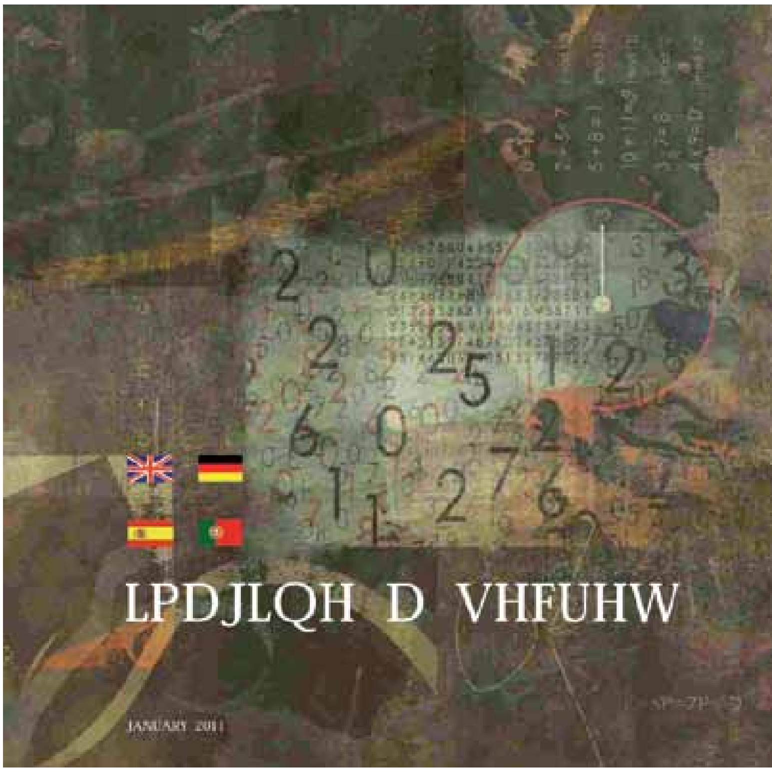
INTERNATIONAL CENTER FOR MATHEMATICS

#29

Bulletin

January 2011

Editorial



2010 continued to be a difficult year for CIM from the financial point of view. However it succeed to pursue new initiatives in collaboration with the associates and to contribute to keep in Portugal a forum for national and international cooperation among mathematicians and other scientists. In spite of the impossibility of using the facilities of the Observatório Astronómico of the University of Coimbra and the current financial crisis in the country, that do not offer optimistic perspectives of a significant change in the next years within the present framework, CIM has been supporting and organizing events in all the country in collaboration with its associates, in particular the Portuguese Mathematical Society (SPM). In particular, CIM has continued to promote and to organize national and international initiatives in the Mathematical Sciences, namely, hosting Maxim Kontsevich that delivered the second Pedro Nunes Lectures at the Universities of Porto and Lisbon, organizing the international conference on "Educational Interfaces between Mathematics and Industry" of the ICMI/ICIAM international study, which proceedings are available on line at the CIM website http://www.cim.pt/files/proceedings_eimi_2010.pdf, offering support to the creation of the *Portuguese Digital Mathematical Library* within the Portuguese coordination of the on going EuDML project, promoting the first *Portugal-Spain MatCampus* (Acampamento Matemático) for high school students, held in Braga and Santiago de Compostela during 18-31 July 2010 and co-organizing the workshop on the Raising Public Awareness in Mathematics, held in Óbidos, in collaboration with the EMS-RPA committee, where the new art and mathematics film on elliptic curves and cryptography LPD-UHW, sponsored by Ciência Viva, was first presented the 26 September 2010 during a join session with the SPM open to the public.

Contents

The CIM Scientific Council, in its meeting held in Lisbon the 25th September 2010, has recognised that CIM has the potential input from the whole national mathematical community through its associates, good international relations, through ERCOM, and should obtain recognition from the Portuguese Fundação para a Ciência e a Tecniologia through seed money to carry out its activities in better conditions. At that meeting it was stressed the CIM character as an umbrella organization, whose associates are research centers, mathematical departments and Societies, as well as its added value of being the unique ERCOM member from Portugal. It was also recognised that CIM does not compete, nor is a threat, to the existing research centers in Portugal, that CIM's mission does not overlap with the centers' projects and activities, but are complementary and, by promoting the cooperation and links among them, CIM has an important actual value to the mathematical community in Portugal.

However, the current situation is untenable for long time and the direction, in order to provoke a change and to facilitate the associates to find a renewed solution for 2012 and beyond, has decided to anticipate the end of its mandate by one year, after securing another limited budget programme for 2011 that you may find in http://www.cim.pt/?q=events and in the pages of this Bulletin. This programme includes, among other activities in association with the CIM associates, the continuation of the *Pedro Nunes Lectures*, in collaboration with the SPM, the organisation of the Summer School on Dynamical Models in Life Sciences in the historical city of Évora, a joint event with the European Mathematical Society and the European Society for Mathematical and Theoretical Biology, the participation at the initiatives of the UTexas/Austin-Portugal programme in Mathematics, the organization of the international conference on Groups and Semigroups: Interactions and Computations, in Lisbon, and pursuing with a second initiative that aims to contribute to the IMU/ICMI Klein Project http://www.mathunion.org/index.php?id=805 with the conference on Elementary Geometry from an Advanced point of View, at the University of Aveiro.

Mathematics of Planet Earth 2013

The initiative consists of holding a year of activities in 2013 under the theme Mathematics of Planet Earth (MPE) www.mpe2013.org

This program will be a great opportunity to showcase the essential relevance of mathematics in planetary issues. We are optimistic that this highly visible activity can create opportunities for additional sponsorship and that, by pooling together resources, a significant level of impact can be achieved.
In this initiative, the project is to hold scientific activities and activities for the public, the media and the schools.

International collaboration is encouraged.

The initiative was first launched by the North American Mathematical Sciences institutes (US and Canada). On their side, the plan is to develop a joint thematic program with the institutes hosting scientific workshops and organizing public lectures. A joint (North American) Scientific Committee chaired by Christiane Rousseau has been formed to encourage joint multi-institutes activities. The invitation is starting to be sent to societies to join and several societies are just waiting for the approval of their board to become official partners.

Spectra in mathematics and in physics: from the dispersion of light to nonlinear eigenvalues

Jean Mawhin
Université Catholique de Louvain

1 Introduction

This lecture deals with the concept of spectrum in various epochs, with various meanings and for various disciplines. Its content can be motivated by a quotation of L.A. Steen [53]:

> Not least because such different objects as atoms, operators and algebras all possess spectra, the evolution of spectral theory is one of the most informative chapters in the history of contemporary mathematics. [...] In 1928 and 1930 Norbert Wiener developed a theory of spectral analysis for functions in an attempt to analyze mathematically the spectrum of the white light, while twenty years later Arne Beurling inaugurated the complementary study of spectral synthesis.

and a quotation of N. Wiener [60]:

> The author sees no compelling reason to avoid a physical terminology in pure mathematics when a mathematical concept corresponds closely to a concept already familiar in physics. [...] The "spectrum" of this book merely amounts to rendering precise the notion familiar to the physicist, and may as well be known by the same name.

We shall return soon to the origin of the use of the word "spectrum" in physics and in mathematics. The mathematical spectrum is partly made of "eigenvalues", a strange word which has not been immediately adopted, as observed by S.H. Gould in [17]:

> The concept of an eigenvalue is of great importance in both pure and applied mathematics. [...] The German word *eigen* means *characteristic* and the hybrid work eigenvalue is used for characteristic numbers in order to avoid confusion with the many other uses in English of the word *characteristic*. [...] There can be no doubt that *eigenvalue* will soon find its way into the standard dictionaries. [...] The English language has many such hybrids: for example *liverwurst*.

Previous work has already been devoted to the development of spectral theory in mathematics, and the reader can find further information and remarks in [28, 31, 36, 59, 61].

2 Light and colours: the chemical unity of the Universe

The first occurence of the word "spectrum" in science seems to be found in a letter of Isaac Newton to the *Royal Society*, in 1672, where he uses the word to denote the oblong colored image, with the colors of a rainbow, produced on a white paper by a beam of Sun light dispersed by a glass prism. The expression is repeated in his book *Opticks* (London, 1704), and, in no case, Newton makes any comment on the choice of the word. "Spectrum" means "vision" in Latin, and comes from "spectare", to look at.

Little progress is made in Newton's experiment in the eighteenth century, except Melvill's observation in 1752 that a flame of salted alcohol only gives a yellow spectrum.

The beginning of the nineteenth century sees the discovery of the infrared and ultra-violet extensions of the spectrum, respectively by William Herschel and Ritter, the crucial discovery by Wollaston of seven dark lines in the solar spectrum, and the association of the color to the frequency in Young's ondulatory theory of light. In 1814, the Bavarian optician Fraunhofer constructs the first spectroscope. This allows him to establish the first map of the Solar spectrum, and to identify the position of one of its dark lines with the bright Natrium D-line.

After some pioneering work of John Herschell, Talbot, Foucault, Kelvin, Stokes and Angström, the mathematical physicist Kirchhoff, associated to the chemist Bunsen, discovers in 1859 the fundamental *laws of spectral analysis*: each line of a spectrum is due to the presence of a given element and conversely, the appearence of a line spectrum can be used as an analytical test for the presence of an element. Furthermore, a substance traversed by a source of light with continuous spectrum gives rise to dark lines having the same position. Consequently, the dark lines in the Solar spectrum reveal the composition of its atmosphere: astrophysics is born and stellar spectroscopy, with the pioneering work of men like Huggins, Miller and Secchi, reveals a fact of fundamental philosophical importance: the chemical unity of the Universe, some two hundred years after Newton's gravitation had shown its physical unity. Let us quote, in this respect, Poincaré [46]:

> Auguste Comte has said, I do not remember where, that it would be vain to try to find the composition of the Sun, because this knowledge would not be useful to Sociology. How could he be so short-sighted ? [...] First, one has recognized the nature of the Sun, that the founder of positivism wanted to forbid us, and one has found there substances which exist on the Earth and had remained unnoticed; for example Helium [...]. This was already for Comte a first flat contradiction. But we owe to spectroscopy a much more precious lesson [...]. We know now [...] that the laws of our chemistry are general laws of Nature, and do not follow from the chance which has made us born on the Earth.

Through the red-shift and the Döppler-Fizeau effect, galaxy spectra have also revealed to expansion of our Universe.

But the importance of spectroscopy is not less in the infinitely small, as spectra appear like signatures of atoms and molecules. After Angström classifies in 1853 the lines of the emission spectrum of Hydrogen in *series,* and after some pioneering work of Mascart, the Swiss teacher Balmer finds heuristically, in 1885, a formula giving the wave numbers ($\nu = 1/\lambda = c\nu'$, ν' the frequency) of one of those series:

$$\nu = R\left(\frac{1}{2^2} - \frac{1}{m^2}\right), \quad (m = 3, 4, \ldots),$$

where $R = 109.677,7 \ cm^{-1}$ is the Rydberg's constant. The lines accumulate near the limit wave number $\nu_l = R/4$, corresponding to the limit wave length $\lambda_l = 3645,6$ Å.

In 1908, Ritz states his *combination principle*: for each type of atom, it is possible to find a sequence of numbers, the *spectral terms,* such that the frequency of any spectral line of this atom is equal to the difference of two of those spectral terms. For example, the Hydrogen atom is characterized by the spectral terms R/n^2, $(n = 1, 2, \ldots)$. This principle implies the *generalized Balmer formula*

$$\nu = R\left(\frac{1}{n^2} - \frac{1}{m^2}\right), \quad (m = n+1, n+2, \ldots, \ n = 1, 2, \ldots),$$

suggesting the existence of Hydrogen lines with new wave numbers, later observed by Paschen, Brackett, and Pfund in the infra-red, and by Lyman in the ultraviolet. The reader can consult [55] for the historical development of spectroscopy and its influence on chemistry and astrophysics.

In 1913, Bohr proposes his quantified planetary model for the Hydrogen atom, from which he deduces mathematically the generalized Balmer formula with $R = 2\pi^2\mu e^4/ch^3$. Here μ is the mass of the electron, e its charge, c the speed of light, h is Planck's constant, and the computation gives a value very close to Rydberg's constant. However Bohr's model is based on some contradictory assumptions, and we may leave again to Poincaré [47] some prophetic conclusion:

> Following the work of Balmer, Runge, Kaiser, Rydberg, those lines are distributed in series, and, in each series, follow simple laws. The first idea is to relate those laws to those of harmonics. In the same way as a vibrating string has infinitely many degrees of freedom, allowing it to produce an infinity of sounds whose frequences are multiple of the fundamental frequency, [...] could the atom produce, for identical reasons, infinitely many different lights ? You know that this so simple idea has failed, because, according to the laws of spectroscopy, it is the frequency and not its square which has a simple expression; because the frequency does not become infinite for the harmonics of infinitely high rank. The idea must be modified or must be abandoned.

It is time to have a look at those vibrating strings to which Poincaré refers.

3 Music and harmonics: the paradigm of the wave equation

The relation between vibrating strings and mathematics can be traced at least to the Pythagorian tradition, but the development of musical theory at the Renaissance has led to physical discussions of the frequency of a vibrating string. Galileo and Mersenne, around 1640, study the dependence of the *fundamental frequency* of vibration with respect to the length, the tension and the mass of the string. At the end of the seventeenth century, Wallis, Robartes and Sauveur describe the connection between the number of nodes and the *overtones* of a vibrating string. See [11, 50] for details and references.

In 1714, Taylor assumes the isochronism of the oscillations for all the points of the string, and their simultaneous passage through the horizontal equilibrium position. He shows analytically that the fundamental frequency is $\nu = (1/2l)(\sqrt{T/\sigma})$, ($T$ is the tension, l the length, and σ the linear density) and the shape of the string is $y = A\sin(\pi x/l)$. In 1732, Jean Bernoulli determines the fundamental frequency of a discrete string made of six masses. None of them mentions the higher modes, which are considered in 1738 by Daniel BERNOULLI for an oscillating suspended string, both in the discrete and in the continuous case (anticipating Bessel functions). Modeling the propagation of sound in the air, EULER obtains in 1750, the characteristic frequencies and the general solution (as a sum of simple harmonic modes)

$$y_k = \sum_{r=1}^{n} A_r \sin \frac{rk\pi}{n+1} \cos\left(2\frac{\sqrt{K}t}{\sqrt{M}} \frac{\sin(\pi r/2)}{n+1} \right),$$

of the discrete model of a horizontal string

$$M\ddot{y}_k = K\left(y_{k+1} - 2y_k + y_{k-1}\right), \quad (k = 1, 2, \dots, n),$$

already written by Jean BERNOULLI in 1727.

