statistical methods. Combines and analyses complex (numerical) information from various sources, can let go the relation to the concrete situation.	Recognizes, interprets and uses connections in complex situations; can analyze and combine different representations of a relation, using mathematical symbols, notations and concepts, and is capable of developing a strategy to solve a practical problem and can, if necessary, let go the relation to t

Figure 2—Summary of the Dutch framework of mathematics (Wijers et al, 2009).

HU and SLO. The other parties functioned as a soundboard. We also organised six pilots on MBO schools to check the 'workability' of the framework (under construction) in practice. In two sessions within each pilot, teachers and coordinators experimented in using the framework to set up a plan for their schools on how to organize the teaching and learning of mathematical literacy. Finally we organised sessions to support the Centres of Expertise in identifying the mathematics in the core tasks and works processes, and to reference the content and level in terms of the framework. These sessions were followed up by continuous support provided by the designers through email, phone or direct contact.

	Number, quantity, measure	Space and shape	Data handling and uncertainty	Relations, change and formulas
Y1	Uses numbers, amounts and measures, and applies familiar procedures and argumentations in simple non-standard situations, is capable of interpreting the results and reporting on them.	Understands and uses geometrical concepts and techniques to create images and constructions in more complex situations, and to calculate and reason with shapes and situations in 2D and 3D.	Interprets and combines (numerical) data from different charts and diagrams, collects numerical data, summarizes the data and can represent it in various way in diagrams or numbers, following known procedures.	Recognizes, interprets and uses connections and relations in somewhat complex (including unfamiliar) situations, can describe a relation between quantities for a concrete task in chart, graph and (word) equation, and can apply known standard procedures in an argued and reasoned manner.
X2	Uses numbers, amounts and measures, performs familiar calculation and measuring tasks in concrete, somewhat complex but orderly situations and can interpret the results.	Understands and uses common geometrical concepts surrounding orientation; understands and uses geometrical concepts and simple prescribed techniques to describe and construct shapes, figures and orderly situations in 2D and 3D.	Reads information from charts, schemes and diagrams, and collects simple numerical data, can represent this in an understandable way, for concrete tasks in familiar situations with little complexity in the personal/ public domain and in the workplace.	Recognizes and uses regularity, patterns and simple connections in familiar situations with little complexity, can compare representations (text, chart, graph, rule of thumb) with each other in simple situations and can do calculations based on simple rules of thumb for concrete tasks.
X1	Uses numbers, amounts and measures, performs simple calculations and measuring tasks in concrete, unequivocal and familiar situations	Reads and understands everyday geometrical concepts on orientation, shapes, figures and situations (2D and 3D) for concrete tasks in unequivocal and familiar situations.	Reads information from simple charts, schemes and diagrams for concrete, explicit tasks in familiar situations with little complexity, will know in this sort of situation whether something is a case of coincidence and uncertainty (chance).	Notices, understands and uses regularity, patterns and relations (connections) in concrete, unequivocal and familiar situations where numbers or quantities are represented in text, chart or graph.

Figure 2—*(Cont.)*

In terms of criterion 4, the institutions involved decided the framework should have a structure similar to the CEFR for the languages, which is widely used in MBO. Instead of the six levels *A1*, *A2*, *B1*, *B2*, *C1* and *C2* of the CEFR, we used the labels *X1*, *X2*, *Y1*, *Y2*, *Z1* and *Z2* were used to avoid confusion and conflation of the two frameworks. We also wanted to avoid that levels of the CEFR would be used for mathematics because a profession might require high language skills but low mathematical ones (receptionist), or vice versa (construction worker). Figure 2 shows the basic structure of the resulting framework for mathematical literacy.

In terms of criteria 1 and 2 we discuss below how we formulated the content domains and levels so as to ensure that all stakeholders could work with it.

Content domains

Formulations in the framework had to be recognizable to mathematical experts but be not so mathematical that they would lead professionals astray. To elaborate on this point we need to address briefly the nature of workplace mathematics. Steen (2003, p. 4) has succinctly characterised it as follows: “Mathematics in the workplace makes sophisticated use of elementary mathematics rather than, as in the classroom, elementary use of sophisticated mathematics.”

From research in workplaces (Hoyles et al., 2002) and our own experiences, we knew that many employees and even employers think they do not use mathematics apart from basic arithmetic, whereas mathematics educators tend to have a much more liberal and comprehensive view on what counts as mathematical. Hence, using terms such as geometry, statistics or algebra might hinder the communication with companies and also with teachers of occupational subjects. On the other hand, mathematicians and mathematics educators still need to be able to identify the mathematical areas to be drawn from in the professions. We analysed existing frameworks for mathematical literacy in various settings: the PISA 2003 assessment framework (OECD, 2003), the numeracy framework for the international Adult Literacy and Life skills Survey (Gal, et al, 1999), Numeracy in the Scottish Core Skills Framework (2003), Equipped for the Future Content Standards (Stein, 2000), Functional Skills Standards for Mathematics (2007) and the Canadian Essential Skills Research Project, which connects mathematics to occupations on a detailed level. Based on the commonalities we found in these frameworks, we decided on having four domains in our framework that would cover basic numeracy (on the lower levels) as well as formal mathematics on a higher level of abstraction (on the higher levels in the framework). We settled on the following domain names: Number, quantity and measurement; Space & shape; Data handling & uncertainty; Relations, change & formulas.

Levels

A next challenge was to decide what adds to the level of mathematical knowledge required. When we distil the mathematics used in concrete situations, it often does not seem to be much more than using basic operations of addition, subtraction, multiplication and division. However, as Kent et al. (2007, p. 79) observed

> The mathematics involved in finance seems superficially similar to what appears in the secondary school mathematics curriculum (for example, calculating compound interest). Yet the effect of the workplace context is to introduce a significant degree of complexity to even the simplest mathematics. No mathematical procedure is an isolated exercise; it is part of a set of decisions and judgments that have to be made about what is a complex process or product. In Lifetime Pensions, the actuarial assistant phrased it thus: "The maths involved is not hard, but it is applied in a very complicated way—there are the company rules and Inland Revenue [tax] rules."

Based on the more general observation that there is an intricate relationship between mathematical and vocational knowledge we decided that contextual reasons should be taken into account when deciding on the level of the use of mathematical knowledge required. For example, a pharmaceutical assistant uses rather basic arithmetic including proportion and percentage when preparing drugs, but she cannot afford a single mistake. Hence the level of mathematical fluency in the realm of Number and Quantity has to be high (e.g., Y2) despite the fact that her calculations without context might seem rather basic. As the reader has seen in Figure 2 we settled on general formulations in the 'can do'- statements, that refer to the importance, complexity, or uniqueness of situations in which mathematical knowledge and skills are used. To formulate this growing complexity between levels we used factors also addressed in the adult numeracy framework of the IALS (Gal et al. 1999). Since MBO level 4 programmes prepare students for higher professional education (HBO), we wanted the framework to apply to mathematical literacy in higher education as well. In analogy to the CEFR we therefore reserved the highest level Z2 for mathematical literacy on a professional level.

Detailed descriptions

What is typical for mathematics in occupations and therefore in vocational education is its contextual character. The framework needed to reflect this, both in the formulation of skills as well as by the use of authentic examples from vocational contexts. These contextual examples help the users of the framework to identify the mathematics that is already implicit in the core tasks and work processes in the qualification files, but so far had remained hidden and had never been emphasized.

