International Center for Mathematics

#30

Bulletin

July 2011

Editorial

Statement of Candidacy
for CIM Board of Directors

We will support the mission of CIM to develop and promote mathematics in Portugal and to strengthen international outreach. We will organize and promote presentations and mini courses given by accomplished mathematicians. We will organize and promote seminars and international conferences in mathematics and emphasize the importance of the interdisciplinary connections with other fields of science, engineering and industry. These events will be targeted mainly to scholars, researchers and Ph.D. students, however, M.Sc.and undergraduate students will be welcome.

We will organize and promote programs we consider important for the future success of Portuguese mathematics. We will create collaborative programs to bring together academic institutes and industry partners to research and develop applications of mathematics for use in industrial and service-oriented sectors. We will promote joint workshops between industry partners and CIM associates to stimulate future collaboration. We will promote mathematics awareness by conducting events directed to the general public, as well as events targeted to more specific groups.

We will take an active role in the European and International organizations where CIM is a member and we will collaborate with other national and international commissions, councils, organizations and associations to develop the broad field of mathematics. We will work to encourage and assist Portuguese mathematicians to become actively engaged in European and International projects.

We are confident we can count on the support and commitment of the CIM membership to achieve our goals. We will invite other recognized organizations to be members and sponsors of CIM.

Submitted by Alberto Adrego Pinto on behalf of the candidates for the Board of Directors of CIM.

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INTERNATIONAL CENTER FOR MATHEMATICS

Coming Events

Elementary Geometry from an Advanced Point of View

Universidade de Aveiro

01–02 September 2011

The aim of this conference is to present several contemporary perspectives on Geometry including, among others, talks on visualization, applications and surveys, both at elementary and more advanced levels. The goal of this meeting, promoted by CIM in collaboration with CIDMA/Univ. Aveiro, CMAT/Univ. Minho and CMAF/Univ. Lisboa, is to contribute to the current international reflection on the ICMI/IMU Klein Project concerning central topics on Geometry, its contents, interdisciplinary connections and approaches for the teaching of this mathematics discipline at senior secondary school and first years at University level.

Organisors: Ana Breda (U. Aveiro), Chair, Ana Pereira do Vale (U. Minho), Tomas Recio (U. Cantabria), Eugénio Rocha (U. Aveiro), José Francisco Rodrigues (U. Lisboa).

NOMA'11: International Workshop on Nonlinear Maps and their Applications

Évora

15–16 September 2011

In the field of Dynamical Systems, nonlinear iterative processes play an important role. Nonlinear mappings can be found as immediate models for many systems from different scientific areas, as Engineering, Economics, Biology, or can be obtained via numerical methods permitting to solve non-linear differential equations. In both cases, the understanding of specific dynamical behaviours and phenomena is of the greatest interest for scientists. This workshop is opened for theoretical studies as well as for applications. We hope that the interaction and the knowledge exchange between Mathematicians, Physicists, Engineers, and other specialists from Nonlinear Sciences will be very fruitful and will give rise to new developments in this area. We invite to present contributions in the fields of Biology, Economics, Electronics, Engineering, Telecommunications, besides more fundamental lectures.

Intelligent Data Analysis Analyzing and Understanding Complex Systems

Porto

21–31 October 2011

When the biennial IDA Symposium series started in 1995, it focused on the problem of end-to-end intelligent support for data analysis. IDA 2011 will refocus on an important and still emerging class of problems: modeling and analyzing complex, dynamical systems such as economic systems, gene regulatory networks, social networks, systems of natural resources, and cognitive systems. The Symposium seeks "first look" papers that might elsewhere be considered preliminary but contain potentially high impact research. The IDA Symposium is open to all kinds of modeling and analysis methods, irrespective of discipline. It is expected to be an interdisciplinary meeting that seeks abstractions that cut across domains. IDA2011 welcomes papers that focus on dynamic and evolving data, models, and structures.

An Interview

with Jacob Palis

by **José Ferreira Alves** [Universidade do Porto]

On the 24th February 2011, after delivering a Pedro Nunes Lecture at the University of Porto.

You have started your undergraduate studies in Electrical Engineering. What made you choose your way in Mathematics?

Well, perhaps we should not say Electrical Engineering, because I didn't finish it. I shifted to Economical Engineering in the meantime. Anyhow, since I came to Engineering School, I had the idea that that was a place for Mathematics — to some extent for Physics too, but specially for Mathematics. This idea was offered to me by my second oldest brother, who was an

Engineer. Actually, very successful that time, later on he became a politician. For a while he was quite a brilliant Engineer and I couldn´t see otherwise. I came from the interior of Brazil, so I went to Engineering because that was ready.

But did you have Mathematics in mind from the very beginning?

Oh, I loved Mathematics since I was a kid! Because of that, my brother convinced me that I should go to Engineering. It was as simple as that. Of course talking about Brazil of the early fifties, that is quite a long time ago. To make it brief, you know, at that point families had the idea that if you like some kind of Biology you should go into Medicine and if you like Mathematics or Physics you should go into Engineering; otherwise, you would go into Law. As simple as that. Me and my four brothers, we went exactly that way. We went to Engineering, to Medicine and one to Law, and that was the picture. In Engineering School I used to ask questions, being sometimes audacious but never being impolite or improper, in the sense of not appreciating the teacher. But I did ask a number of times embarrassing questions, apparently. And so by the end of the Engineering course, although enjoying the course, and enjoying specially the last part on economical engineering, I developed the taste for Mathematics. I went to participate in seminars, at IMPA already and at some centers for physics also. I formed the idea that I should study Mathematics and Physics and then go back to Engineering to see it better and to know how to answer some of my questions. I was not able to get good answers from the professors at the time. Anyhow, my family was very surprised with the idea, because at that point I already had several offers to work in Engineering, but I convinced them that it was reasonable to get a fellowship and go to the United States to study Mathematics. So, that is how the story started.

How did you get in touch with Steve Smale?

Since I have decided to go to do Mathematics in the United States, I inquired who was the best mathematician that had visited Brazil in recent times. Then I was told it was Steve Smale, and I wrote him asking if he would be my advisor in the University of Columbia. Looking back it is amazing how I dared to do that myself. Anyhow, the answer was yes. He certainly consulted people he knew in Brazil, but I don´t know about that. Then at the last moment he moved to Berkeley. That again was interesting, because there was no more time to apply for Berkeley and he had to negotiate the acceptance from Columbia University to Berkeley. He succeeded and I went there. Well, I was not a mathematician, my education was not complete, there were some fronts where I felt extremely well and others where I felt I had to start from scratch. Somehow I survived in the three years I completed my PhD, together with the master degree, which at that point was not much.

Did you make a personal choice on Steve Smale, not conditioned by the area he was working in?

Your question is very appropriate. Of course, among the courses I had at IMPA, one by Peixoto was on Dynamical Systems and I liked that. Peixoto at one point said: "This topic will not be covered because is too hard". So, I went home and I did it. It was the Unstable Manifold Theorem. Later on I learned it in more sophisticated ways, but I did it with the instruments I had at the time. I liked that. Of course, I had some good courses in Algebra, good courses in Topology and Differential Topology, but somehow I liked the fact that he had visited IMPA in Brazil.

Steve Smale was already working in Dynamical Systems?

He was definitely working in Dynamical Systems. When he visited Brazil he was in a transition from Differential Topology to Dynamical Systems. Anyhow, the reasons sound like not very deep ones, but that was the way I would go ahead. I went to the United States and I did it quite well. My thesis was well accepted and immediately it was generalized by myself and my advisor and we formulated the conjecture. That was interesting. This shows how I was, I proposed it to Smale and he said: "Well, let´s do it". What was called the Stability Conjecture became really one of the main sources of research for the next 20 years, and it was finally well solved in some interesting way — not completely, but... — by a student of mine: Ricardo Mañe. So that was the first big question I was involved in asking.

Did you think about staying in the United States?

Then I got some good news from Brazil in terms of more support for science. The National Bank for Development initiated a new program in Economical Engineering that was precisely this way. Some of the economists convinced the bank to put a percentage of its budget — this was quite huge — into basic science and basic engineering. This was good news and then there was the organization of a graduate program — sometimes we call postgraduate program —, another interesting fact. I did get some offers to stay in the United States but I was not really available and with some good news from Brazil, together with the bad one that we had a dictatorship in 1964. But, on the other hand, the organization of the masters and PhD programs in Brazil was done in a superb way, because they did focus in the best groups not on institutions. Usually institutions are very heavy. I was surprised because I thought the University of São Paulo and all its doctorate courses would be immediately approved, but it was not that way. The choice was made on the existence of good groups of researchers.

That made you feel even more confident about returning to Brazil.

Sure. At that point, another good memory I have is that going to bookshops in Berkeley I found this little book called *The Double Helix*, by James Watson, which was the story of how DNA structure was found. A little book I was reading, it was not long and it let me curious for a few days. There were some gossips inside and whatsoever, but that was not the point that took me. It was the fact that he described quite well — very well in my opinion — the atmosphere in Cavendish Laboratory. This was really what I always talked: one should have some kind of magic in the ambience for the young talents to pursue science. Perhaps we could do so in Mathematics and other areas in Brazil. So I decided to go back and try to contribute with some creation of a magical place for Mathematics in Brazil.

I must say that this fits perfectly well the opinion I have about you. An informed opinion, since I got my PhD at IMPA.

Thank you. I didn´t even listened to the offers that were made to me. I was not available. But I stayed ten more months in the United States, in the East coast. To make brief a long story, I went to visit Brown and MIT, specially these two places (I also went to Harvard...), mostly in Brown to some extent. In February I returned to Berkeley, they had offered me an Assistant professorship which I took. But I kept saying that I was going to quit in August and return to Brazil.

There was always in your mind the idea of returning to Brazil…

Basically yes, but as I told you, around 1967, when I was about to finish — I finished in September — I heard some good news from Brazil and as soon as I finished I also got an offer to go back to IMPA and another offer to go to the Federal University of Rio de Janeiro. The final decision was really at that point. I had the idea to go back, but I didn´t play any game in either way if I would stay or not. But when I got this, basic science was starting to get better funding for the construction of graduate programs in Brazil. Then this little book... I thought I could contribute to create such a magical place in Brazil. I decided to go. Probably I had more chances in Brazil than in the United States, where to some extent this idea was already there in more developed places. Brazil was a bigger challenge and my country. I did return in August 1968 to IMPA and to the University, but quickly I saw that my dream could perhaps be better achieved in IMPA, not so easy to do so at the University, too big. So I quit the University, but people were not happy.

IMPA was not as it is now…

No, but on the other hand it is important to say that IMPA was founded in the right way by three people in 1952. Three good people: Leopoldo Nachbin, Maurício Peixoto and another one more senior, Lélio Gama, a good guy in Astronomy and Mathematics. So it started very well. The point is that IMPA was very good from the beginning, but both Peixoto and Nachbin would travel a lot. They had positions abroad later on. It was indeed a more stable situation in institutions where they could do very good research as a routine. So, that was very important that it started very well with this people playing a very important role. However, it was clear to me when I decided to go back — I started convincing Manfredo do Carmo who came for a postdoc — we had to do very good research on the day by day basis, do it continuously.

You kept a strong collaboration with some American universities with constant visits in both directions.

That is right. It is not a criticism, just a fact: Peixoto was connected to Brown University and Nachbin to New York University — not the State University. My idea was that we should have a Program working regularly. That was the key word: regularity. I should say quickly that we set up a new PhD Program at IMPA in the seventies. Not alone, of course, with do Carmo, Lima and Peixoto that finally came back from the United States. To my surprise, I had immediately wonderful students. In fact, in two and a half years three of them had concluded the thesis. Among them was Welington de Melo, an excellent mathematician, and Ricardo Mañe. That was very fulfilling to have such bright people concluding in record time. There was also Pedro Mendes, very good too. The first one was Welington, the second was Mañé and this was incredibly fulfilling to me, corresponding to this idea of having a regular program, and that went on without stopping.

Let us talk about Mañé. Is it true that he wrote you a letter saying he solved some problems?

Yes, absolutely! Unfortunately I am not good at archives. This letter was lost when we moved from one building to the other. We were downtown and we moved to the new building in the Botanical Garden neighborhood. The most precious part of my correspondence was in a unique box — which was stupid — and this box was lost. I searched it back and forth and, my God, I was desperate! Inside there was this beautiful letter by Mañé. He didn´t even have the master degree, he wrote me a letter saying that he had solved five questions in Dynamics. The first one maybe he had it, I don´t know, and that was certainly correct because I had done it before. It was a good question, not a great question. But the other four, each of them would grant him a permanent position essentially in any place. A number of them are open until now, including the Stability Conjecture. These questions are still open in a more general way, but he solved them in some particular cases.

How old was him?

It was the year of 1970, so he was 22. This letter came in an interesting moment. Well, my life is full of special moments. To initiate this new Program I had talked to Peixoto about it. And Elon Lages Lima — a very good mathematician in Topology, always very helpful — was there all the time. Anyhow, we decided to organize an international meeting in Dynamics, in 1971, in the middle of the year. In a certain way, to stimulate students. It was thought to be that way. We would work hard since late 1969 on until 1971. It would be a good time to have very good people to come to Brazil in Dynamics, and more broadly in Geometry also. So, we start preparing these students, among them Welington de Melo, also very audacious. He came from Minas Gerais to get Master. He came to my seminar and I said "my God, like I did in Berkeley". I came directly to the seminar of Smale, which was a seminar really about recent research. I had big gaps like Probability Theory and Welington did the same in Rio. He insisted he could do it, and I agreed. "If you can stand it, amazing". I´ll never forget this fact. I told him: "I did survive, if you can survive…"

The case of Mañé is similar.

Mañé in some sense yes, also. Anyhow, in that letter he showed such a maturity! One of the questions I remember is open until now. Not solved, except in very few cases: the Stability Conjecture. It is much settled by him, in the C^1 topology is complete. Another one was about Anosov systems: if the periodic points would be dense or not. It is open until now. It is amazing, but he stated the questions extremely well. He claimed that he had good ideas to solve them. I remember specially this two, but there were four or five. I was very impressed. That also showed how I would react toward this situation, certainly not conservative about these things. I took that letter — I was enthusiastic about it — and I convinced Peixoto and Lima that we should invite him for the meeting in the following year.

The famous meeting in Bahia!

Yes, in 1971, the first time I met Mañé. And he was invited without even completing the undergraduate studies. He was about to complete them. People reacted in different ways. Jorge Lewowicz didn´t like it because he was in a sense the supervisor of Mañé for what they call *tesina* — it´s common in Spanish speaking countries. He didn´t like it at all, but I loved it! I convinced people they should know this guy. I was taken by the maturity of the statements, but one has to have good will. Peixoto and Lima told me: "OK, we agree". Then he came and we discussed one of the topics that I have done but he improved. He asked me what I thought about him going to NYU to work with Moser, and I said: "Well, wonderful!" But instead he wrote me in September of 1971 saying that he wanted to come to IMPA and asking me if I could be his advisor. Similar to what I did. I said: "Sure! I bet on you." And he came.

Presently, do you still see young students using that method?