Through a limit process, D'ALEMBERT deduces from it, in 1746, the one-dimensional *wave equation*

$$\frac{\partial^2 y(t,x)}{\partial t^2} = a^2 \frac{\partial^2 y(t,x)}{\partial x^2},$$

where $a^2 = T/\sigma$. He determines the solutions, satisfying the boundary conditions

$$y(t,0) = 0, \quad y(t,l) = 0, \quad (t \in \mathbb{R}),$$

through the change of independent variables still used to-day. In 1752, he introduces the method of *separation of variables.*

In 1749, EULER mentions that all possible motions of the vibrating string are periodic with the period of the fundamental mode, and that individual modes whose period is half, third,... of the fundamental one can occur. He writes those particular solutions in the form

$$y(t,x) = \sum a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l}$$

for the initial configuration

$$y(0,x) = \sum a_n \sin \frac{n\pi x}{l},$$

without precising if the sum is finite or not.

After reading the papers of d'Alembert and Euler on wave equation, DANIEL BERNOULLI claims in 1755 that there are enough free constants a_n to represent all the possible initial shapes as

$$f(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l},$$

and that all the subsequent motions are given by

$$y(t,x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi a t}{l}.$$

Those conclusions are refuted by EULER and D'ALEMBERT for different reasons, and the warm quarrell between those three giants lasts for some ten year, without conclusion, despite some deep comments of LAGRANGE in 1759 (see [50]). For references to the original sources and historical development, see [4, 8, 29, 56, 57].

4 HEAT AND POTENTIAL: THE IRRESISTIBLE ASCENT OF FOURIER SERIES

FOURIER modelizes the conduction of heat in a memoir submitted to the *Académie des Sciences de Paris* in 1807, rejected by the referees LAGRANGE, LAPLACE and LEGENDRE, revised in 1811, awarded the *Grand Prix de Mathématiques de l'Académie* in 1812, and only published in 1824-26 in its *Mémoires*, after Fourier has became its permanent secretary. In the meantime, Fourier has published a variant as a book, the famous *Théorie mathématique de la chaleur* (Paris, 1822).

Fourier establishes that the temperature $T(x,y,z,t)$ in a point (x,y,z) of a homogeneous and isotropic body satisfies the *heat equation*

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = k^2 \frac{\partial T}{\partial t},$$

where the constant k^2 depends upon the material. He studies several special cases by separation of variables, raising again the question of representing an arbitrary function by a trigonometric series, and obtaining, in a complicated way, the formula relating the coefficients of the series to the function. For references on Fourier's work, see [9, 18, 32].

Fourier's results motivate STURM and LIOUVILLE [34, 35, 54] to study in 1836-37 the general problem of eigenvalues and eigenfunctions for an arbitrary second order ordinary linear differential equation

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \lambda \rho(x) y = 0, \quad (p(x) > 0,\ \rho(x) > 0),$$

with the boundary conditions

$$y'(a) - h_1 y(a) = 0,\ y'(b) + h_2 y(b) = 0,$$

where $h_1 \geq 0$, $h_2 \geq 0$ and $a < b$. They prove that the problem has a nontrivial solution only when λ takes one of the values of an increasing sequence of positive numbers λ_n tending to infinity (*eigenvalues*), that the solutions v_n corresponding to the eigenvalue λ_n (*eigenfunctions*) are orthogonal in the sense

$$\int_a^b v_m(x)v_n(x)\rho(x)\,dx = 0, \quad (m \neq n),$$

and that each C^2 function satisfying the boundary conditions can be developed into a uniformly convergent series $f(x) = \sum_{n=1}^{\infty} c_n v_n(x)$, where the generalized Fourier coefficients $c_n = \int_a^b f(x)v_n(x)\rho(x)\,dx$ satisfy the *Parseval equality*

$$\int_a^b f^2(x)\rho(x)\,dx = \sum_{n=1}^{\infty} c_n^2$$

For the first time, general results are obtained which do not depend upon some explicit form of the solution of the differential equation. For studies of the work of Sturm and Liouville, see [2, 39, 40].

The Sturm-Liouville theory motivates of course the obtention of similar conclusions for the simplest partial differential equation case, namely the *eigenvalue problem for the Laplacian* on a general planar or spatial domain Ω (excluding the use of separation of variables)

$$\Delta u + \lambda u = 0 \quad in \quad \Omega, \quad u = 0 \quad on \quad \partial\Omega.$$

Schwarz [52] proves in 1885 the existence of the first eigenvalue and eigenfunction, and shows that a smaller Ω gives a larger λ_1. Picard [43] obtains in 1893 the existence of the second eigenvalue, and Poincaré [45] proves in 1894 the existence and the essential properties of all the eigenvalues and eigenfunctions, by showing that the solution of

$$\Delta u + \lambda u = f \quad in \quad \Omega, \quad u = 0 \quad on \quad \partial\Omega,$$

can be expressed as a meromorphic function of λ, whose poles are real and are the eigenvalues. Physically, f can be considered as a force applied to the vibrating membrane or body, and its free oscillations are those for which the forced oscillations become infinite. See [10, 41] for more details.

Motivated by Poincaré's work, Fredholm [15] publishes in 1903 a systematic study of the *integral equations of second type*

$$u(x) + \int_0^1 K(x,\xi)u(\xi)\,d\xi = f(x).$$

Following an idea of Volterra, he approximates the integral equation by the finite algebraic linear system

$$u_n\left(\frac{i}{n}\right) + \sum_{j=1}^{n} K\left(\frac{i}{n},\frac{j}{n}\right)u_n\left(\frac{j}{n}\right)\frac{j}{n} = f\left(\frac{i}{n}\right), \ (i = 1, 2, \dots, n).$$

Using finite-dimensional linear algebra and a limit process, he obtains the necessary and sufficient conditions of solvability, but does not emphasize the corresponding eigenvalue problem

$$u(x) + \lambda \int_0^1 K(x,\xi)u(\xi)\,d\xi = 0.$$

This will be done by Hilbert, as we shall see later. This finite-dimensional linear algebra has been developed in the nineteenth century, to answer some questions raised by analytical and celestial mechanics in the eighteenth century.

5 Stability of the Solar system: the origin of the secular equation

Motivated by elasticity problems, Euler anounces in 1743 the resolution of the linear ordinary differential equations with constant coefficients

$$Ay + B\frac{dy}{dx} + C\frac{d^2y}{dx^2} + D\frac{d^3y}{dx^3} + \dots + L\frac{d^ny}{dx^n} = 0.$$

Functions of the form $y(x) = e^{rx}$ are solutions if and only if r is a solution of the *characteristic* or *indicial* or *auxiliary* equation

$$A + Br + Cr^2 + \dots + Lr^n = 0,$$

and the general solution is obtained as a linear combination of the n special solutions associated to its roots.

d'Alembert, in his *Traité de dynamique* (Paris, 1743), studies second order systems of the form

$$y_i'' + \sum_{k=1}^{n} A_{ik}y_k = 0, \quad (i = 1, 2, \dots, n),$$

with $n = 2, 3$ and, for $n = 3$, special values of the coefficients A_{ik}, and, in 1750, first order systems of the type

$$\begin{aligned} x' + ax + by + cz &= 0, \\ y' + ex + fy + gz &= 0, \\ z' + hx + my + nz &= 0, \end{aligned}$$

but he obtains only partial results.

In 1766 Lagrange considers the general second order system above, and, through the substitution $y_i = x_i e^{\rho t}$, $(i = 1, 2, \dots, n)$, shows that

$$x = (x_1, x_2, \dots, x_n)$$

must verify the linear system (in modern matrix notations, with $A = (A_{ik})$, $A^T = (A_{ki})$)

$$\rho^2 x + A^T x = 0.$$

The elimination of the x_i in this system implies that ρ^2 must verify an algebraic equation of degree n. Lagrange is interested in modeling situations where the equilibrium $x = 0$ is stable, and concludes from this *a priori* physical stability and from the form of the solutions that the roots ρ^2 must be real, negative, and simple! In 1778, Lagrange obtains, in linearizing some equations of celestial mechanics, the first order system

$$h_i' + \sum_k A_{ik} l_k = 0, \quad l_i' - \sum_k A_{ik} h_k = 0, \quad (i = 1, 2, \dots, n),$$

where $h_i^2 + l_i^2 = e_i^2$, the square of the excentricity of the orbit, and uses the same approach, ending with the same physical "proof" for the properties of the roots of the corresponding algebraic equation, he calls *secular equation* [30]:

> One must notice that although we have supposed the roots [...] of the [secular] equation [...] real and distinct, it can happen that imaginary [=complex] ones exist; [...] we only observe that the quantities will increase with t; consequently, the above solution will stop to be exact after some time; but happily those case do not seem to occur in the system of the world.

Laplace is convinced that a mathematical proof of the properties of the secular roots should be preferred. In 1787, he deduces the *a priori* boundedness of the solutions from a first integral, first obtained in an approximate way, but rigorously proved two years later. Laplace's arguments are used by Lagrange, in his *Mécanique analytique* (Paris, 1788), for the study of small motions around an equilibrium, using this time the well known energy integral.

In his *Leçons sur les applications du calcul infinitésimal à la géométrie* (Cours de l'Ecole polytechnique, Paris, 1826), Cauchy associates the reduction of a quadric to its axes to an eigenvalue problem and its characteristic equation, invariant for any orthogonal change of coordinates, and proves rigorously that all eigenvalues are real. Sturm uses in 1829 his theorem of the number of real zeros of a real polynomial to prove the reality of the roots of the secular equations introduced by Lagrange and Laplace. The same year, Cauchy [5] gives another proof, shows the analogy of the problem of characteristic values in problems of analytic geometry, differential equations, solid and celestial mechanics, introduces the term *characteristic polynomial* or *equation*, which will finally overcome earlier or even later terminologies like *S-equation, determining, secular* or *latent equation.*

In the second half of the XIX^{th} century, the study of this equation becomes a topics of pure algebra, considered in the language of matrices or forms. For example, Hermite [25] gives in 1854 the standard proof of the reality of the characteristic roots of a *Hermitian form.* All this is carefully described in [19, 20, 21, 22, 29], with references to the original papers.

6 Algebra and geometry in infinite dimensional space: the birth of functional analysis

Motivated by Fredholm's theory, Hilbert publishes, between 1904 and 1910, a series of six articles, later reproduced in book form, under the title *Grunzüge einer allgemeinen Theorie der linearen Integralgleichungen* (Leipzig, 1912). He first follows essentially Fredholm's approach but considers an integral equation containing explicitely the complex parameter λ

$$u(x) - \lambda \int_0^1 K(x,y) u(y)\, dy = 0.$$

Hilbert supposes then $K(s,t)$ *symmetrical* ($K(s,t) = K(t,s)$), and uses the theory of finite quadratic forms to prove the reality of the eigenvalues and the orthogonality of the eigenfunctions. He generalizes to this setting the theorem of principal axes of analytical geometry, the variational characterization of eigenvalues due to Liouville-Weber-Poincaré that we shall consider later, and proves the *Hilbert-Schmidt expansion theorem.* In his own words:

> The method [...] consists in starting from an algebraic problem, namely the problem of the orthogonal transformation of quadratic forms in n variables in a sum of squares, and, through a rigorous limit process for $n = \infty$, to succeed in solving the considered transcendental problem.

Hilbert then forgets the initial motivation by integral equations and considers directly the infinite bilinear form in the sequences $x = (x_j), y = (y_k)$

$$B(x,y) = \sum_{p,q=1}^{\infty} k_{pq} x_p y_q,$$

when $\sum_{j=1}^n |x_j|^2$ et $\sum_{j=1}^n |y_j|^2$ converge, and B is bounded on the corresponding unit ball. He again generalizes the theorem of principal axes. For this, he must introduce, in addition to the discrete spectrum make of the eigenvalues, a *continuous* or *band spectrum,* a word first used in a mathematical setting by Wirtinger [62] in 1897, by analogy with spectra of molecules, in his discovery of band spectra for Hill's equation. Let us quote Dieudonné [10]:

> We now return to the most original part of Hilbert's 1906 paper, in which he discovered the entirely new phenomenon of the "continuous spectrum". [...] In 1897 Wirtinger developed similar ideas for Hill's equation
>
> $$y'' + \lambda q(x) y = 0.$$
>
> [...] The similarity with the optical spectra of molecules leads him to speak of the "Bandesspectrum" of Hill's equation. [...] Although Hilbert does not mention Wirtinger's paper, it is probable that he had read it (it is quoted by several of his pupils), and it may be that the name "Spectrum" which he used came from it.

To eliminate the continuous spectrum, Hilbert defines the concept of *completely continuous* quadratic forms. He applies his results to integral equations, introducing explicitely the notion of *complete orthogonal system* of functions.

Hilbert's work is simplified and geometrized by Ehrard Schmidt in 1907, who introduces the concept of *orthogonal projector;* the same year F. Riesz extends Hilbert theory to $L^2(0,1)$ and shows its isomorphism with l^2.

In 1908, H. Weyl considers *singular integral equations*

$$u(x) + \lambda \int_I K(x,y) u(y)\, dy = 0,$$

where integration is made on an unbounded interval I, and shows the existence of band spectra. His work is generalized by Carleman in 1923. The famous two volumes monograph *Methoden der mathematischen Physik* (Berlin, 1924) of Courant-Hilbert describes the state of the art of the mathematical tools of classical physics, before becoming the bible for the new physics. As noticed by C. Reid [48]:

> The Courant-Hilbert book on mathematical methods of physics, which had appeared at the end of 1924, before both Heisenberg's and Schrödinger's work, instead of being outdated by the new discoveries, seemed to have been written expressly for the physicists who now had to deal with them.

Excellent surveys of the development of Hilbert's ideas can be found in [1, 23, 24, 42, 49].

7 Quantum mechanics: unifying the physical and mathematical spectra

In 1923, L. de Broglie recovers Bohr's formula for hydrogen atom by associating to each particle a wave of some frequency and identifying the stationary states of the electron to the stationary character of the wave.

As the observable lines of an atomic spectrum can be represented by the infinite matrix $(\nu_{nm} = T_n - T_m)$ of the differences between the spectral terms, Heisenberg proposes in 1925 to replace the position q of an electron by an infinite matrix $q_{nm} e^{2\pi i \nu_{nm} t}$, and similarly for its momentum p. The diagonal elements of the corresponding Hermitian infinite matrices correspond to a stationary state and the other ones to corresponding transitions. The matrices q and p satisfy the *Born-Jordan non-commutativity condition* $pq - qp = \frac{h}{2\pi i} I$ and Hamilton-type canonical equations of motion. In 1926, Pauli deduces the Bohr formula for Hydrogen atom from matrix mechanics.

