INTERNATIONAL CENTER FOR MATHEMATICS

#34

Bulletin

JULY 2013

Editorial

As the International Center for Mathematics (CIM) celebrates its twentieth anniversary this month, it is also the perfect opportunity to look back on this past year, which has undoubtedly been one of the most ambitious, eventful, and impactful years in the association's history. With the support of our associates from 13 leading Portuguese universities, our far-flung partners at the University of Macau, and member institutions such as the Portuguese Mathematical Society, CIM in 2013 showed yet again the importance of a forum such as this to bring together leading Portuguese-speaking scientists and researchers around the world.

The hallmark project of the year was the UNESCO-backed International Program Mathematics of Planet Earth (MPE) 2013, which CIM participated in as a partner institution. This ambitious and global program was tasked with exploring the dynamic processes underpinning our planet's climate and man-made societies, and laying the groundwork for the kind of mathematical and interdisciplinary collaborations that will be pivotal to addressing the myriad issues and challenges facing our planet now and in the future. CIM embraced the MPE call to action by organizing two headline conferences in March and September of this year. In the spring CIM held the "Mathematics of Energy and Climate Change" conference in Lisbon, with the autumn conference was titled "Dynamics, Games, and Science." Both were held at the world-renowned Calouste Gulbenkian Foundation in Lisbon, one of more than 15 respected Portuguese foundations and organizations that enthusiastically supported the CIM conferences. As well as the conferences themselves, well attended "advanced schools" were held before and after each gathering: in the spring at the Universidade de Lisboa and in the fall at the Universidade Técnica de Lisboa.

These conferences succeeded in bringing together some of the most accomplished mathematical and scien-

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tific minds from across the Portuguese-speaking world and beyond, while also serving as a launchpad for one of CIM's most exciting endeavors in many years. Just recently the center announced the new CIM Series in Mathematical Sciences, to be published by Springer-Verlag, which will include lecture notes and research monographs. "The collaboration with Springer will bring mathematics developed in Portugal to a global audience," CIM President Alberto Adrego Pinto said at the time of the announcement, "and will help strengthen our contacts with the international mathematics community."

The first volume in the series, consisting of review articles selected from work presented at the "Mathematics of Energy and Climate Change" and "Dynamics, Games, and Science" conferences, already make a powerful case for CIM's international profile and reach. There's the impressive roster of mathematicians and researchers from across the United States, Brazil, Portugal and several other countries whose work will be included in the volumes. Then there's the editorial board responsible for this first installment, a world-renowned quartet comprised of: president of the European Research Council starting on January 1, 2014, Jean Pierre Bourguignon, of École Polytechnique; former Société Mathématiques Suisse and European Mathematical Society president Rolf Jeltsch, of ETH Zurich; current Sociedade Brasileira de Matemática president Marcelo Viana, of Brazil's Instituto Nacional de Matemática Pura e Aplicada; and current CIM president, Alberto Pinto, of Universidade do Porto. This series represents a very real scientific and reputational achievement for the center.

While the MPE program was a key focus of CIM's activities this year, the center did organize a number of other events aimed at fostering closer ties and collaboration between mathematicians and other scientists, principally in Portugal and other Portuguese-speaking countries. In May, CIM held the 92nd European Study Group with Industry meeting, part of a vitally important series held throughout Europe to encourage and strengthen links between mathematics and industry. As the MPE program made clear, humanity faces all manner of challenges, both manmade and from nature, many of which industry is attempting to solve but that mathematics and science are most well-equipped to tackle. Yet it is often industry that spawns the kinds of innovative ideas that will launch the next great scientific and technological revolutions, which academia must engage with. The potential for dialogue and cooperation between academia and industry is, in fact, so great that I have actually made it one of the core initiatives of my presidency of the Society for Industrial and Applied Mathematics (SIAM), based in the United States of America.

The center also put on a number of seminars, summer schools, and workshops on subjects ranging from game theory to nonlinear mapping to stochastic dynamics in finance to representation theory. These were held at many of CIM's associate institutions, including universities in Porto, Évora, Coimbra, Lisbon, Aveiro, and even Macau, the better to support a regular exchange of ideas and build a lasting network among Portuguese mathematicians and researchers.

As we look back at the successful year CIM has had in 2013, we should also think on the dramatic changes taking place in the world at this moment, changes that put the mathematical sciences — and I include here statistics, operational research, and computer science — front and center. Foremost among these is the rise of Big Data, especially where it relates to national security, finance, medicine, and the Internet (among other fields), which has come to dominate research in many scientific sectors and requires new analytical tools that mathematics can provide. This new landscape will require an unparalleled partnership between science and industry, and is why the European Commission recently announced its Europe 2020 Growth Strategy, which calls for investment in groundbreaking research, innovation in industry, and the cultivation of a new generation of scientists. It is no coincidence that these three pillars are at the core of CIM's own mission.

Irene Fonseca
President of CIM Scientific Council

INTERNATIONAL CENTER FOR MATHEMATICS

Coming Events

Coupled cell networks and dynamics

3-5 February 2014

Porto

http://www.fc.up.pt/cmup/coupledcells

Coupled-cell systems are formed by interacting individual dynamical systems (the cells). The network associated with a coupled-cell system codifies information concerning the types of cells and the interactions involved. Coupled-cell networks are a natural tool for the mathematical modeling of a wide range of problems in fields such as biology, physics, economics, and social sciences, among others. For pure and applied scientists it is a challenge to understand the interplay between the structure of the network and the dynamics of coupled-cell systems. Real- world networks pose challenges for mathematicians. In particular, phenomena that would be nongeneric in an arbitrary dynamical system can become generic when constrained by a particular network topology. Also interesting is how much of the dynamics can be derived from the network structure independently of the specific equations chosen to describe specific phenomenon. The workshop will focus on recent theoretical developments on the dynamics of coupled cell systems inspired by real-world applications as well as on the understanding of real- world networks using the dynamics of networks of dynamical systems.

Invited Speakers

Paulo Aguiar. Faculty of Sciences, University of Porto, Porto, Portugal.

Peter Ashwin. College of Engineering, Mathematics and Physical Sciences, University of Exeter, Exeter, UK.

Fathican Atay. Max Planck Institute for Mathematics in the Sciences, Germany.

Michael Field. Department of Mathematics, University of Rice, Houston, USA.

Isabel Labouriau. Faculty of Sciences, University of Porto, Portugal

Maria Conceição Leite, Department of Mathematics and Statistics, College of Natural Sciences and Mathematics, University of Toledo, USA.

Raoul-Martin. Memmesheimer, Neuroinformatics department, Donders Institute, Radboud University Nijmegen, Netherlands.

José Fernando Mendes. Department of Physics, University of Aveiro, Portugal.