No. It is not so common. I think it became more standard. Anyhow, I know these two cases. Maybe it happens still… In a certain sense it happens, but not exactly in the same way. You see these young fellows, Artur Avila and Gustavo Moreira, it is a different story because they won the Olympiads and they came to IMPA very early. They were audacious too, both Avila and Moreira, but they came through courses. Also Carlos Matheus. This two guys [de Melo and Mañé] were more at the level of research, but I agree it still happens.

Just to finish about Mañé: looking back and knowing about his fantastic work from the very beginning, it's strange that he didn´t win the Fields Medal. Do you think nowadays it would be different?

I do hope so.

Latin American has no Fields Medal.

No, but I think Artur already deserved it.

This last time?

Sure, a wonderful candidate. Maybe he didn´t get it because he was too young.

Still very young…

He is 33 now. Well, the only chance of Mañé, because of age, was 1986. I did talk to Moser about him, and Moser was the chair of International Mathematical Union at that time. You know, not being at a main center is not easy. He was certainly considered. But you have different explanations. That year there were three people, I think, all with outstanding work, so there was a place for a fourth person. Anyhow, one had to do with the solution of Fermat's Problem, not the complete solution, but quite spectacular. Another one was the Poincaré Conjecture in dimension four.

Always Poincaré Conjecture…

Smale had done it for dimension five or more. That was also wonderful without no questions about it. But, you know, it depends a lot on the committee and it depends also on visibility. In terms of visibility, of course the main centers in America win. The Russians took quite a while to get Fields Medal. After Sergei Novikov it became more common. But Sergei, I think, he was the first one, which is strange in some sense because they had very good schools before. Sergei was in 1970, I believe, there were no Russians before. It is a question of visibility, of people knowing also.

There was a wall!

Being outside the main centers is not such an easy task. Now Avila, in some sense we are very happy to have Avila half of the time. Then the idea of being also in Paris I think is very good for visibility too. Not only that, the French school is wonderful. So it is nice. In the case of Mañé, either he would win in 86 or not. The result was spectacular, but also he obtained this result perhaps too close to date of decision. That was only a question of time, it was very short.

Bad luck…

Certainly, I´m sure he was considered. Then there was the case of Marcelo Viana in the year of 2002, very disappointing.

Marcelo has also been considered?

Certainly very much considered. I would say he was on a short list. Then it was given to two algebrists, it's too much. As a secretary to IMU, eight years before we approved certain obvious principles on how to get the Fields Medal. One was diversity. It's bad for science to repeat the same field and there was no justification for that. I was the president, so I was very disappointed because the chair was Sinai and they did not follow the principle. Now I hope this will disappear with Avila. When they are forming committees, the most natural tendency is to have people from the main centers. Anyhow, to conclude that, certainly Mañé was in the level of Fields Medal, and I insist on saying the same about Marcelo. Now there is Artur and I have no questions about that.

Let us change topic. 1982 was the year of your first scientific visit to Portugal. Tell me about that experience.

Well, I got this invitation by a group, a kind of international institution, a network called Mathematicians of Latin Languages. I didn´t quite understand what was that. I knew Lisbon, I got to Lisbon a couple of times but just for vacation, returning from some other places in Europe, but I didn´t get in touch with the mathematical community. So I got this invitation for a meeting to be held in Coimbra. I got several invitations before. Some colleagues in Portugal wanted me to visit their own institutions, but I never set any date. Finally in 1982 I said: "Ok, I say yes to his invitation". It's funny, because latter I was told they were not expecting me to accept this time. Coimbra was very attractive, a visible place, the history... Then to my surprise when I got there in Pousada de São Marcos — to me it was a reproduction of a castle, a palace very austere but at the same time with good taste — and there was a very nice cocktail to welcome participants. There was a number of participants to give main talks of the meeting and I was one of them. Then, toward the end of the cocktail one of the main organizers told me: "You know, Leopoldo Nachbin criticized us for inviting you" [laughs]. Because Nachbin was a member of this network and he said someone had a better name than mine. I said: "Well, I am already here!" And he said: "No, no, it´s OK. I just mention this for you to know". I said: "I don't care, I'm here, I´m happy". It was like that. In the day after there was the idea of the special group giving the main talks dining there at the same place — the table for dinner reminded me the Tavola of King Arthur! Then we had some activities and I met the young people. I decided to stay with them, in particular with Marcelo and Jorge Rocha. Marcelo was giving a talk, not one of the main talks. He was very young but had very nice results. At the end of the lecture I told him it was very good and so on.

He was already a good speaker, I guess.

Good speaker. Good results for a kid. It was quite impressive. I thought "my god, I go to get dinner with senior people and here we have bright young people". So I decided to stay with them and I said: "Look, it is a pleasure for me to stay with you, having dinner together, but you have to take me back to this place, otherwise I have to go to a hotel and I offend the organizers". But when we came back I was sure the way it looked the place was closed. So I suggested throwing little stones at the windows in the back. Luckily, there was someone in the kitchen that opened the door. Absolutely true story! This was very good, because I think we started a very strong connection with Portuguese young people. Maybe a year or two later — I don't remember exactly — three of them went to IMPA.

Maria Carvalho went to work with Mañé, Marcelo and Jorge to work with you.

That is right. It was a nice story. Again it shows that uncertainty is a very precious thing in life. It's hard for people; it was hard for me to accept the idea of uncertainty as part of everyday life. All these things I am telling you fit perfectly well this idea that uncertainty is part of live. I think that if you accept this idea, in many cases you can turn it into very good things. Certainly, one of the best things in my life was to accept that invitation.

That´s fantastic! It was almost 30 years ago. How do you see the development of the Portuguese Mathematics in the last 30 years.

Immense! It is absolutely another world. Potentially, of course, people were here, but the number of good researchers in Mathematics now is very impressive. It was a very successful development. It is not a question of criticizing the past; these things were like that in Brazil. But now here is much better. I come and feel at home. I saw people in the lecture today and how people reacted. I think you went a long way. I am not saying that nothing was there in the beginning, it is not true, good people, but much more dense now, much more visible. And you have young people. It is a beautiful thing.

A good way of measuring that is perhaps looking at the quantity of young people involved.

Like the one you introduced me today. He is doing very good things.

Jorge Freitas…

We were talking nice Mathematics just over the coffee, which certainly was not here before. This is almost the proof of big changes. It is much more active the present caring for young people.

I'm glad to hear that. Let us talk about Mathematics in general. Nowadays we feel a big pressure in Mathematics, as if one should have applications almost immediately. We hear very often: "what is the use of this?" What is your opinion about that?

That is terrible. We should not pursue this kind of topic. I think there is a certain confusion about more basic science — sometimes more pure — or applied sciences. They should live together. I think it is very important to be creative in basic sciences as well as in the Industry or in applications. It´s part of a complex, you have to do both. I think there is a wrong vision often about pressure to have people doing basic science to move to applications. That is completely nonsense. It is obvious that basic science turns into good applications, not necessary by the same people that have created basic science. Then you have to have those things together. No pressure. The pressure does not solve anything. It depends on the talent. For instance, it is very important to have research and development in the Industry. Otherwise, the Industry will be offering the same products over and over again. We know that you need creativity, things move on. Human beings like novelties, new tv sets, very thin, now we have iPods, taking over computers, smartphones and so on. However, you have to have an ambience of freedom, stimulation, and magic as I said, in both sectors, both are very precious and important in a community. They both should be supported and stimulated. You cannot do this part and forget the other one. I cannot understand people being nervous about economies and then say: "We are going to support only applications and patents". This is nonsense. On the other hand, we do need good people in that part too. Industry should respond to that, should stimulate researchers to come and to be creative. Nowadays the magic word is innovation.

Another place where we see pressure and numbers measuring everything is in the scientific production, as the impact factor and so on. How do you see that?

Very interesting question. Not so easy, because there is a lot of that now. Again, my view is that if you apply, first of all, indicators across different areas you have big distortion, because different areas have different cultures. In Mathematics we tend to produce fewer papers, but they are more complete. It is not better or worse than in other areas, but different culture. In Biology the tendency is to produce shorter results and publish more. Shorter results are not necessarily

José Ferreira Alves and Jacob Palis

less important. So if you go across different areas you commit a big mistake. And that is what is common nowadays; we have the H factor or the number of citations... On the other hand, if you look more globally and don´t mix areas it makes more sense. My view is not for individuals, if you apply this for individuals you are again about to make a mistake, a serious mistake. I like the idea — I think it is a reasonable idea, not wonderful — to have indicators like number of citations in certain areas, for instance, Mathematics, the average in certain countries with respect to the world average. This makes some sense to me. There is some logic in that. Again, indicators always have high degree of uncertainty. Anyhow, this seems to be reasonable to say that more citations mean better journals because better journals are more visible and so it is a tendency to correlate that.

That can also be increased artificially.

Not if you are talking about countries. If you talk about individuals, I don´t think any of this things make any sense. But if you talk about a global community, the tendency is to be more reasonable the indicator comparing with the world average in the same area. That is more concentrated in more advanced countries. Everything I am saying makes sense to me, reasonably.

Ok, Jacob, thanks a lot for this wonderful conversation.

It was my pleasure!

Wavelets and Economics

by **Maria Joana Soares*** and **Luís Aguiar-Conraria****

The use of wavelet analysis is very common in a large variety of disciplines, such as signal and image processing, quantum mechanics, geophysics, medicine, biology, etc. In economics, however, wavelets are still a mysterious, but colorful, tool for time-series analysis. The pioneering work of Ramsey and Lampart [26] is unknown to the majority of economists. Among the exceptions to this rule, one can point to [4], [14], and [12]. See [6], for a recent survey of wavelet applications to economic data. Probably, wavelets are not more popular among economists, because wavelet multivariate analysis is still incipient. Recently, however, Gallegati [11] — using the maximum overlap discrete wavelet transform — and Crowley and Mayes [5] and Aguiar-Conraria and Soares [1] — using the continuous wavelet transform — showed how the cross-wavelet analysis could be fruitfully used to uncover time-frequency interactions between two economic time-series. Still, most surely, wavelets will not become very fashionable in economics until a concept analogous to the spectral partial-coherence is developed. On this regard, the proficient reader may be interested in our most recent working-paper [2].

We present a brief and self-contained introduction to the wavelet tools used, namely the continuous wavelet transform, the wavelet coherency and the wavelet phase-difference. Then we apply these tools to a real world economic problem — the study of the synchronization of the Portuguese and Spanish economic cycles, in the last 5 decades. Decades that include the democratic transition in both countries (mid-1970s), the European Union membership of both countries (1986), and the adoption of a single currency, the Euro (1999).

Time-frequency localization

In what follows, $L^2(\mathbb{R})$ denotes the set of square integrable functions, i.e. the set of functions defined on the real line and satisfying $\int_{-\infty}^{\infty} |x(t)|^2\, dt < \infty$, with the usual inner product

$$\langle x, y\rangle = \int_{-\infty}^{+\infty} x(t)\overline{y(t)}dt$$

and associated norm $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$. By influence of the signal process literature, this space is usually referred to as the space of finite energy signals, the energy of a signal x being simply its squared norm.

Given a function $x \in L^2(\mathbb{R})$, $\hat{x}$ will denote its Fourier transform, here defined as:

$$\hat{x}(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-i\omega t}dt. \text{ [1]}$$

Note 1.—With the above convention of the Fourier transform, ω is an *angular* (or radian) frequency. The relation to the usual Fourier frequency f is given by $f = \omega/2\pi$.

Note 2.—We use the symbol x to denote a general function, since this is a more common notation for time-se-

[1] As it is well known, for a function in $L^2(\mathbb{R})$, the above formula must be understood as the result of a limiting process, e.g. $\hat{x}(\omega) = \text{l.i.m.} \int_{-n}^{n} x(t)e^{-i\omega t}dt$, with l.i.m. denoting the limit in the mean, i.e. the limit in the L^2 sense; we will use this type of abuse of notation frequently in these notes.

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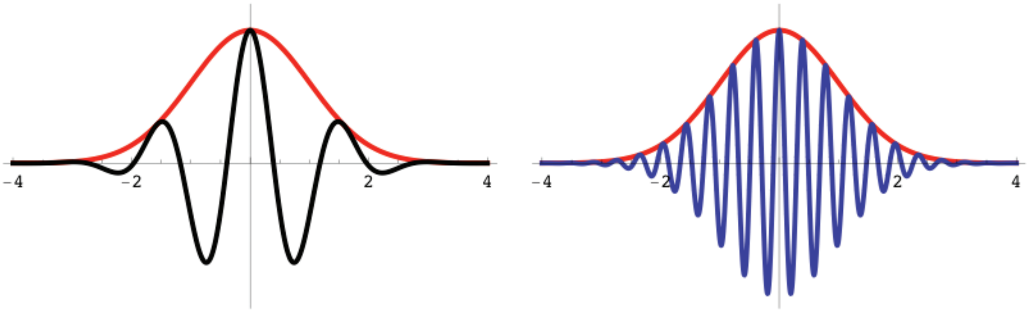

FIGURE 1.—A Gaussian function g (in red) and the real part of two functions $g_{\tau,\omega}$: $g_{0,4}$ (in black) and $g_{0,16}$ (in blue).

ries, which are our main objects of interest in this paper.

The spectral representation of a function given by its Fourier transform determines all the spectral components embedded in the function, but does not provide any information about when they are present. To overcome this problem, Denis Gabor, the Hungarian-born Nobel laureate in physics, proposed, in his fundamental paper on communication theory [10], the use of a modified version of the Fourier transform which became known as a windowed Fourier transform (or short time Fourier transform). The idea is simple: we first choose a window function g, i.e. a well localized function in time;[2] by multiplying the function x by translated copies of g, we are able to selects "local sections" of x, whose Fourier transforms are then computed. We thus obtain a function of two-variables, τ (the translation parameter) and ω (the angular frequency), given by

$$\mathscr{F}_{g;x}(\tau,\omega) = \int_{-\infty}^{+\infty} x(t)\overline{g(t-\tau)}e^{-i\omega t}dt. \qquad (1)$$

We can also view the above procedure in a different manner: starting with a basic window function g, a two-parameter family of functions $g_{\tau,\omega}$ is generated, via translation by τ and modulation by ω, $g_{\tau,\omega}(t) = g(t-\tau)e^{i\omega t}$, and the inner products of x with all the member of this family are then computed: $\mathscr{F}_{g;x}(\tau,\omega) = \langle x, g_{\tau,\omega}\rangle$. The principal limitation of this technique is that it gives us a fixed resolution over the entire time-frequency plane. In fact, the functions $g_{\tau,\omega}$, being obtained by simple translations in time and modulations (i.e. translations in frequency) of the window function g, all have the same "size" as g; see Figure 1.

The main idea of the continuous wavelet transform is again to compute the inner products ofthe function x with members of a two-parameter family of functions $\psi_{\tau,s}$. In this case, however, the functions $\psi_{\tau,s}$ are obtained from a given window function ψ — the so-called *mother wavelet* — which is already oscillatory (and hence, in a certain way, can be seen as a function of a given frequency), by a dilation by a scaling factor s and a translation by τ,

$$\psi_{\tau,s}(t) = |s|^{-1/2}\psi\left(\frac{t-\tau}{s}\right);$$

see Figure 2. For $|s| > 1$, the windows $\psi_{\tau,s}$ become larger (hence, correspond to functions with lower frequency) and when the scales satisfy $|s| < 1$, the windows become narrower (hence, become functions with higher frequency). The main advantage of the continuous wavelet transform, as opposed to the windowed Fourier transform, is now clear: it provides us a time-scale (or time-frequency) representation of a function with windows whose size automatically adjusts to frequencies.