Independently and the same year, Schrödinger proposes to express the Bohr's quantification conditions as an eigenvalue problem [51]:

> In this communication I wish first to show in the simplest case of the Hydrogen atome (nonrelativistic and undistorted) that the usual rules for quantization can be replaced by another requirement, in which mention of 'whole numbers' no longer occur. Instead the integers occur in the same natural way as the integers specifying the number of nodes in a vibrating string. The new conception can be generalized, and I believe it touches the deepest meaning of the quantum rules. [...] The equation contains a "proper value parameter" E, which corresponds to the mechanical energy in macroscopic problems [...]. In general the wave or vibration equation possesses *no* solutions, which together with their derivatives are one-valued, finite and continuous througout the configuration space, *except* for certain special values of E, the *proper values.* These values form the "proper value spectrum" which frequently includes continuous parts (the "band spectrum", not expressly considered in most formulae [...]) as well as discrete points (the "line spectrum"). The proper values either turn out to be identical with the "energy levels" [...] of the quantum theory as hitherto developed, or differ from them in a manner which is confirmed by experience.

Starting from Hamilton-Jacobi equation

$$H\left(q, \frac{\partial S}{\partial q}\right) = E,$$

Schrödinger sets $S = K \log \psi$ (K is an action) and obtains

$$H\left(q, \frac{K}{\psi}\frac{\partial \psi}{\partial q}\right) = E.$$

For the electron of the Hydrogen atom this equation is

$$\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 - \frac{2\mu}{K^2}\left(E + \frac{e^2}{r}\right)\psi^2 = 0,$$

where $r^2 = x^2 + y^2 + z^2$. Schrödinger introduces the problem of finding an extremum of

$$\int_{\mathbb{R}^3} \left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2 - \frac{2\mu}{K^2}\left(E + \frac{e^2}{r}\right)\psi^2\right] dx\, dy\, dz,$$

among all sufficiently smooth functions ψ tending zero at infinity. The correspond Euler-Lagrange equation

$$\Delta\psi + \frac{2\mu}{K^2}\left(E + \frac{e^2}{r}\right)\psi = 0,$$

is *Schrödinger's equation.* Using spherical coordinates and separation of variables ($\psi(r,\theta,\varphi) = u(r)v(\theta)w(\varphi)$), Schrödinger reduces the problem to finding nontrivial solutions tending to 0 when $r \to \infty$ for the ordinary differential equations

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} + \left(\frac{2\mu E}{K^2} + \frac{2\mu e^2}{K^2 r} - \frac{n(n+1)}{r^2}\right)u = 0,$$

where $n = 0, 1, 2, \ldots$.

With the help of Weyl, Schrödinger shows that this equation has solutions with the required asymptotic properties if and only if $E > 0$ or

$$E < 0 \text{ and } \frac{\mu e^2}{K\sqrt{-2\mu E}} = j, \quad (j = n+1, n+2, \ldots).$$

For $n = 0$, those conditions become

$$E_j = -\frac{\mu e^4}{2K^2 j^2}, \quad (j = 1, 2, \ldots),$$

and reduce to Bohr's ones by taking $K = h/2\pi$. Schrödinger has reduced the problem of finding the energy spectrum of the Hydrogen atom to an eigenvalue problem on $\mathbb{R}^3$

$$L\psi + \frac{2\mu}{K^2}E\psi = 0,$$

for some differential operator L. Its mathematical spectrum exactly corresponds to the physical spectrum. Poincaré's program is realized.

Schrödinger gives later the now classical derivation of his equation, based upon the analogy between mechanics and optics, and closer to de Broglie's ideas. He also develops a *perturbation method,* inspired by the work of Rayleigh in acoustics, gives the *time-dependent Schrödinger's equation*

$$\frac{4\pi i}{h}\frac{\partial\psi}{\partial t} = \Delta\psi - \frac{8\pi^2}{h^2}V(t,x,y,z)\psi.$$

and proves the equivalence between his wave mechanics and Heisenberg's matrix mechanics. Mathematically, this fact is linked to the isomorphism between l^2 and L^2. Indeed, as observed by Condon [6], physicists could have saved some time and energy if they had taken Hilbert more seriously:

> I remember that David Hilbert was lecturing on quantum theory that fall [1926], although he was in very poor health at the time. [...] Hilbert was having a great laugh on Born and Heisenberg and the Göttingen theoretical physicists because when they first discovered matrix mechanics they [...] had gone to Hilbert for help and Hilbert said the only time he had ever had anything to do with matrices was when they came up as a sort of by-product of the eigenvalues of the boundary-value problem for a differential equation. So if you look for the differential equation which has these matrices you can probably do more with that. They had thought it was a goofy idea and that Hilbert didn't know what he was talking about. So he was having a lot of fun pointing out to them that they could have discovered Schrödinger's wave mechanics six months earlier if they had paid a little more attention to him.

See [26, 27] for the development of quantum mechanics.

Quantum theory gives in return a huge impetus to the mathematical development of spectral theory for unbounded linear operators. According to L.A. Steen [53]:

> The mathematical machinery of quantum mechanics became that of spectral analysis and the renewed activity precipitated the publication by Aurel Wintner of the first book devoted to spectral theory in 1929.

In 1927, von Neumann defines axiomatically the concept of *abstract Hilbert space* and developes, between 1927 and 1929, a spectral theory for *unbounded self-adjoint operators* in a Hilbert space. He synthetizes his approach in the epoch-making book *Mathematische Grundlagen der Quantenmechanik* (Berlin, 1932), and, the same year, Stone publishes his *Linear Transformations in Hilbert Spaces* (Providence, 1932), the first systematic treatise on the spectral theory of unbounded linear operators.

8 Variational characterization of eigenvalues: the way to a nonlinear spectral theory

Using Lagrange multipliers, Lagrange and Cauchy (1829-30) are already well aware that the smallest and the largest eigenvalue of a symmetric quadratic form

$$Q(u) = \sum_{j,k=1}^{n} a_{jk}u_j u_k, \quad (a_{jk} = a_{kj}),$$

can be obtained my minimizing and maximizing it on the unit sphere $\sum_{j=1}^n u_j^2 = 1$. If the corresponding extremum is reached at u^*, then u^* is an associated eigenvector, an approach later developed by Rayleigh.

In the setting of integral or partial differential equations, Liouville, in unpublished papers written around 1850, H. Weber [58] in 1869, and Poincaré [44] in 1890, independently propose a *recursive variational method* to determine all eigenvalues $\lambda_1 \leq \lambda_2 \leq$

$\ldots \leq \lambda_n$ and corresponding eigenvectors $u^1, u^2, \ldots, u^n$ of Q:

$$\lambda_1 = \min_{\|u\|=1} Q(u) \quad (= Q(u^1)),$$

$$\lambda_j = \min_{\|u\|=1, \langle u, u^1 \rangle = 0, \ldots, \langle u, u^{j-1} \rangle = 0} Q(u) \quad (= Q(u^j)), \quad (j = 2, \ldots, n).$$

Further considerations of Poincaré lead to a non-recursive *minimum-maximum principle* explicitely given by Fischer [13] in 1905:

$$\lambda_j = \min_{\{X^j \subset \mathbb{R}^n : dim X^j = j\}} \max_{\{u \in X^j : \|u\|=1\}} Q(u).$$

Weyl introduces in 1912 a *maximum-minimum principle*:

$$\lambda_j = \max_{\{p_1, \ldots, p_{j-1} \in \mathbb{R}^n\}} \min_{\{\|u\|=1, \langle u, p_i \rangle = 0, 1 \leq i \leq j-1\}} Q(u),$$

and Courant widely uses those principles in various existence and approximation questions of mathematical physics (see the survey [7]). The principles are easily extended to the abstract setting of symmetric bilinear forms in Hilbert spaces.

In 1930, Lusternik and Schnirel'mann [37, 38] extend this theory by replacing Q by an differentiable function f and the unit sphere by a finite dimensional compact differentiable manifold M. To replace the dimension of vector spaces, they introduce the concept of *category* $cat_X(A)$ of a closed set A in a topological space X, namely the least integer k such that A can be written as $\cup_{j=1}^k A_j$, with closed subsets A_j contractible in X. Lusternik and Schnirel'mann prove that the number of critical points of f on M is at least $cat_M(M)$, and that the corresponding values of f at the critical points (*critical values*) are given by

$$c_k = \inf_{A \in A_k} \sup_{u \in A} f(u),$$

where $A_k = \{A \subset M : A \text{ closed}, \mathrm{cat}_M(A) \geq k\}$ for $k = 1, 2, \ldots$.

Of course one has to check that A_k is non empty, which requires topological considerations. In particular they prove that if $F : \mathbb{R}^n \to \mathbb{R}$ is of class C^1 and even, then the system

$$F'(u) = \lambda u$$

has at least n pairs of solutions $[(\lambda, u), (\lambda, -u)]$ with $\|u\| = 1$. A version of this result is given in 1939 by Lusternik when $\mathbb{R}^n$ is replaced by a real infinite-dimensional separable Hilbert space, F' is compact and satisfies some other conditions.

In the fifties and the sixties, further extensions of Lusternik-Schnirel'mann theory to some infinite-dimensional problems are given by Krasnosel'skii, J.T. Schwartz and Palais (see references in [63]). F. Browder [3] refines and extends them to study nonlinear spectral problems in a Hilbert or a suitable reflexive Banach space X, which are of the form

$$F'(u) = \lambda G'(u),$$

where $F, G : X \to \mathbb{R}$ are suitable sufficiently smooth even nonlinear functionals. He finds conditions upon F and G which insure the existence of infinitely many critical levels.

The special case of $X = W_0^{1,p}(\Omega)$, $p > 1$, Ω a bounded domain of $\mathbb{R}^N$, $F(u) = \int_\Omega |\nabla u(x)|^p \, dx$ and $G(u) = \int_\Omega |u(x)|^p \, dx$ leads to the eigenvalue problem for the so-called *p-Laplacian operator* Δ_p, defined by

$$\Delta_p u(x) := \operatorname{div} \left(|\nabla u(x)|^{p-2} \nabla u(x) \right),$$

with the Dirichlet boundary conditions

$$u = 0 \text{ on } \partial\Omega.$$

An *eigenvalue for* $-\Delta_p$ with the Dirichlet boundary conditions is a λ such that the problem

$$-\Delta_p u = \lambda |u|^{p-2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

has a nontrivial solution.

The Lusternik-Schnirel'mann technique implies the existence of a sequence of eigenvalues given by a minimax characterization.

When $N = 1$, it follows from direct computations that this sequence constitutes the whole spectrum, but the problem remains open for $N \geq 2$. For the corresponding *ordinary vector p-Laplacian* $u \mapsto \left(\|u'\|^{p-2} u' \right)'$ where $u : [0, 1] \to \mathbb{R}^m$, $m \geq 2$, the spectrum is completely known in the case of Dirichlet conditions $u(0) = u(1) = 0$, but not in the case of periodic boundary condition $u(0) - u(1) = u'(0) - u'(1) = 0$.

The corresponding forced problem is always solvable (although not uniquely) when λ is not an eigenvalue, but solvability conditions at an eigenvalue remain almost *terra incognita.*

9 Spectra for asymmetric nonlinear operators: a possible tool for suspension bridges

The above extensions preserve the $\mathbb{Z}_2$-symmetry of the linear situation. Motivated by some asymmetric asymptotically linear boundary value problems, Fučik and Dancer have independently introduced in 1976 the study of problems of the form

$$-\Delta u = \mu u^+ - \nu u^- \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $u^+ = \max(u, 0), u^- = \max(-u, 0)$. An *eigenvalue* of this problem is now a couple (μ, ν) of reals such that the above problem has a nontrivial solution, and the set of eigenvalues is usually called the *Fučik* or the *Dancer-Fučik spectrum* of the corresponding Dirichlet problem. Abstract extensions in suitable ordered Banach spaces exist as well. Lazer and McKenna have emphasized the possible interest of those problems in modeling suspension bridges and explaining their possible instability (see [33]).

In the ordinary differential case ($N = 1$ and $\Omega =]0, 1[$), the Fučik spectrum is completely determined and made of the family of hyperbolic type curves

$$\frac{m}{\sqrt{\mu}} + \frac{n}{\sqrt{\nu}} = \frac{1}{\pi}, \quad (m, n = 0, 1, 2, \ldots),$$

whose intersection with the diagonal reproduces of course the standard spectrum

$$\{k^2\pi^2 : k = 1, 2, \ldots\}.$$

Very little is know in contrast when $N \geq 2$, except some properties for the first non trivial curve, some information on the shape near the diagonal points (λ_k, λ_k) associated to the classical eigenvalues, and some generic results about the structure in curves of the spectrum.

Here again the solvability of the forced problem is rather well understood when (μ, ν) is not in the Fučik spectrum, but much remains to be done in finding solvability conditions when (μ, ν) belongs to the spectrum.

Needless to say that the situation is still less developed in the study of the *Fučik spectrum of the p-Laplacian*

$$-\Delta_p u = \mu|u|^{p-2}u^+ - \nu|u|^{p-2}u^- = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

References can be found in the monographs [14, 12].

10 Conclusion

Many other modern aspects of spectral theory could have been discussed here, like bifurcation theory, Gelfand C^*-algebras, the spectrum of Riemannian manifolds, inverse spectral problems, perturbation theory or the relation between the spectrum of Schrödinger equations and the solution of some nonlinear partial differential equations. This would have taken the lecture beyong its time schedule, and the author beyond his abilities.

I hope that the story above has revealed the immensively creative power of unplanified research, as well as its unavoidable tortuous development. According to the Chinese tradition, only devils follow straight lines.

The conclusion will be left, like the Introduction, to some quotations, one from the middle, and one from the end of this century. They may convey some changes in mentalities in the fifty years period. The first one is due to R. Godement [16]:

> We believe that the human mind is a "meteor" in the same way as the rainbow – a natural phenomenon; and that Hilbert realizing the "spectral decomposition" of linear operators, Perrin analyzing the blue color of the sky, Monet, Debussy and Proust recreating, for our wonder, the scintillation of the light on the see, all worked for the same aim, which will also be that of the future: the knowledge of the whole Universe.

The second one is due to M. Zworski [64]:

> Eigenvalues of self-adjoint operators describe, among other things, the energies of bound states, states that exist forever if unperturbed. These do exist in real life [...]. In most situation however, states do not exist for ever, and a more accurate model is given by a decaying state that oscillates at some rate. [...] Eigenvalues are yet another expression of humanity's narcissic desire for immortality.