Célia Sofia Moreira. Centre of Mathematics, University of Porto, Portugal.

Bob Rink. Department of Mathematics, VU University Amsterdam, The Netherlands.

Francisco C. Santos. Department of Computer Science and Engineering, Instituto Superior Técnico, University of Lisbon, Portugal.

Yunjiao Wang. Department of Mathematical Sciences, University of Houston, USA.

Dynamical Systems Applied to Biology and Natural Sciences (DSABNS Anual Workshop)

10-12 February 2014

University of Lisbon–CMAF, Lisbon

http://ptmat.ptmat.fc.ul.pt/dsabns2014/

The Fifth Workshop DSABNS will be held at the Centro de Matemática e Aplicações Fundamentais (CMAF), Lisbon University, in Portugal, from February 10 to 12, 2014.

The Fifth Workshop Dynamical Systems applied to Biology and Natural Sciences will be held at the Centro de Matemática e Aplicações Fundamentais (CMAF), Lisbon University, in Portugal, from February 10 to 12, 2014. The workshop has both theoretical methods and practical applications and the abstracts included in the program will cover research topics in population dynamics, eco-epidemiology, epidemiology of infectious diseases and molecular and antigenic evolution.

Participants are kindly requested to register by 3 February 2014. If you intend to participate. Looking forward to see you there.

Workshop Organizers

Maíra Aguiar [Lisbon University]
Bob Kooi [Vrije Universiteit Amsterdam
Luis Mateus [Lisbon University]
Filipe Rocha [Lisbon University]
Urszula Skwara [Lisbon University]
Nico Stollenwerk [Lisbon University]
Ezio Venturino [Turin University]

Plenary Talks

Konstantin Blyuss [University of Sussex, UK]
Nick Britton [University of Bath, UK]
Bernard Cazelles [Ecole Normale Supérieure, France]
Alvaro Corral [Universitart Autònoma de Barcelona, Spain]
Laurent Coudeville [Sanofi Pasteur, France]
Bob Kooi [Vrije Universiteit, The Netherlands]
Andrea Parisi [Lisbon University, Portugal]
Andrea Pugliese [Università de Trento, Italy]
Mario Recker [University of Exeter, UK]
José Francisco Rodrigues [Lisbon University, Portugal]
Francisco C. Santos [Lisbon University, Lisbon University, Portugal]
Anavaj Sakuntabhai [Pasteur Institut, France]
Gauthier Sallet [INRIA, France]
Nico Stollenwerk [Lisbon University, Portugal]
Ezio Venturino [Turin University, Italy]

Invited Talks

Isabel Rodríguez Barraquer [Johns Hopkins University, USA]
Carlos Braumman [Évora University, Portugal]
Fabio Chalub [Nova University, Portugal]
Teresa Faria [Lisbon University, Portugal]
José Martins [Polytechnic Institute of Leiria, Portugal]
Maria Teresa T. Monteiro [Minho University, Portugal]
Luis Mier-y-Teran-Romero [Johns Hopkins University, USA]
Paulo Pimenta [Centro de Pesquisas René Rachou, Brazil]
Paula Rodrigues [Nova University, Portugal]
Ana Clara Silva [Instituto de Administração da Saúde e Assuntos Sociais, Portugal]
Max Souza [Fluminense Federal University, Brasil]
Hyun Mo Yang [Campinas University, Brasil]

3rd International Conference on Dynamics, Games and Science

17-22 February 2014

University of Porto — Porto
http://www.fc.up.pt/DGS2014/

Following the 1st and 2nd International Conference Dynamics, Games and Science - DGS I 2008 and DGS II 2013, we invite the Academic Community including MSc and PhD students and researchers to participate and to present their research work. If you would like to present a paper you are working on, please register at

http://www.fc.up.pt/DGS2014/registration.html

The 3rd International Conference Dynamics Games and Science 2014 — DGS III 2014, aims to bring together world top researchers and practitioners from the fields of Dynamical Systems, Game Theory and its applications to such areas as Biology, Economics and Social Sciences.

DGSIII represents an opportunity for MSc and PhD students and researchers to meet other specialists in their fields of knowledge and to discuss and develop new frameworks and ideas to further improve knowledge and science.

Keynote speakers

Albert Fisher [University of São Paulo, Brazil]
Alberto Pinto [University of Porto, Portugal]
Athanassios Yannacopoulos [AUEB, Greece]
Bruno Oliveira [INESC TEC, Portugal]
Carlos Braumann [University of Evora, Portugal]
David Rand [University of Warwick, UK]
David Zilberman [University of California, USA]
Diogo Pinheiro [Brooklyn College, USA]
Elvio Accinelli [UASLP, Mexico]
Filipe Martins [INESC TEC, Portugal]
Frank Riedel [Bielefeld University, Germany]
Gabrielle Demange [EHESS, France]
Jérôme Renault [Université de Toulouse, France]
João Gama [University of Porto, Portugal]
João Paulo Almeida [INESC TEC, Portugal]
José Martins [INESC TEC, Portugal]
Mohamad Choubdar [University of Porto, Portugal]
Nico Stollenwerk [University of Lisbon, Portugal]
Nigel Borroughs [University of Warick, UK]
Onesimo Hernandez-Lerma [IPM, Mexico]
Penelope Hernandez [University of Valencia, Spain]
Rabah Amir [University of Arizona, USA]
Renato Soeiro [University of Porto, Portugal]
Ricardo Cruz [University of Porto, Portugal]
Robert MacKay [University of Warwick, UK]
Rolf Jeltsch [ETH Zurizh, Switzerlan]
Saber Elaydi [Trinity University, USA]
Sebastian van Strien [Imperial College London, UK]
Tenreiro Machado [ISEP, Portugal]

Interview

with Francesco Brenti, Christian Krattenthaler and Vic Reiner

by **Olga Azenhas** [CMUC, University of Coimbra] and
António Guedes de Oliveira [CMUP, University of Porto]

The *Summer School on Algebraic and Enumerative Combinatorics*, sponsored by *Centro Internacional de Matemática* (http://www.cim.pt), took place in July, 2–13, 2012, at the *Centro de Estudos Camilianos*, S. Miguel de Seide, in a building of Álvaro Siza, the 1992 Laureate of the *Pritzker Architecture Prize*. It was also financially supported by the *Fundação para a Ciência e a Tecnologia* (http://www.fct.pt) by the *Centro de Estruturas Lineares e Combinatórias* (Universidade de Lisboa), the *Centro de Matemática da Universidade de Coimbra* (Universidade de Coimbra) and the *Centro de Matemática da Universidade do Porto* (Universidade do Porto).