The functional use of mathematics was stressed in the framework by taking care that always the situation in which the mathematics is used is indicated in the 'can do' statement. When one clicks on a cell on the Dutch website of the Framework the 'can do'-statement is elaborated and illustrated using a fixed format consisting of: a set of sub skills; exam-

Description – number, quantity, measurement – Y1

SKILL—Uses numbers, quantities and measures, and applies familiar procedures and arguments in complex and simple non-standard situations, is able to interpret the results and report on them.

SET OF SUB SKILLS

- Applies (familiar) mathematical procedures in complex and simple non-standard situations to solve a problem or achieve a desired outcome, can do this through estimation, mental calculation, on paper or using a calculator.
- Reads (unfamiliar) measuring instruments, is capable of skillful interpolation, uses the system of measure units and can convert measures within the system (for example, convert 0,5 dl to 50 ml in a recipe).
- Is able to work with ease with decimal numbers, percentages and related fractions and measures that occur in familiar situations (for example, calculating VAT), while making use of their mutual relations
- Is able to verify whether the result of a calculation is in the right order of magnitude and what the 'margin for error' is.
- Is able to present calculations and their results in a clear and structured manner.

EXAMPLES/SITUATIONS

- Procedure to calculate/formulate VAT
- Being treasurer for one's own (small) sports club without subsidies
- Calculate the amount of material for an order or assignment and bring enough materials
- Use percentage as a factor in depreciation and compound interest
- Determine extra costs for a non-standard repair on a car
- Give advice on the use of fertilizer
- Make, control and follow a budget for the cost of repairing defects and/or damage, and the cost of maintenance work
- Calculate a patient's liquid balance

Underlying mathematics

- Arithmetical procedures
- Decimal numbers, percentages, relations and fractions
- Measures and units

Updated: 20090108

Figure 3—The description behind the general statement of a Y1 cell of the framework

ples from citizenship and professions; and — in general terms — the mathematical 'background' (see Figure 3).

The large set of examples from various occupations should assist Centres of Expertise and teachers in deciding on the levels required and achieved. The mathematical background is there to facilitate communication with the mathematics community and to make it possible to refer to mathematical standards and curriculum documents.

	Number, Quantity, Measurement	Space and Shape	Data Handling and Uncertainty	Relationships, Change, Formulas
Y2				
Y1	✓	✓	✓	
X2	✓	✓	✓	✓
X1	✓	✓	✓	✓

Figure 4—Mathematical level for the cobbler, level 4

Space & shape	Y1	The traditional cobbler uses mathematical concepts and techniques to make illustrations and constructions, and to calculate and reason about shapes and situation in two and three dimensions. He uses the skills for instance in: • reading working plans and sketches • experimenting with shapes • making construction drawings • biomechanics • copy techniques • making detailed patterns • copying patterns on to material

Figure 5—The justification for the level Y1 on Space and Shape for the cobbler

Has the mathematical literacy required become more visible and accountable?

Visibility

The mathematics required in the occupations has indeed become more visible in the qualification files. The eighteen Centers of Expertise inserted the levels of required mathematical literacy into the 241 qualification files for 2009/2010 in the form of a matrix as shown in Figure 4. They did this for 614 out of the 642 occupational profiles.

About two thirds of the Centers of Expertise also provided a justification of the levels of mathematical literacy. These justifications give insight into the relationships between the mathematical skills and the core tasks and work processes of the occupational profile as can be seen in Figure 5.

The occupational work processes for which the cobbler needs a certain mathematical skill are listed with this skill. An interesting detail is the fact that in the text describing the skill in general terms, the cobbler is added as the acting person. In this way not only the content and level of the required mathematical literacy is made visible but also how and where it is used. This on its turn informs education. We also analysed a sample of the justified profiles and found a high degree of consistency across the Centres of Expertise in the way similar levels were attributed to similar core tasks. We take this as an indication that we succeeded in designing the framework as an instrument facilitating communication (criterion 1) and allowing non-mathematical stakeholders to identify the domains and levels of mathematical literacy required for specific occupations in a reliable way (criterion 2).

Accountability

We hoped that increased visibility of mathematical literacy in the qualification files would lead to more explicit attention in day-to-day education as well as in assessment in relation to students' future occupations. However, the Ministry of Education decided to introduce central examinations for mathematics (arithmetic), a measure that the majority of the MBO stakeholders consider to be at odds with CBVE. Because there is only one national exam per level for all MBO students, it is impossible to have connections to the specific occupations or even to the sectors of MBO. In sum, this means that the accountability for mathematics in MBO has increased but not in the way that the framework intended, i.e. taking account of the specificities of the different occupational. Instead there is a risk that mathematics again, like in previous times, will be taught and assessed as a separate subject with no clear relevance for the occupations.

Lessons learned

The framework proved a useful instrument in CBVE and served as an interface between various communities. In that sense the framework can be seen as 'boundary object' (Star & Griesemer, 1989), an artefact that is used in different communities and serves the communicative purpose of each of them. What turned out crucial was the set of clear examples from work situations. They function as 'two-sided' objects that have two faces at the same time: a mathematical one and an vocational one, which helped recognition by the various types of communities involved.

It is perhaps tempting to focus on characteristics of the framework itself, but the communicative processes supporting the use of such a framework are also very important. The support by designers was of critical value so as to help non-mathematical people from the Cen-

	Number , Quantity, Measurement	Space and Shape	Data Handling and Uncertainty	Relationships, Change, Formulas
Y2	22	14	13	12
Y1	153	78	149	72
X2	17	68	25	80
X1	2	17	1	7
blank	—	17	6	23

Figure 6—Levels per content domain of 194 mathematical profiles summarised

tres of Expertise in judging and valuing the content and especially the levels of the mathematics they had identified in the qualification files. Because the first author functioned as the main resource consistency across different parties was ensured.

The matrices with the levels for mathematical literacy we collected from all qualification profiles allowed us to make comparisons across them. Figure 6 gives a sense of the distribution of levels identified for 194 profiles on level 4. The Z-levels were never used, but might become relevant to higher professional education (HBO).

The marked cells indicate the levels occurring most frequently in the corresponding content domain. For the majority of these intermediate-level occupations a typical level on Number, Quantity and Measurement as well as on Data Handling and Uncertainty is required, whereas these occupations differ in the level of mathematics needed for Space and Shape and Relationships, Change and Formulas. It thus seems that the levels of mathematical literacy required for the first and third domain are rather generic, and that the levels of the second and fourth domains are more specifically connected to certain occupations. This raises questions about the focus in general secondary and pre-vocational education, in which the emphasis is on the second and fourth domain. In the Netherlands, hardly any attention is paid to arithmetic, although many vocational students still tend to find it hard and need it. Also little time is spent on handling data. A shift in emphasis in these types of education towards these two content areas of mathematical literacy seems preferable to better prepare students for MBO.

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Mathematical Modeling Courses and Related Activities in China Universities

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Abstract Mathematical modeling courses and related activities are the most important events in China university mathematical education in the last thirty years, which significantly changed, and will continue to change the contents and forms of mathematical education in China universities. This paper summarizes the history and current status of mathematical modeling courses and related activities (in particular, the mathematical contest in modeling) in China universities.