WAVELET TOOLS

The Wavelet

The minimum requirement imposed on a function $\psi \in L^2(\mathrm{R})$ to qualify for being a *mother* (*admissible* or *analyzing*) *wavelet* is that it satisfies the following technical condition, usually referred to as the *admissibility condition* (AC):

$$0 < \int_{-\infty}^{+\infty} \frac{|\widehat{\psi}(\omega)|}{|\omega|}d\omega < \infty; \qquad (1)$$

see [7, p.22]. In this case, the constant given by the value of the above integral,

$$C_\psi = \int_{-\infty}^{+\infty} \frac{|\widehat{\psi}(\omega)|}{|\omega|}d\omega,$$

is called the *admissibility constant*. The wavelet ψ is usually normalized to have unit energy, which we always assume here. We should point out that the square integrability of ψ is a very mild decay condition and that, in practice, much more stringent conditions are imposed. In

[2] Gabor, in his paper, used Gaussian functions as windows and the transform with this particular class of functions is now called a Gabor transform.

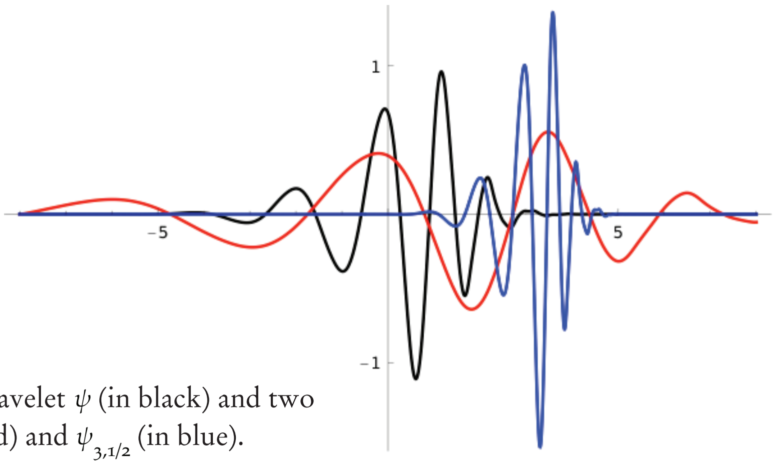

Figure 2.—A mother-wavelet ψ (in black) and two functions $\psi_{\tau,s}$: $\psi_{0,3}$ (in red) and $\psi_{3,1/2}$ (in blue).

fact, for the purpose of providing a useful time-frequency localization, the wavelet must be a reasonable well localized function, both in the time domain as well as in the frequency domain. For functions with sufficient decay, imposing the AC (1) is equivalent to requiring that

$$\widehat{\psi}(0) = \int_{-\infty}^{+\infty} \psi(t)dt = 0;$$

again, [7, p.24]. This implies that the function ψ has to wiggle up and down the t-axis, i.e. it must behave like a wave; this, together with the assumed decaying property, justifies the choice of the term wavelet (originally, in French, *ondelette*) to designate ψ.

The Continuous Wavelet Transform

As referred before, starting with a mother wavelet ψ, a family $\psi_{\tau,s}$ of "wavelet daughters" can be obtained by simply scaling ψ by s and translating it by τ

$$\psi_{\tau,s}(t) := \frac{1}{\sqrt{|s|}}\psi\left(\frac{t-\tau}{s}\right) \qquad \tau, s \in \mathbb{R}, s \neq 0.$$

The parameter s is a scaling or dilation factor that controls the length of the wavelet (the factor $1/\sqrt{|s|}$ being introduced to guarantee preservation of the unit norm, $\|\psi_{\tau,s}\| = 1$) and τ is a location parameter that indicates where the wavelet is centered.

Given a function $x \in L^2(\mathbb{R})$, its *continuous wavelet transform* (CWT) with respect to the wavelet ψ is a function of two-variables, $W_{x;\psi}$, obtained by projecting x, in the L^2 sense, onto the over-complete family $\{\psi_{\tau,s}\}$:

$$W_{x;\psi}(\tau, s) = \langle x, \psi_{\tau,s}\rangle = \frac{1}{\sqrt{|s|}}\int_{-\infty}^{\infty} x(t)\,\overline{\psi}\left(\frac{t-\tau}{s}\right)dt. \quad (2)$$

Note 3.—When the wavelet ψ is implicit from the context, we abbreviate the notation and simply write W_x for $W_{x;\psi}$.

Inversion of CWT

The importance of the admissibility condition (1) is due to the fact that its fulfilment guarantees that the energy of the original function x is preserved by the wavelet transform, i.e., the following Parseval-type relation holds:

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{C_\psi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left|W_x(\tau, s)\right|^2 \frac{d\tau ds}{s^2}.$$

In other words, the operator defined by

$$\mathscr{W}_\psi : L^2(\mathbb{R}) \longrightarrow L^2\left(\mathbb{R}\times\mathbb{R}\setminus\{0\}, \frac{d\tau ds}{s^2}\right)$$
$$x \mapsto \frac{1}{\sqrt{C_\psi}}W_{x;\psi}$$

is an isometry. Also, if ψ satisfies (1), it is possible to recover x from its wavelet transform. In fact, due to the high redundancy of this transform (observe that a function of one variable is mapped into a bivariate function), many reconstruction formulas are available. For example, when the wavelet ψ and x are real-valued, it is possible to reconstruct x by using the formula

$$x(t) = \frac{2}{C_\psi}\int_0^{\infty}\left[\int_{-\infty}^{+\infty} W_x(\tau, s)\psi_{\tau,s}(t)d\tau\right]\frac{ds}{s^2},$$

showing that no information is lost if we restrict the computation of the transform only to positive values of the scaling parameter s, which is a usual requirement, in practice; see e.g. [7].

Wavelet Power and Wavelet Phase

When the wavelet ψ is a complex-valued function, the wavelet transform $W_{x;\psi}$ is also complex-valued and can, therefore, be expressed in polar form as

$$W_x(\tau, s) = |W_x(\tau, s)|e^{i\phi_x(\tau,s)}, \quad \phi_x \in (-\pi, \pi].$$

The square of the amplitude, $|W_x(\tau, s)|^2$ is called the *wavelet power* and the the angle $\phi_x(s, \tau)$ is known as the *(wavelet) phase*.

For real-valued wavelet functions, the imaginary part is constantly zero and the phase is, therefore, uninformative. Hence, in order to obtain phase information about a time-series, it is necessary to make use of complex wavelets.

Analytic Wavelets

When a complex wavelet ψ is to be used, it is convenient to choose it as *analytic* (or *progressive*), where by this we mean that its Fourier transform is supported on the positive real-axis only, i.e. $\widehat{\psi}(\omega) = 0$ for $\omega \leq 0$.[3] In fact, when ψ is analytic and x is real, reconstruction formulas involving only positive values of the scale parameter s are still available; in particular, if the wavelet satisfies $0 < |K_\psi| < \infty$ where

$$K_\psi = \int_0^\infty \frac{\overline{\widehat{\psi}(\omega)}}{\omega} d\omega,$$

then one can use the following reconstruction formula, known as the *Morlet formula*, which is particularly useful for numerical applications:

$$x(t) = 2\Re\left(\frac{1}{K_\psi}\int_0^\infty W_x(t,s)\frac{ds}{s^{3/2}}\right),$$

where $\Re(.)$ denotes real part; see, e.g. [8] or [17]. For other useful features of analytic wavelets, we refer the reader to [27], [25], [20], [21] and also [24].

Note 4.—In what follows, we assume that all the wavelets considered are analytic and hence, that the wavelet transform is computed only for positive values of the scaling parameter s. For this reason, in all the formulas that would normally involve the quantity $|s|$, this will simply be replaced by s.

The Morlet Wavelets

The admissibility condition (1) is a very weak condition. In fact, it can be shown easily that the set of wavelet functions, $\{\psi \in L^2(\mathbb{R}) : \psi \text{ satisfies AC(1)}\}$, is dense in $L^2(\mathbb{R})$; see, e.g. [23, p. 5]. In practice, however, the choice of which wavelet to use is an important aspect to be taken into account, and this will be dictated by the kind of application one has in mind.

To study the synchronism between different time-series, it is important to select a wavelet whose corresponding transform contains information on both amplitude and phase, and hence, a complex-valued analytic wavelet is a natural choice.

The most popular analytic wavelets used in practice belong to the so-called *Morlet wavelet* family. This is a one-parameter family of functions, first introduced in [15], and given by

$$\psi_{\omega_0}(t) = \pi^{-1/4} e^{i\omega_0 t} e^{-\frac{t^2}{2}}.$$

Strictly speaking, the above functions are not true wavelets, since they fail to satisfy the admissibility condition.[4] In fact, since the Fourier transform of the Morlet wavelet $\widehat{\psi}_{\omega_0}$ is given by $\widehat{\psi}_{\omega_0}(\omega) = \sqrt{2}\pi^{1/4} e^{-\frac{1}{2}(\omega-\omega_0)^2}$, one has $\widehat{\psi}_{\omega_0}(0) = \sqrt{2}\pi^{1/4} e^{-\omega_0{}^2/2} \neq 0$. However, for sufficiently large ω_0, e.g. $\omega_0 > 5$, the values of $\widehat{\psi}_{\omega_0}(\omega)$ for $\omega \leq 0$ are so small that, for numerical purposes, ψ_{ω_0} can be considered as an analytic wavelet; see [9].

The popularity of the Morlet wavelets is due to their interesting properties. First, for numerical purposes, as we have just seen, they can be treated as analytic wavelets. Second, since the wavelet ψ_{ω_0} is the product of a complex sinusoidal of angular frequency ω_0, $e^{i\omega_0 t}$, by a Gaussian envelope, $e^{-t^2/2}$, it makes perfect sense to associate the angular frequency ω_0 — i.e. the usual Fourier frequency $f_0 = \omega_0/(2\pi)$ — to this function; in this case, the wavelets at scale s can be associated with frequencies $f_s = \omega_0/2\pi s$; in particular, for the very common choice of $\omega_0 = 6$, we have $f_s \approx 1/s$ and hence the period (or wavelength) is $p_s \approx s$, which greatly facilitates the interpretation of the wavelet analysis as a time-frequency analysis. Finally, the wavelet ψ_{ω_0} is a function with optimal joint time-frequency concentration, in the sense that it attains the minimum possible value of uncertainty associated with the Heisenberg uncertainty principle.

All our numerical results were obtained with the Morlet wavelet ψ_{ω_0} for the particular choice $\omega_0 = 6$.

Computational Aspects

In practice, when one is dealing with a discrete time-series $x = \{x_k : k = 0, \dots, T-1\}$ of T observations with a uniform time step δt, the integral in (2) has to be discretized and is, therefore, replaced by a summation over the T time steps; also, it is convenient, for computational efficiency, to compute the transform for T values of the parameter τ, $\tau = n\delta t$; $n = 0, \dots, T-1$. Naturally, the wavelet transform is computed only for a selected set of scale values $s \in \{s_m : m = 0, \dots, F-1\}$ (corresponding to some frequencies f_m; $m = 0, \dots, F-1$). Hence, our computed wavelet spectrum of the discrete time-series x will simply be a $F \times T$ matrix $W_x = (w_{mn})$ whose (m,n) element is given by

[3] Functions with a positive frequency spectrum were introduced in signal analysis by D. Gabor in [10]. He called them "analytic signals", because they can be extended analytically to the upper-half complex plane.

[4] In order to fulfill the admissibility condition, a correction term has to be added, as: $\psi_{\omega_0}(t) = \pi^{-1/4}\left(e^{i\omega_0 t} - e^{-\omega_0^2}\right) e^{-t^2/2}$.

$$w_{mn} = \frac{\delta t}{\sqrt{s_m}} \sum_{k=0}^{T-1} x_k \overline{\psi}\left((k-n)\frac{\delta t}{s_m}\right).$$

Although it is possible to calculate the wavelet transform using the above formula for each value of m and n, one can also identify the computation for all the values of n simultaneously as a simple convolution of two sequences; in this case, one can follow the standard procedure and calculate this convolution as a simple product in the Fourier domain, using the Fast Fourier Transform algorithm to go forth and back from the time and spectral domains.

As with other types of transforms, the CWT applied to a finite length time-series inevitably suffers from border distortions; this is due to the fact that the values of the transform at the beginning and the end of the time-series are incorrectly computed, in the sense that they involve missing values of the series which are then artificially prescribed. Since the "size" of the wavelets $\psi_{\tau,s}$ increases with s, these edge-effects also increase with s. The region in which the transform suffers from these edge effects is called the *cone-of-influence* (COI). In this area of the time-frequency plane the results are less reliable and have to be interpreted carefully.

We also compute the so-called *wavelet ridges* which are simply the "local maxima" of the wavelet power matrix $|W_x|$. These are computed in the following manner: in each column, every element is compared with the neighbors located up to a specified distance, and the values which are larger than a given factor of the "global maximum" of $|W_x|$ (i.e. its largest value) are selected.

NOTE 5.—Although, numerically, we compute the wavelet transform in a discrete grid of the time-scale plane, the time and scale discretizations are so fine that we still refer to this as the continuous wavelet transform. The so-called *discrete wavelet transform* (DWT) often used in practice, but which we do not consider in this paper, corresponds to a very specific choice of s and τ — the dyadic grid $s = 2^j$, $\tau = 2^{jk}$; j, $k \in \mathbb{Z}$ — and makes the transform non-redundant. This, however, imposes far more stringent conditions on the choice of the mother wavelet ψ; see, e.g. [7].

Example: time-frequency localization of the CWT

We have argued that the main advantage of wavelet analysis over spectral analysis is the possibility of tracing transitional changes across time. To illustrate this, we now consider an example, with simulated data, taken from [2]. We generate 100 years of monthly data, according to the following data generating process:

$$y_k = \cos\left(\frac{2\pi}{10}t_k\right) + \cos\left(\frac{2\pi}{p_k}t_k\right) + \varepsilon_k, \quad k = 1, \dots, 1200,$$

with $t_k = k/12$, $k = 1, \dots, 1200$, and where

$$p_k = \begin{cases} 5, & \text{if } 480 \le k \le 720 \\ 3, & \text{other values of } k \end{cases}$$

and ε_k is a white noise. The above time-series is the sum of two periodic components with a random error term. The first periodic component represents a 10-year cycle, while the second periodic component shows some transient dynamics. It represents a 3-year cycle that, between the fourth and and sixth decades, changes to a 5-year cycle. Figure 3 displays some results related with this example.

The change in the dynamics of the series is nearly impossible to spot in Figure 3 (a), which contains a simple representation of the time-series. Furthermore, if we use the traditional spectral analysis, the information on the transient dynamics is completely lost, as we can see in Figure 3 (d). The power spectral density estimate is able to capture both the 3-year and the 10-year cycles, but it completely fails to capture the 5-year cycle that occurred in the fifth and sixth decades.