References

[1] M. Bernkopf, The development of function spaces with particular reference to their origins in integral equation theory, *Arch. Hist. Exact Sci.* 3 (1966), 1-96

[2] M. Bocher, *Leçons sur les méthodes de Sturm,* Gauthier-Villars, Paris, 1917

[3] F.E. Browder, Infinite-dimensional manifolds and nonlinear elliptic eigenvalue problems, *Annals of Math.* (2) 82 (1965), 459-477

[4] J.T. Cannon and S. Dostrovsky, *The evolution of dynamics. Vibration theory from 1687 to 1742,* Springer, New York, 1981

[5] A. Cauchy, Sur l'équation à l'aide de laquelle on détermine les inégalités séculaires des mouvements des planètes, *Exercices de math.* 4 1829 (*Oeuvres* (2), IX, 174-195)

[6] E.U. Condon, 60 years of quantum physics,*Physics Today* 15 (Oct. 10 1962), 37-49

[7] R. Courant, Variational methods for the solution of problems of equilibrium and vibrations, *Bull. Amer. Math. Soc.* 49 (1943), 1-23

[8] S.S. Demidov, Création et développement de la théorie des équations différentielles aux dérivées partielles dans les travaux de J. d'Alembert, *Revue Hist. Sci. Applic.* 35 (1982), 3-42

[9] J. Dhombres and J.B. Robert, *Fourier, créateur de la physique mathématique,* Belin, Paris, 1998

[10] J. Dieudonné, *History of functional analysis,* North Holland, Amsterdam, 1981

[11] S. Dostrovsky and J.T. Cannon, Early vibration theory: physics and music in the seventienth century, *Arch. History Exact Sci.* 154 (1975), 169-218

[12] P. Drábek, *Solvability and bifurcations of nonlinear equations,* Longman, Harlow, 1992

[13] E. Fisher, Ueber quadratische Formen mit reellen Koeffizienten, *Monatsh. f. Math. und Phys.* 16 (1905), 234-249

[14] S. Fučik, *Solvability of nonlinear equations and boundary value problems,* Reidel, Dordrecht, 1980

[15] I. Fredholm, Sur une classe d'équations fonctionnelles, *Acta Math.* 27 (1903), 365-390

[16] R. Godement, L'avenir des mathématiques concretes, in *Les grands courants de la pensée mathématique,* F. Le Lionnais ed., Cahiers du Sud, Paris, 1948, 321-326

[17] S.H. Gould, *Variational methods for eigenvalue problems,* Toronto Univ. Press, Toronto, 1957; 2nd ed. 1966

[18] I. Grattan-Guinness, *Joseph Fourier 1768-1830,* MIT Press, Cambridge, 1972

[19] Th. Hawkins, Cauchy and the spectral theory of matrices, *Historia Math.* 2 (1975), 1-29

[20] Th. Hawkins, The theory of matrices in the XIXth century, in *Proc. Int. Congres Math. Vancouver,* vol. 2, 1975, 561-570

[21] Th. Hawkins, Another look at Cayley and the theory of matrices, *Arch. Intern. Hist. Sci.* 26 (1977), 82-112

[22] Th. Hawkins, Weierstrass and the theory of matrices, *Arch. Hist. Exact Sci.* 17 (1977), 120-163

[23] E. Hellinger, Hilberts Arbeiten über Integralgleichungen und unendliche Gleichungssystem, in *Hilberts Gesammelte Abhandlungen,* vol. 3, Berlin, 1935, 94-140. Reprinted Chelsea, New York 1953

[24] E. Hellinger and O. Toeplitz, *Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten,* Teubner, Leipzig, 1928. Reprinted Chelsea, New York, 1953

[25] C. Hermite, Remarques sur un théorème de M. Cauchy, *C.R. Acad. Sci. Paris,* 41 (1855), 181-183 (*Oeuvres,* I, 459-461)

[26] B. Hoffmann, *L'étrange histoire des quanta,* Seuil, Paris, 1967

[27] M. Jammer, *The conceptual development of quantum mechanics,* McGraw-Hill, New York, 1966

[28] M. Janet, Harmoniques et spectres. Idées de Volterra. Equation de Fredholm. Espace de Hilbert. Physique classique et physique moderne, in *Les grands courants de la pensée mathématique,* F. Le Lionnais ed., Cahiers du Sud, Paris, 1948, 413-421

[29] M. Kline, *Mathematical thought from ancient to modern time,* Oxford University Press, Oxford, 1972

[30] J.L. Lagrange, Recherches sur les équations séculaires des mouvements des noeuds, et des inclinaisons des orbites des planètes, *Histoire Acad. Sci.*, 1774 (publ. 1778) (*Oeuvres,* 6, 635-709)

[31] R.E. Langer, What are Eigen-werte ? *Amer. Math. Monthly* 50 (1943), 279-287

[32] R.E. Langer, Fourier series. The genesis and evolution of a theory, *Amer. Math. Monthly* 54 (1947) Supplem. 1-86

[33] A.C. Lazer and P.J. McKenna, Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis, *SIAM Review* 32 (1990), 537-578

[34] J. Liouville, Mémoires sur le développement des fonctions ou parties de fonctions en séries dont les divers termes sont assujettis à satisfaire à une même équation différentielle du second order contenant un paramètre arbitraire, *J. Math. Pures Appl.* (1) 1 (1836), 253-265; 2 (1837), 16-35; 418-436

[35] J. Liouville and C. Sturm, Extrait d'un mémoire sur le développement des fonctions en séries dont les différents termes sont assujettis à satisfaire une même équation différentielle linéaire, contenant un paramètre variable, *J. Math. Pures Appl.* (1) 2 (1837), 220-223

[36] E.R. Lorch, The spectral theorem, in *Studies in Modern Analysis,* Math. Ass. America, Washington, 1962, 88-137

[37] L. Lusternik, Topologische Grundlagen der allgemeinen Eigenwerttheorie, *Monatsh. Math. Phys.* 37 (1930), 125-130

[38] L. Lusternik and L. Schnirelmann, *Topological methods in variational problems* (Russian), Trudy Scient. Invest. Inst. Math. Mech., Moscow, 1930. French translation: *Actualités scientifiques et industrielles* No. 188, Hermann, Paris, 1934

[39] J. Lützen, Sturm and Liouville's work on ordinary linear differential equations. The emergence of Sturm-Liouville theory, *Arch. History Exact Sci.* 29 (1984), 309-376

[40] J. Lützen, *Joseph Liouville (1809-1882). Master of pure and applied mathematics,* Springer, New York, 1990

[41] J.B. Pécot, Les théories spectrales de Poincaré, *Sciences et techniques en perspective* 26 (1993), 173-205

[42] J.B. Pécot, Les théories spectrales de Hilbert et de Schmidt, *Sciences et techniques en perspective* 26 (1993), 206-249

[43] E. Picard, Sur l'équation aux dérivées partielles qui se présente dans la théorie de la vibration des membranes, *C.R. Acad. Sci. Paris* 117 (1893), 502-507

[44] H. Poincaré, Sur les équations aux dérivées partielles de la physique mathématique, *Amer. J. Math.* 12 (1890), 211-294

[45] H. Poincaré, Sur les équations de la physique mathématique, *Rend. Circolo Mat. Palermo* 8 (1894), 57-155

[46] H. Poincaré, *La valeur de la Science,* Flammarion, Paris, 1905

[47] H. Poincaré, *Dernières pensées,* Flammarion, Paris, 1913

[48] C. Reid, *Hilbert,* Springer, New York-Berlin, 1970

[49] F. Riesz, *Les systèmes d'équations à une infinité d'inconnues,* Gauthier-Villars, Paris, 1913

[50] J.F. Rodrigues, *A Matemática e a música, Colóquio/Ciências. Revista de Cultura Científica,* Número 23

[51] E. Schrödinger, Quantisation as a problem of proper values, English translation of *Annalen der Physik* (4) 79, 80, 81 (1926), in *Collected Papers on Wave Mechanics,* Blackie and Son, London, 1928

[52] H.A. Schwarz, Ueber ein die Flächen kleinsten Inhalts betrffendes Problem der Variationsrechnung, *Festschrift zum Jubelgeburtstage des Herrn Weierstrass, Acta Soc. Fennicae* 15 (1885), 315-362 (*Ges. Math. Abh.* 1, 223-269)

[53] L.A. Steen, Highlights in the history of spectral theory, *Amer. Math. Monthly* 80 (1973), 359-381

[54] C. Sturm, Sur les équations différentielles linéaires du second ordre, *J. Math. Pures Appl.* (1) 1 (1836), 106-186; 373-344

[55] R. Taton éd., *Histoire généale des sciences,* 4 vol., Presses Universitaires de France, Paris, 1957-1964

[56] C.A. Truesdell, The rational mechanics of flexible or elastic bodies, 1638-1788, Introduction to Leonhardi Euleri Opera Omnia, vol. X et XI seriei secundae, in *Leonhardi Euleri Opera Omnia,* ser. 2, Opera Mechanica at Astronomia, vol. XI part 2, Zürich, 1960

[57] C.A. Truesdell, The theory of aerial sound, 1687-1788, in *Leonhardi Euleri Opera Omnia,* ser. 2, XIII, VII-CXVIII, Lausanne, 1955

[58] H. Weber, Ueber die Integration der partiellen Differentialgleichung $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$, *Math. Ann.* 1 (1869), 1-36

[59] H. Weyl, Ramifications, old and new, of the eigenvalue problem, *Bull. Amer. Math. Soc.* 56 (1950), 115-139

[60] N. Wiener, *The Fourier integral and certain of its applications,* Cambridge Univ. Press, Cambridge, 1933

[61] N. Wiener, The historical background of harmonic analysis, *AMS Centennial Public.* 2, Amer. Math. Soc., Providence, 1938, 56-68

[62] W. Wirtinger, Beiträge zu Riemann's Integrationsmethode für hyperbolische Differentialgleichungen, und deren Anwendungen auf Schwingungsprobleme, *Math. Ann.* 48 (1897), 364-389

[63] E. Zeidler, *Nonlinear functional analysis,* vol. III, Springer, New York, 1985

[64] M. Zworski, Resonance in physics and geometry, *Notices Amer. Math. Soc.* 46 (1999), 319-328

Coming Events

International Center for Mathematics

Pedro Nunes Lectures — 2011

Open questions leading to a global perspective dynamics by Jacob Palis

Pedro Nunes Lectures is an initiative of Centro Internacional de Matemática (CIM) in cooperation with Sociedade Portuguesa de Matemática (SPM), with the support of the Fundação Calouste Gulbenkian, to promote visits of notable mathematicians to Portugal. Each visitor is invited to give two or three lectures in Portuguese Universities on the recent developments in mathematics, their applications and cultural impact. Pedro Nunes Lectures are aimed to a vast audience, with wide mathematical interests, especially PhD students and youth researchers.

Jacob Palis (IMPA)
February 23, 2011 (15:00) Universidade do Porto.
March 02, 2011 (16:30) Universidade de Lisboa.

Open questions leading to a global perspective in dynamics

Abstract.—We will address one of the most challenging and central problems in dynamical systems, meaning flows, diffeomophisms or, more generally, transformations, defined on a closed manifold (compact, without boundary or an interval on the real line): can we describe the behavior in the long run of typical trajectories for typical systems? Poincaré was probably the first to point in this direction and s tress its importance. We shall consider finite-dimensional parameterized families of dynamics and typical will be taken in terms of Lebesgue probability both in parameter and phase spaces. We will discuss a conjecture stating that for a typical dynamical system, almost all trajectories have only finitely many choices, of (transitive) attractors, where to accumulate upon in the future. Interrelated conjectures will also be discussed.

3rd Porto Meeting on Mathematics for Industry, Porto, Portugal, April, 28-30, 2011

The purpose of this meeting is to focus the attention on the many and varied opportunities to promote applications of mathematics to industrial problems. Its major objectives are:

Development and encouragement of industrial and academic collaboration, facilitating contacts between academic, industrial, business and finance users of mathematics.

Through "bridging the industrial/academic barrier" these meetings will provide opportunities to present successful collaborations and to elaborate elements such as technology transfer, differing vocabularies and goals, nurturing of contacts and resolution of issues.

To attract undergraduate students to distinctive and relevant formation profiles, motivate them during their study, and advance their personal training in Mathematics and its Applications to Industry, Finance, etc.

The meeting will be focused on short courses, of three one-hour lectures each, given by invited distinguished researchers, which are supplemented by contributed short talks by other participants and posters of case studies.

The meeting will be followed by the 81th European Study Group that will take place in Lisbon between the 23th and the 27th of May 2011.

Summer Course and Workshop on Optimization in Machine Learning, Austin, Texas, USA, May 31 - June 7, 2011

The Summer Course on Optimization in Machine Learning (May 31 - June 4, 2011) will introduce a range of machine learning models and optimization tools that are used to apply these models in practice. For the students with some Machine Learning background the course will introduce what lies behind the optimization tools often used as a black box as well as an understanding of the trade-offs of numerical accuracy and theoretical and empirical complexity. For the students with some Optimization background this course will introduce a variety of applications arising in Machine Learning and Statistics as well as novel optimization methods targeting these applications.

This course will be followed by the Workshop on Optimization in Machine Learning (June 6-7, 2011) which will bring a number of experts in the area and further present the state-of-art.

CoLab Mathematics Summer School and Workshop, Instituto Superior Técnico, June 13-24, 2011

The CoLab Mathematics Summer School and Workshop is a yearly event, organized by the UTAustin|Portugal Program that aims at bringing together Ph.D. students and junior faculty with well known experts in the several areas of mathematics.This event will be held in the Mathematics Department of Instituto Superior Técnico in Lisbon, from June 13–24, 2011.

The school will be between June 13–17, 2011, and this year main topic is Aubry Mather Theory and Optimal Transport. The faculty for this school (L. Ambrosio, P. Bernard, Y. Brenier, and A. Figalli) are internationally known experts in these fields. We believe that their courses will be extremely useful for Ph.D. students, postdocs and established researchers who wish to broaden their knowledge in this very active area of research.

The Nonlinear PDEs workshop aims at bringing together researchers in several areas of nonlinear partial differential equations and its applications, and will be held in the week of June 20-24.

This event is also part of the program of the Portuguese International Center for Mathematics (CIM).

Dynamical Models in Life Sciences, University of Évora , July 24-30, 2011

The aim of this summer school is to bring together specialists and students in mathematics, biology, physics and engineering with a common goal: the better understanding of the mathematics behind life sciences and the better understanding of life sciences using a deeper knowledge of mathematics.
This will be a joint event of the portuguese Centro Internacional de Matemática, the European Society for Mathematical and Theoretical Biology and the European Mathematical Society in the historical city of Evora, Portugal, providing the perfect environment for the interaction between senior scientists in the field and students.

It will consist of 7 mini-courses aimed to Ph.D. students and junior post-docs, in mathematics, biology and related areas.

Optimization 2011, Universidade Nova de Lisboa, July 24-27, 2011

Optimization 2011 aims to bring together researchers and practitioners from different areas and with distinct backgrounds, but with common interests in optimization.

This meeting has international recognition as an important forum of discussion and exchange of ideas. It is the seventh edition of a series of international conferences in optimization organized in Portugal under the auspices of APDIO (the Portuguese Operations Research Society).