Introduction

In the last thirty years, due to carrying out Reform-and-Open Policy in the economic field, China's national economy has gained great achievements. However, the reform in China education system lags behind and is not as fruitful as in the economy field. As a result, the education system is oftentimes, especially in recent years, attacked and criticized in China by pointing out that the students graduating from colleges and universities lack innovation consciousness and creative abilities (Dan & Xie, 2009).

For example, in 1999, the Ministry of Education of China and the China Youth League cosponsored a survey for China students' creative thinking abilities among 19,000 students in 31 provinces (Ban, 2001). The survey revealed that only 4.7 percent students considered themselves to have curiosity, confidence, perseverance and imagination. Only 14.9 percent students hoped to cultivate their exploring spirits for new things, and to enhance their abilities of information collection and imagination. Only 33 percent students participated in practicing activities during their study life in schools. The proportions of the students with the initial creativity personality and creativity characteristics are as low as being 4.7 percent and 14.9 percent respectively. In addition to this, if a student raised an objection to his/her teacher in the class, 48.1 percent students thought that most students would keep silent, and 16.5 percent students even thought that most students would criticize the objector.

The findings from the survey clearly reveal that most students in China lack innovation consciousness. The reasons for that are surely complicated. One of the most important reasons might be that the schools and teachers in China put their attention on teaching the students only about the knowledge and skills, but neglect cultivating students' creative thinking ability. As a result of this kind of teaching styles, the knowledge and skills are the only pursue for students. The second reason lies in that the evaluation criteria in China schools neglect the student's individuality and personality development. For a long time, one thinks a good student is the only one who gets very high grades in his/her class courses, and the students with lower grades but more creative ideas are considered to have ridiculous thoughts. The evaluation criteria have suppressed students' personality development (Dan & Xie, 2009).

In more recent years, it is widely recognized in China that in order for the country to be able to obtain sustainable development, it is crucial to construct China into an innovative country. Since the education system shoulders the special mission in cultivating national spirit of innovation and the cultivation of creative talents, reforming the teaching styles and the evaluation criteria for students has attracted more and more attention in China. The primary objective of the reform is to regard the cultivation of the innovation spirit and practicing ability as the key of the education system (Dan & Xie, 2009).

As one component of the education reform in China tertiary education, mathematical modeling courses and related activities are highlighted as the breakthrough in mathematical education reform (Jiang, 1998; Xiao, 2000 & 2002). The reason behind this is that more and more mathematical teachers in China have recognized the importance and value of mathematical modeling teaching process and related activities. Therefore, mathematical modeling is gradually becoming the best bonding point to enhance students' mathematical knowledge and application ability. It is also an important way to enlighten innovative consciousness and thinking, as well as to train innovative talents (Dan & Xie, 2009).

In my own opinion, mathematical modeling courses and related activities are the most important events in China university mathematical education in the last thirty years, which significantly changed, and will continue to change the contents and forms of mathematical education in China universities. This paper summarizes the history and current status of mathematical modeling courses and related activities (in particular, the mathematical contest in modeling) in China universities.

Mathematical Modeling Courses

Origination and development

Before 1980s, the university mathematical education in China followed the 1950s' system of former Soviet Union in all respects, with the objective of passing on the knowledge of pure mathematics - from definitions and theorems to reductions and proofs. The teachers' teaching processes and the students' learning processes were both "examination-oriented", while the students' ability of using mathematics to solve real-world problems was highly neglected. As a result, most of the students, who usually loved mathematics very much at their primary and middle schools, were weary of the university mathematics as it was tedious and uninteresting (Jiang, 1998; Xiao, 2000 & 2002).

In early 1980s, as China opened to the world again, more and more mathematical educators in China began to pay attention to such phenomena and were eager to change the situation by learning from modern education system of foreign countries. Recognizing that mathematical modeling is a bridge between the real world and mathematics, some top universities in China started to offer mathematical modeling as an optional course. The contents of the course revealed intimate connections between mathematics and real-world industrial problems. The teaching process of the course was to create an environment to arouse students' desire to learn and develop their ability of self-study, and to enhance their application and innovation ability. In order to improve the students' quality in mathematics, the emphasis was put on the students' ability of acquiring new knowledge and the processes of problem solving, rather than only knowledge and skills in pure mathematics.

Not surprising, the mathematical modeling course was very welcomed by students, and the students' interests in mathematics were recalled lively. Therefore the course obtained forceful support from the universities' administrations. However, due to lack of a strong motivation to change the conventional education style in most of the other universities, as well as lack of qualified mathematical modeling teachers and suitable textbooks, only about thirty top universities offered such courses before 1990, with only eight Chinese textbooks being published during this period (Jiang, 1998).

Popularization

In order to train teachers qualified for mathematical modeling courses and exchange the teaching experience among these teachers, China Conference on the Teaching of Mathematical Modeling and Application (CCTMMA) was organized every two or three years since 1986. In 1990, CSIAM (China Society for Industrial and Applied Mathematics) was founded, and within it a sub-society of mathematical modeling was established. CCTMMA was co-sponsored by this sub-society and the educational committee of CSIAM since 1991. They became powerful players in motivating and organizing the teaching of mathematical modeling. In 2009, more than six hundred teachers participated in the 11th CCTMMA, which was thirty times than the twenty participants in the first CCTMMA of 1986. CSIAM also has been organizing a national annual mathematical contest in modeling for undergraduates since 1992 (more details about the contest will be provided in next Section).

Currently, mathematical modeling courses are offered in about one thousand universities in different forms, which are more than half of all the universities in China. In the last decade, more than 110 Chinese textbooks on mathematical modeling were published, which are suitable to be used in courses for universities of different levels and students from different majors. In these textbooks, various industry problems are modeled as mathematical problems which can be solved with the students' knowledge learnt from their fundamental mathematics courses. Most of the university students, no matter which majors they enrolls in, can get some training in mathematical modeling. Furthermore, more than 200 universities have their own students' societies on mathematical modeling, and the societies are very active in the campuses for organizing extra-curriculum mathematical modeling activities by the students themselves.

Prospects

The teaching of mathematical modeling in China universities has been developing quickly. Below are some new trends we can see in the near future.

Mathematical experiments course

This course is related to mathematical modeling courses, but redesigned to make use of the mathematical experimentation on computers (Jiang, 2001). While the mathematical modeling courses usually consist of various cases from real-world industry, the focus of the mathematical experiments course is put on training the students to learn and use modern mathematical technologies. For example, in my university, this new course tries to integrate mathematical modeling and mathematical software with fundamental mathematical techniques in numerical analysis, optimization, and statistics. Popularization of mathematics software on personnel computers enhances students' numerical computing functions and image processing functions, which provides technical feasibility of the course and ensures the students' learning efficiency for such a course. More and more universities in China begin to offer this kind of courses recently, although up to now there are in different forms and with different contents in different universities (Jiang & Xie, 2007). In order to facilitate the course developments, more than 200 universities set up mathematical experiments laboratories dedicated to the courses.

Merging the idea and the method of mathematical modeling into the main mathematical courses in universities and colleges

This was a project supported by the Ministry of Education of China since 2002. There are usually three main (traditional) mathematical courses in China universities: Calculus, Algebra and Geometry, and Random Mathematics. They prefer to present everything rigorously and systematically, but usually with neither motivations nor applications. Reforming the main mathematical courses to merge the idea and the method of mathematical modeling is a much harder and more important task. The project's emphases are put on designing and writing feasible mathematical modeling modules, which include the explanation of the whole mathematical modeling process from real world problems so that they can be embedded or effectively used for the teaching of the main courses. Most importantly, the use of the modules will not disturb instructors' regular teaching but will stimulate and raise students' interest in studying the main mathematical courses (Jiang et al., 2007b). The project has already lasted for several years, but it is still just at the beginning, and there should be many efforts to be done in this direction.