Comparing with Figure 3 (c), we observe that spectral analysis gives us essentially the same information as the global wavelet power spectrum (GWPS), which is an average, across all times, of the wavelet power spectrum. On the other hand, Figure 3 (b) shows the wavelet power spectrum itself. On the horizontal axis, we have the time dimension (in years) and the vertical axis gives us the periods.[5] The intensity of power is given by the color. The color code for power ranges from blue (low power) to red (high power), with regions with warm colors thus representing areas of high power. The cone-of-influence is shown with a thick grey line. The white lines show the local maxima (or ridges) of the wavelet power spectrum, giving us a more precise estimate of the cycle period. We observe a white line on period 10 across all times, meaning that there is a permanent cycle with this period. We are also able to spot the 3-year cycle that occurs up to year 40 and, again, between years 60 and 100. Finally, we are also able to identify a yellow/orange region between the years 40 and 60, with the white stripes indicating a cycle of period five. This means that a cycle of roughly 5-year periodicity, relatively important in explaining the total variance of the time-series and taking place between years 40 and 60, was hidden by the Fourier power spectrum estimate.

[5] Note that, since we are using a ψ_6 Morlet wavelet, the periods are almost identical to the scales.

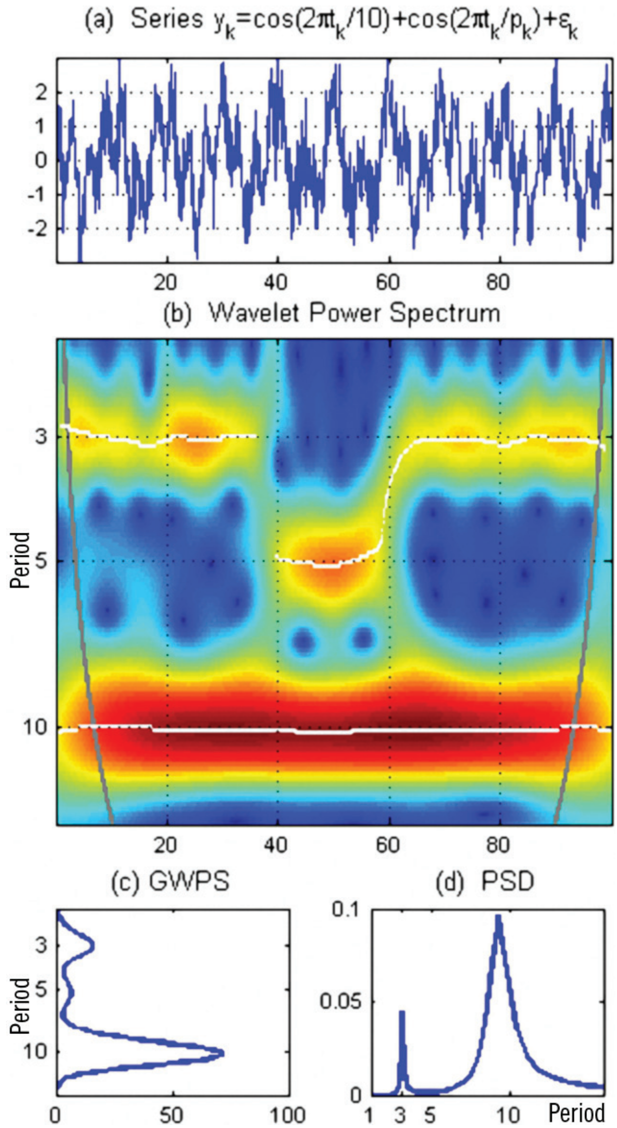

Figure 3.—(a) Series $y_k = \cos(2\pi t_k/10) + \cos(2\pi t_k/p_k) + \varepsilon_k$. (b) Wavelet power spectrum of y— The cone-of-influence is shown with a grey line; the color code for power ranges from blue (low power) to red (high power); the white lines show the local maxima of the wavelet power spectrum. (c) Global wavelet power spectrum, i.e. average wavelet power (over all times) for each frequency. (d) Fourier power spectral density.

Figure 3 (b) clearly illustrates the big advantage of wavelet analysis over spectral analysis. While the Fourier transform is silent about changes that happen across time, with wavelets we are able to estimate the power spectrum as a function of time and, therefore, we do not loose the time dimension. The wavelet power spectrum is able to capture not only the 3-year and 10-year cycles, but also to capture the change that occurred between years 40 and 60.

Cross-Wavelet Analysis

In many applications, one is interested in detecting and quantifying relationships between two non-stationary time-series. The concepts of cross-wavelet power, wavelet coherency and wavelet phase-difference are natural generalizations of the basic wavelet analysis tools that enable us to appropriately deal with the time-frequency dependencies between two time-series.

Cross-Wavelet Transform, Cross-Wavelet Power and Phase-Difference

The *cross-wavelet transform* (XWT) of two time-series, x and y, first introduced by Hudgins, Friehe and Mayer [18], is simply defined as

$$W_{xy}(\tau,s) = W_x(\tau,s)\overline{W_y(\tau,s)},$$

where W_x and W_y are the wavelet transforms of x and y, respectively. The modulus of the XWT, $|W_{xy}(\tau,s)|$ is known as the *cross-wavelet power*.

As for the wavelet transform, if the wavelet ψ is complex-valued, the cross-wavelet transform is also complex-valued and can be written as

$$W_{xy}(\tau,s) = |W_{xy}(\tau,s)|e^{i\phi_{xy}(\tau,s)},$$

where

$$\phi_{xy}(\tau,s) = \phi_x(\tau,s) - \phi_y(\tau,s),$$

with ϕ_x and ϕ_y denoting the phases of x and y respectively,[6] is the *phase-difference* of x and y (also called the *phase-lead* of x over y). A phase-difference of zero indicates that the two time-series move together at the specified (τ,s) value; if $\phi_{xy} \in (0,\pi/2)$, then the series move in-phase, but the time-series x leads over y; if $\phi_{xy} \in (-\pi/2,0)$, the series also move in-phase, but, in this case, is the series y that is leading; a phase-difference of π indicates an anti-phase relation; if $\phi_{xy} \in (\pi/2,\pi)$, then the series are out-of-phase, and y is leading; finally, if $\phi_{xy} \in (-\pi,-\pi/2)$, the series are out-of-phase and x is leading.

Complex Wavelet Coherency

In analogy with the concept of coherency used in Fourier analysis, given two time-series x and y one can define their *complex wavelet coherency*, ϱ_{xy}, by:

$$\varrho_{xy}(\tau,s) = \frac{\mathcal{S}\left(W_{xy}(\tau,s)\right)}{\left[\mathcal{S}\left(|W_x(\tau,s)|^2\right)\mathcal{S}\left(|W_y(\tau,s)|^2\right)\right]^{1/2}}, \quad (3)$$

[6] More correctly, we have $\phi_{xy} = \phi_x - \phi_y (\text{mod } 2\pi)$.

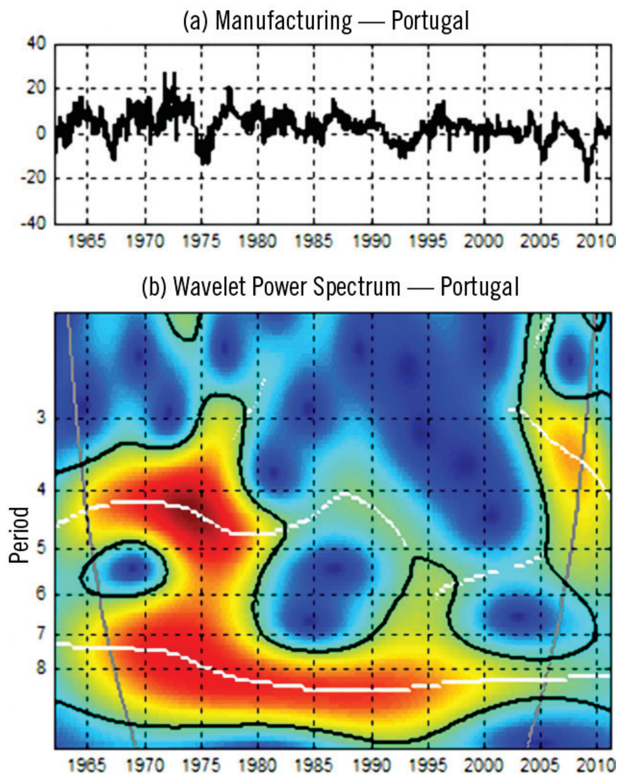

Figure 4.—Total Manufacturing for Portugal (a) and corresponding wavelet power spectrum (b). Color codes are as in Fig.3. The thick black contour represents the 5% significance level.

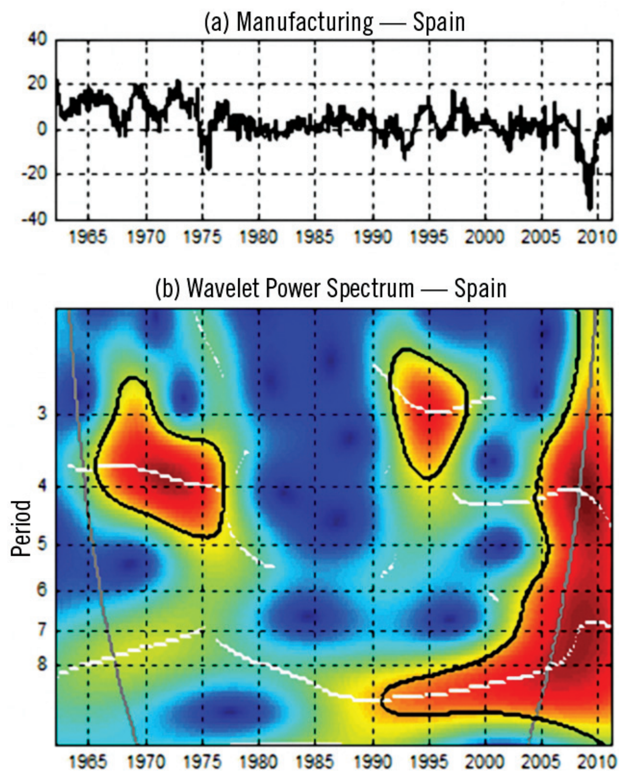

Figure 5.—Total Manufacturing for Spain (a) and corresponding wavelet power spectrum (b). Color codes are as in Fig.3. The thick black contour represents the 5% significance level.

where $\mathscr{S}$ denotes a smoothing operator in both time and scale; smoothing is necessary, because, otherwise, coherency would be identically one at all scales and times. Time and scale smoothing can be achieved, e.g. by convolution with appropriate windows; see [3] or [16], for details.

The absolute value of the complex coherency is called the *wavelet coherency* and is denoted by $R_{x,y}(\tau, s)$. As in the case of the usual Fourier coherency, wavelet coherency satisfies the inequality $0 \leq R_{xy}(\tau, s) \leq 1$, whenever the ratio (3) is well defined. At points (τ, s) for which $\mathscr{S}\left(|W_x(\tau, s)|^2\right) \mathscr{S}\left(|W_y(\tau, s)|^2\right) = 0$, we will define $R_{xy}(\tau, s) = 0$.

As referred by Liu [22], the advantage of these "wavelet-based" quantities is that they may vary in time and can detect transient associations between studied time-series.

Significance Tests

As with other time-series methods, it is important to assess the statistical significance of the results obtained by wavelet analysis. The seminal paper by Torrence and Compo [28] is one of the first works to discuss significance testing for wavelet and cross-wavelet power. However, more work needs to be done on this area. For reasonable general processes, like an ARMA process, one has to rely on bootstrap techniques or Monte Carlo Simulations. To our knowledge, there is no good way of assessing the statistical significance of the phase-difference. In fact, Ge [13] argues that one should not use significance tests for the wavelet phase-difference. Instead, its analysis should be complemented by inspection of the coherence significance.

Business Cycle Synchronization Between Portugal and Spain

In this section we describe the application of the continuous wavelet tools — more specifically, wavelet coherency, phase, and the phase-difference — to the study of the synchronization of the economic cycles of Portugal and Spain, before and after these two countries joined

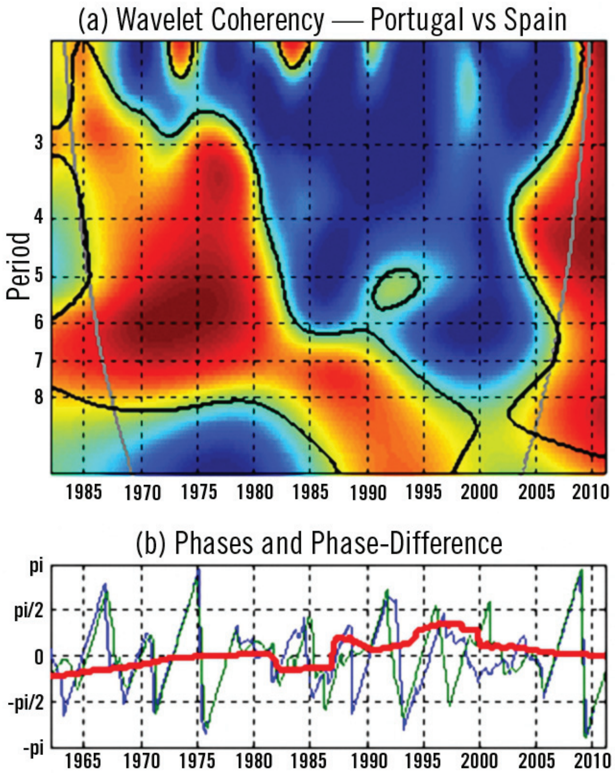

Figure 6.—(a) Wavelet coherency — coherency ranges form blue (low coherency) to red (high coherency); the black thick contour designates the 5% significance level. (b) Phases and phase-difference at the 3–8 year frequency band — The green line represents the phase for Spain, the blue line the phase for Portugal and the red line represents the phase-difference between Spain and Portugal.

the Euro Zone. The data used consist of the total manufacturing year-to-year growth rates for Portugal and Spain. We gathered monthly data from January of 1962 until February of 2011. The results are displayed in Figures 4–6. Figures 4 (a) and (b) represent the time-series for Portugal and the corresponding wavelet power spectrum and Figures 5 (a) and (b) the same quantities, but for Spain. The color codes are as in Figure 3. The black thick contour indicates the 5% significance level.

Wavelet coherency is shown in Figure 6 (a). Figure 6 (b) displays the average values of the phases and phase-difference at the 3–8 year frequency band: the green line represents the phase for Spain, the blue line the phase for Portugal and the red line the phase-difference between Spain and Portugal. Figure 4 (b) tells us that for the entire period of analysis Portugal has a very active business cycle around the 8-year frequency. A cycle with a similar period is also present in Spain — Figure 5 (b) — however, it is statistically significant only after 1990. Cycles associated with shorter periods are also statistically significant in both countries until 1980. It is interesting to note that at the end of the sample, probably because of the world financial and economic crisis, both countries display a large wavelet power spectrum. This is particularly true in the case of Spain, one might add. We recall that, although the precise details are different, these two countries, during the decade of 1960 and in the first half of the decade of 1970, both had proto-fascist regimes ([19]) and, in the second half of that decade, they both turned into democratic regimes.