Together with Marc Noy (Universitat Politécnica de Catalunya), Francesco Brenti (Universitá di Roma *Tor Vergata*), Christian Krattenthaler (Universität Wien) and Vic Reiner (University of Minnesota) were in S. Miguel de Seide for the Summer School, where they lectured courses on Combinatorics of Coxeter Groups, on Map Enumeration, and on Reflection Group counting and q-counting. After the school, they have kindly accepted to answer some questions we posed.

Francesco Brenti

We were pleasantly surprised by the success of the school. What was your own impression? How do your previous summer school experiences compare with this one?

F. Brenti.—The impression was very good. Students were able to solve problems that usually require a few days of thinking. Of course they worked on them all together, which certainly helped, but they were impressive just the same, also considering that many of them, though combinatorialists, came from very different areas of combinatorics, and had never worked on combinatorics of Coxeter grups before. I was also extremely pleased by how effectively the students picked up the basic and fundamental techniques that are used for research in combinatorics of Coxeter groups.

Concerning comparison to other summer schools, from the point of view of the lectures there was not much difference, but there was a huge difference for what concerns the exercises. In Luminy, for example, we were asked to assign "homework" problems to the students, and the students would work on them in the afternoons, in groups, with the teacher walking around among them and being available for questions and explanations. Then, after the afternoon coffee break, we would all gather in the lecture room and the students (or, if no one had solved the problem, the professor) would explain a solution to the others. In Guimarãaes we were given complete freedom on how to use the recitation time, so I used it as I usually do in my own courses, namely I propose a problem and then wait for input from the students, trying to follow all the threads of reasoning that they propose. I think this method is more effective for a couple of reasons. First, if the professor explains an exercise then the student thinks: "OK, he's the professor, of course he knows how to do it", but if a fellow student solves the problem then the student thinks "Gee, she could do it, why can't I?". Secondly, if I explain a solution to the problem, it is usually the simplest solution, but the students often find solutions that I would have never thought of, and which often involve many more concepts and theorems than the simplest solution, and are therefore more effective in making them learn the material, besides, false starts and mistakes are also instructive. Whenever you use this system of doing exercises, there is always a fear that we might all be staring at each other for an hour, but this never happened to me, and it did not happen also this time, in fact, quite the opposite (once, at some point one of the organizers came up to me saying that we should really all go to the restaurant for lunch, as they were waiting for us!). I like this way of doing exercises because the performance of students at exams showed to me beyond any doubt its effectiveness.

C. Krattenthaler.—I have participated now in several summer schools for PhD students and postdocs as a lecturer. This is always a pleasure — and has also been so this time — since one talks to young, motivated people who want to learn something from you and therefore are extremely interested in your lectures: they are open to absorb material which is absolutely new to them (whether they always assimilate this with ease, this is a different matter …), and they are willing to put significant effort to master the material taught to them, with the motivation that, in order to become — and be (!) — a true researcher, one has to constantly enlarge one's own expertise and perspective. If I am to compare this experience with previous ones, then I would say that the level of enthusiasm and commitment of the young people in São Miguel de Seide has been the same as at previous schools, as it should be!

V. Reiner.—I was very pleased with the willingness of the students to ask questions during lectures, and to really grapple with exercises during the problem sessions. Perhaps I shouldn't have been surprised. My one previous experience as a summer school lecturer was at an ACE (Algebraic Combinatorics in Europe) Summer School in Vienna 2005, run by Christian Krattenthaler. Looking back, the two summer school experiences were quite similar. Except in 2005, I seem to remember eating more *schnitzel*, less *bacalhau*.

How important do you think that summer schools like this one, in S. Miguel de Seide, are for students wishing to work in mathematics and in combinatorics, in particular?

F. Brenti.—I think that they are very important, in fact almost essential, because Ph.D. programs in mathematics in Europe (except

at the very top schools) are not as comprehensive as they are in the U.S. So, it is common for a Ph.D. student in Europe to be able to take only one graduate course in combinatorics, even if this is the field in which he/she wants to work in, as opposed to the 3 or 4 that would be available in the best U.S. schools. This is certainly true in Italy.

C. Krattenthaler—I have already mentioned the important point in my answer to the first question: in order to become — and be — a true researcher, one has to constantly enlarge one's own expertise and perspective. Consequently it is particularly important for young people to attend such schools, where they are together with international senior and young scientists, where they are offered instruction in material or views which may not be presented at their home institution, and where they can profit from the expertise of the other participants of the school. Moreover, this is also an ideal place for building up scientific (and non-scientific …) contacts and collaborations.

V. Reiner.—They are very important, as an easier path into topics that might otherwise seem mysterious and forbidding. In addition, I think they give an invaluable opportunity to meet other students, postdocs, and faculty in combinatorics, in a setting that is closer and friendlier than a typical conference.

How did you start working in combinatorics? Could you tell us briefly about your mathematical beginnings, and subsequent career development? Who (or which event) influenced you most?

F. Brenti.—I've been reading math books ever since I was 12. At age 16 I stumbled upon a book that was a collection of essays, each one about some areas of modern mathematics, that had been translated into Italian by U.M.I. (the Italian Mathematical Society). I read essentially all of them but the one that definitely fascinated me the most was the one written by G. C. Rota (that was the first time I had ever heard this name) about combinatorics. The simplicity of the problems discussed and yet at the same time their extreme difficulty fascinated me. But life is often different from how we imagine it so when I was a graduate student working under the direction of Rota I did not like the mathematics that he was doing at the time, and I remember that I was studying Richard Stanley's papers in my spare time (!). After about a year of this, I decided that it couldn't go on, and I switched to Stanley. I have never regretted this. After M.I.T. I was a Hildebrandt Assistant Professor at the University of Michigan in Ann Arbor for 3 years and then I was a member of the Institut Mittag-Leffler in Sweden for a year. That was when I started working in combinatorics of Coxeter groups. I had no teaching duties in Sweden, so I had a lot of free time, and one day in the new books section of the Library I found this book written by J. E. Humphreys entitled "Reflection groups and Coxeter groups". It was so well written that I started reading it and eventually ended up working in the subject. After Sweden I came back to Italy, where I have remained except for several years of leave spent in various places.