Mathematical Contest in Modeling

Origination and development

Mathematical Contest in Modeling (MCM) first appeared in USA, which was organized by Consortium for Mathematics and its Applications (COMAP) in 1985. COMAP also initiated Interdisciplinary Contest in Modeling (ICM) in 1999. Teams from China universities have participated in the contest every year since 1989, and recently more than half of the teams of MCM and ICM are from China (Figure 1).

Recognizing the contest is beneficial to the students and helpful to the mathematics education reform in universities, CSIAM organized the first China Undergraduate Mathematical Contest in Modeling (CUMCM) in 1992. CUMCM is co-organized by CSIAM and the Ministry of Education of China since 1994, and from 1999, the contest has been divided into two categories — Group A for four-year university students, and Group B for two or three-year college students. The aim of the contest is to give students exposure to modeling process and to improve students' understanding of mathematics, mathematical modeling and experimentation, thereby providing an opportunity for the students to cultivate their creativity and problem solving ability (Jiang et al., 2007a).

Registered teams download the contest problems at the prescribed time through the CUMCM Website. In this three-day long (72 hours) contest, teams of up to three undergraduates students will investigate, model, and submit a solution to one of two modeling problems, which simulate real-word problems in engineering, management, etc. During the contest, teams are permitted to refer to any data source they wish, but they must cite all sources. Failure to credit a source will result in a team being disqualified from the competition. Team members may not seek help from or discuss the problem with their advisor or anyone else, except other members of the same team. When the contest ends, each team should submit a solution paper to the contest organizer for judging. Currently, only top 2% of the total solution papers will be awarded as the first prizes, with about next 6% as the 2nd prizes.

Popularization

Because of the very challenging nature of the contest, it attracts the most competitive students in China in an ever-increasing manner. Currently, CUMCM has become the most widespread extra-curriculum scientific activity for undergraduates in China. In 2009, there was participation by 15042 teams from 1135 institutions in the contest, representing almost all of the most prominent institutions and more than 50% of all institutions in China. It is

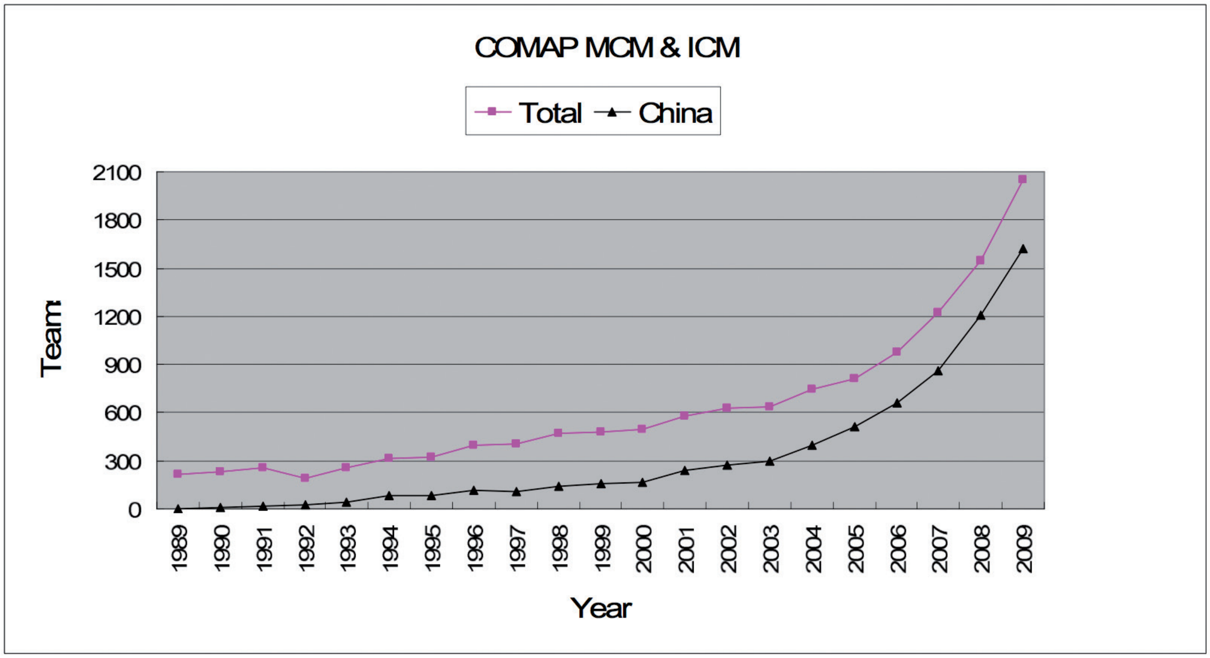

Figure 1—Number of teams participating in COMAP contests.

also interesting that more than 80% of the participants are engineering, economics, management, and even humanities majors, other than mathematics majors one might expect. Figure 2 plots the statistics on the numbers of institutions and teams participating in CUMCM. More details about CUMCM can be found in Li (2008), or from the Web site http://www.mcm.edu.cn (including all the contest problems used in previous years in English). We also welcome teams from other countries to join the contest.

In order to celebrate its 10th and 15th anniversaries, CUMCM organized university students' summer camps on mathematical modeling in 2001 and 2006 respectively. Many similar contests, with smaller sizes and dedicated scopes, are also organized by various organizers in China.

Influence

The contest is a real challenge to its participants and is very welcomed by the students. The special experience students have got during the contest is greatly helpful to tap their innovative potential and strengthen their cooperative spirit. The contestants conclude their experience with one sentence "Once participated, benefit for life long". The whole contest process consists of three stages, namely, the training and preparation before the contest, the hard work during the three-day contest, and the summing up of students' own experience and doing further work on the contest problems after the contest. Through these stages students' creativity and overall ability are greatly improved. In deed, most of the winners