In this edition, we are happy to announce a special session celebrating the 60th anniversary of our dear colleague Joaquim João Júdice, the founder of the Optimization series and a well-known researcher in the field of Optimization.

Groups and Semigroups: Interactions and Computations, University of Lisbon, July, 25-29, 2011

The aim of this conference is to deepen the existing interactions between group theory and semigroup theory. The main themes that the conference cover include, not exclusively, the application of permutation group theory in the theory of transformation semigroups; computational techniques in group theory and semigroup theory; and combinatorial methods in group theory and semigroup theory.

Elementary Geometry from an Advanced Point of View, University of Aveiro , September 1-2, 2011

The aim of this conference (http://egapv2011.glocos.org) is to present several contemporary perspectives on Geometry including, among others, talks on visualization, applications and surveys, both at elementary and more advanced levels. The goal of this meeting, promoted by CIM in collaboration with CIDMA/Univ. Aveiro, CM/Univ. Minho and CMAF/Univ. Lisboa, is to contribute to the current international reflection on the ICMI/IMU Klein Project concerning central topics on Geometry, its contents, interdisciplinary connections and approaches for the teaching of this mathematics discipline at senior secondary school and first years at University level.

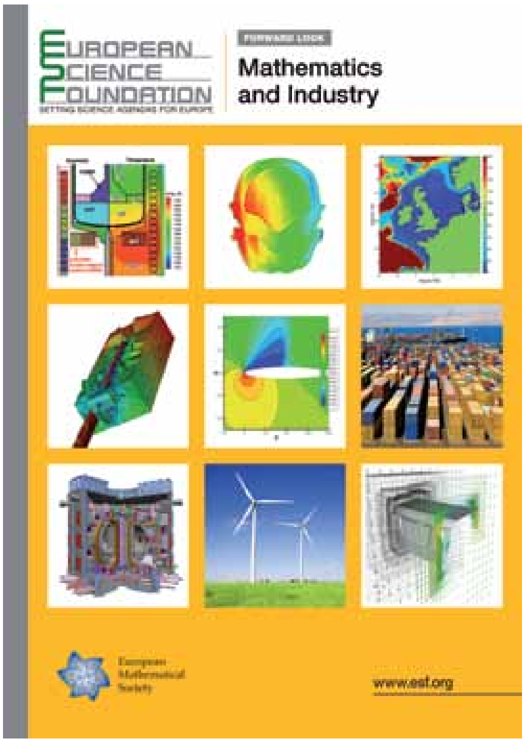
Organisors—Ana Breda (U Aveiro)—Chair, Ana Pereira do Vale (U Minho),
Tomas Recio (U Cantabria), Eugénio Rocha (U Aveiro), José Francisco Rodrigues (U Lisboa)

Encontro CIM-SPM-SPE

O Caos e o Acaso

Forward Look Mathematics and Industry

European Science Foundation



The European Science Foundation published the report "Forward Look Mathematics and Industry". The document can be accessed via the link http://www.esf.org/publications.html. We reproduce here the conclusion of the report:

"The basic message of this report is that if Europe is to achieve its goal of becoming the leading knowledge-based economy in the world, mathematics has a vital role to play. In many industrial sectors the value of mathemat- ics is already proven, in others its potential contribution to competitiveness is becoming apparent. The benefits resulting from a dynamic mathematics community interacting actively with industry and commerce are considerable and certainly far outweigh the rather modest costs required to support such a community. Nevertheless, such benefits will not be realised unless action is taken to develop mathematics and a coordinated community of industrial and applied mathematicians needed for the future success and global competitiveness of the European economy and prosperity."

—*Forward Look Mathematics and Industry*, p. 5

Editors: Assis Azevedo (assis@math.uminho.pt); António Fernandes (amfern@math.ist.utl.pt)
Address: IIIUL-Universidade de Lisboa; Av. Prof. Gama Pinto, 2; 1649-003 Lisboa

The CIM Bulletin is published twice a year. Material intended for publication should be sent to one of the editors.
The Bulletin is available at www.cim.pt
The CIM acknowledges the financial support of FCT–Fundação para a Ciência e a Tecnologia

The first group photo at the headquarters of Regimento de Cavalaria nº6 de Braga

MatCampus2010 — A completely new experience

by Paula Mendes Martins and Victoria Otero-Espinar

1 The meeting

If you had visited the University of Minho and the University of Santiago de Compostela last July, you would find a different environment. Forty teenagers from Portugal and Galiza, together with four high school teachers participated in the meeting MatCampus 2010 — a mathematical camp between Portugal and Spain, from the 18th to the 31st of July.

MatCampus 2010 was an initiative of the International Mathematical Centre (CIM) and was jointly organized by CIM, the Faculty of Mathematics at the University of Santiago de Compostela (USC) and the Department of Mathematics and Applications at the University of Minho (UM). The main objective of this meeting was to give galician and portuguese 17 year old students the opportunity to gain hands-on experience and to discuss mathematics. This initiative, pioneered in Europe, has developed a unique environment for cultural exchange. The institutions involved are recognized for their high quality research in mathematics, with well qualified teachers with previous experience in the out-reach activities of mathematics in different contexts. These include Math Olympiad, the project ESTALMAT to stimulate mathematical talent and the National Championship of Mathematical Games.

Several aims were proposed for this activity. These included

- to encourage positive attitudes and special capabilities of the participants in the field of mathematics;

Group photo in Aula Magna, USC

- to broden the mathematical background of the participants on issues that are motivating and not part of the school curriculum;
- to deepen the knowledge and use of new technologies (graphing calculators, mathematics software, internet, ...) as a source of information, updating and as a learning environment;
- to consolidate and increase the stimulation of curiosity, the formulation of interesting questions, knowledge and practice of methods, techniques and work processes specific to mathematics;
- to provide a more humane vision of mathematics through reports, readings and other activities;
- to provide tools and enhance personal resources to continuous learning, self-oriented study and through teamwork.

All these purposes were achieved.

With in MatCampus 2010, the organizers also wanted to establish an international collaboration that may serve as a reference for mathematical european campus for youth. The organizers hope that this event may be continued and be repeated in other European countries or regions, thereby enhancing the image of Portugal and Galicia in European landscape of mathematics education. In this sense, the activities undertaken in 2010 were recorded on the MatCampus own website.[1]

The event had the scientific support of the *Real Sociedad Matemática Española* (RSME), of the *Sociedad de Estadística e Investigación Operativa* (SEIO), of the *Sociedad Española de Matemática Aplicada* (SEMA) and of the *Sociedade Galega para a Promoción da Estatística e a Investigación de Operacións* (SGAPEIO) and members from many different universities were involved.

[1] http://www.matcampus2010.org

2 The activities

MatCampus 2010 was a two week event. The first week took place in Braga and the second one in Santiago de Compostela. The main idea was to combine scientific and pedagogical activities with leisure ones.

The mathematical sessions took place in the Department of Mathematics and Applications of UM and in the Faculty of Mathematics of USC, which provide all the facilities necessary to carry out the proposed activities. In Braga, the participants stayed in the headquarters of the Regimento de Cavalaria n° 6 and had their meals in the university canteen. In Santiago de Compostela, they slept and had their meals in the university residence Monte de la Condesa, near the faculty building.

The participants were never left without supervision. Alexandre Cortés Ayaso and Rafael Fernández Casado, from Galiza and members of the Organising Committee of MatCampus, Paula Gomes and João Paulo Gonçalves, from Portugal, are four high school teachers that accompanied the forty students during the two weeks. These four theachers were fundamental for the success of MatCampus 2010.

2.1 in Braga

Day 1.—The first activity was led by the project *Atractor*, coordinated by Professor Arala Chaves, from

One moment of the mathematical Logic session

University of Porto. This activity, focused on the dvd "Symmetries" recently launched by Atractor, presented to students, in a playful and attractive way, the seventeen plane symmetry groups and seven frieze patterns. In the afternoon, the students went to Campus of Azurém, in Guimarães, which is the second campus of UM. There, they were able to observe the work developed in the area of robotics by scientists of UM, from the Departments of Industrial Electronics and of Mathematics and Applications. This trip to Guimarães finished with a visit to the town castle, the so-called birth place of Portugal.

Day 2.—On Tuesday morning, a session on Mathematical Computation was presented by two members of the Mathematics and Applications Department of UM, Joana Torres and Ricardo Severino. In this course some computationally simple mathematical models were explored that show how you can lose the ability to predict temporal evolution. The concepts of fractal and deterministic chaos were presented. After lunch, the participants worked with the software Surfer in an activity organised by Stephan Klaus, from the Mathematische Forschungsinstitut Oberwolfach, connected with the project Imaginary 2008 (http://www.imaginary2008.de/).

In the evening, the students participated in a Photo-paper at the town center, which allowed them to have a different view of Braga.

Day 3.—The morning of day 3 was dedicated to astronomy, with a talk presented by Cacilda Moura, from Physics Department of UM, and Mathematical Logic, with a working session. In this session, coordinated by three members of the Organising Committee of MatCampus, Paula Mendes Martins, Cláudia Mendes Araújo and Suzana Mendes Gonçalves, all from UM, the participants were invited to answer some intriguing problems, using the concepts of Mathematical Logic first introduced. The afternoon was dedicated to a visit to Museu D. Diogo de Sousa and a boat ride on Rio Cávado.

Day 4.—The participants had the opportunity to relate mathematics and music. The musical activities were streamlined by Ana Pereira do Vale (University of Minho) and Maria Helena Albuquerque (University of Coimbra). Using mathematical concepts, the students were able to compare styles of music, and conclude that the mathematical structure behind every style of music is the same. To complete the schedule in the morning, António Machiavelo, from University of Porto, gave a talk about cryptography.

One of the classrooms where the students attended sessions

The officer in charge of the headquarters where the students stayed during the first week kindly offered the participants a pleasant afternoon, showing them the facilities, along with some military exercises and allowing each student to have a ride on their magnificent horses.

Day 5.—This was the last day at the University of Minho. The morning began with a working session on graph theory, led by Assis Azevedo, from UM, CIM and also a member of the Organising Committee. The software Grin was the tool used for the entire session. Isabel Leite, a high school teacher from Escola Secundária de Vila Verde and also a member of the Organising Committee, led the following session, based on sensors. Although some portuguese students had worked with this kind of material in their schools (the high school mathematics plan encourages the use of sensors as much as possible), this was new for the spanish students. The day ended with mathematical games. Games such as chess, hex, traffic lights, slime trail and oware kept the participants thinking throughout the afternoon. Some puzzles including Rubik cube and Rubik cube revenge were also explored.

During the first week, the students had the opportunity to visit an interactive exhibition on Recreational Mathematics, which included 15 different activities (such as squaring polygons and soma cube) to give school students the opportunity to gain hands-on experience and to discuss mathematics. Some of the students used every break during the activities in Braga to visit and re-visit the exhibition.

On the 24th, the group went to Ponte da Barca for an orienteering event, organised by the *.COM – Clube de Orientação do Minho*. The day ended with a dinner in a town restaurant. And the participants said goodbye to Braga.

2.2 in Santiago de Compostela

On the 25th of July all was ready to welcome the participants in Santiago. Victoria Otero Espinar, Dean of the Faculty of Mathematics, and Rosa Crujeiras Casais, members of the Organising Committee, were waiting for the students in the University Residence. Other members of the welcoming committee, tutors that permanently accompanied the students during their stay in Santiago, also participated: Carmela Rodríguez Alvarez, Rosalia Rodriguez Couceiro, Cibrán Santos Bouza and Jose Luis Villarino Barja. After the break and lunch all participants visited the city.

Day 1.—The week started with a welcoming ceremony in the Aula Magna of the Faculty of Mathematics, with the participation of academic authorities, sponsors and many of the professors of the Faculty.

After an obligatory visit to the Faculty's facilities, the students went to Caixa Galicia CIEF Center to attend the Lecture given by the well-known geneticist Angel Carracedo, Director of the Institute of Legal Medicine of USC. In this talk, they were able to check the interdisciplinary nature of the sciences and the necessity of mathematical knowledge in the development of sciences. The genetic laws of Mendel introduced mathematical models based on the observation of biologic experiments. Today, genetics is a science in its own right, whose results are directly related to improvements in stochastic modeling.

To complete the topic, a session on Mathematics of genetics was presented by two members of the Department of Statistics and Operational Research of USC, Manuel Magariños and Rosa Crujeiras Casais. In this course, using genetical basis, some simple mathematical models that show population evolution were explored.

Photo Composition by Ana Rita Gomes, from
Escola Secundária da Gafanha da Nazaré (Portugal)

In the afternoon, the participants visited the Centro de Supercomputación de Galicia. The CESGA is the center for high-performance computing, communications and advanced services used by the Scientific Community of Galicia including the University System of Galicia and the Spanish National Research Council (CSIC). The students were able to simulate the previously explained mathematical genetic models by using the facilities of the Cesga's high performance computer, Finisterrae.

A good way to finish a hard day was the visit to the Astronomical Observatory Ramón María Aller of the USC. In this visit, participants were immersed in the science of astronomy. They dreamed once more with the planets, comets, stars, galaxies, etc ... helped by the interesting presentation and nocturnal observations of Professor José A. Docobo Durántez, Director of the Observatory, and Professors Josefina Ling Ling and Pedro Pablo Campo Diaz.

Day 2.—To show that mathematics are essential to an understanding of the world around us, the day was devoted to mathematical modeling of real problems. In the workshop Numerical simulation of industrial problems: an acoustic structural problem was presented by Andrés Prieto Aneiros, from Faculty of Mathematics of the USC, who showed how numerical simulation tools can be applied to the resolution of industrial and business processes.

The thematic day addressed to mathematical simulation techniques ended with a visit to the company Castrosúa, which collaborates with a researchers's team of the Faculty of Mathematics.

Day 3.—Wednesday morning was dedicated to visiting Science museums in the city of A Coruña. The students were introduced to scientific novelties in the *Casa das Ciencias*, and in an interactive journey through the human body in DOMUS.

Taking advantage of the proximity of the sea, the participants enjoyed a pleasant afternoon on the beach.

Day 4.—In the session dedicated to calendars, the mathematician José María Barja, Rector of the University of A Coruña, showed how the history of calendars is strongly linked to the history of mathematics. In this activity, various ways of measuring time throughout history were reviewed and exercises on calculations related to calendars were carried out.

To complete the schedule in the morning, the Delegate to Galicia of the Olympic Committee of the Real Sociedad Matemática Española, Felipe Gago Couso, member of the Faculty of Mathematics of the USC, gave a talk about working skills for problem solving: strategies, techniques for simplification, generalization and analysis of problem formulation.

In the afternoon, there was a mathematical tour of Santiago de Compostela, coordinated by a member of the Organising Committee of MatCampus, Pilar García Agra.