C. Krattenthaler.—How did I start to work in combinatorics? This began at the age of 14 or so, when I became interested in figuring out what the probability was that, if you threw — say — n dice, the sum of the scores added up to — say — S. I remember that I computed long tables in small cases (no computers yet!), discovered partial results, then, at some point, learned about factorials and binomial coefficients (which was really helpful …), and in the end (I believe roughly two years later) I figured out the formula which (now I know that!) one can easily get by inclusion-exclusion. However, at the time, I had no clue how to prove my formula …

Around that time, one of the mathematics teachers at our school started to give voluntary mathematics sessions, in which I participated, where he introduced us to material which was not covered by the standard school *curriculum*. This teacher could have had an academic career (but probably his parents did not allow him to pursue such a career …), but he was entirely inappropriate as a school teacher. The latter fact was no problem during these mathematics sessions, since the participants wanted to be there and to learn something (and we could learn a lot from him). But when he had to deal with a crowd of pupils in an ordinary class, then

Vic Reiner

things were entirely different: his strongest "weapon" against the noise coming from the various not so interested pupils consisted in standing in front of the classroom, smiling helplessly …

At age 17, these mathematics sessions turned into preparation courses for the mathematics olympiad. In the first year I participated in the "beginner's competition" of the Austrian mathematics olympiad (earning a gold medal), and in the second year in the "competition for the advanced" (earning a silver medal). I thus qualified for the International Mathematics Olympiad (which that year, 1977, took place in Beograd), where I ended up at a place which will not be mentioned explicitly here …

It was then "clear" that, when I would enter university, I would go for mathematics (and piano, as a matter of fact), which I did at the University of Vienna (and the University for Music and Performing Arts in Vienna). (This was less clear for my father, but he did not object …)

In the first year of my mathematics studies, Johann Cigler (who later became my advisor) taught a combinatorics course, which I attended with enthusiasm. I also followed several other courses and seminars given by Cigler. These were always very fascinating since Cigler did not just present the material like that, but instead did it always in original ways, and he would always present us his own thoughts and ideas he had on the subject, including (open) questions for us students. Answers and solutions to Cigler's questions and problems that I found became in the end (more or less) my thesis.

V. Reiner.—I figured out a bit later than many mathematicians that I really liked math. I was supposed to go to medical school! That's the nature of my family background.

In college, I chose a math major as a pre-medical student, and quickly realized two things: (1) Math classes were more interesting, and taught better than at my high school. (2) Math people were — really — smart! I soon realized how frequently dumb I would feel if I were to go into mathematics, but I just started enjoying the material more and more.

Once in math graduate school, I feel lucky to have received excellent advice from my older office-mate, Maciej Zworski (now at UC Berkeley), who recommended Richard Stanley as an advisor. This turned out to be a great choice for me.

Combinatorics, as a systematic study of discrete configurations that encode complex structures and, in particular, the enumeration of objects according to certain restrictions, is now widely recognised as an integral area of pure mathematics. It also has an increasingly important interface with neighbouring areas such as physics, computer science and molecular biology, for example. What is your personal opinion about the impact of combinatorics on these areas, and vice-versa?

F. Brenti.—I think that, as always in mathematics and in science, connections between different areas of mathematics (and of science) are mutually beneficial. Many problems in my current area of research (combinatorics of Coxeter groups) come from algebra, and you would not have considered them if it wasn't for this connection. Similarly for many combinatorial problems that have arisen from research in geometry, physics, or computer science. Regarding the mutual impact of combinatorics and the areas with which it interfaces, I think that in algebra and geometry the influence has been mainly on using results from these areas to solve combinatorial problems but not much on using combinatorial results to solve algebraic and geometric problems. I think this is due to the fact that combinatorics is a much younger subject than both algebra and geometry, and therefore has at its disposal tools that are not yet as advanced as in those areas. I think that this will naturally change as combinatorics matures and discovers deeper and deeper theorems, and that we have already seen some examples of this.

C. Krattenthaler.—Interplay between different disciplines is always extremely fruitful for *all* the involved disciplines, and therefore I am very excited about the interactions between combinatorics and (statistical) physics, computer science and molecular biology. The interplay between combinatorics and computer science is the one which is ongoing longest — here one has to mention the development of efficient algorithms for the solution of combinatorial problems (of daily life), and the analysis of algorithms — and it is manifest in the fact that many researchers in combinatorics are employed in computer science departments. In one part of my research work I am involved in the interplay of combinatorics and statistical physics: one can measure my excitement for this interplay if one knows that, during my studies, I had never entered a physics lecture (because I did not feel that I wanted to do learn more about physics than what I already knew from high school ...). I shall say more on this interplay in my answer to Question 5. The interplay between combinatorics and molecular biology comes from the fact that computers are now strong enough to scan through data read from genomes — and extract useful information! — provided one applies sufficiently efficient algorithms for the extraction of information. Interesting combinatorial problems arise from there, but I am somewhat sceptical how useful the theoretical results which one is able to find on these problems really are for the actual biological questions. My impression is that — at least for now — computer power is more important than theoretical ideas. That may change of course.

V. Reiner.—I don't know about our impact on these other areas, but we in combinatorics owe them a lot. As my friend Mark Shimozono once modestly claimed, "I'm not smart enough to know which combinatorics is going to be interesting on its own — I need algebra as a crutch, to point me toward the right objects to study." What is true for algebra is also true for physics, computer science, biology. I would even include subjects like economics. Look at this year's Nobel Prize in economics, which was awarded essentially for matching theory.

If you had to give a synopsis of the current state-of-the-art in combinatorics, which challenging open or recently solved problems would you choose to mention?

F. Brenti.—Such a list is necessarily biased by my own preferences, taste, and expertise! Regarding recently solved problems I would definitely mention the Strong Perfect Graph Theorem of Chudnovsky, Robertson, Seymour and Thomas, the combinatorial proof of Schur-positivity of Macdonald polynomials by Assaf, the Polya-Schur Master Theorems of Borcea and Branden, and the recently announced nonnegativity of Kazhdan-Lusztig polynomials of Elias and Williamson. Regarding open problems, I would mention the combinatorial invariance conjecture for Kazhdan-Lusztig polynomials by Dyer and Lusztig, the problem of finding a combinatorial interpretation for Kazhdan-Lusztig polynomials, and the conjectured nonnegativity of the complete **cd**-index of a balanced digraph by Ehrenborg and Readdy (even just in the special case of Bruhat graphs).

C. Krattenthaler.—This is an impossible task because "Combinatorics" is a vast subject whose branches include Enumerative Combinatorics, Analytic Combinatorics, Algebraic Combinatorics, Probabilistic Combinatorics, Geometric Combinatorics, Algorithmic Combinatorics, Design Theory, Graph Theory, Extremal Combinatorics, Combinatorial Optimization (and I am sure that I forgot something). There have been exciting developments in practically in all of these branches during the past years.

So let me content myself with a corresponding commentary relating to the topic of my course, "Map Enumeration". An exciting development which has taken place in this part of combinatorics (and also related parts) is the growing interaction between (enumerative) combinatorialists, probabilists, and statistical physicists. Over a long time, these three communities worked by themselves and sometimes in parallel, (re)discovering the same things without knowing that these were also considered by the other communities (not to mention which results were known by these other communities). This has changed dramatically during the past 15 years. Researchers in these three communities have understood that the developments in the other communities are also relevant for themselves. Many interactions have now taken place, with physicists becoming interested in purely combinatorial problems, combinatorialists becoming interested in problems of probability and physics, etc. Among the recent culmination points one has to mention the solution of several notorious enumeration problems on alternating sign matrices by methods coming from statistical physics, the beautiful and deep asymptotic theory for the behaviour of large tilings (actually: perfect matchings) which contributed to the award of a Fields medal, and the proof of the so-called Razumov-Stroganov conjecture on the ground state of a certain Hamiltonian by purely combinatorial methods (carried out by two physicists).