In 1982, Portugal had a severe Current Account crisis that led to an IMF intervention in 1983. The two countries applied together to be part of the European Economic Community, which they joined in the first of January of 1986. It is very interesting to note how this historical evolution of the countries is reflected in the evolution of the synchronization of their economic cycles and how this can be read with the wavelet tools.

In Figure 6 (a), we see that until early 1980's, the two time-series are highly coherent and in Figure 6 (b), we observe that their phases, at business cycle frequencies 3–8 years period cycles, were well aligned in this period of time, with a slight lead from Portugal. From the early 1980s to about 1986, coinciding with the period immediately after the IMF intervention, we clearly see a de-synchronization between the two countries business cycles. Between 1986 and 1995, the two countries became more synchronized again, in particular at lower frequencies, as we can see in Figure 6 (a), but the phases were not aligned anymore. Instead, the phase-difference between Spain and Portugal (red line) tells us that the Portuguese business cycle was lagging the Spanish one. After 1999, when both countries joined the Euro, the phase-difference started approaching zero. After 2002, the phase-difference became almost zero, suggesting that the business cycles became aligned again. After 2004, we also observe a region of high coherency, which reinforces our previous conclusion. Therefore, coinciding with the adoption of a common currency, the business cycles became more synchronized.

Note 6.—The pictures and the numerical results given in the paper were obtained using a matlab toolbox developed by the authors, the ASToolbox, freely available at http://sites.google.com/site/aguiarconraria/joanasoares-wavelets.

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A statistical perspective on the identification of geophysical trends

by **Susana M. Barbosa***

Introduction

In the diverse geoscience problems investigated at the Instituto Dom Luiz (IDL), a partner of CIM, the identification and quantification of trends is one of the most ubiquitous activities. From the analysis of the outputs of complex meteorological numerical models in a climate change context to the exploitation of geophysical resources and renewal energy sources, accurate knowledge of trends and corresponding uncertainties is fundamental for answering most scientific and societal questions.

Although the concept of trend, as a general direction and tendency, is physically intuitive and apparently simple, its mathematical formulation is far from trivial. In fact, there's no formal definition of trend, which makes trend quantification a delicate, despite common, activity. The wide range of time scales involved in most geophysical problems, and the usually very short period for which reliable data are available, further hinders the identification and quantification of geophysical trends. Here the mathematical aspects of trend assessment in a geophysical context are briefly described. Specific details can be found in Fatichi et al (2009) and Barbosa (2011).

Stochastic models

Different types of stationary and non-stationary processes can originate sequences of observations with trend-like features. Even purely random processes can generate time series exhibiting visually appealing trends, particularly for relatively short records. Some of the most common generating models assumed for geophysical time series are described below.

Autoregressive model

A first order autoregressive process X_t is defined as $X_t = c + \phi X_{t-1} + \varepsilon_t$ with $0 < \phi < 1$, c = constant and $\varepsilon_t \sim \mathcal{N}(0, \sigma^2)$. It is also called a red noise since its spectrum decreases as frequency increases, similarly to red light in the range of visual radiation. This is a purely random, stationary process, with constant mean and variance, but red-noise time series can exhibit an apparent monotonic temporal structure that can be misleadingly taken as indication of non-stationary behavior.

Trend-stationary model

A trend-stationary process X_t is defined as $X_t = a + bt + \varepsilon_t$ with a, b = constant and $\varepsilon_t \sim \mathcal{N}(0, \Sigma)$. It is a non-stationary process, since the mean evolves in time. This is the model implicitly considered in the majority of geophysical contexts, though the purely linear approximation can be inadequate (e.g. Miranda & Tomé, 2009).

Difference-stationary model

A random walk or difference stationary process X_t is defined as $X_t = c + \phi X_t - 1 + \varepsilon_t$ with $\phi = 1$ and describes a process whose value at a time t is equal to its value at the previous instant plus a random shock, similarly to the path of a drunken man whose position at a given time is its position at the previous time plus a step in a random direction. It corresponds to a 1st order autoregressive process with $\phi = 1$ and is also called an integrated process of order 1, since its 1st derivative is stationary. This is a non-stationary process, since both the mean and the variance evolve in time.

Long-memory model

A process X_t is a stationary long range dependent or long memory process if its autocovariance function γ_X decays as a power law, such that observations widely separated in time can still have a non-negligible covariance: $\lim_{\tau \to \infty} \gamma_X(\tau) = C\tau^{-\alpha-1}$, where C and α are constants satisfying $C > 0$ and $-1 < \alpha < 0$. Long-memory time series are characterised in the time domain by persistent-

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in-time autocorrelations, decaying as a power law, and in the frequency domain by high spectral content at frequency zero.

Trend assessment

A fundamental aspect in the study of geophysical trends is the possible underlying mechanism generating the observed sequence of observations. Though inherently unobtainable, understanding the underlying generating process is the ultimate aim of any trend analysis.

The conventional approach in the study of geophysical trends is to assume a trend-stationary model, estimate the parameters a, b of the regression, and then test the adequacy of the model from the statistical significance of the resulting estimates. However, even if the slope of a linear regression model is statistically significant, the underlying stochastic model may not be a reasonable assumption. In fact, all of the stochastic models mentioned in section 2 are able to generate finite sequences with statistically significant linear trends.

Assessment of whether the monotonic behavior exhibited by a geophysical time series is better characterised by a trend-stationary model, a difference-sttaionary model or a long-memory model has both conceptual and practical implications. For example, while both trend-stationary and difference-stationary time series exhibit a tendency behavior, the former is characterised by a deterministic trend tendency with stable variance, while the latter is characterized by a stochastic tendency with increasing variance. The distinction between the two kinds of nonstationary behavior has not only practical implications (e.g. forecasting) but more importantly on the physical interpretation of the identified trend: in the case of a trend-stationary model the trend can be interpreted as deterministic and due to some forcing factor, while in the case of a difference-stationary model the apparent trend is the result of stochastic fluctuations.

A possible approach to trend assessment is based on parametric statistical tests, developed in econometrics contexts for discriminating between difference-stationary and trend-stationary time series. The PP test (Phillips & Peron, 1988) tests the null hypothesis of a difference stationary random walk process against a trend stationary alternative. It is based on the model $X_t = \eta + \beta_t + \pi X_{t-1} + \psi_t$ where ψ_t is a stationary noise process and η and β are the parameters of a first-order polynomial regression. The null hypothesis is expressed by $H_0 : \pi = 1$ against the alternative $H_1 : \pi < 1$ The KPSS test (Kwiatkowski et al, 1992) tests the null hypothesis of a trend-stationary process against a difference-stationary alternative. The KPSS test is based on the model $X_t = \beta t + r_t + \nu_t$, where r_t is a random walk, $r_t = r_{t-1} + \varepsilon_t$, $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ and ν_t is a stationary noise process.

The joint application of the KPSS and PP allows to assess whether a deterministic linear trend is a reasonable assumption for the data considered. If only the null hypothesis of the KPSS test is rejected, the time series is difference stationary. Then its long-term variability should not be characterized by the slope of a linear regression model (even if it is statistically significant), since the assumption of a deterministic trend is not itself plausible. Conversely, if only the null of the PP test is rejected, the time series is trend-stationary. If both tests reject the respective null hypothesis, alternative behaviors (such as long range dependence) should be considered.

Concluding remarks

The identification of trends is one of the most common activities in geosciences and one with the highest societal implications, since policy makers require information on tendencies to sustain environmental policies, for example in a climate change context. Different kind of stochastic processes can originate finite temporal sequences with visually appealing (and statistically significant!) trends. Trend assessment is therefore a fundamental activity, that can be performed by the joint application of parametric statistical tests of hypothesis.

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Some mathematical aspects of mountain waves: work carried out at Instituto Dom Luiz

by **Miguel A. C. Teixeira***

Research in many areas of the Geosciences is pursued at Instituto Dom Luiz (IDL), an Associated Laboratory of the University of Lisbon that is a partner of CIM. In the Atmospheric and Climate Modelling group at IDL, research on basic fluid dynamics relevant to meteorology is carried out, using numerical and analytical methods. As a member of this research group with a particular interest in analytical and semi-analytical mathematical techniques, the author will try to describe in this paper some aspects of research on mountain waves, which is perhaps one of the most interesting areas from an applied mathematics point of view.

Introduction

Mountain waves are a type of internal gravity waves forced by airflow over mountains. Internal gravity waves exist, not at an interface, like ocean waves, but in the interior of the atmosphere. They require an atmosphere with stable stratification, where air parcels that are displaced vertically tend to oscillate. These waves are fairly common, but can only be visualized when the atmosphere has enough moisture for clouds to form in the regions of ascending motion. Associated with mountain waves there is a pressure distribution at the surface which causes a drag force on the mountains (Smith, 1980). To this corresponds a reaction force acting on the atmosphere, which must be represented in some way (parametrized) in global climate and weather prediction models. This is required because the dominant contributions to this force come from mountains of width ≈10km, which are typically not resolved by these models. Current research at IDL aims to understand how mountain wave drag varies with input parameters of the incoming large-scale flow, in order to contribute to the improvement of existing parametrizations of this process.

Mountain wave equations

Mountain waves, like other meteorological phenomena, are governed by a set of partial differential equations comprising the Navier-Stokes equation, the conservation of mass, a heat balance equation and an equation of state for ideal gases. In the following equation set, the rotation of the Earth is neglected, because the scale of the motions is relatively small, yet viscosity is also neglected because the scale is larger that that of viscous boundary layers.

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \vec{g}, \qquad (1)$$

$$\frac{\partial T}{\partial t} + \vec{v}\cdot\vec{\nabla}T = \frac{1}{\rho c_p}\left(\frac{\partial p}{\partial t} + \vec{v}\cdot\nabla p\right), \qquad (2)$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0, \qquad (3)$$

$$p = R_a \rho T. \qquad (4)$$

In (1)–(4), $\vec{v} = (u, v, w)$ is the velocity vector, ρ is the density, p is the pressure, T is the absolute temperature, $\vec{g}$ is the acceleration of gravity, R_a is the ideal gas constant for air and c_p is the corresponding specific heat at constant pressure.

Equation (2) results from the first law of thermodynamics for adiabatic processes, because the motions associated with mountain waves are fast enough for heat transfer to be insignificant (except when there is cloud formation).

For simplicity, 2D motion (in an $x - z$ vertical plane) is considered. The flow is also assumed to be stationary, because the waves are generated by a fixed topographic forcing. Additionally, the Boussinesq approximation is assumed. This is combined next with linearization of the equations of motion to obtain a final simplified equation set. In the Boussinesq approximation, the thermo-

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dynamic dependent variables of (1)–(4) are decomposed as $\rho = \overline{\rho}(z) + \rho'$, $p = \overline{p}(z) + p'$ and $T = \overline{T}(z) + T'$, where the overbar denotes a reference state that depends only on height and the primes denote perturbations associated with the mountain waves. The reference state is assumed to be in hydrostatic equilibrium:

$$\frac{d\overline{p}}{dz} = -\overline{\rho}g. \tag{5}$$

Additionally, constant reference values of the density and other flow variables (denoted by a zero subscript) are assumed to exist such that $\overline{\rho} \approx \rho_0$ and $\rho'/\overline{\rho} \approx \rho'/\rho_0$ and similarly for the other variables. The Boussinesq approximation also assumes that

$$\frac{\rho'}{\rho_0} \approx -\frac{\theta'}{\theta_0}, \tag{6}$$

where $\theta = \overline{\theta}(z) + \theta'$ is the potential temperature. This is defined as

$$\theta = T\left(\frac{p_0}{p}\right)^{R_a/c_p}, \tag{7}$$

where p_0 is a reference pressure (generally assumed to be $p_0 = 10^5$Pa). θ is a very important quantity in meteorology because it is conserved in adiabatic processes. Equation (6) amounts to assuming that the density is a much weaker function of pressure than of temperature, which is acceptable for motions much slower than the speed of sound.

Linearization of the equations of motion goes one step further by assuming the same kind of decomposition also for the velocity vector: $\vec{v} = (U(z) + u', V(z) + v', w')$ (where the capital letters correspond to the reference wind, which is only a function of height), and neglecting all products of perturbations. With all these simplifications, which are valid for waves over relatively low mountains, the equation set (1)–(4) becomes:

$$U\frac{\partial u'}{\partial x} + w'\frac{\partial U}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}, \tag{8}$$

$$U\frac{\partial w'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + b, \tag{9}$$

$$U\frac{\partial b}{\partial x} + N^2 w' = 0, \tag{10}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0, \tag{11}$$

where $b = g(\theta'/\theta_0)$ is a buoyancy perturbation and $N^2 = (g/\theta_0)(d\overline{\theta}/dz)$ is the mean static stability.

Through differentiation and summation, these equations may be combined into one single equation for w':

$$\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2} + l^2(z)w' = 0, \tag{12}$$

where

$$l^2 = \frac{N^2}{U^2} - \frac{1}{U}\frac{d^2U}{dz^2}. \tag{13}$$

$l(z)$ is called the Scorer parameter.

Since the boundary conditions are most conveniently applied in wavenumber space and the waves are expected to be confined near an isolated topography, Fourier analysis is adopted to express all flow variables, including w':

$$w'(x,z) = \int_{-\infty}^{\infty} \widehat{w}(k,z)e^{ikx}dk, \tag{14}$$

where $\widehat{w}$ is the Fourier transform of w'. Then, (12) can be written

$$\widehat{w}'' + \left[l^2(z) - k^2\right]\widehat{w} = 0, \tag{15}$$

where the primes denote differentiation with respect to z. Despite its simplicity, in general this equation has no analytical solution. Two exceptions occur when $l(z)$ is either a slow function of z, or is piecewise constant. These two cases will be addressed next in turn.

Slowly varying Scorer parameter profile

When the Scorer parameter varies relatively slowly with height, the WKB approximation can be used to solve (15). This entails defining a new rescaled vertical coordinate as $Z = \varepsilon z$, where ε is a small parameter (Bender and Orszag, 1999), so that (15) becomes:

$$\varepsilon^2\ddot{\widehat{w}} + \left[\frac{N^2}{U^2} - \varepsilon^2\frac{\ddot{U}}{U} - k^2\right]\widehat{w} = 0, \tag{16}$$

where the dots denote differentiation with respect to Z. Additionally, a solution of the form

$$\widehat{w} = \widehat{w}(Z=0)e^{i\varepsilon^{-1}\int_0^Z [m_0(\varrho)+\varepsilon m_1(\varrho)+\varepsilon^2 m_2(\varrho)+\ldots]d\varrho} \tag{17}$$

is adopted. In this equation, the vertical wavenumber of the mountain waves is expanded as a power series of ε. By inserting (17) into (16), and considering terms only up to second-order in ε, the following set of algebraic equations for m_0, m_1 and m_2 is obtained:

$$-m_0^2 + \frac{N^2}{U^2} - k^2 = 0, \tag{18}$$

$$i\dot{m}_0 - 2m_0 m_1 = 0, \tag{19}$$

$$i\dot{m}_1 - 2m_0 m_2 - m_1^2\frac{\ddot{U}}{U} = 0. \tag{20}$$

If the wave motion itself can be considered hydrostatic, i.e.

$$\frac{1}{\rho_0}\frac{\partial p'}{\partial z} = b, \qquad (21)$$

then the k^2 term in (18) must be neglected. This corresponds to relatively wide mountains, which are those that give a dominant contribution to the drag. In this case, the definitions for m_0, m_1 and m_2 found from (18)-(20) are:

$$m_0 = \frac{N}{U}\mathrm{sgn}(k), \qquad (22)$$

$$m_1 = -\frac{1}{2}i\frac{\dot{U}}{U}, \qquad (23)$$

$$m_2 = -\frac{1}{8}\frac{U}{N}\mathrm{sgn}(k)\left(\frac{\dot{U}^2}{U^2} + 2\frac{\ddot{U}}{U}\right). \qquad (24)$$

In (22)–(24) it was assumed that N is constant, because generally the vertical variation of U is more important in the atmosphere. The sign function has been included in m_0 (and as a consequence appears also in m_2) so that the wave energy propagates upward, as is logical for waves forced topographically. It can be shown that if the vertical wavenumber of these waves (and thus m_0) has the same sign as Uk, the group velocity of the waves has a positive vertical component, as required.