What is a mathematical tour? Do you need to know a lot of maths to follow a route through the streets and squares of Santiago de Compostela? No, in this walk, participants had the opportunity to discover the elements and mathematical properties where least expected, in addition to the many attractions of the city. They realized the beauty that can be generated with appropriate use of shapes and geometric properties, and the students were able to train their eyes to capture the mathematical relationships that are sometimes hidden in the most unexpected objects. This was shown subsequently in photographs that the participants took for a Mathematical Photograph Competition, organised by three members of the Organising Committee of MatCampus, Elena Vazquez Abal, Rosa Crujeiras Casais and Victoria Otero Espinar.

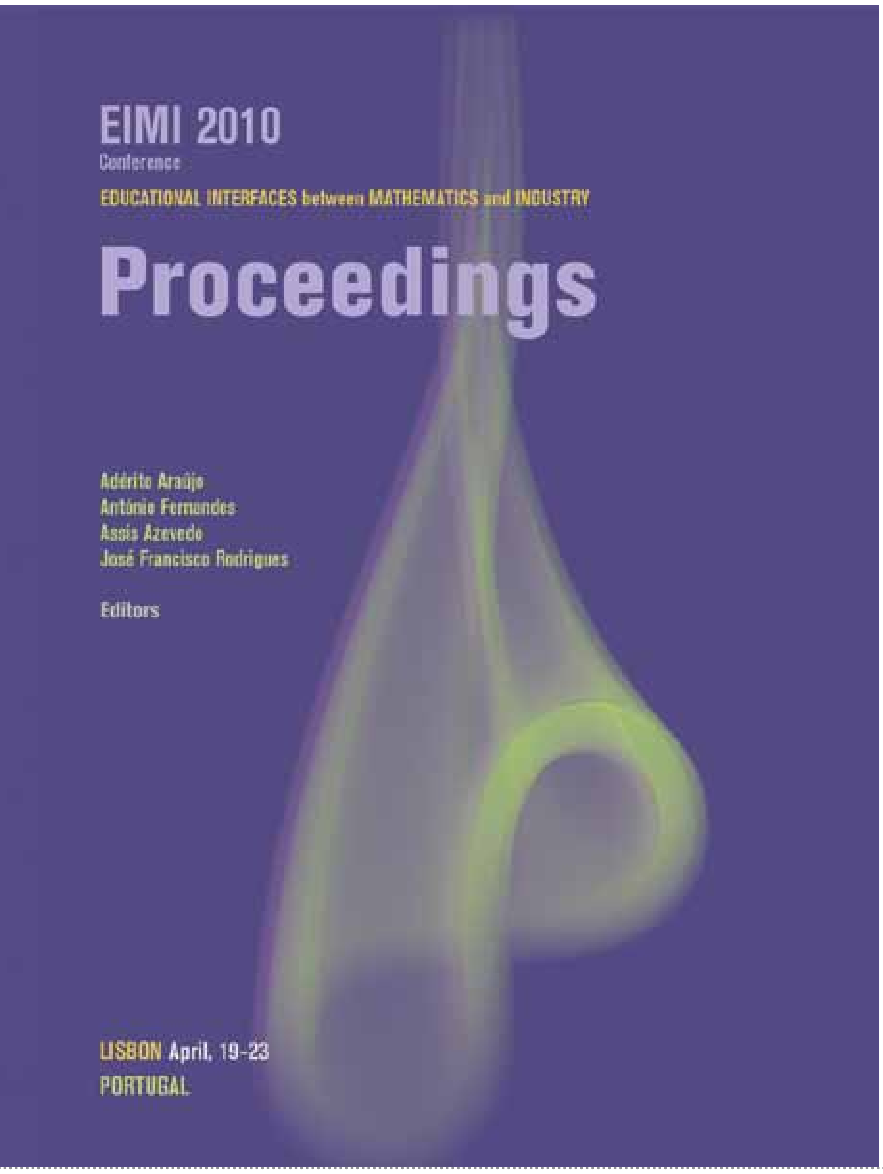
Day 5.—The whole morning was occupied by a working session on Origami, led by Teresa Otero Suárez and Miguel A. Vidal Martín, high school teachers. In this activity, origami was used as a didactic resource in mathematics. The students used modular origami to allow a physical representation and to test properties of polyhedra .

The mathematical camp MatCampus 2010 ended on Friday with a closing ceremony that took place in the Aula Magna of the Faculty of Mathematics. The program of activities of these last two weeks ended with a play, “Innúmeros Números”, by César Goldi and Vicente de Souza.

3 Was it worthwhile?

When an event like MatCampus 2010 comes to an end, the obvious question arises: was it Worthwhile? In the case of MatCampus 2010, the answer is also obvious: Yes, it was.

When, by suggestion of José Francisco Rodrigues, the Organising Committee (Paula Mendes Martins, Assis Azevedo, Cláudia Mendes Araújo, Suzana Mendes Gonçalves, Isabel Leite, María Victoria Otero Espinar, María Elena Vázquez Abal, Rosa María Crujeiras Casais, Pila García Agra, Rafael Fernández Casado and Alexandre Cortés Ayaso) began to prepare MatCampus 2010, the organisers could not imagine how successful this mathematical camp was going to be. At the end of the two weeks, the forty participants were unanimous in affirming that they were very lucky to be chosen to take part of MatCampus 2010. Everyone asked for MatCampus new edition in 2011 because they were willing to repeat the experience. This was a completely new experience that should be repeated for many years and by as many universities as possible.



EIMI 2010 Proceedings

In partnership with COMAP, on behalf of CIM, A. Araújo, A. Fernandes, A. Azevedo and J. F. Rodrigues have edited the EIMI PROCEEDINGS of the international conference on “Educational Interfaces between Mathematics and Industry” of the ICMI/ICIAM international study, which are available on line at the CIM website http://www.cim.pt/files/proceedings_eimi_2010.pdf and were available for the associates in book form at the CIM annual general assembly in May.

An Interview with Sir Michael Atiyah

by **Ana Cristina Ferreira** [Universidade do Minho]

Sir Michael Atiyah is one of the most well-known and important mathematicians of the past century. He has had a long and distinguished career, receiving both the Fields Medal in 1966 and the Abel Prize in 2004. Some of his most significant contributions include the Atiyah-Singer Index Theorem and topological K-theory, and he remains active in mathematical research to this day. His planned visit to Portugal to deliver the Pedro Nunes lectures was disrupted by the Icelandic volcano eruption last spring and is now rescheduled for the period between 27 March and 7 April 2011.

You have once said that your father thought you'd be a mathematician when he realized you were making money out of exchanging foreign currency. How/when would you say you first become interested in pursuing mathematics as a career?

I seriously focused on mathematics at the age of 16. When we moved to England in 1945 we selected the school (Manchester Grammar School) that had the reputation as the best in the country for Mathematics.

Could you tell us a bit about your undergraduate/PhD experience, and what some of the difficulties you encountered were?

I had no difficulties as an undergraduate, coming out top of the university in my first year. As a graduate student I had a bad period at the beginning of my second year, and I took to attending classes in other subjects (architecture and archaeology). But then I got a prize and things improved. Many years later Serre told me he had a similar experience and almost decided to give up mathematics.

Graduate students

You had graduate students who became very well-known mathematicians, for example, Simon Donaldson, Nigel Hitchin and Graeme Segal. Do the benefits to a researcher of taking on PhD students outweigh the costs?

The best research students cause least work and are the most rewarding, but others cause much work and there is certainly a limit on how many students one should take on at a time.

What advice would you give to people who are doing a PhD now or planning to do one in the near future?

My advice is that one should not start research in mathematics unless one is dedicated to the subject. One has to have a passion for it in order to surmount the difficulties and disappointments that always arise.

Collaborators and other mathematicians

You have had some notable collaborators such as Hirzebruch, Singer, and most recently Vafa and Witten. Can you tell us a bit about these collaborations, and how important they were/are to your work? Do you they really help you peer around the corner?

I find collaboration natural and very useful. Another person has a different point of view and usually a different background. Moreover collaboration often develops into friendship and makes research a less solitary undertaking.

Which mathematicians do you most admire/respect, and why?

Among past mathematicians Hermann Weyl is the one I admire most. The breadth of his interests and the elegance of his style have been my model.

The index theorem

The Atiyah-Singer index theorem is a landmark of the 20th century mathematics. Could you tell us a bit about how it all begun?
With Gelfand's conjecture that the index of an elliptic operator expressed as topological invariants or with the proof of the integrability of the Â-genus? What made you think that the Â-genus of a spin manifold was the index of a Dirac operator?

The story of the index theorem is one I have recounted often and I recommend you read the introduction to my collected works volume 3.

There are at least three different proofs of the index theorem, one involving K-theory, one involving the cobordism theory and the other using heat kernels. Why was it so important to the mathematical community to have different proofs? Do they give more insight on the theory?

Different proofs have different merits. They lead to different generalizations and they connect up with other subjects in different ways. One could say that the number of different proofs of a theorem is one measure of its significance. Gauss is said to have had 8 proofs of the law of quadratic reciprocity.

Awards

You were awarded the Fields Medal in 1966. How do you think receiving this award affected your career? Could you also tell us about the Abel prize and other awards? What kind of impact did these have in your research?

Awards provide encouragement, but are not the motivation of mathematicians. The Fields Medal, coming when one is young (under 40) has the most effect. Others, such as the Abel Prize, come later in life, are gratifying and compensate for the passage of youth!

A bit of controversy

People often wonder what goes on inside a brilliant mind. Do you agree with the sentence "genius is 1% inspiration and 99% perspiration?"

Brilliant ideas are rather rare and have to be backed up by detailed work, but the ratio of 99 to 1 sounds excessive. It depends on what measure one is using. It also depends on the individual and the style of mathematics.

Do you still have to wear a bulletproof vest because of your comment on the classification of finite simple groups?

I am not afraid to speak my mind, though I may not always be diplomatic.

You are still a very active mathematician. Would you disagree with Hardy's famous quote that "mathematics is a young man's game"?

I agree with Hardy that the main advances and innovations in mathematics are made when people are young, but they can continue to produce worthwhile mathematics into later life.

Interactions with different areas of Mathematics, Physics and other sciences

You once said you were a jack of all trades. Would you said the future of mathematics lies in finding bridges between different areas or is mathematics getting so specialized that it will eventually be fragmented?

My personal interest has always been in the interactions between different areas of mathematics (and also physics). This also helps to keep mathematics together. But others have a different style and concentrate on one field. We need all types.

Could you tell us a bit more about your recent work in physics and its interactions with mathematics? Was the time ripe for topological quantum field theory?

The new interaction between topology and quantum physics came as something of a surprise. In retrospect the time seems to have been auspicious, but this is hindsight.

What would you say has been the impact of String theory on Mathematics?

The impact of physics on mathematics has been unexpectedly wide. Almost all branches have been dramatically affected. In fact it can be seen as a revolution.

Is physics the lifeblood of mathematics or is it that physics just happens to be written in the language of mathematics? Can one exist without the other?

Physics and mathematics have a long history in common, and have affected each other. It is not possible to do modern physics without the language and techniques of mathematics. The converse is not true.

Have you ever done any research in other sciences, like Biology or Engineering for example? If so, how did it affect your outlook on Mathematics?

I have a modest interest in neurophysiology (and have been part author of a paper in the field). I have also published papers on historical or philosophical themes.

Positions held

You had different positions in several universities around the world. How do you see the duality between a Professor's duties as a teacher and as a researcher? Do you think that teaching just "gets in the way" or do you find that there is something to be gained by teaching undergraduate students?

My undergraduate teaching career was limited to my early years, but I taught graduate courses for longer and I regard supervision of graduate students as teaching. Too much teaching can wear one down but some teaching is a good discipline and contact with students is essential.

You were involved in the creation of the Isaac Newton Institute for Mathematical Sciences in Cambrigde and you were its first director. Are you happy with its development?

I was involved in setting the Newton Institute on its path and in determining its general policy of taking a very wide view of mathematical science. I think it has been a success and has influenced similar institutes in other countries.

You had other administration positions, such as Master of Trinity College. Could you tell us about that experience?

My other main administrative positions have been as Master of Trinity College, President of the Royal Society (of London) and more recently President of the Royal Society of Edinburgh. At a suitable age I felt it was my duty to take on such positions, so as to contribute to society, in return for the privilege I have had of a life mainly devoted to research. They have all been interesting and have widened my experience.

You also contributed to the Foundation of the European Mathematical Society. How important was it for Europe to have a Mathematical Society and how do you think it helps promoting the exchange between mathematicians?

The European Mathematical Society had a slow start but eventually took off at an opportune time, when Europe was getting together. It is now finding its own feet and developing in new directions.

You were also Savilian Professor of Geometry at Oxford. How do you compare Oxford to Cambridge?

To an outsider Oxford and Cambridge seem identical. They have a similar history and are unique establishments in many ways. However they do differ significantly in detail. Cambridge has always been stronger in mathematics and science,with Oxford focusing more on the humanities. But they have converged and these differences are now less marked.

Visit to Portugal

It was very unfortunate that you couldn't deliver the Pedro Nunes lectures in April due to the incident with the volcanic ash cloud. Would this have been your first visit to Portugal? If not, what did you think of the country in your previous visits?

40 years ago I had a family holiday in Portugal, including Lisbon and the Azores, so I was disappointed that the volcano prevented my trip this spring.

When can we expect your visit to be rescheduled?

It will now be rescheduled for next spring.

The workshop on “Raising the Public Awareness of Mathematics” in Óbidos (Portugal)

This year Óbidos hosted again an international workshop: “Raising the Public Awareness of Mathematics” (organizers: E. Behrends, Berlin; N. Crato, Lisbon; J.F. Rodrigues, Lisbon; see http://c2.glocos.org/index.php/RPAM/rpam2010).

The opening was on September 26, 2010, it took place in connection with a “mathematical afternoon” organized by the Portuguese Mathematical Society (SPM) in cooperation with the town of Óbidos. At this event mathematical films and lectures for a general public were presented.

One of these lectures was given by G.-M. Greuel, the current president of ERCOM (the EMS committee of the European Research Centres on Mathematics), the other by H. Leitão about mathematics in the Age of Discoveries.

Later, one could participate in a reception for an itinerant mathematical exhibition (“Medir o Tempo, o Mundo, o Mar”) on the use of geometry to measure the universe and help astronomical navigation, jointly organised by the SPM and the Museum of Science of the University of Lisbon. The exhibition and a reception took place at a local art gallery.

Also, at the occasion of this public awareness event, the website www.mathematics-in-europe.eu of the EMS was “officially” launched. And the fact that many members of the EMS/RPA committee were present in Óbidos was used to discuss in a separate meeting the next steps in connection with the realization of this website.