V. Reiner.—My absolutely favorite open problem currently is this: Understand the relation between the notions of noncrossing partitions and nonnesting partitions for finite reflection groups and Weyl groups, and in particular, why they obey the same beautiful enumeration formula.

In this summer school, we have seen a number of different cultures represented within the field of combinatorics, each with its own particular set of tools. How would you describe the essence of your own research to a young student in search of a research topic?

F. Brenti.—The essence of my own research is to study combinatorial problems that arise from algebra and to use algebraic techniques to solve combinatorial problems.

C. Krattenthaler.—The search of an appropriate research topic is an extremely delicate question (in particular for advisors of PhD students ...). If I am asked how I proceeded to find "my" problems then the answer is that the only criterion has always been the appeal that a problem had to me, and I did not care whether this was an "important" problem or not. Whether this is a good (or sufficient) advice for young people, I cannot tell. I believe that it is if it is interpreted in the following way: it is obvious that it makes no sense to work on problems which one actually does not like, only for the sake that somebody said that this is an "important" problem (where the notion of "importance" would require a separate discussion). A problem should somehow appeal to somebody, otherwise one will not feel the motivation to solve it. However, whether a problem appeals to somebody or not should also be implicitly guided by this question of "importance". This is something which should go together automatically: appeal and "importance" or "interest". There is no way to put this in any framework of rules: we all know that problems which have been regarded as important at some point were not regarded so at a later point, and that there were problems which were not looked at by almost anybody at some point but which turned out to be of crucial importance at a later point. One has to develop one's own feeling for this by listening to others (at such summer schools, for example) and to one's own inner voice.

V. Reiner.—I like to understand how classical theory of reflection groups and invariant theory points us toward beautiful objects to count, and how to insightfully add the q's when we q-count them.

Acknowledgment.—Many thanks to Ethan Cotterill whose remarks greatly improved the English of our initial version for the interview text.

92nd European Study Group with Industry

by Deolinda M. L. D. Rasteiro*

The 92nd European Study Group with Industry took place from 6 to May 10, 2013 at ISEC, the Coimbra Engineering Institute of Coimbra's Polytechnic, organized by the Department of Mathematics and Physics, ISEC-DMF (http://dfm.isec.pt/esgi92/) and LCM – Laboratory for Computational Mathematics (http://www.uc.pt/uid/lcm) of Centre for Mathematics of the University of Coimbra (https://cmuc.mat.uc.pt/rdonweb/).

The meeting has counted with the participation of several experts with a large experience in this type of events. By the 7th consecutive year, Portuguese researchers and academics tried to strength the links between Mathematics and Industry by using Mathematics to tackle industrial problems that were proposed by industrial partners (see: http://www.ciul.ul.pt/~freitas/esgip.html).
For the participants, these problems were mathematically interesting challenges. For the companies, those were open-problems that had not yet been solved with their own (and/or consulting) resources.

The set of problems proposed had a wide variety of features due to theirs origin and also due to the extent of application.

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In this edition there were selected 4 problems proposed by different companies namely, Critical Software (http://www.criticalsoftware.com), TULAES (http://www.tulait.eu/Tulaes/servi%C3%A7os.html), Active Space Technologies (http://www.activespacetech.com/EN/home.htm) and Sonae MC (http://www.hipersuper.pt/tag/sonae-mc/). The problems that were proposed were: Model to Estimate and Monitor the progress of System Testing Phase, Customer's Expected

Energy Consumption, Modelling percolation and fractal structure in aerogels, Picking optimization respectively. The mathematical subjects used during the event comprehended probability theory, statistics, classification, optimization, numerical analysis or partial differential equations, just to name a few.

In this year's Portuguese ESGI, especially due to the current economic situation, the results overwhelmed the organizers (and the companies') best expectations. For the organizers, some of them involved since 2007 when the first Portuguese ESGI edition took place, the objective is to spread mathematical knowledge and use it to help the industrial tissue. According to them, the success of ESGI's in Portugal may be measured by the growing number of participants, proposed problems, and by the fact that some companies are submitting new problems after their first participation. The comments from the companies' representatives were very positive.

Graphs of polyhedra and the theorem of Steinitz

by António Guedes de Oliveira*

The theorem of Steinitz characterizes in simple terms the graphs of the polyhedra. In fact, the characteristic properties of such graphs, according to the theorem, are not only simple but "very natural", in that they occur in various different contexts. As a consequence, for example, polyhedra and typical polyhedral constructions can be used for finding rectangles that can be decomposed in non-congruent squares (see Figure 1). The extraordinary theorem behind this relation is due to Steinitz and is the main topic of the present paper.

Steinitz's theorem was first published in a scientific encyclopaedia, in 1922 [16], and later, in 1934, in a book [17], after Steinitz's death. It was ignored for a long time, but after "being discovered" its importance never ceased to increase and it is the starting point for active research even to our days. In the middle of the last century, two very important books were published in Polytope Theory. The first one, by Alexander D. Alexandrov, which was published in Russian in 1950 and in German, under the title "Konvexe Polyeder" [1], in 1958, *does not* mention this theorem. The second one, by Branko Grünbaum, "Convex Polytopes", published for the first time in 1967 (and dedicated exactly to the "memory of the extraordinary geometer Ernst Steinitz"), considers this theorem as the "most important and the deepest of the known results about polyhedra" [9, p. 235].