The lower boundary condition, which requires that, in inviscid conditions, the flow is tangential to the topography, can be expressed, in the linearized approximation, as

$$w'(z=0) = U_0\frac{\partial h}{\partial x} \Rightarrow \widehat{w}(z=0) = iU_0k\widehat{h}, \qquad (25)$$

where $U_0 = U(z=0)$, $h(x)$ is the surface elevation and $\widehat{h}(k)$ is its Fourier transform. This completely specifies the solution to the problem. If an explicit solution is required, the integrals in the exponent of (17) must be calculated. This is possible analytically for the term involving m_1, but not in general for those involving m_0 and m_2. Nevertheless, for calculating mountain wave drag this is not necessary, since, the drag per unit spanwise length of the mountain is defined as

$$D = \int_{-\infty}^{+\infty} p'(z=0)\frac{\partial h}{\partial x}dx$$
$$= 2\pi i\int_{-\infty}^{+\infty} k\widehat{p}^*(z=0)\widehat{h}\,dk, \qquad (26)$$

where the asterisk denotes complex conjugate and $\widehat{p}$ is the Fourier transform of the pressure perturbation. Using (8) and (11), this quantity can be expressed as

$$\widehat{p} = i\frac{\rho_0}{k}(U'\widehat{w} - U\widehat{w}'), \qquad (27)$$

which means that at the surface, using (17), it becomes

$$\widehat{p}(z=0) = i\rho_0U_0^2\Big[m_0(z=0) + \varepsilon m_1(z=0) + i\frac{U_0'}{U_0} + \varepsilon^2 m_2(z=0)\Big]\widehat{h}, \qquad (28)$$

where $U_0' = U'(z=0)$. If (28) is used in (26), and (22)–(24) are also employed, the drag normalized by its value D_0 for a constant mean wind U_0 is given by (Teixeira and Miranda, 2004)

$$\frac{D}{D_0} = 1 - \frac{1}{8}\frac{U_0'^2}{N^2} - \frac{1}{4}\frac{U_0''U_0}{N^2}, \qquad (29)$$

correct to second-order in ε, where

$$D_0 = 4\pi\rho_0 NU_0\int_0^{+\infty} k|\widehat{h}|^2\,dk. \qquad (30)$$

Thus the WKB approximation allows one to obtain a closed-form analytical expression for the correction to the drag due to the variation of the wind with height. Something analogous could be done if N was assumed to be a function of height as well.

Two-layer atmosphere

Consider now that the atmosphere has a two-layer structure, with different (constant) values of l in each layer: l_1 near the surface ($0 < z < H$) and l_2 aloft ($z > H$). It will be assumed that $l_1 > l_2$, since unlike the opposite possibility, this allows wave trapping near the surface, which affects mountain wave drag in an interesting way (Scorer, 1949). In this situation, (15) has solutions of the form:

$$\widehat{w} = a_1e^{im_1z} + b_1e^{-im_1z} \text{ if } k^2 < l_1^2, \qquad (31)$$

$$\widehat{w} = c_1e^{-n_1z} + d_1e^{n_1z} \text{ if } k^2 > l_1^2, \qquad (32)$$

in the lower layer, where $m_1^2 = l_1^2 - k^2$ and $n_1^2 = k^2 - l_1^2$. In the upper layer, on the other hand,

$$\widehat{w} = a_2e^{im_2z} \text{ if } k^2 < l_2^2, \qquad (33)$$

$$\widehat{w} = c_2e^{-n_2z} \text{ if } k^2 > l_2^2, \qquad (34)$$

where $m_2^2 = l_2^2 - k^2$ and $n_2^2 = k^2 - l_2^2$. The first solutions in (31)–(32) and (33)–(34) correspond to waves whose energy propagates vertically, while the second solutions are evanescent. In the upper layer (33) corresponds to an upward propagating solution, whereas (34) corresponds to a wave that decays with height. This makes physical sense for topographically generated waves. a_1, b_1, c_1, d_1, a_2 and c_2 are functions of k which are determined by the boundary conditions. These prescribe that the waves either propagate upward or decay as $z \to +\infty$ (this is already included in (33)–(34), as mentioned above), that the flow is tangential to the topography at the surface, (25), and that the streamline slope and pressure perturbation are continuous at $z = H$. For simplicity, it is assumed next that the discontinuity of l is due to N and not to U, which is taken as constant. This slightly simplifies the boundary conditions, but other possibilities could be accommodated without too much effort, if required.

Then it can be shown that

$$a_1 = \frac{iUk\hat{h}(m_1 + m_2)e^{-im_1 H}}{2m_1 \cos(m_1 H) - 2im_2 \sin(m_1 H)}, \tag{35}$$

$$b_1 = \frac{iUk\hat{h}(m_1 - m_2)e^{im_1 H}}{2m_1 \cos(m_1 H) - 2im_2 \sin(m_1 H)}, \tag{36}$$

if $k^2 < l_2^2$. If $l_2^2 < k^2 < l_1^2$ instead, then:

$$a_1 = \frac{Uk\hat{h}(im_1 - n_2)e^{-im_1 z}}{2m_1 \cos(m_1 H) + 2n_2 \sin(m_1 H)}, \tag{37}$$

$$b_1 = \frac{Uk\hat{h}(im_1 + n_2)e^{im_1 z}}{2m_1 \cos(m_1 H) + 2n_2 \sin(m_1 H)}. \tag{38}$$

When the waves are evanescent in both layers, i.e. $k^2 > l_1^2$, it can be shown that no mountain wave drag is produced, since the pressure perturbation is symmetric with respect to the orography. When $l_1^2 > l_2^2$, two possibilities exist: either the waves propagate vertically in both layers (when $k^2 < l_2^2$), or they propagate in the first layer but not in the second, i.e. are trapped ($l_2^2 < k^2 < l_1^2$). In both cases, the drag is given by (26) with

$$\hat{p}(z = 0) = \frac{\rho_0 U m_1}{k}(a_1 - b_1) \tag{39}$$

which results from (27) and (31). If the necessary calculations are performed, the drag is found to be given by two contributions: one from wavenumbers between 0 and l_2 (D_1) and the other from wavenumbers between l_2 and l_1 (D_2). These two contributions can be written:

$$D_1 = 4\pi\rho_0 U^2 \int_0^{l_2} k|\hat{h}|^2 \frac{m_1^2 m_2}{m_1^2 \cos^2(m_1 H) + m_2^2 \sin^2(m_1 H)} dk, \tag{40}$$

$$D_2 = \mathrm{Re}\left[4\pi i\rho_0 U^2 \int_{l_2}^{l_1} k|\hat{h}|^2 m_1 \times \frac{m_1 \sin(m_1 H) - n_2 \cos(m_1 H)}{m_1 \cos(m_1 H) + n_2 \sin(m_1 H)} dk\right]. \tag{41}$$

The integral in (40), which gives the drag due to mountain waves that propagate in the two atmospheric layers, must be evaluated numerically. Since the integrand in (41) is real, contributions to the drag from this integral only come from singularities along the real axis. These correspond to the modes of the trapped lee waves. These modes are given by the condition that the denominator in the integrand of (41) vanishes, that is

$$\tan(m_1 H) = -\frac{m_1 H}{n_2 H}. \tag{42}$$

The wavenumber of each lee wave mode, say k_i (with $i = 1, 2, \ldots$), can be found by solving (42) numerically. Then, the lee wave drag, which is produced by waves which are trapped in the lower atmospheric layer, can be calculated by finding the imaginary part of the integral in (40). In this calculation, which only receives contributions from the singularities on the real axis, the integration path must be indented above each singularity. This is because, with the addition of Rayleigh damping frictional terms to the governing equations, the singularities move to the negative imaginary semi-plane. Then, it can be shown that the lee wave drag takes the form:

$$D_2 = 4\pi^2\rho_0 U^2 \sum_i |\hat{h}|^2(k_i) \frac{m_1^2(k_i) n_2(k_i)}{1 + n_2(k_i) H}. \tag{43}$$

where the sum is performed over all lee wave modes.

Concluding remarks

Two examples of mathematical methods that can be profitably employed in the study of mountain waves have been described. Additional asymptotic techniques, such as the method of multiple scales, or matched asymptotic expansions, to give only two examples, are routinely used in fluid mechanics, and can be applied to appropriate problems in meteorology, or in the geosciences in general, as long as small parameters exist, of which the researcher may take advantage.

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Stratified Models in First-Order Logic

by **José Roquette***

The various nature of the mathematical objects in what concerns their complexity, our knowledge of them or the possibility to make them explicit (for example, infinitesimal or ilimited real numbers) is a strong motivation to consider their distribution into *levels* or *strata*. The *stratification* depends on the selected property (or properties) of the mathematical objects that are the subject-matter of our study.

Stratified Models

Along this article $\mathscr{L}$ is a first-order language with equality, no constant symbols, no function symbols and the logical symbols:

parentheses	")" and "("
variables	$v_0, v_1, \ldots, v_n, \ldots$
0-ary connective	$\perp$ (falsity, *falsum*, *absurdum*)
binary connective	$\Rightarrow$ (implication)
universal quantifier	$\forall$

The basic definitions and conventions of $\mathscr{L}$ are as usual (see [2]); in particular, $\neg\phi$ ("not" ϕ), $\phi \vee \psi$, (ϕ "or" ψ), $\phi \wedge \psi$ (ϕ "and" ψ) and $(\exists v_i)\phi(v_i)$ ("there is a" v_i "such that" $\phi(v_i)$) abbreviate, respectively, $\phi \Rightarrow \perp$, $(\neg\phi) \Rightarrow \psi$, $\neg(\neg\phi) \vee (\neg\psi))$ and $\neg(\forall v_i)\neg\phi(v_i)$.

The expressions $\mathrm{Term}(\mathscr{L}')$, $\mathrm{Term}_C(\mathscr{L}')$, $\mathrm{Atom}(\mathscr{L}')$, $\mathrm{Form}(\mathscr{L}')$, $\mathrm{Sent}(\mathscr{L}')$, $\mathrm{At}(\mathscr{L}')$ denote, respectively, the classes of the *terms*, the *closed terms*, the *atomic formulae*, *the formulae*, the *sentences* and the *atomic sentences* of whatever first-order language $\mathscr{L}'$ we are using.

Definition.—Let P be a set and $\leq$ a total, dense, preordering relation on P. The expressions "$p < q$" and "$p =_{\leq} q$" abbreviate, respectively: "$p \leq q$ and $q \nleq p$" and "$p \leq q$ and $q \leq p$", for each $p, q \in P$.

We will be interested on total dense preorderings $\mathbb{P} = (P, \leq)$ having a $\leq$-minimal element $\mathbb{0}_{\leq}$ (which we denote simply by $\mathbb{0}$ when no confusion arises) and no $\leq$-maximal element; more explicitely: $\mathbb{0} \leq p$, for every $p \in P$; and given $q \in P$ there is a $p \in P$ such that $q \leq p$ and $p \nleq q$.

Fix $\mathbb{P} = (P, \leq, \mathbb{0})$ as above and consider a class valued function D defined on P such that, for each $p \in P$, the image of p under D is a non-empty class. For $\mathbb{P}$, D as before, a sequence $\mathscr{F} := (P, \leq, \mathbb{0}, D) = (\mathbb{P}, D)$ is called a *stratifying frame*. The elements of P are the *nodes* of $\mathscr{F}$ and for each $p \in P$, the set $D(p)$ is the *domain* of $\mathscr{F}$ at the node p.

To each $a \in D(p)$ we associate a constant symbol $\bar{a}$ (using different constant symbols for different elements of $D(p)$). If $a \in D(p) \cap D(q)$, then the constant symbol associated with a is the same.

At this point it is convenient to introduce some extensions of the original first-order language $\mathscr{L}$.
By $\mathscr{L}_*$, we understand the first-order extension of $\mathscr{L}$ defined as $\mathscr{L}_* := \mathscr{L} \cup \{\mathbf{0}, \sqsubseteq\}$, where $\mathbf{0}$ is a *constant symbol* and $\sqsubseteq$ is a new *binary relation symbol* called *precedence of level*.

For each $p \in D(p)$ we denote by $\mathscr{L}_*^p$ the first-order extension of $\mathscr{L}_*$ given by $\mathscr{L}_*^p := \mathscr{L}_* \cup \{\bar{a} \mid a \in D(p)\}$.

Finally, $\mathscr{L}_*^+$ is the first-order extension of $\mathscr{L}_*$ defined

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as: $\mathscr{L}_*^+ := \cup_{p \in P} \mathscr{L}_*^p$. The language $\mathscr{L}_*$ is the *stratifying language* associated with $\mathscr{L}$.

So, the class of all closed terms of $\mathscr{L}_*^+$ is:

$$\mathrm{Term}_C(\mathscr{L}_*^+) = \{\bar{a} \mid (\exists p \in P) a \in D(p)\} \cup \{\mathbf{0}\}.$$

For any terms $t_1, t_2 \in \mathrm{Term}(\mathscr{L}_*^+)$, the expression $t_1 \sqsubset t_2$ abbreviates "$t_1 \sqsubseteq t_2$ and $t_2 \not\sqsubseteq t_1$". (The relations $t_1 \sqsubseteq t_2$ and $t_1 \sqsubset t_2$ must be read, respectively, as "t_1 *precedes* t_2" and "t_1 *strictly precedes* t_2".)

Consider a function V defined on $\mathrm{Term}_C(\mathscr{L}_*^+)$ and with values in P such that $V(\mathbf{0}) = \mathbb{0}$, $a \in D(V(\bar{a}))$ and for each $p \in P$ if $a \in D(p)$, then $V(\bar{a}) \leq p$ and $a \notin D(q)$, for arbitrary $q < V(\bar{a})$ in P. ($V(\bar{a})$ must be thought as "the"{1} first level of interpretation of $\bar{a}$.)

Having described how closed terms are interpreted we will now make the necessary preparatory steps towards the description of the semantics in *stratified models* (a concept to be introduced later). Consider a function Σ defined on P such that, for each $p \in P$, the value of p under Σ is a set of atomic sentences of $\mathscr{L}_*^p$. (The set $\Sigma(p)$ establish, for each $p \in P$ the "basic truths" at p.)