“Raising the Public Awareness of Mathematics” was a joint initiative of CIM (Centro Internacional de Matemática, Portugal) and the rpa (“raising publing awareness”) committee of the EMS. About 40 participants from Europe and the USA attended this workshop. In more than 30 lectures information concerning various rpa activities was presented. Four aspects were of particular importance:

1. National experiences
2. Exhibitions / Mathematical Museums
3. Popularization activities
4. Popularization: why and how?

A number of talks was of a more “fundamental” character. It should be noted that the results of this workshop will be published as a book: “Raising Public Awareness of Mathematics” (Springer, 2011). Everyone who wants to realize rpa projects in the future is invited to profit from the experience of the experts who met in Óbidos.

LPDJLQH D VHFUHW

A FILM OF ART AND MATHEMATICS ON ELLIPTIC CURVES AND CRYPTOGRAPHY

by José Francisco Rodrigues

Pythagorean triples such as (3, 4, 5) or (4961, 6480, 8161) were well known by ancient Babylonians around 1600 B.C. They were also aware of their correspondence to right triangles with integer sides and to the problem of splitting a given square number into two squares. Although such triples have been studied in detail since the time of Euclid, around 300 B.C., it was only in the middle of the XVII century that Pierre de Fermat stated the famous observation: "No cube can be split into two cubes, nor any biquadrate into two biquadrates, nor generally any power beyond the second into two of the same kind".

This became the famous "Fermat's Last Theorem", stating that the equation $A^N + B^N = C^N$ has no nonzero integer solutions when N is greater than 2. It was completely proven in 1994, about three and a half centuries later, using the XX century theory of elliptic curves!

Elliptic curves have deep and beautiful properties. They are plane curves of the type

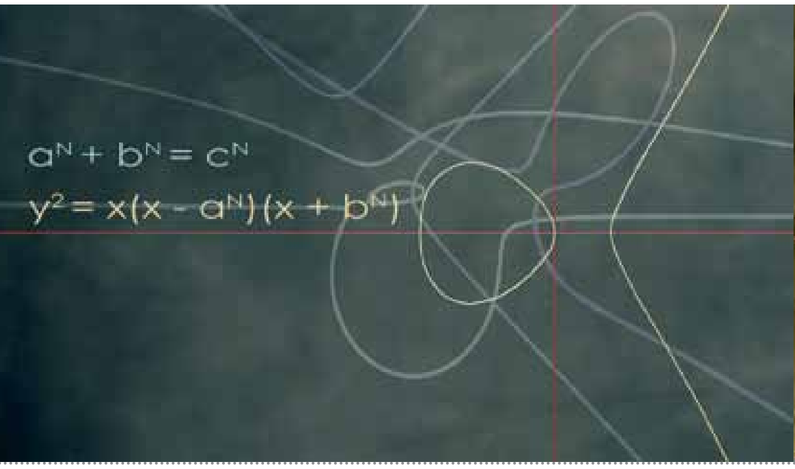
$$y^2 = x^3 + ax + b$$

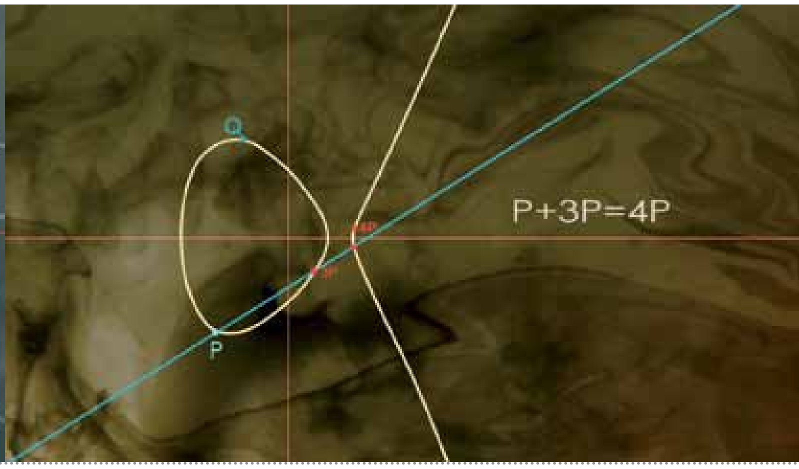
that have been studied since the XIX century. That equation in the affine plane corresponds to the homogeneous equation

$$y^2z = x^3 + axz^2 + bz^3,$$

which describes in space a family of algebraic surfaces with two parameters a and b. The computational variation of these equations generates beautiful animations that stimulate our imagination and evoke our mathematical creativity.

Cryptography refers to secure methods to transmit and safeguard secret and valuable information. Since 1977 the RSA public key system has been widely used. It is based on prime number theory and on the difficulty of factoring very large integers. With





The first public presentation of the movie LPDJLQH D VHFUHW was done the 26 September 2010 in Óbidos during the opening of the Workshop "Raising the Public Awareness in Mathematics". The movie exists in four languages, English, German, Portuguese and Spanish, and can be freely downloaded from http://www.cim.pt/LPD-UHW

the impact of the elliptic curve method for integer factorization, Elliptic Curve Cryptography (ECC) was invented by mathematicians in 1985, and since then the mathematical sophistication of cryptography has been raised to a whole new level.

The security of the ECC algorithms is based on the discrete logarithm problem of elliptic curves, which seems to be a much harder problem in finite field arithmetic. Recent mathematical advances imply that a certain desired security level can be attained with significantly smaller keys, for instance, a 160-bit ECC key provides the same level of security as a 1024-bit RSA key.

The theory of elliptic curves illustrates the beauty of the links between number theory, algebra and geometry and provides a powerful mathematical tool to strengthen security of e-commerce and secure communications. The old and unreliable method of the Caesar cipher of using only the simple arithmetic operation to encipher a message in the usual Latin alphabet by means of the formula $d = c - 3 \pmod{26}$ is outdated. But, it gives us the key to decipher the title of this film:

_ _ _ _ _ _ _   _   _ _ _ _ _ _

Credits

Movie: LPDJLQH D VHFUHW

Initiative: Centro Internacional de Matemática; Casa da Animação; Mathematisches Forschungsinstitut Oberwolfach

Original Idea: José Francisco Rodrigues

Conception: Victor Fernandes; Stephan Klaus; Armindo Moreira; José Francisco Rodrigues

Realization and Production: Victor Fernandes; Armindo Moreira

Surfer Movies: Andreas Matt; Bianca Violet

Original Music: Victor Fernandes; Armindo Moreira

Acknowledgements: CMAF/Universidade de Lisboa; Fundação Calouste Gulbenkian; IMAGINARY exhibition; Vila de Óbidos

Sponsor: Ciência Viva

Looking forward EuDML — The European Digital Mathematics Library

André de Oliveira Gomes
Master Student in Mathematics of the FC/Universidade de Lisboa
IST /Universidade Técnica de Lisboa fellow for the EuDML project

The present article intends to give some information about EuDML project, an European project that aims to "deliver a truly open, sustainable and innovative framework for access and exploitation of Europe's rich heritage of mathematics", and stress the importance of building an emerging PtDML (Portuguese Digital Mathematics Library) in this context of change [1]. EuDML is funded by the ICT Policy support Programme of the European Commission, coordinated by a research team of the Computer Science Department of IST (Instituto Superior Técnico) at the Technical University of Lisbon, led by José Luís Borbinha, the project scientific coordinator.

1. A Dream of a Unified Digital Mathematics Library

"Les mathématiciens se content de mettre leur production à la dispositon de tous comme sur des etagères où l´on peut venir se servir."

—Jean-Pierre Serre

The Digital Era, the emergence of the internet, the webmail, the new ways of communicate and the new channels of communication, changed the ways that researchers and scientists produce, publish and disseminate their scientific work. The birth of the electronic communication at the end of the 20th century has opened new opportunities for easier, faster dissemination, and more powerful discovery of scientific new results [2].

Mathematics is a basic science for a wide range of other branches of human knowledge, which needs quantitative though, with a huge variety of applications in all other sciences and technologies. So, building a central data service of mathematical knowledge has not only importance for the mathematical community, but also for all users of that knowledge. The main goal of a DML (Digital Mathematics Library) is to provide all the possible mathematical literature online and easily available through a central service or repository to anyone who has an internet connection in an electronic device, such as computer, cell phone of new generations or other type of machines [3].
The dream and vision were scripted long time ago [4]:

"In light of mathematicians' reliance on their discipline's rich published heritage and the key role of mathematics in enabling other scientific disciplines, the Digital Mathematics Library strives to make the entirety of past mathematics scholarship available online, at reasonable cost, in the form of an authoritative and enduring digital collection, developed and curated by a network of institutions."

—NSF DML project, Cornell 2002, CEIC 2004, IMU 2006

The benefits of building EuDML as part of a whole DML are unquestionable. EuDML is going to make the European heritage easily available from everywhere, since the laboratory of a mathematician is composed almost by its library and its richness is weighted by the availability of the mathematical knowledge required by its owner. This was previously emphasized by several workshops [5].

2. Previous work

A lot of work has already been done to transfer the past and current mathematical content into digital files (retro digitization process). Much of the current literature born in a digital way, i.e., created electronically and available online since its publication. Mathematics scope and the dimension of its scientific corpus are huge with all of its branches and interlinked areas in theoretical and applied subjects, so following all the mathematics is impossible for a single individual. Mathematicians find themselves navigating the literature, moving from one article or book to another, pursuing results and proofs and relying on references in order to link to the next item. The linking process has become more important with the literature growth and it is one of the reasons why electronic publication has great potential benefit for mathematical research. The stress nowadays should be on integrating this dispersed content into one distributed electronically virtual library of mathematics.

Before the beginning of the EuDML project, several individual and satellite projects built their "mini-DML's" and central repositories or central data services, such as (see [6])

- NUMDAM (NUMérisation des Documents Anciens Mathematiques) [7],[8];
- Project Euclid [9];
- EMIS (European Mathematical Information Service) [10];
- CZ-DML (Czech DML) [11];
- E-DML (Biblioteca Digital Española de Matématicas) [12];
- Ulf Rehmann´s Collection (Bielefeld University) [13];
- JSTOR [14];

But alone, these single projects don't have the global scope or the empowered efforts that EuDML can achieve.

3. Presenting EuDML [1]

> "The EuDML strives to make the significant corpus of mathematics scholarships published in Europe available online, in the form of an authorative and enduring digital collection, whether a researcher needs to follow a subtle pyramid of reasoning through a chain of related articles, an engineer needs to find results related to a particular concept, or a scholar project studies the history of a specific mathematical issue, there is a common need for an integrated interconnected gateway to the body of preserved mathematical literature."
>
> —EuDML Document of Work [15]

The project, partially funded with the EC funds, through its "Competitiveness and Innovation Programme", started on 1 February 2010, and will last for three years, until 31 January 2013 and intends to design and build a collaborative digital library service, through a network of academic libraries, aggregating a central metadata repository and endowing each item and a pointer to the associated full text and bibliographical databases [15]. EuDML will help the long term preservation of digital mathematical literature and will put the European Mathematical Community at the leading edge of the global drive towards a World Digital Mathematics Library, helping to maintain Europe's foremost position in mathematical research. Bringing together the dispersed heritage of digital mathematical literature, EuDML will provide explicit support, using the latest technologies for visually impaired users as well as automatic language translation support, in order to overcome language barriers, using the multilingual mathematical knowledge already developed by the institutional partners.

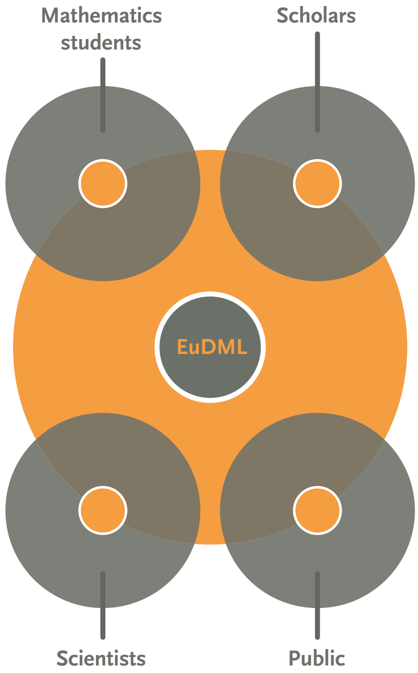


Figure 1

Targets

The foremost target (see Fig. 1) users are students and scholars, whose studies depend on the validity and the availability of previous and original mathematical literature. Working mathematicians have a prominent place among the target, since their production and the validation of their knowledge and of new results depends often on pre-existing literature. But not only mathematicians are part of the target users. Since mathematics has ramifications and interconnections with all other scientific disciplines, all the users of quantitative methods and logic-deductive reasoning are among this group of possible users.

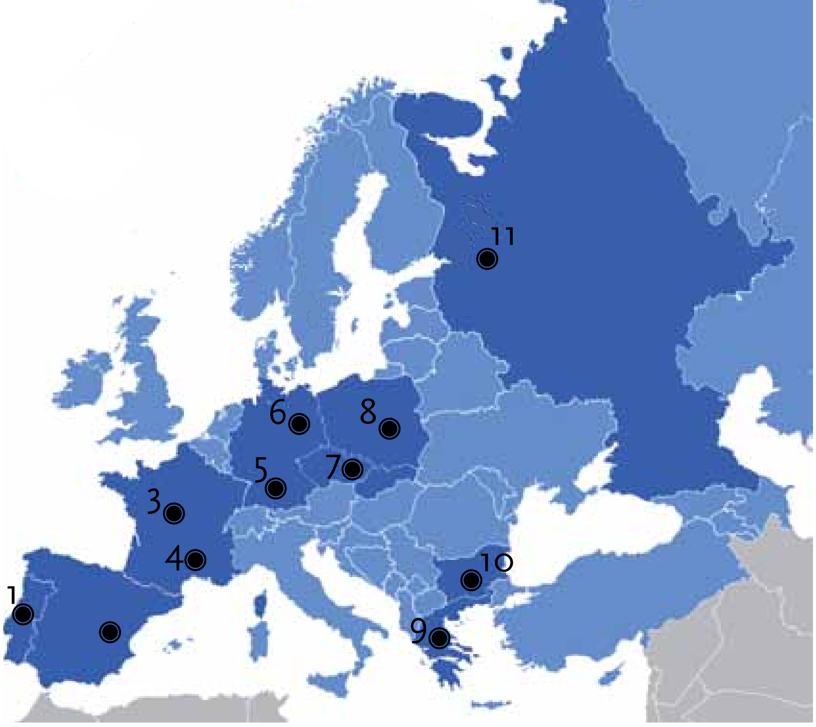


Figure 2

EuDML Body

Responsible for achieving sustainability and success in the end of the project, the EuDML consortium, is composed by 14 institutional partners, which include mathematicians, librarians, digital library experts, publishers, professional information service and document engineering specialists, and computer scientists, under the supervision of an external multidisciplinary scientific advisory board, formed under the auspices of the EMS (European Mathematical Society). The partners of the Consortium are:

- IST/UTL(Instituto Superior Técnico): Computer Science and Engineering Department (Portugal)
- UJF/CMD Université Joseph Fourier, Grenoble 1 (France)
- UB University of Birmingham (United Kingdom)
- FIZ Fachinformationszentrum (FIZ) Karlsruhe (Germany)
- MU Masarykova Univerzita (Czech Republic)
- ICM University of Warsaw, ICM (Poland)
- CSIC Consejo Superior de Investigaciones Cientificas ES (Spain)
- EDPS Edition Diffusion Presse Sciences (France)
- USC Universidade de Santiago de Compostela: Instituto de Matemáticas ES (Spain)
- IMI-BAS Institute of Mathematics and Informatics, BAS BG (Bulgaria)
- IMAS Matematický ústav AV ČR, v. v. i. (Czech Republic)
- IU Ionian University (Greece)
- MML Made Media (United Kingdom)
- CNRS/CMD Centre National de la Recherche Scientifique (France)

Content providers

EuDML is designed to eventually integrate all of the DML projects from every country in Europe, encouraging additional countries to embrace the project, creating a global DML in the future, and pursuing European publishers to cooperate with the library, with policies of cooperation designed properly (open-access, moving wall…). In the lifetime of the project there will be organized some workshops in order to introduce EuDML to the community of data providers that could be interested in the project.