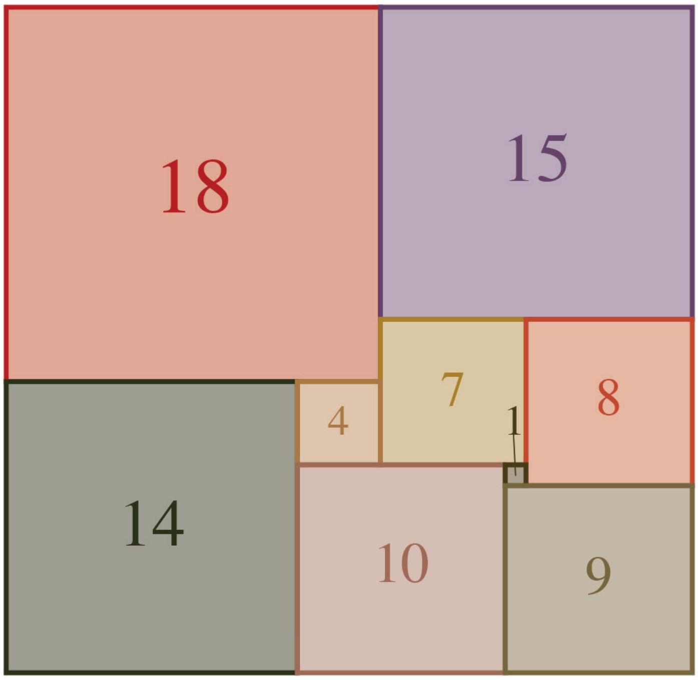

Figure 1.— Decomposition of a rectangle in non-congruent squares

Polyhedra are polytopes of dimension 3 and polygons are polytopes of dimension 2; a polytope of dimension 1 is an edge and a polytope of dimension 0 is a vertex. Every polytope of dimension greater than 1 has a related graph, formed by its edges and vertices. Yet, for dimensions greater than 3, we do not know which graphs arise, and which do not, as the graph of a polytope. The quest for properties that would characterize these graphs, in the line of Steinitz's theorem but for any dimension greater than 3, has been in fact a long and important line of research.

This paper was written as an invitation: we invite the reader, a student, perhaps, to visit an old but very vivid area of research, of which we are happy to present some glimpses, including of a few personal contributions.

* CMUP and University of Porto [The drawings were made with Mathematica™ by the author]

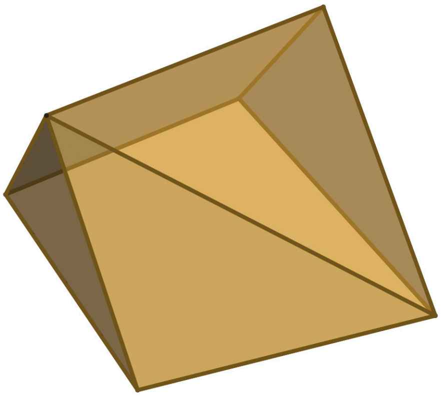

Figure 2.—A polyhedron — $\mathcal{P}$

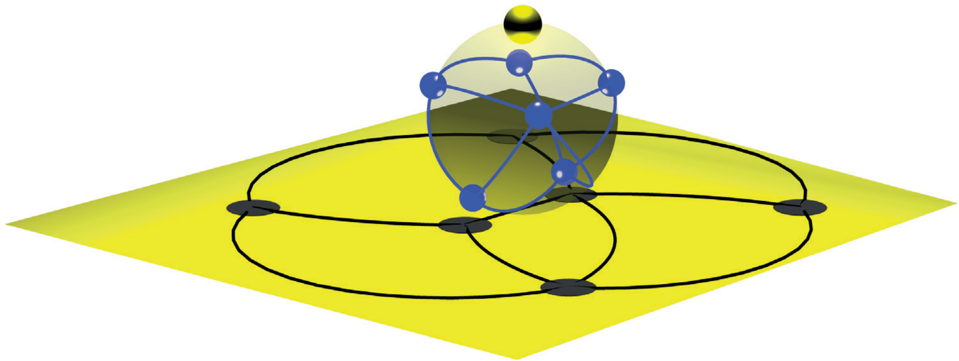

Figure 3.—Planarity of the graph of a polyhedron ($\mathcal{P}$)

Before stating and commenting this theorem, let us introduce some basic notions that are perhaps not familiar to the reader.

Generalities

Given a non-empty finite set V and given a set

$$E \subseteq \left\{\{u,v\} \subseteq V \mid u \neq v\right\},$$

we say that (V,E) is a *graph*, the elements of V being the *vertices* of G and the elements of E the *edges* of G. We write $e = uv$ for $e = \{u,v\}$ and call u and v the *vertices incident with* e. If we are given a set E' disjoint from V and an injective function $\varphi : E' \to \left\{\{u,v\} \subseteq V \mid u \neq v\right\}$, we also consider (V,E') as a graph by naturally identifying E' with $\varphi(E')$.

For example, given a polygon or a polyhedron, the vertices and the edges of the polygon, or of the polyhedron, form obviously a graph, for which the vertices are points (in the plane or in space), the edges are line segments and incidence is inclusion.

For another example, we may consider the following graph G underlying the decomposition presented in Fig. 1, which we call *the graph of the decomposition*, where the vertices are line segments and the edges are rectangles:

- the vertices of G are the maximal horizontal segments that contain the sides of the squares of the decomposition. Note that these segments, together with the maximal vertical segments defined similarly, determine completely the decomposition;
- the edges are the squares of the decomposition and the rectangle being decomposed.
- any of these squares, as well as the full rectangle, has two sides contained in two horizontal segments; as an edge, these are the vertices incident with it.

Every graph $G = (V,E)$ may be *represented geometrically in the plane* by another graph $G' = (V',E')$, where V' is a set of points in the plane in bijection φ with V and E' is a set of Jordan arcs in the plane in bijection ψ with E, in such a way that, for every $e = uv \in E$, $e' = \psi(e)$ connects $\varphi(u)$ to $\varphi(v)$. We say that G' *is plane* — in which case we say that G *is planar* — if, given any two edges (two arcs, hence) $e', f' \in E'$, if $A = e' \cap f'$ then either $A = \{v'\}$ for a vertex $v' \in V'$ (incident with e' and f') or $A = \varnothing$. See Figure 3 for an example. In general, we call *topological graph* to a graph obtained as G' above, either in the plane, in the sphere, in the torus, etc. Similarly to plane graphs, we may have then *spheric graphs* or *toroidal graphs*. In particular, spheric graphs are planar and plane graphs can be represented in the sphere. To see this, in one direction, consider the stereographic projection of the spheric

[1] By a face we mean one of the connected components of the complement in the sphere of the union of the edges. The same notion applies to the torus, for example, or to the plane, where (exactly) one of the regions is unbounded.