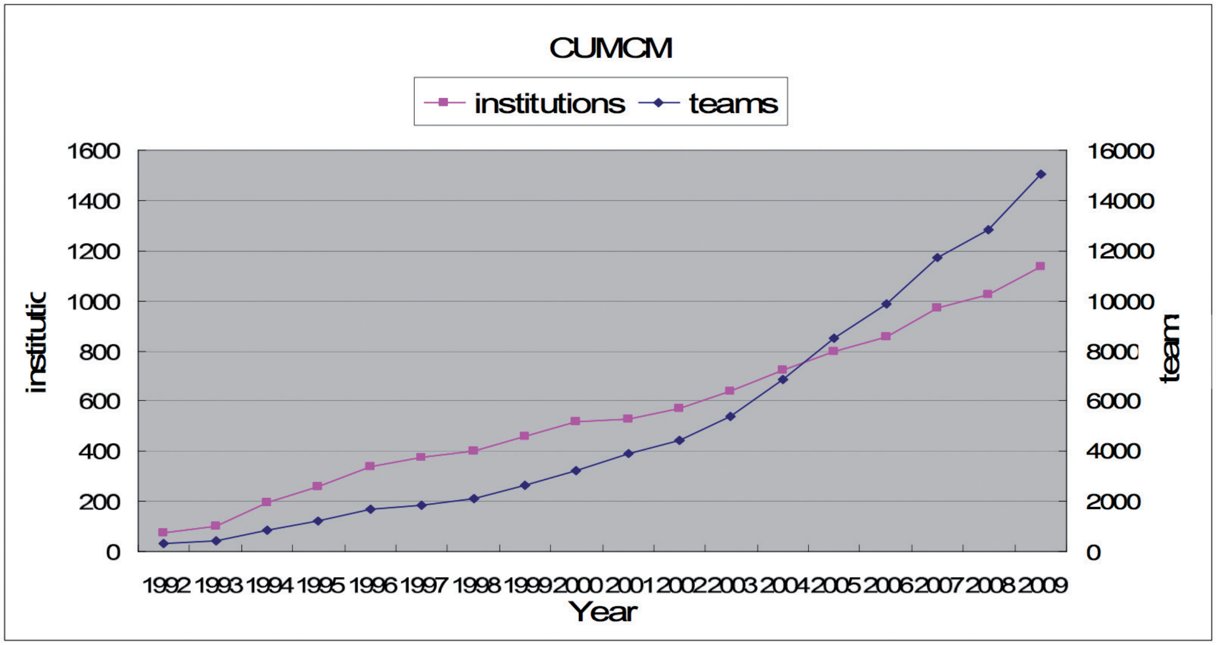

Figure 2—Numbers of institutions and teams participating in CUMCM.

of CUMCM have done very well in their successive courses and final-year projects before their graduation. The scientific and industry communities are getting to know more and more about CUMCM, and they are glad to accept the students who have the experience of the contest when they go to graduate schools or find jobs after their graduation. Some industry corporations, such as World-Sky Group, Netease Corporation, Higher Education Press, Wolfram Research Inc. and The MathWorks Inc. also sponsor the contest or related activities.

Difficulties and prospects

CUMCM does encounter some difficulties as the participation continuous to grow. First of all, the good contest problems are vital to the success of the contest. Contributing a good modeling problem, which is both a meaningful real-world problem and also a solvable problem by most teams within three days, is a real challenging task to the organizers.

Another difficulty the organizer faced is how to ensure the equity and fairness of the contest. Since the contest lasts for three days and it is essentially a completely open contest, it is not easy to supervise if some teams violate the contest rules. The organizer emphasizes the very importance of self-discipline.

Due to the success of CUMCM, a similar contest for the graduated students also appeared in China just several years ago. We are also considering the feasibility to organize similar contests for the primary and secondary students. Actually, in USA, COMAP has organized the contest for high school students for many years. But in China, most of the primary and secondary students only care about getting high marks in the entrance examination so that the primary students can enter into high-quality secondary schools, and the secondary students can enter into top universities. This situation should be changed before mathematical modeling courses and related activities get into their campuses.

Summary

In order to promote a close relation between mathematics and the world outside mathematics (other sciences, industry and high tech, social and human life etc.), mathematical modeling plays a crucial role. It now becomes an important part of contemporary industrial and applied mathematics and an absolutely necessary step for connecting mathematics and applications. For university students, it should emphasize and organize their formation and training on mathematical modeling. Mathematical modeling courses and related activities are very successful in China universities, which play an important role in reforming the contents and forms of mathematical education in China universities.

Acknowledgement

This paper is compiled and re-organized based on the references list in below (In particular, Dan & Xie, 2009; Jiang, 1998; Jiang and Xie, 2007; Jiang et al. 2007a & 2007b). In fact, it can be regarded as a compound and compressed version of these references.

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Appendix – Discussion document of the joint ICMI/ICIAM study on educational interfaces between mathematics and industry (in L'Enseignement Mathématique, 55 (2009), pp 197–209)

1. Introduction

The ICMI/ICIAM-Study on *"Educational Interfaces between Mathematics and Industry"* (EIMI-Study) starts from two assumptions, namely:

(1) There are intimate connections between innovation, science, mathematics and the production and distribution of goods and services in society. In short: there are intimate connections between mathematics and industry;

(2) In view of these connections, there is a need for a fundamental analysis and reflection on strategies for the education and training of students and maybe the development of new ones.

The EIMI-Study, organised jointly by the International Commission on Mathematical Instruction (ICMI) and the International Council for Industrial and Applied Mathematics (ICIAM), seeks to better understand these connections and to offer ideas and suggestions on how education and training can contribute to enhancing both individual and societal developments.

1.1 Tentative description of the Field

Historically, there have been productive interactions between mathematics and industry in generating and solving problems associated with the development of humankind, economically and socially. In a modern, technological world, mathematics is said to be used almost everywhere. However, these uses are not generally visible except to specialists. Even people using mathematics in their workplaces may not recognize its presence.

There have been many studies of the mathematics used in the workplace – ranging from descriptive lists of traditional school-based topics to sociological studies of workplace activities set in context. There have been many collections of applications of problem solving and modelling based on or informed by practical industrial problems, especially at higher levels of mathematics, in fields such as the natural & physical sciences, engineering, and finance.

Internationally, there are frequent articles and debates in the popular media citing employer dissatisfaction with the perceived quality of mathematics education. Graduates from schools, vocational colleges, and universities often appear unable to draw upon and use mathematics in work situations as opposed to classroom or examination contexts. At all educational levels, students typically have been taught the tools of mathematics with little or no mention of authentic real world applications, and with little or no contact with what is done in the workplace (be it the classical engineering situations or other more recent activities like biotechnology, biomedicine, the financial, insurance and risk sector or consulting engineering companies).

Nowadays, highly complex problems need to be solved and, hence, some training to solve such problems – in particular, real life problems – is necessary. Increasingly, powerful computers make it possible to treat such complex problems and this is achieved not only using off-the-shelf software but with innovation, often mathematical innovation requiring insight and analysis.

In order to better understand these phenomena, the Study starts from a broad definition of INDUSTRY (from the Organisation for Economic Co-operation and Development) *"... broadly interpreted as any activity of economic or social value, including the service industry, regardless of whether it is in the public or private sector" (OECD 2008, p. 4). The term "industry"* obviously refers to a diverse range of activities, producing goods and services. Under constraints such as time and money, these activities generally attempt to optimize limited-sometimes scarce-resources, both material and intellectual. The overarching goal is to maximize benefits for certain groups of people while, ideally, minimizing harm to other groups and the natural environment.

"MATHEMATICS (*or the mathematical sciences, here the two terms are used interchangeably) comprises any activity in the mathematical sciences, including mathematical statistics*" (OECD 2008, p. 4). Workers at all levels utilize mathematical ideas and techniques, consciously or unconsciously, in the process of achieving the desired workplace outcome. In other words, mathematics is just one part of a repertoire of tools and strategies of a practical nature. However, as a major factor in decision-making and communication processes, it is crucial that mathematics be used appropriately, accurately, and with confidence. For this Study, we start from the assumption that professional mathematicians are located in the academia, in industry, sometimes in both. The discourse of mathematics in all its various specializations involves certain ways of thinking and acting. Traditionally, Mathematicians consider axioms and definitions and make logical deductions. In mathematical modelling, one formulates problems in mathematical terms. However, the mathematical solution needs to take into account the industrial context.