The functions D, Σ and V satisfy the following conditions:

1. If $p \leq q$, then $D(p) \subseteq D(q)$.
2. $\perp \notin \Sigma(p)$, for every p.
3. If $p \leq q$, then $\Sigma(p) \subseteq \Sigma(q)$.
4. The formula $\bar{a}_1 = \bar{a}_2$ is in $\Sigma(p)$ iff $V(\bar{a}_1) \leq p$, $V(\bar{a}_2) \leq p$ and $a_1 = a_2$.
5. the formula $\bar{a}_1 \sqsubseteq \bar{a}_2$ is in $\Sigma(p)$ iff $V(a_1) \leq V(\bar{a}_2) \leq p$.
6. If R_i is a n_i-ary relation symbol of $\mathscr{L}$ and $a_1, \dots, a_{n_i}, b_1, \dots, b_{n_i} \in D(p)$ with $a_1 = b_1, \dots, a_{n_i} = b_{n_i}$, then: $R_i(\bar{a}_1, \dots, \bar{a}_{n_i})$ in $\Sigma(p)$ implies that $R(\bar{b}_1, \dots, \bar{b}_{n_i})$ is in $\Sigma(p)$. (Evidently, if R_i is a n_i-ary relation symbol of $\mathscr{L}$ and $R_i(\bar{a}_1, \dots, \bar{a}_{n_i})$ is in $\Sigma(p)$, then: $V(a_j) \leq p$ [for $1 \leq j \leq n_i$] since $\Sigma(p)$ consists of atomic formulas of $\mathscr{L}_*^p$.

Definition.—For $P, \leq, D, \Sigma, V, \mathbb{0}$ as described previously let $\mathscr{F} := (P, \leq, D, \mathbb{0})$ be a stratifying frame. A *stratified model* for $\mathscr{L}_*$ is a sequence $\mathscr{S}_* := (\mathscr{F}, \Sigma, V, \mathrm{o})$ such that,

1. $\mathrm{o} \in D(\mathbb{0})$;
2. $\bar{\mathrm{o}}$ is identified with $\mathbf{0}$.

The *nodes* of $\mathscr{S}_*$ and the *domain* of $\mathscr{S}_*$ at each $p \in P$ are those of $\mathscr{F}$.

Remark.—If $\mathscr{S}_* = (\mathscr{F}, \Sigma, V, \mathrm{o})$ is a stratified model for $\mathscr{L}_*$, then it is easy to prove that P is an infinite set and, at each $p \in P$, D and Σ determine a classical structure (see [1]) $\mathfrak{A}_p$ whose domain (which we denote by $|\mathfrak{A}_p|$) is $D(p)$ and:

1. if $p \leq q$, then $|\mathfrak{A}_p| \subseteq |\mathfrak{A}_q|$.
2. The interpretations $R_i^{\mathfrak{A}_p}$ of a n_i-ary relation symbol R_i of $\mathscr{L}$ and $\sqsubseteq^{\mathfrak{A}_p}$ of $\sqsubseteq$ are: $R_i^{\mathfrak{A}_p}(a_1, \dots, a_{n_i})$ iff the formula $R_i(\bar{a}_1, \dots, \bar{a}_{n_i})$ belongs to $\Sigma(p)$, and $a_1 \sqsubseteq^{\mathfrak{A}_p} a_2$ iff the formula $\bar{a}_1 \sqsubseteq \bar{a}_2$ belongs to $\Sigma(p)$.

   So, for $p \leq q$ we have that $R_i^{\mathfrak{A}_p} \subseteq R_i^{\mathfrak{A}_q}$ (i.e. $R_i^{\mathfrak{A}_q}$ extends $R_i^{\mathfrak{A}_p}$) and $\sqsubseteq^{\mathfrak{A}_p} \subseteq \sqsubseteq^{\mathfrak{A}_q}$ (i.e. $\sqsubseteq^{\mathfrak{A}_q}$ extends $\sqsubseteq^{\mathfrak{A}_p}$).
3. $\bar{a}^{\mathfrak{A}_p} := a$ for every $p \in P$ and for every $a \in D(p)$;
4. in particular, since $\mathbf{0}$ is $\bar{\mathrm{o}}$ we obtain, $\mathbf{0}^{\mathfrak{A}_p} := \mathrm{o}$ for every $p \in P$.

Proposition.—Let $\mathscr{S}_* := (\mathscr{F}, \Sigma, V, \mathrm{o})$ be a stratified model for $\mathscr{L}_*$. Then,

1. if $\bar{a}_1, \dots, \bar{a}_n$ are closed terms of $\mathscr{L}_*^+$, then there is a $p \in P$ such that:
   $a_1, \dots, a_n \in D(p)$ and $a_1, \dots, a_n \in D(q)$ implies $p \leq q$, for every $q \in P$
   (we refere to this last proposition as "(*)").
2. If $\bar{a}_1, \dots, \bar{a}_n$ are closed terms of $\mathscr{L}_*^+$, then there is a $p \in P$ such that: $V(\bar{a}_j) \leq p$ (for $1 \leq j \leq n$) and if $V(\bar{a}_j) \leq q$ (for $1 \leq j \leq n$), then $p \leq q$, for every $q \in P$.
3. If $p_1, p_2 \in P$ satisfy proposition (*), then $p_1 =_{\leq} p_2$.

If $\mathscr{S}_* := (\mathscr{F}, \Sigma, V, \mathrm{o})$ is a stratified model for $\mathscr{L}_*$ and $\bar{a}_1, \dots, \bar{a}_n$ are closed terms of $\mathscr{L}_*^+$, we define $V(\bar{a}_1, \dots, \bar{a}_n)$ as "the" p (unique modulo $=_{\leq}$) satisfying the proposition (*). (The level $V(\bar{a}_1, \dots, \bar{a}_n)$ is "the" *first level of interpretation of all the* $\bar{a}_1, \dots, \bar{a}_n$.)

Stratified Semantics

In order to completely characterize the semantics in stratified models, we need to extend the considerations made in the previous section to arbitrary formulae. The next proposition fully describes the situation. In fact, by induction on the complexity of formulae we can prove the following,

Proposition.—Let $\mathscr{S}_* := (\mathscr{F}, \Sigma, V, \mathrm{o})$ be a stratified model for $\mathscr{L}_*$. Then there exists a unique function Σ^*, defined on P, such that for each $p \in P$, $\Sigma(p)$ is a subset of $\Sigma^*(p)$ which consists of sentences of $\mathscr{L}_*^p$ and

1. if ϕ is an atomic formula of $\mathscr{L}_*^p$ and $\phi \notin \Sigma(p)$, then $\phi \notin \Sigma^*(p)$.

{1} The definite article refers to the binary relation $=_{\leq}$.

2. if $\phi \Rightarrow \psi$ is a formula of $\mathscr{L}_*^+$ then $\phi \Rightarrow \psi$ belongs to $\Sigma^*(p)$ iff $\phi \notin \Sigma^*(p)$ or $\psi \in \Sigma^*(p)$ and both ϕ and ψ are sentences of $\mathscr{L}_*^p$.
3. if $(\forall v_i)\phi(v_i)$ is a formula of $\mathscr{L}_*^+$ then $(\forall v_i)\phi(v_i)$ belongs to $\Sigma^*(p)$ iff $\phi(\bar{a})$ belongs to $\Sigma^*(p)$, for every $a \in D(p)$.

Notation.—[2] We write $p \Vdash \phi$ for $\phi \in \Sigma^*(p)$ (read "p *forces* ϕ").

So, we have, for each $p \in P$:

1. $p \Vdash \bar{a}_1 = \bar{a}_2$ iff $V(\bar{a}_1) \leq p$, $V(\bar{a}_2) \leq p$ and $a_1 = a_2$.
2. $p \Vdash \bar{a}_1 \sqsubseteq \bar{a}_2$ iff $V(\bar{a}_1) \leq V(\bar{a}_2) \leq p$.
3. $p \nVdash \bot$.
4. $p \Vdash \phi \Rightarrow \psi$ iff $p \nVdash \phi$ or $p \Vdash \psi$, for all sentences ϕ and ψ of $\mathscr{L}_*^p$.
5. $p \Vdash (\forall v_i)\phi(v_i)$ iff $p \Vdash \phi(\bar{a})$, for every $a \in D(p)$.

As a direct consequence of the proposition above we can derive a few more properties of the forcing relation:

6. $p \Vdash \neg\phi$ iff $p \nVdash \phi$.
7. $p \Vdash \neg\neg\phi$ iff $p \Vdash \phi$, for every sentence ϕ of $\mathscr{L}_*^p$.
8. $p \Vdash \phi \vee \psi$ iff $p \Vdash \phi$ or $p \Vdash \psi$.
9. $p \Vdash \phi \wedge \psi$ iff $p \Vdash \phi$ and $p \Vdash \psi$, for all sentences ϕ and ψ of $\mathscr{L}_*^p$.
10. $p \Vdash (\exists v_i)\phi(v_i)$ iff there is an $a \in D(p)$ such that $p \Vdash \phi(\bar{a})$.
11. $V(\bar{a}_1) \Vdash \bar{a}_1 \sqsubseteq \bar{a}_2$ iff $V(\bar{a}_1) =_{\leq} V(\bar{a}_2)$.
11. $V(\bar{a}_2) \Vdash \bar{a}_1 \sqsubseteq \bar{a}_2$ iff $V(\bar{a}_1) \leq V(\bar{a}_2)$.
12. $p \Vdash \bar{a}_1 \sqsubset \bar{a}_2$ iff $V(\bar{a}_1) < V(\bar{a}_2) \leq p$.

Definition.—If $\mathscr{S}_* := (\mathscr{F}, \Sigma, V, o)$ is a stratified model for $\mathscr{L}_*$ and ϕ is a formula of $\mathscr{L}_*$,[3] we define: $p \Vdash \phi$ iff $p \Vdash \mathrm{cl}(\phi)$, where $\mathrm{cl}(\phi)$ is the *universal closure* of ϕ .

Proposition.—Let $\mathscr{S}_* := (\mathscr{F}, \Sigma, V, o)$ be a stratified model for $\mathscr{L}_*$. Then, if v_i and v_j are different variables of $\mathscr{L}$:

1. $p \Vdash v_i \sqsubseteq v_i$, for each $p \in P$.
2. $p \Vdash v_i \sqsubseteq v_j$ iff $V(\bar{a}) =_{\leq} V(\bar{b})$, for every $a, b \in D(p)$.
3. $p \Vdash v_i \sqsubseteq \mathbf{0}$ iff $V(\bar{a}) =_{\leq} \mathbb{0}$, for every $a \in D(p)$.
4. $p \Vdash \mathbf{0} \sqsubseteq v_i$, for each $p \in P$.

In certain circumstances truth is preserved when moving to an upper strata. The next definition isolates classes of formulae for which this is indeed the case.

Let $\mathscr{F} := (P, \leq, D, \mathbb{0})$ be a stratifying frame. The classes of the *elementary progressive* and the *elementary regressive* sentences of $\mathscr{L}_*^+$, denoted, respectively, by $\mathrm{Prg}_0(\mathscr{L}_*^+)$ and $\mathrm{Rgr}_0(\mathscr{L}_*^+)$, are defined inductively as follows:

P1 If ϕ is an atomic formula of $\mathscr{L}_*^+$, then ϕ is elementary progressive.

P2 If ϕ_1 and ϕ_2 are elementary progressive, then $\phi_1 \wedge \phi_2$ and $\phi_1 \vee \phi_2$ are elementary progressive.

R1 $\bot$ is elementary regressive and if ϕ is an atomic sentence of $\mathscr{L}_*^+$, different from $\bot$, then $\neg\phi$ is elementary regressive.

R2 If ϕ_1 and ϕ_2 are elementary regressive, then $\phi_1 \wedge \phi_2$ and $\phi_1 \vee \phi_2$ are elementary regressive.

PR If ϕ_1 is elementary progressive and ϕ_2 is elementary regressive, then $\phi_1 \Rightarrow \phi_2$ is elementary regressive.

RP If ϕ_1 is elementary regressive and ϕ_2 is elementary progressive, then $\phi_1 \Rightarrow \phi_2$ is elementary progressive.

We may now define the classes of the *extended elementary progressive* and the *extended elementary regressive* sentences of $\mathscr{L}_*^+$, denoted, respectively, by $\mathrm{Prg}(\mathscr{L}_*^+)$ and $\mathrm{Rgr}(\mathscr{L}_*^+)$:

Pi If ϕ is elementary progressive, then ϕ is extended elementary progressive.

Pii If $\phi(v_i)$ is a formula of $\mathscr{L}_*^+$ such that for each $p \in P$ and $a \in D(p)$, the formula $\phi(a)$ is elementary progressive, then $(\exists v_i)\phi(v_i)$ is extended elementary progressive.

Ri If ϕ is elementary regressive, then it is extended elementary regressive.

Rii If $\phi(v_i)$ is a formula of $\mathscr{L}_*^+$ such that for each $p \in P$ and $a \in D(p)$, the formula $\phi(\bar{a})$ is elementary regressive, then $(\forall v_i)\phi(v_i)$ is extended elementary regressive.

For these classes of sentences the weak monotonicity of the forcing relation holds, i.e. if $\mathscr{S}_* = (\mathscr{F}, \Sigma, V, o)$ is a stratified model for $\mathscr{L}_*$, then $p \leq q$ implies that if $p \Vdash \phi$,

{2} For a *modal view* of *forcing*, see [6].
{3} Every sentence of $\mathscr{L}_*$ is also a sentence of $\mathscr{L}_*^p$ and every formula of $\mathscr{L}_*$ is also a formula of $\mathscr{L}_*^p$.

$$(\mathrm{PLR}_1)\frac{}{v_i \sqsubseteq v_i} \qquad (\mathrm{PLR}_2)\frac{}{v_i \sqsubseteq v_j \wedge v_j \sqsubseteq v_k \Rightarrow v_i \sqsubseteq v_k} \qquad (\mathrm{PLR}_3)\frac{}{v_i \sqsubseteq v_j \vee v_j \sqsubseteq v_i} \qquad (\mathrm{PLR}_4)\frac{}{\mathbf{0} \sqsubseteq v_i}$$

Figure 1. Precedence of Level Rules

then $q \Vdash \phi$ for every extended elementary progressive sentence ϕ of $\mathscr{L}_*^p$ and $p \leq q$ implies that if $q \Vdash \phi$, then $p \Vdash \phi$, for every extended elementary regressive sentence ϕ of $\mathscr{L}_*^p$.

The following notions are in part borrowed, in part adapted from first order-logic:

1. If ϕ is a sentence of $\mathscr{L}_*$ and $\mathscr{S}_* := (\mathscr{F}, \Sigma, V, 0)$ is a stratified model for $\mathscr{L}_*$ we define $\mathscr{S}_* \Vdash \phi$ (read "$\mathscr{S}_*$ *forces* ϕ" or "$\mathscr{S}_*$ *is a stratified model of* ϕ") as: $\mathscr{S}_* \Vdash \phi$ iff $p \Vdash \phi$, for every $p \in P$.