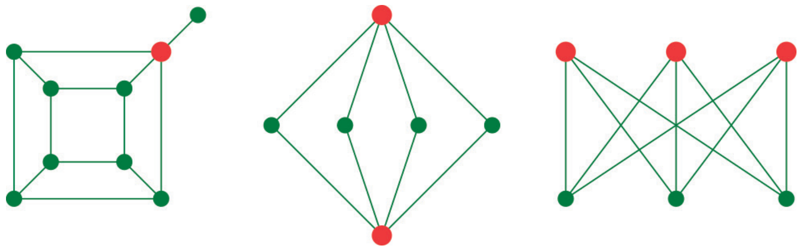

Figure 4.—Connected graph that is *not* 2-connected, 2-connected graph that is *not* 3-connected and 3-connected graph that is *neither* planar *nor* 4-connected

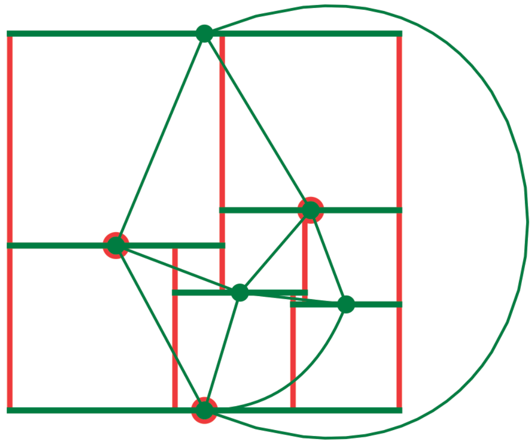

Figure 5.—Graph of the decomposition of a rectangle

graph, taking as pole of projection a point interior to a face[1] (see Figure 3). The other direction works similarly.

The graphs of the polygons, in the precise sense defined above, are obviously *plane*, and the edges are even *straight* line segments. It is easy to see that the graphs of the polyhedra can be represented as spheric graphs and hence are *planar* (again, see Figure 3). Finally, it can be proved [7] that the graph of the decomposition of a rectangle is also planar. In Figure 5 we can see the plane graph drawn over the decomposition.

A *path in a graph* G is a sequence of pairwise distinct vertices of G, $c = [v_0, v_1, \ldots, v_k]$, such that $v_0 v_1, v_1 v_2, \ldots, v_{k-1} v_k$ are edges of G. The *endpoints* are v_0 and v_k and c is said *to connect* them. A *cycle* is defined like a path, except that $v_0 = v_k$. In both cases, k, the number of edges, is the *length*.

We say that G is connected (or 1-*connected*) if there is a path connecting any two different vertices u and v. It is 2-*connected* if, given a vertex x and two vertices u and v, different from each other and from x, there is a path that does not contain x connecting u to v. In other words, G is 2-connected if, for every $x \in V(G)$, the graph G' obtained from G by excluding x from $V(G)$ and by deleting all edges incident with x from $E(G)$ is still connected. In general, it is *n-connected* if, for every $x \in V(G)$, G' is $(n-1)$-connected. In Figure 4 we show different examples of connectivity.

For an example of a non-planar graph, consider the last graph of Figure 4, usually called $K_{3,3}$. In fact, if we suppose that the graph can be represented in a sphere, since we need a cycle to close a face and there are no cycles in the graph with length less than 4, and since every edge belongs exactly to the boundary of two faces, we see that the number of faces is at most half the number of edges. So, we must have 6 vertices, 9 edges and at most 4 faces. But this is in contradiction with Euler's formula.

We note that the plane graph drawn in Figure 5 over the decomposition of the rectangle is also a representation of the graph of the polyhedron of Figure 2. The same graph appears in black in Figure 3. But whereas the unbounded face is adjacent to 4 edges in the graph of Figure 3, in the graph of Figure 5 it is adjacent to 3 edges. But it is important to note that the edges adjacent to any face in one representation correspond exactly to the edges adjacent to a face in the other representation. In fact, by an important theorem of Whitney, a cycle C is the boundary of a face in any representation in the plane (or in a sphere) if and only if the graph obtained from G by removing the edges of C is connected.

It is not difficult to see that this particular graph is 3-connected: although it can be disconnected by deleting the 3 marked vertices (and the incident edges), *it cannot be disconnected by deleting only 2 vertices*. The same happens with the graphs of all the polyhedra, according to the theorem that is central in this paper: they are all 3-connected.

Theorem of Steinitz.—The graph of every polyhedron is *planar* and 3-*connected*. Conversely, any graph with more than 3 vertices that is both planar and 3-connected is the graph of a polyhedron.

We will come back to Steinitz's theorem. Before, let us consider briefly the connection between the theorem of Steinitz and the decomposition of rectangles. Our plan is to state afterwards this theorem, to "explain" things

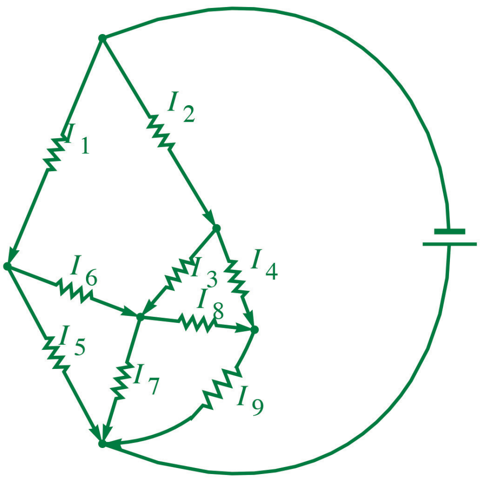

Figure 6.—Electric circuit associated with the graph of the decomposition of a rectangle

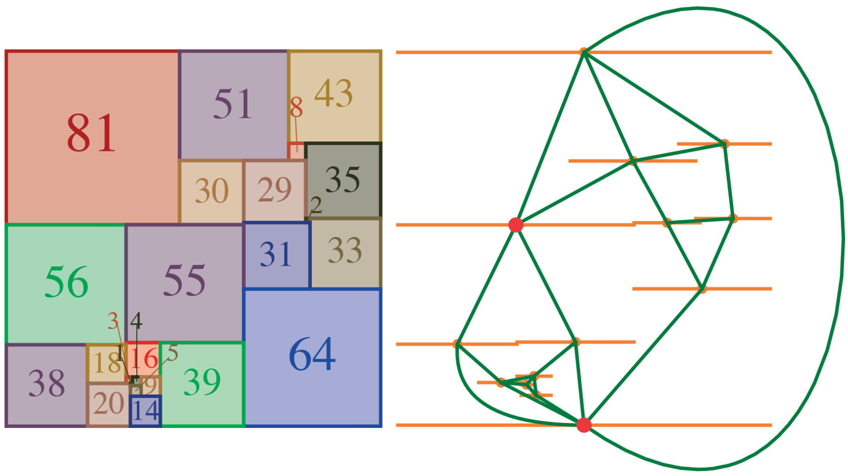

Figure 7.—*Non-simple* decomposition of a square and connectivity of the graph

when they can be easily "explained" (although we do not prove them …) and to present some more modern consequences of the theorem and of its various, modern or not so modern, proofs.

Decomposition of a rectangle in non-congruent squares

The rectangle in Figure 1 is "almost a square", in that its dimensions are 33×32. But it is not a true square, and for a long time no one knew whether a square could be decomposed in squares pairwise non-congruent squares. In an attempt to solve this question, four students of the Trinity College, Cambridge, Roland Brooks, Cedric Smith, Arthur Stone e William Tutte [7], defined and studied the graph of a decomposition. They not only presented *perfect squares*, as they called the squares that can be decomposed in pairwise non-congruent squares, but proved that there are infinitely many different (non-similar) perfect squares.