This Study will examine the implications for education at the intersection of two communities of practice – industrialists and mathematicians or industry and mathematics. We wish to emphasize that there should be a balance between the perceived needs of industry for relevant mathematics education and the needs of learners for lifelong and broad education in a globalised environment. In other words, learners should be equipped for flexibility in an ever-changing work and life environment, globally and locally.

1.2 Rationale for the Study

WHO ARE THE INTENDED BENEFICIARIES OF THIS STUDY? They include, among others:

- students enrolled in formal education systems across all sectors, including vocational, secondary, tertiary, and even primary;
- pre-service teachers [teacher students] and practising teachers involved in continuing education or professional development programs;
- teacher educators for the above categories;
- learners undertaking workplace education, from low-skilled workers through to management [and their workplace teachers/trainers];
- industry decision makers;
- mathematicians working in industry;
- policy makers.

What are the aims of the Study? The aims of the Study are:

- to broaden the public awareness of the integral role that mathematics plays in society with respect to low- and high-technology industries;
- to broaden the awareness of industry with respect to what mathematics can and cannot realistically achieve under current circumstances;
- to broaden the awareness of industry with respect to what school and university graduates can and cannot do realistically in terms of mathematics;
- to broaden the awareness of mathematics teachers and educators with regard to industrial practices and needs with respect to education;
- to enhance the appropriate usage of mathematics in society and industry (e.g., by presenting examples of good practice);
- to attract and retain more students, encouraging them to continue their mathematical studies at all levels of education through meaningful and relevant contextualized examples;

- to improve mathematics curricula at all levels of education.

Why is there a need for this Study? This Study is needed:

- to create new and innovative educational practices and support existing good practices;
- to ensure that, when used as an employment selection tool, Mathematics is used appropriately;
- to develop in learners the mathematical reasoning and logical thinking needed in industry;
- to enhance the dialogue and understanding between the communities of mathematicians, workers and industry decision makers, politicians, and educators.

2. Thw role of mathematics – visibility and black boxes

We all use mathematics every day; to predict weather, to tell the time, to handle money. Mathematics is more than formulas or equations; its logic, its rationality have for a long time gone beyond just numbers. However, people are often not aware of the importance of the role of mathematics in modern technologies. Many people have a restricted view of what mathematics is and does. We need to make the use of mathematics in modern society more visible.

If young people are not aware of the importance of mathematics and have not personally experienced its applicability, they may not want to study mathematics in school. This may limit their career and educational opportunities later on. Many societies have had some kind of selection process, such as Classics (be it studying Chinese, Greek or Latin). Today, mathematics serves this purpose in many countries. This can be progressive, in that it gives children an opportunity for upward social mobility through studying mathematics. However, it can also be repressive in that it can limit the opportunities of students with problems in mathematics. Some people will manage to compensate for gaps in their mathematical training, but others will not. The consequences, political, cultural and educational, are important.

The role of mathematics is twofold. It can give people highly developed skills in abstraction, analysis of underlying structures, and logical thinking. It can also give them experience with the best tools for formulating and solving problems. We will refer to this as analysis. This is in comparison with applying "black boxes", which refers to the packaging mathematics with other conceptual and material tools into (hopefully) automatic solutions to problems, with the consequence of hiding the mathematics from the immediate view of the users. This packaging can be anything from a fast food cash register, where the keys show only pictures of the items instead of numbers, to the search algorithm in *Google™*. We believe strongly that most people will need a combination of the skills listed above. Just knowing how to apply black boxes has many shortcomings:

- It limits innovation, critical analysis and adjustments to the techniques;
- It does not allow analysis in case of failure of the black box;
- It makes it harder for people to judge the appropriateness of various techniques and the validity of the output;

The exact balance of emphasis between analysis and black box techniques and the various levels of description of the inner workings of the black boxes will depend on the nature of the application.

Questions

1.—How can mathematics, especially industrial mathematics, be made more visible to the public at large?

2.—How can mathematics be made more appealing and exciting to students and the professionals in industry?

3.—How can mathematics serve a progressive rather than a restrictive role in education and training for the workplace?

4.—What is the best way to teach analytical skills to various groups of students?

5.—To what extent is it necessary or desirable to describe the inner workings of black boxes?

6.—What are the social implications of not explaining the inner workings of black boxes?

3. Examples of use of technology and mathematics

Modern workplaces are characterised by the use of very different types of technology. 'Technology' is understood in the broadest sense, including traditional machinery, modern information technology, and workplace organisation. Examples include a turning lathe, paper forms for reporting on production, technical drawing packages such as Computer Aided/Assisted Design/Drafting (CAD), and the assembly line as a means of organising production in contrast to other forms of workplace organisation.

The conceptual basis of most "modern" information technology is obviously some sort of (often discrete) mathematics. Reports like *Mathematics in Industry* (OECD, 2008) mention university-level examples of uses in the chemical industry, oil exploration, medical imaging, micro- and nano-electronics, logistics & transportation, finance, information security, and communications and entertainment as areas of industrial use of mathematics. It lists mathematical themes such as complexity, uncertainty, multiple scales, large-scale simulations and data & information (see p. 10). As a consequence, one could think that the growth in technology use implies an increasing presence of mathematics in the workplace. Research findings and anecdotal evidence seem to point to the contrary: Mathematics is said to disappear from the workplace. At least, mathematics is less visible in modern workplaces than in traditional ones. Why?

One way to understand this paradox is the following interpretation of the situation: As discussed above, technology is a means of packaging mathematics with other conceptual and material tools into ideally automatic solutions of problems, thus hiding mathematics from the immediate view of the users. One side-effect of these black boxes is to secure a certain distribution of technological power, giving the control of a whole range of situations to those few who understand the inner working and intricacies of the boxes.

Questions

1.—How is it possible to describe and analyse the role of Mathematics within technology? What are insightful examples of the role of technology in showing and/or hiding mathematics in the workplace?

2.—Does the existence of special types of technology hiding mathematics from the view of the user imply a change in the mathematical demands on the user? How?

3.—To be more precise and to give one example for a more detailed question: With the idea of exact measurement furthered by modern technology, do old competencies like estimation of results and reading of different scales become obsolete when using modern technology? Or, do they become more important?

4.—What are the social and political consequences of the 'crystallising' and 'hiding' of mathematics in black boxes – this question is pertinent not only to modern technology like computers, but also to software and hybrid technology.

4. Communication and collaboration

In the workplace, mathematics is seldom undertaken as an individual activity. Mathematical work, mostly on modelling and problem solving, is almost always a group activity and frequently the groups involved are made up of individuals with diverse expertise and expectations. Communication between and among such groups is crucial. By communication we include listening, writing, speaking and the use of communication technologies as essential skills. A societal or industrial problem may be concerned with making a process faster, cheaper, more robust or, in a general sense, more efficient. But when is the problem amenable to mathematical analysis and solution? Communication at this level is often difficult because managers, mathematical and non-mathematical team members may come with different training, different goals, and use different languages. They may have to cope with highly complex situations, often marked by uncertainty, the use of multiple scales, sometimes relying on large-scale simulations. Nonlinearity, data and information are important aspects of modern industry (OECD 2008).

In industries of all sizes, small, medium and large, good communication is extremely important in understanding the nature of a problem and its mathematical components. Communication is clearly essential between team members in "solving" a problem. However, translating a mathematical solution into a workable solution can be challenging as there may be confounding social and political considerations.

It is also necessary to understand that in some industrial settings, intellectual property rights and secrecy may be an issue restricting open communications.