   We also define $\Vdash \phi$ (read "ϕ *is universally valid*" or "ϕ *is valid*") as: $\Vdash \phi$ iff $\mathscr{S}_* \Vdash \phi$, for every stratified model $\mathscr{S}_*$ for $\mathscr{L}_*$.

   If Δ is a set of sentences of $\mathscr{L}_*$, we define $\mathscr{S}_* \Vdash \Delta$ (read "$\mathscr{S}_*$ *forces* Δ" or "$\mathscr{S}_*$ *is a stratified model of* Δ") as: $\mathscr{S}_* \Vdash \Delta$ iff $\mathscr{S}_* \Vdash \phi$, for every $\phi \in \Delta$.

2. If $\Gamma \cup \{\phi\}$ is a set of sentences of $\mathscr{L}_*$, we define $\Gamma \Vdash \phi$ (read "ϕ is a stratified logical consequence of Γ") as: $\Gamma \Vdash \phi$ iff for every stratified model $\mathscr{S}_*$, if $\mathscr{S}_* \Vdash \Gamma$, then $\mathscr{S}_* \Vdash \phi$.

   If $\Gamma \cup \{\phi\}$ is a set of fomulas of $\mathscr{L}_*$ and the free variabes of the formulas in $\Gamma \cup \{\phi\}$ are among $v_{i_1}, \dots, v_{i_n}$, we define $\Gamma \Vdash \phi$ (read "ϕ *is a stratified logical consequence of* Γ") as: $\Gamma \Vdash \phi$ iff for every stratified model $\mathscr{S}_*$, for every $p \in P$ and for every $a_1, \dots, a_n \in D(p)$, if $p \Vdash \Gamma(\bar{a}_1, \dots, \bar{a}_n)$, then $p \Vdash \phi(\bar{a}_1, \dots, \bar{a}_n)$; where $p \Vdash \Gamma(\bar{a}_1, \dots, \bar{a}_n)$ abbreviates: "$p \Vdash \psi(\bar{a}_1, \dots, \bar{a}_n)$, for every $\psi \in \Gamma$.

Soundness, Conservativeness and Completeness

Along the two preceding sections we described models (and the corresponding semantics) for stratified first-order logic. But a logical system is not completely described before we introduce a notion of formal proof—a syntactical device conceived to capture truth. Soundness and completeness in a certain extent measure the adequacy of this formal device to its purpose. In this case the notion of proof generalizes the usual one in the case of first-order logic with equality (see [2]). We will skip the details here being enough to know that a formal proof of a formula ϕ is a tree, each node of it is a formula obtained from nodes that are immediate successors of it by applying some basic rule of inference, the bottom node of that tree being the formula ϕ. As a matter of fact proofs are well-founded trees, a fact that allows a form of induction on "proof complexity".

As we said before our basic rules of inference are those of first-order logic and four more rules that deal with the novelty relatively to first-order logic—the binary *precedence of level* predicate (see Figure 1). We present them here just for the sake of completeness. The non-specialist can safely ignore them since the understanding of their meaning is of no importance for the sequel.

The soundness theorem establishes precisely the fact that if a formula ϕ can be formaly proved or derived from hypothesis on a set of $\mathscr{L}_*$-formulas Γ, then ϕ is true in every stratified model of Γ. Using the notation $\Gamma \vdash \phi$ to indicate the fact that there is a derivation of ϕ with hypothesis in Γ the soundness theorem is usually restated as

$$\text{if } \Gamma \vdash \phi \text{ then } \Gamma \Vdash \phi.$$

(The soundness theorem can be proved using induction on the derivation of ϕ from Γ.)

It is typical of mathematical reasoning to adopt different frameworks to represent the same objects just for the sake of making these objects more understandable or making easier to establish relations between them. It is well known that Hilbert tought that this was the case of the use of infinitary notions. Hilbert was correct only to a certain extent. But this attitude revealed fruitful in fields of mathematics such as non-standard analysis or more generaly in the field of mathematical logic via non-standard models.

The stratified first-order logic which we have been describing is relatively to first-order logic in this exact relation. In fact we can prove a result of conservativeness, more precisely: if $\Gamma \cup \{\phi\}$ are formulas of $\mathscr{L}$, then $\Gamma \vdash \phi$ in the context of first-order logic iff $\Gamma \vdash \phi$ in the context of stratified first-order logic.

The semantic analog of this relation (the semantic extension property) can be obtained from this, the completeness theorem for first-order logic and stratified soundness. The semantic extension property establishes that if ϕ is a logical consequence of Γ, then it is a stratified consequence of Γ. In fact, by completeness of first-order logic if ϕ is a logical consequence of Γ then there is a proof of ϕ from Γ. By conservativeness there is also a stratified proof of ϕ from Γ. And using soundness, we can conclude that ϕ is a stratified consequence of Γ.

Definition.—We denote by $\mathscr{P}_c(\mathsf{Sent}(\mathscr{L}_*))$ the set of all $\Gamma \subseteq \mathsf{Sent}(\mathscr{L}_*)$ such that whenever Γ as a first-order model, then Γ has a stratified model.

Using the previous definition and the completeness theorem for first-order logic we can easily prove the following result.

Theorem [Completeness].—If Γ is a consistent{4} subset of $\mathsf{Sent}(\mathscr{L}_*)$ and $\Gamma \cup \{\neg\phi\} \in \mathscr{P}_c(\mathsf{Sent}(\mathscr{L}_*))$, then: if ϕ is a stratified consequence of Γ, then there is a proof of ϕ with hypothesis in Γ.

Conclusion

The *stratification* presented in this work may be applied to any theory (in the usual, informal sense, of this word) formalizable in a first-order language like $\mathscr{L}$. So, we may stratify ZFC{5} or even such theories as *Nelson internal set theory*, IST, or *Hrbăcek set theory*, HST, that are largely used in *nonstandard analysis* (see[5]).

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{4} Here as elsewhere in this work, "consistent" has the usual meaning in first-order logic.

{5} For a different approach, see [3], [4].

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The Planet Earth System, a challenge to mathematicians

by **José Francisco Rodrigues***

In 2000 the World Mathematical Year offered an occasion for a collective reflection on the great challenges of the 21st Century, on the role of Mathematics as a key for the development and on the importance of the image of Mathematics in the public understanding. The countless repeated phrase "*the Universe is written in mathematics language*", written by Galileu in 1614, is truer than ever but it raises new challenges in the current age of data-intensive science driven, in particular, by the information and communication technologies. The "rising tide of scientific data" created by the digital revolution provides new possibilities to face some of society's great challenges of energy and water supply, global warming and healthcare.

During the last centuries, Mathematics has developed a "universal method for the study of the systems". In particular, for the Planet Earth System, Jacques-Louis Lions synthesized in his book "*El planeta Tierra. El papel de las matemáticas y de los superordenadores*" (Madrid, 1990) that universal method in three parts: the mathematical modelling; the analysis and the simulation; and the control of the systems.

In 2007 a scientific workshop on "Climate Change: From Global Models to Local Action", organized by the Mathematical Sciences Research Institute, in Berkeley, identified several mathematical research topics that might contribute to resolving problems whose solutions have a large societal impact. From high dimensional systems to model reduction, from multiscale computations to data assimilation, from uncertainty quantification to economics and societal aspects, the areas of mathematics that might have a significant role in those problems vary from dynamical systems and nonlinear differential equations to asymptotic and numerical analysis, from computational science to statistics and operations research, or from stochastic processes to game and control theories.

During the 2010 International Congress of Mathematicians, held in Hyderabad, India, at the meeting of delegates of the International Mathematical Sciences Institutes, Christiane Rousseau has presented an invitation to Institutes and Societies in Mathematical Sciences around the World: *Mathematics of the Planet Earth — 2013* www.mpe2013.org. This initiative, first launched in USA and Canada, has now many partners in Europe and around the world and consists of holding a year of activities in 2013 under that theme. The project is to hold scientific activities, research programmes and activities for the public, the media and the schools.

Some institutions already announced activities or made calls for proposals. For instance, the *Centre de Recerca Matemàtica*, Barcelona, will be organizing in the summer 2012 a special activity entitled "The Mathematics of Biodiversity". The *Centre de recherches mathématiques*, Montréal, is considering to organize a thematic semester on "Biodiversity and Sustainable Development", during the fall of 2013, and to partner with other Canadian institutes in the organization of a program on "Models and Methods in Ecology, Epidemiology and Public health". The Portuguese *Centro Internacional de Matemática* (CIM), in collaboration with two research associates of the University of Lisbon, the *CMAF* and the *Instituto Dom Luiz*, has made a first public session the 6th May 2011 to present the MPE2013 initiative and to call for collaboration and initiatives in Portugal under the theme Mathematics of the Planet Earth. In addition, several mathematical societies, including the European Mathematical Society and the Portuguese Mathematical Society are also planning to participate with initiatives related to that theme.

On the other hand, it has been suggested for the year 2013 the organization of *A Global Exhibition on Mathematics of Planet Earth* of a new type. The proposal is to have an Open Source Exhibition with modules that could be reproduced and utilized by many users around the world from science centres and museums to schools. The realization will not be centralized. It will rather be split among many partners around the world, possibly with collaborative networks of participants. Some coordination by an international committee on exhibits and museums associated to the MPE2013 initiative is under preparation. The exhibition will have a virtual part, as well as several material parts. Copies of the material parts could be recreated or travel around the world and the virtual modules could be available on the basis of creative

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commons licenses. If possible, a global opening coordinated at the same day in all countries could be planned in order to amplify the visibility of the mathematics of the planet Earth. The idea was presented by the CIM director at the annual ERCOM meeting held the 9th April 2011 in the Mathematical Institute of Oberwolfach, and is currently under development.

The Mathematics of the Planet Earth – 2013 initiative will be another great occasion for showing the essential relevance of mathematics in planetary issues at research level for resolving some of the greatest challenges of the 21st Century, as well as at the level of raising the public awareness of mathematics and at the educational and cultural level.

Four themes with potential examples of modules for a virtual exhibition on the "Mathematics of Planet Earth"

by **Christiane Rousseau***

A planet to discover: *oceans; meteorology and climate; mantle processes, natural resources; celestial mechanics; cartography*.

1.—Crystallography: The crystallographic groups describe the different possible symmetries of the tilings of 3-dimensional space which are invariant under three independent translations. There are applications in the structure of crystals inside rocks. More generally, in chemistry, crystallography is the science of the arrangements of atoms inside a solid. Some arrangements are denser than others and the density of the packings is related to the chemical properties of the chemical elements. On the mathematical side, density of packings is linked with Kepler conjecture on the densest packing of spheres. Different densities can be studied: the densest one, the random density when spheres are packed at random. The same questions can be asked for objects with other shapes than spheres. More recently, mineralogical finding offered evidence that quasicrystals might form naturally under suitable geological conditions.

2.—Fractals provide models for the shapes of nature: Rocky coasts, ferns, the networks of brooks and rivers, for instance deltas. The fractal dimension is a measure of the "density" of a fractal which allows to compare the density of different fractals.

3. The movements of the Earth and the planets in the Solar system: The inner planets (Mercury, Venus, the Earth and Mars) have chaotic motions. Simulations show a 1% chance that Mercury be destabilized and encounters a collision with the Sun or Venus. There is a much smaller chance that all the inner planets be destabilized and that there could be a collision between the Earth and either Venus or Mars in ~3.3Gyr.

4.—The role of the Moon to stabilize the axis of the Earth. If we remove the Moon, then simulations show that the Earth's axis would undergo large oscillations and we would not experience the climates that we now have. In the same spirit there are recent studies making the link between the changes of the parameters of the Earth: angle of the axis, eccentricity of the orbit, etc. and the past climates of the Earth (glaciations periods).

5.—Why the seasons? Why the length of the day is different at different dates, depending of the latitude? Theses themes are very standard. But, in many countries, they disappear from basic science education and needs to be taught independently.

6.—The eclipses. Two types of eclipses: Sun eclipses of Moon eclipses; Explanation of the phenomenon; Previsions of the eclipses.

7.—Weather previsions: The use of models; The butterfly effect.

8.—Remote sensing for exploring the Earth. It could be the use of aerial photographs to discover resources or the use of seismic waves to discover resources in the underground.

9.—Localizing events: Earthquakes, thunderstorms, etc. This is done through triangulation when several distant stations note the time when they register the event.

10.—The Global Positioning System (GPS).

11.—Elements of cartography. It is not possible to draw a map of the Earth respecting ratios of distances.

12.—The use of tools in geography to measure the Earth: How to measure the height of a mountain? How to draw maps of a region?

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13. — Geometric grids on the Earth to make numerical computations. When making computations on the surface of the Earth, it is natural to divide the surface in small surface elements. If these are determined by small increases in longitude and latitude, then there are singularities at the poles. Geodesic grids could be more convenient for such calculations. They are linked to polyhedra inscribed in the sphere.

14. — Movement of tectonic plates, continental drifts, rifts. Mathematics studies the dynamics of the planet mantle as an application to geosciences.

A planet supporting life: *ecology, biodiversity, evolution.*

15. — The phylogenetic tree of life. In biology, phylogenetics is the study of evolutionary relatedness among species. A phylogenetic tree or evolutionary tree is a branching diagram or tree showing the inferred evolutionary relationships among various biological species or other entities based upon similarities and differences in their physical and/or genetic characteristics. Computational phylogenetics is concerned by applying algorithms to assemble a phylogenetic tree representing a hypothesis about the evolutionary ancestry of a set of genes or species. It is a tool for taxonomy which is the science of classifying organisms

16. — Population models. Models of epidemics. Invasive species.

A planet organized by humans: *political, economic, social and financial systems; organization of transport and communications networks; management of resources; energy.*

17. — Transport systems: How to organize transport systems in efficient ways.

18. — The web graph: It is a way of connecting the planet together.

19. — Management of resources.

20. — Game theory and applications in economics and in biology. In economics we have Nash equilibrium. In biology, game theory is useful for modeling some interactions of populations.

21. — The economy of solidarity and how to fight poverty around the world: For instance the microcredit.

A planet in the balance: *climate change, sustainable development, epidemics; invasive species, natural disasters.*

22. — Climate models: How to use chaotic weather previsions where no prevision is valid past 14 days to long term climate previsions. One technique is to consider an average of many simulations with close initial conditions.

23. — Hydrographic previsions: What will be the quantity of rain expected to be received on a given region? This is important for agriculture, but also for filling the reservoirs of electric dams.

24. — Evolution of the human population over history. Previsions for the next centuries.

25. — Percolation: At each occupied node of a vertex there is a probability p that an adjacent node becomes occupied. Depending on p, the final configuration can have very different properties. Percolation models are useful to study the diffusion of liquids including pollutants inside soils. They also provide models for the diffusion of epidemics.

26. — The age of the Earth. The first serious attempts were done by Lord Kelvin around 1840. Kelvin used Fourier's law of heat the gradient of temperature measured empirically and some very strong hypotheses simplifying the problem. He gave an interval of 24 to 400 million years. This estimate was in contradiction with the observations of the geologists and it was incompatible with the new theory of evolution of Darwin which required a much older planet. Kelvin has neglected to take into accounts the convection movements inside the Earth which slow down considerably the cooling of the mantle.

27. The rising of the sea level with the melting of the glaciers.