Actually the group of the content providers is spatially configured in the European territory such as the picture shows (see Fig. 2).

Website [1]

At this moment there is not yet much information in the website, but at the time the project evolves there will be available more information, such as public deliverables and documents of the project and information concerning activities, such as workshops and public demo services (Fig. 3).

Workshop with content providers (Prague 15 October 2010)

EuDML is alive and is evolving and making itself known by the community. In the October 15, 2010, the partners UJF/CMD and IMAS organized a workshop with content providers, held at the Institute of Mathematics of the Academy of Sciences of the Czech Republic, in Prague. Besides project partners, it was attended by representatives of content providers, Ulf Rehman from IMU (International Mathematical Union),Ari Laptev, president of EMS (European Mathematical Society), publishers and further stakeholders, by librarians, such as Springer Services+Business Media representatives, mathematicians, the London Mathematical Society representatives, and public authorities.

The goals of the workshop (see [16]) were the presentation of the project and its purposes, a brief summary of the policies regarding content selection, archiving and access; plans for system architecture and releases schedule and the promotion of the dialogue between the participants, ending with an open panel discussion.

The program committee was formed by some partners of the project:

- Thierry Bouche (UJF/CMD chair)
- Jiri Räkosnik (IMAS/ local organizer)
- Enrique Macias Virgós (USC)
- Marie Louise Chaix (EDPS)

Representatives of the IMU (Ulf Rehman), the EMS (Ari Laptev), Universitäts Bibliothek Göttingen, and the Serbian Academy of Sciences presented their views and contributions to the global DML effort, and more specifically their suggestions to the EuDML project.

The final discussion was conducted by the questions of publishing and the power in the future of EuDML project in the definition of commercial policies of publishing of big companies, such as Springer or Elsevier, defining policies and agreements of open access, moving wall or some kind of individual users contributions, stressing not only the importance of contacting the important commercial publishers, but also how the EuDML objectives could help the strengthen of small and medium publishers, democratizing therefore the access to the mathematical publications.

Another EuDML workshop with data providers will be scheduled in next July, presenting EuDML to a broader audience and showing the goals achieved at the time.

4. It's time for the Portuguese DML (PtDML)

Meanwhile EuDML is growing and preparing its place as the main delivery service of a "truly open, sustainable and innovative framework for access and exploitation of Europe's rich heritage of mathematics" ([17], [18]) , it was formed a Portuguese Committee on Electronic Information and Communication (CEIC-Pt) in a previous meeting in the University of Minho (April 9, 2010), composed by Eugénio Rocha (Universidade de Aveiro), José Luís Borbinha (IST/UTLisboa), José Francisco Rodrigues (CMAF/ Universidade de Lisboa), Pedro J. Freitas (ULisboa and SPM-Portuguese Mathematical Society), Pedro Quaresma (Universidade de Coimbra) and Pedro Patricio (Universidade do Minho) with the mission to develop a portal that may congregate all the Portuguese mathematical heritage and all the publications of all the Portuguese mathematicians, linking it with EuDMl in the future. This is an important issue not only to study of the History of Portuguese Mathematics, but also to the current Portuguese mathematicians, which works will be accessible through a single and central place, in order to facilitate the search and the access.

There are several important publications and works of the past that are not yet collected digitally and it will be important to give access in a reference site such as PtDML may be in the future. For example, the Pedro Nunes works , the "História das Matemáticas em Portugal", of Francisco Gomes Teixeira (available in a personal website [19]), the mathematical works of José Anastácio da Cunha, Daniel da Silva, Gomes Teixeira, J. Vicente Gonçalves, Mira Fernandes, Ruy Luís Gomes and J. Sebastião e Silva, among others, which are dispersed by the several libraries, such as the National Library, the libraries of the University of Coimbra, Lisbon and Porto, the Library of the Academy of Sciences of Lisbon and several private collections. The electronic availability of these works is very important to the divulgation of the Portuguese mathematical heritage. One important source of interesting works of the mathematicians of the Portuguese past is the "Memoirs" ([20], [21]) of Academy of Sciences of Lisbon. The histor-

Figure 3

ical interest of the "Memoirs" is important since it contains a few important texts of Portuguese mathematicians.

But not only should the past be the focus of attention. It is important to have easy access to the currents publications of the Portuguese mathematical community. Articles, papers, books, chapters in proceedings are different types of written mathematical data and content, but should not be the only one to be considered. In the Digital Era, the multimedia content shouldn't be neglected; for instance it is relevant to make better known the Pedro Nunes Lectures held by the initiative of CIM (accessible through [22]). In what concerns journals and periodicals, the first journal digitized and currently available in the webpage of the SPM (Portuguese Mathematical Society) is "Portugaliae Mathematica" [23], the research journal of this society. But other periodicals should also be accessible in digital format, like the "Boletim" [24] and the "Gazeta da Matemática" [25], both of the SPM, or the "Boletim" [26] of the SPE (Statistical Portuguese Society).

At this time the initiative is still in a first stage, collecting data for cataloging and planning what and how to congregate the information, but it aims to construct a website that hopefully with collect and connect all the available content [27].

References

[1] EuDML website http://www.eudml.eu/

[2] J. Borbinha, Digital Libraries and the Rebirth of Printed Journals, in pp 97–110, of [28].

[3] John Ewing, Twenty Centuries of Mathematics: Digitizing and Disseminating the Past Mathematical Literature, Notices of the AMS, August 2000, volume 49, 771–779

[4] The Digital Mathematics Library, Notices of the AMS, volume 50, number 8, 918-923

[5] European Science Foundation Workshop on the European Virtual Library in Mathematics: http://www.usc.es/~esfmaths

[6] World Digital Mathematics Library: http://www.mathunion.org/WDML/

[7] The mini-DML: http://www.numdam.org/minidml/

[8] NUMDAM: http://www.numdam.org

[9] Project Euclid: http://projecteuclid.org

[10] The European Mathematical Information Service (EMIS): http://www.emis.de/

[11] Czech DML: http://dml.cz/

[12] "Biblioteca Digital Española de Matematicas": http://dmle.cindoc.csic.es/

[13] Ulf Rehmann's Collection: http://www.math.uni-bielefeld.de/~rehmann/DML/dml_links.html

[14] JSTOR: http://www.jstor.org/

[15] EuDML technical documents and Document of work

[16] Deliverable 3.3. EuDML , Thierry Bouche; Jiri Rákosník (soon available in EuDML website)

[17] E.M. Rocha, J.F. Rodrigues, Disseminating and Preserving Mathematical Knowledge, Communicating Mathematics in Digital Era, in 3–23, [28] and also in Bol. Soc. Port. Mat. 53 (2005), 1–21.

[18] T. Bouche, Towards a Digital Mathematical Library, Communicating Mathematics in Digital Era, in 47–74, [28]

[19] http://www.mac.uc.pt/~jaimecs/livrogt/livrogt.html

[20] F. Gomes Teixeira, História das Matemáticas em Portugal, Academia das Ciências, Lisboa, 1934.

[21] J.F.Rodrigues, Portuguese Mathematical Journals. Some aspects of (almost) periodical research publications, in pp. 601–627, The Practice of Mathematics in Portugal. Acta Universitatis Conimbrigensis Coimbra,(2004); (Eds. Luís Saraiva, Henrique Leitão) and also in Bol. Soc. Port. Mat. 50 (2004).

[22] CIM website: http://www.cim.pt

[23] Portugaliae Mathematica at the Portuguese National Library http://purl.pt/index/pmath/vol/PT/index.html

[24] Bulletim of the Portuguese Mathematical Society: http://nautilus.fis.uc.pt/bspm/

[25] Gazeta of the Portuguese Mathematical Society: http://www.mat.uc.pt/~gazeta/GazetaOnline/online.php

[26] SPE webpage: http://www.spestastistica.pt

[27] Assis Azevedo, J.L. Borbinha, Pedro J. Freitas, Eugénio Rocha, The Challenges of the Pt-DML, pp. 2–5, CIM Bulletin #27, January 2010

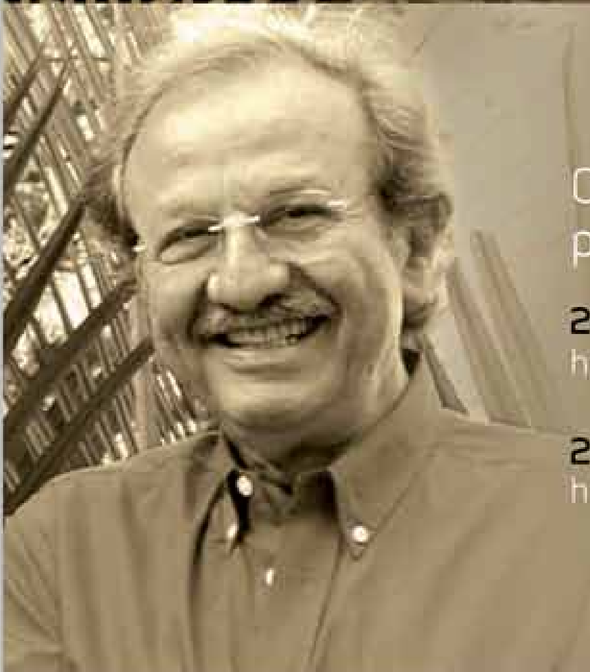
[28] J. Borwein, E.M. Rocha, J.F. Rodrigues (Eds.) Communicating Mathematics in Digital Era, A K Peters,Wellesley MA, 2008.

PEDRO NUNES LECTURES

Centro Internacional de Matemática

SPM SOCIEDADE PORTUGUESA DE MATEMÁTICA

Jacob Palis



Open questions leading to a global perspective in dynamics

24th February/ 2011/ 15.00 h/ Porto
http://www.fc.up.pt/cmup

2nd March/ 2011/ 16.30 h/ Lisboa
http://cmaf.ptmat.fc.ul.pt

ABOUT PEDRO NUNES LECTURES

Pedro Nunes Lectures *is an initiative of* ***Centro Internacional de Matemática (CIM)*** *in cooperation with* ***Sociedade Portuguesa de Matemática (SPM)****, with the support of the* ***Fundação Calouste Gulbenkian****, to promote visits of notable mathematicians to Portugal. Each visitor is invited to give two or three lectures at Portuguese Universities on the recent developments in mathematics, their applications and cultural impact.* ***Pedro Nunes Lectures*** *are aimed to a vast audience, with wide mathematical interests, especially PhD students and youth researchers.*

Webstreaming available at:
www.cim.pt/?q=glocos-pedronunes

For further information:
CENTRO INTERNACIONAL DE MATEMÁTICA
www.cim.pt

SOCIEDADE PORTUGUESA DE MATEMÁTICA
www.spm.pt

Jacob Palis is a Brazilian mathematician and professor. He obtained a degree in Engineering at the Federal University of Rio de Janeiro and a Ph.D in Mathematics at the University of California, Berkeley. Since 1973 he has held a permanent position as professor at the Instituto Nacional de Matemática Pura e Aplicada (IMPA) in Rio de Janeiro, from which he was director between 1993 and 2003.
He is the President of the Academy of Sciences for the Developing World since 2007. He is also a foreign member of several academies of sciences, including the American and French academies and he is currently the President of the Academia Brasileira de Ciências. He was also president of the International Mathematical Union from 1999 to 2002. Palis has received numerous medals and prizes. He has 42 Ph.D students, including several current professors of the Universities of Lisbon and Porto, and he was recently elected foreign member of the Academia de Ciências de Lisboa.
In 2010 he was awarded the Balzan Prize for his fundamental contributions in the mathematical theory of dynamical systems that has been the basis for many applications in various scientific disciplines (such as in the study of oscillations). His research interests are mainly dynamical systems and differential equations. Some themes are: global stability and hyperbolicity, bifurcations, attractors and chaotic systems.

SPONSORS:

FUNDAÇÃO CALOUSTE GULBENKIAN · CMAF UL · Grupo de Física Matemática da Universidade de Lisboa · CMUP Centro de Matemática · CAMGSD

DYNAMICAL MODELS IN LIFE SCIENCES

EMS-ESMTB-CIM Summer School

University of Évora . Portugal . July 24-30 . 2011
Colégio Luís António Verney

Mini-courses

Dynamical models of Cancer
David Dingli (Mayo Clinic College of Medicine, Rochester, USA)

Adaptive dynamics and the evolution of pathogens
Eva Kisdi (University of Helsinki, Finland)

Modelling spatio-temporal phenomena in Biology
Philip Maini (University of Oxford, UK)

Modelling Meso-evolution: adaptive dynamics and beyond
Hans Metz (University of Leiden, The Netherlands)

Stochastic and Deterministic Processes in Spatial Population Dynamics
Sergei Petrovskii (University of Leicester, UK)

Mathematical Models in Hemodynamics
Adélia Sequeira (Technical University of Lisbon, Portugal)

Ecology and Eco-epidemiology
Ezio Venturino (University of Torino, Italy)

Scientific Organizing Commitee

Fernando Carapau (Universidade de Évora)
Fabio Chalub (Universidade Nova de Lisboa)
Francisco Santos (Universidade Nova de Lisboa)
Nico Stollenwerk (Universidade de Lisboa)

www.cim.pt/ssmtb2011

Sponsors

CMAF · FCT · FUNDAÇÃO CALOUSTE GULBENKIAN