Questions

1.—How to identify which societal and/or industrial problems should be worked on?

2.—How to better communicate within multi-disciplinary working groups?

3.—How to communicate the underlying mathematics to the problem owners and/or general public?

4.—How to achieve greater quantitative literacy among school leavers, workers, and the general population?

5. Teaching and learning of industrial mathematics – making industrial mathematics more visible

In section 1, we described the somewhat paradoxical situation of Mathematics being used more and more extensively in modern society, while it is progressively disappearing from societal perception. Let us consider some examples:

- To many primary school students the long division algorithm is a black box;
- Most students have never thought about how calculators compute transcendental functions such as the exponential;
- Google's Page Rank algorithm is a powerful application of eigenvectors in very high-dimensional linear algebra;
- GPS, the Global Positioning System, involves diverse technologies, and its understanding requires geometry, linear algebra and coding theory among others;
- Cryptography is fundamental to modern society. Despite its high sophistication, it involves mathematical ideas that can be presented in a simplified way to a broad audience;
- Weather forecasting shows the power of combining mathematical modelling and high-speed computers;
- Statistics is a useful tool, but many people do not understand how to interpret statistical results properly;
- Computer animation involves many mathematical issues, such as using quaternions to parameterize rotations.

Logistics and decision making are broader fields, which also imply the use of simple and/or sophisticated mathematical technology. In all fields, most of the mathematical technologies are used without realising that mathematics is used; mathematics is normally not even mentioned in relation to them. Teaching and learning of mathematics has to cope with this fact by either hiding mathematics from the view of the learner or deliberately showing the use of mathematics even if this is not obvious. The (in)visibility of mathematics, especially industrial mathematics implies a major problem in education and training. Nevertheless, describing the inner workings of these procedures can be a great way to motivate teaching and learning of various topics.

Questions

1.—Who decides what will be explained and to whom?

2.—How to decide the level of explanation for various groups?

3.—How to organise teaching and learning in order to make industrial mathematics visible – if this is wanted/necessary?

4.—How much is it appropriate to explain for educational purposes in order to generate interest and excitement without overwhelming the learner?

6. Using technology and learning with technology: modelling & simulation

For a better understanding of the role of technology in the educational interfaces between industry and mathematics, it may be helpful to distinguish between technology created and used to help industry do its job – often called *"indutech"* – and technology created especially to foster teaching and learning mathematics and its use for instance in industry – called *"edutech"*. Using either kind of technology, especially new technology, usually requires special efforts to become acquainted with it, to develop routines and practice. To put it in workplace words, a special effort (in terms of time set aside) must be made to teach and learn using a specific technology. This can be an obstacle to switching to a more modern technology as long as the older one still "does the job". On the other hand, change and innovation are necessary in industry (especially in times of globalisation).

There are a number of issues related to using technology in industry (*indutech*) and teaching and learning with technology (*edutech*). We name just a few:

- Technology may be a reason to make obsolete certain competencies by means of routines packed into black boxes – e.g., simple arithmetic and more advanced handheld calculators may change the role of traditional arithmetic performed with paper and pencil or in the head;
- Technology taken as an unquestionable "god" may call for a critical evaluation of the results, both in industry and in education;
- Technology can be a means to enhance learning by simulating the workplace situation (virtual workplace simulation). This can be technology especially created for teaching/learning purposes as well as the usual workplace technology used in a typical educational way. Simulation – especially of unusual, even dangerous situations – for educational purposes can reveal the inner workings of black boxes. Modelling and simulation may be seen as a way to create opportunities to better understand an input- output system, to see the consequences of certain input variables, and even, perhaps, to understand the mechanisms inside the system.

Questions

1.—How should one decide on the level of detailed mathematics expected to be taught/learned in a given vocational black box situation?

2.—How can mathematics help the transfer of technological procedures and/or solutions between different fields of industry?

3.—What criteria should be used to judge the appropriateness of simulation in the teaching & learning of industry related practice?

4.—How can one compensate for the "standardising effects" of any technology that is in widespread use?

7. Teaching and learning for communication and collaboration

As mentioned in section 3, communication and collaboration form an integral and important part of the industrial use of mathematics. Because of the importance of communication skills for work in industry, is it desirable to have these skills taught and learned in all parts of education and training. But there are additional justifications for including teaching and learning communication skills within mathematics education. To cite a few:

- Modelling industrial problem settings at each level of education can be used to teach mathematics in the context of its contemporary uses. This modelling relies on effective communications throughout the process;
- Teaching communication skills is a natural part of the mathematics learning process (listening, writing, speaking and using communication technologies);
- Assessing communication skills (listening, writing, speaking and using communication technologies) is a natural part of the mathematics assessment process;
- Collaborative learning should begin in primary education and be continued as part of a life-long learning process. Examples include: planning and making healthy soft drinks in primary school, analysing strategic games or simulating running a small business in secondary education, and working on real industrial problems in tertiary education.

Questions

1.—What communication skills are specific to mathematics?

2.—Are there specific skills for use in relation to industrial mathematics?

3.—How do we teach mathematics as a "second language"?

4.—What is the role of mathematical contests and competitions in developing and assessing communications skills in mathematics?

8. Curriculum and syllabus isues

A partnership between mathematics and industry requires adjustments of the mathematics curriculum in order to prepare students for both the needs of mathematics and the requirements of industry. This can also support the teaching of mathematics in general. Students are often taught as if mathematics were a dead science and a finished product. Giving students an opportunity to experience the excitement of realistic applications may enliven their view of the subject.

Questions

1.—What are the advantages and disadvantages of identifying a core curriculum of mathematics for industry within the general mathematical curriculum at various levels and for various professions?

2.—What are useful ways to introduce mathematics for industry into vocational education?

3.—What are the advantages and disadvantages of creating specific courses on mathematics for industry vs. including the topic in the standard mathematical courses at various levels?

4.—What are the advantages and disadvantages of treating mathematics for industry as an interdisciplinary activity or as part of the traditional mathematics syllabus?

9. Teacher training

Teachers must be trained in new mathematical content, pedagogy and assessment and to recognize the presence of mathematics in society and industry. This should take place in schools of education for mathematics teachers of all levels, in-service programs, and industrial training programs.

There may be a special need to act on teacher training in areas like statistics, discrete math (recursion, graph theory, matrices), operations research and mathematical modelling in the context of real applications. These curricular changes must be complemented by changes and innovation in the pedagogy offered to teachers, such as stressing collaborative learning, making decisions on appropriate use of educational technology and fostering communication skills. Assessment modes and practices should include informal assessment techniques, portfolio and group assessment.

These changes may be fostered by industry visits or longer-term placements of students who want to become teachers and by practising teachers. Understanding the educational interface between mathematics and industry can be enhanced by current and future teachers actually going into the workplace to observe and talk to workplace personnel as they experience real industrial problems.

Questions

1.—What level of understanding of this new content in relation to EIMI is appropriate for each grade level?

2.—What are good practices that support this new direction in teacher training?

3.—How to implement these changes in an efficient way?

10. Good practices & lessons to be learned

In all sectors of education there are examples of good practice in relation to this Study. To give an example, we mention engineering programs that seem to foster creativity and innovation by final-year projects in a better way than school and university courses in the mathematical sciences. One important aspect of the Study is to communicate and exchange examples of good practices in order to make concrete the ideas already discussed in this document. We would like to collect outstanding examples of innovative practices that are suitable for use and adaptation by teachers at the various levels of education.

Possible examples may include:

- Creating *optimal routes* for garbage collection (appropriate at different levels of sophistication starting with 4 year-olds); graph/network theory;
- Analysing the *probability concepts* which underpin different games of cards; probabilistic decision making;
- *Analysing data* related to the environment and patterns of consumption such as energy use, clothing sales (appropriate at different levels of sophistication starting with 10 year-olds); statistical decision making;
- *project management* for an event of relevance to the particular students; total project management skills;

- *game theory* (decision making);
- *continuous optimization* – e.g., finding an optimal path on a downhill ski slope.

We are also looking for good examples of how to integrate industry into the educational process. For example, through:

- study groups which address problems brought into the classroom from industry;
- internships (for students and teachers into industry; for industrialists into education);
- summer/winter schools, where industry representatives work together with students on problems of relevance to industry rather than the school curriculum;
- competitions involving mathematics used to solve industry-related problems under realistic conditions;
- the documentation of existing case studies.

Lessons to be learned from failures are of the same interest as those from successes. The examples should include the intended outcomes, the pedagogical practices involved, and a critical evaluation of the process.

11. Research and documentation

Launching this Study, we start from the assumption that, from a global perspective, there is a need to research existing practices; no coverage of the whole field currently exists. Pertinent topics are diverse – with diverse methodologies being appropriate to the field of educational interfaces between mathematics and industry.

Within this research field, mathematical interfaces can include mathematics "proper", but are usually interdisciplinary in practice. Mathematics is part of the method of coping with industrial situations, but can also be the object of study (especially in studies on societal and industry needs and educational practice and innovation). For example, there is a need for studies on the curricular consequences for vocational, secondary, tertiary and primary education.

Compared to other topics, it seems obvious that national and trans-national documentation is widely missing in the field of mathematics and industry (even with CEDEFOP and other institutional databases). Suggestions and contributions describing existing and future research and documentation of activities in the field of Educational Interfaces between Mathematics and Industry will be most welcome.

Participation in the Study

Design of the Study — The ICMI/ICIAM joint Study on *Educational Interfaces between Mathematics and Industry* is designed to enable researchers and practitioners around the world to share research, theoretical work, projects descriptions, experiences and analyses. It will consist of two components: the *Study Conference* and the *Study Volume*.

1) The *Study Conference* will be held in Lisbon, Portugal, on April 19-23, 2010, the number of participants to be invited being limited to approximately 100. It is hoped that the Conference will attract not only "experts" but also some "newcomers" to the field with interesting and refreshing ideas or promising work in progress, as well as participants from countries usually under-represented in mathematics education research meetings.

Participation in the Study Conference will be by invitation only, based on a submitted contribution. Proposed contributions will be reviewed and selections made according to the quality of the work, the potential to contribute to the advancement of the Study, with explicit links to the themes and approaches outlined in this Discussion Document, and the need to ensure diversity among the perspectives. Accepted papers will appear in the Conference Proceedings that will be published by ICMI and ICIAM as a CD- ROM and on the Internet (open access), and will form the basis of the Study's scientific work. An invitation to the conference does not imply that an oral presentation of the submitted contribution will be made during the Conference, as the International Program Committee (IPC) may decide to organize it in other ways that facilitate the Study's effectiveness and productivity.

The Study Conference will be a working one, every participant being expected to be active. We therefore hope that the participants will represent a variety of backgrounds, expertise, experience and nationalities that will lead to a suitable coverage of the Study theme, its different topics and the related questions. Such attendance should be drawn broadly from the mathematics and mathematics education communities, as well as from industry. It is the IPC's hope that the Conference will attract applied mathematicians, representatives from industry, mathematics educators, in particular those involved in the preparation of school teachers, practitioners in the teaching of applications of mathematics and researchers in mathematics education – both experienced people and young researchers entering the field.

Unfortunately an invitation to participate in the Conference does not imply financial support from the organisers, and participants should finance their attendance at the Conference. It is hoped that this invitation will help participants to get appropriate support from their own countries. Funds are being sought to provide partial support for participants from non-affluent countries, but the number of such grants will be limited.

2) The *Study Volume*, a post-conference publication, will appear in the New ICMI Study Series (NISS), published by Springer. Acceptance of a paper for the Conference does not ensure automatic inclusion in this book. The Study Volume will be based on selected contributions as well as on the outcome of the Conference. The exact format of the Study Volume has not yet been decided but it is expected to be an edited coherent book that can hopefully serve as a standard reference in the field for some time.

A report on the Study will be presented during the 7th International Congress on Industrial and Applied Mathematics (ICIAM 2011, to be held on July 18-22, 2011, in Vancouver, Canada), as well as at the 12th International Congress on Mathematical Education (ICME-12), to be held in Seoul, Korea, on July 8-15, 2012.

The aim is to present the Study Volume itself at ICIAM 2011.

Call for Contributions — The International Programme Committee hereby invites individuals or groups to submit original contributions on specific questions, problems or issues related to the topic of the Study for consideration by the Committee. A submission should represent a significant contribution to knowledge about the Study topic and may address questions from one or more of the Study themes (see sections 1 to 10 above), or further issues relating to these, but it should clearly identify its primary focus. The IPC welcomes high-quality proposals from researchers and practitioners (both from education and industry) who can make solid practical and scientific contributions to the Study. New researchers in the field and participants from countries under-represented in mathematics education research meetings are especially encouraged to submit contributions. To ensure a rich and varied scope of resources for the Study, participation from countries with different economic levels or with different cultural heritage and practices is encouraged. We hope that applied mathematicians, as well researchers and mathematics educators from the early years to tertiary level, will come up with new insights and guidelines for future work. Contributions from vocational education and industrial Mathematics are highly welcome.

Those who would like to participate should prepare a 6-10 page paper (Times 14- point font, single spaced lines) addressing matters raised in this document or other issues related to the topic of the Study. All papers must be written in English, the language of the Conference, and include a 200-word abstract. Papers concerning work that is ongoing or yet to be carried out are also welcome. Research questions should be carefully stated and expected results should be formulated, if possible with reference to earlier and related work.

These documents should be submitted no later than September 15, 2009, to both co- chairs of the Study by e-mail. All such documents will be regarded as input to the planning of the Study Conference and will assist the IPC in issuing invitations no later than November 15,

2009. All submissions must be uploaded electronically to the Study website (http://www.cim.pt/eimi/) together with appropriate personal data, in the form of a PDF, RTF or DOC file not exceeding 1.5 MB.

Study Timeline

- Submissions for participation in the Study should be uploaded to the website by October 15, 2009;
- Submissions will be reviewed and decisions made about inclusion in the Conference Proceedings. Notifications about these decisions will be sent by November 15, 2009 to all those who sent in submissions. In the case of papers accepted for the Conference, some suggestions for changes may be sent to the authors;
- Final versions of papers accepted for the Conference Proceedings must be received by March 15, 2010.

International Programme Committee

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Inquiries

Inquiries on all aspects of the Study and suggestions concerning the content of the Study Conference should be sent to both co-chairs:

- Alain Damlamian, damla@univ-paris12.fr
- Rudolf Sträßer, Rudolf.Straesser@math.uni-giessen.de.

The official website for the joint ICMI/ICIAM Study is http://www.cim.pt/eimi/

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