They proved that the graph of a decomposition is always planar, as we have seen in our example. At the same time, they noted they could see this graph as the diagram of an electric circuit (see Figure 6), where vertices represent junctions, edges represent wires with resistors of unitary resistance or, in a unique case, a power source, and where the Kirchhoff's current and voltage laws hold true.

In fact, in such a circuit, on the one hand, the sum of the currents that enter a junction or vertex (the sum of the sides of the squares above the maximal horizontal segment that is the vertex) must equal the sum of the currents that leave the junction, or the sum of the sides of the squares below the vertex. On the other hand, considering any face and the upmost and downmost vertices and the two different paths between them, the sum of the sides of the squares that are the edges of one path is of course equal to the sum of the sides of the squares that are the edges of the other path. Then, the theorem of Kirchhoff implies that, up to the total voltage V of the circuit, the values of all currents and voltages are uniquely determined.

This gave them the means to start to construct decompositions, just by considering suitable graphs and by "electrifying" them. We consider an example based in the graph of the polyhedron $\mathscr{P}$ of Figure 2, namely as represented in Figure 5, but "electrified".

In Figure 6, we have five independent "current equations", $I_1 + I_2 = V$, $I_2 = I_3 + I_4$, $I_4 + I_8 = I_9$, $I_3 + I_6 = I_7 + I_8$ e $I_1 = I_5 + I_6$ and four "voltage equations" (remember the resistances are unitary): $I_1 + I_6 = I_2 + I_3$, $I_3 + I_8 = I_4$, $I_8 + I_9 = I_7$ e $I_6 + I_7 = I_5$. Hence, in the solution of the system,

$$\begin{cases} I_1 & = 6\,V/11 \\ I_2 & = 5\,V/11 \\ \vdots & \vdots \\ I_9 & = 3\,V/11 \end{cases}$$

Making $V = 33$, the width of the rectangle, we find the sides of the squares of the decomposition of Figure 1.

This construction was based in the graph of a polyhedron. What happens if the starting graph is not 3-connected?

In Figure 7 we consider a decomposed square and its graph. The graph is not 3-connected, since it can be disconnected by deleting the two marked vertices. But

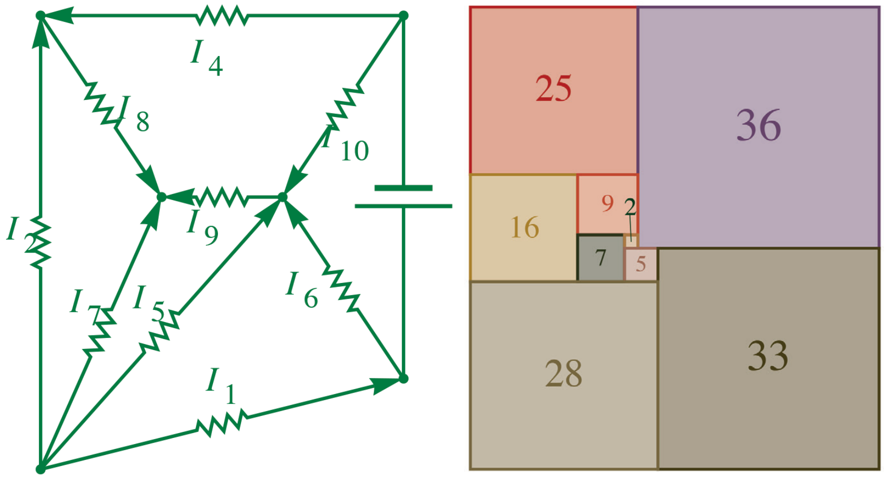

Figure 8.—Same polyhedron, different decomposition

these vertices are the horizontal sides of a "subrectangle" (in the southwest corner of the decomposition) and it can be proven that the fact that, when deleted, they disconnect the graph means, in terms of the decomposition, that the subrectangle is already decomposed in mutual non-congruent squares. So, we have a "subdecomposition" of the decomposition. If we do not want this to happen, we must consider only *planar, 3-connected* graphs, that is, graphs of polyhedra.

We have used this idea [10], by considering eight particular polyhedra with six vertices, from which all the 30 possible decompositions with eleven or less squares can be obtained. Note that the way we draw the graph in the plane (or the choice of the "electrified" edge, more precisely) may determine different decomposed rectangles. For example, the electrification in Figure 8 of the graph represented in Figure 3 leads to the solution

$$(I_1, I_2, I_4, I_5, I_6, I_7, I_8, I_9, I_{10}) =$$
$$= \left(\frac{28\,V}{61}, -\frac{16\,V}{61}, \frac{25\,V}{61}, -\frac{5\,V}{61}, -\frac{33\,V}{61}, -\frac{7\,V}{61}, \frac{9\,V}{61}, \frac{2\,V}{61}, \frac{36\,V}{61}\right),$$

which determines the new decomposition (see Figure 8).

Theorems of Steinitz and Tutte

Let us consider a little further the "easy part" of the theorem of Steinitz, that claims that the graph of a polyhedron is always planar and 3-connected. We start exactly by the connectivity, but having in mind a theorem by Balinski [3], which claims that the graph of any d-polytope is d-connected, for any $d \geq 2$, and the author's proof.

Consider a polyhedron $\mathscr{P}$, two vertices, P and Q, and let us "tell why" deleting these points and the incident edges from the graph G of $\mathscr{P}$ does not result in a disconnected graph. Note that, in the general case, the number of *deleted points* should be $d - 1$.

So, let A, B and R be three points different from the deleted points so that R is adjacent to one of them, P, say, and let π be the (hyper)plane defined by R and all the deleted points. For simplicity sake, we only consider here the case where A and B are not in π. Then, obviously, either they are in the same side of π or they are in opposite sides. We want to connect A and B by a path that does not include either P or Q.

In the first case, note that either A is at maximal distance to π or there exists a vertex, *adjacent to* A, at greater distance: just consider the (hyper)plane χ parallel to π through A and the part of $\mathscr{P}$ (a new polytope with vertex A) that lies in the side of χ opposite to π. This means that A is connected to a vertex A' at maximal distance to π, and so is B, to B'. If they are equal, we are done. If not, they both lye in a face of $\mathscr{P}$, and can be connected in the graph of $\mathscr{F}$, which is also a polytope.

In the second case, note that there are two vertices adjacent to R, in the opposite side of π of each other. By the previous argument, one of these vertices can be connected to A and the other one to B, and so these points are connected through R.

Figure 2 suggests a proof of planarity of G, the graph of a polytope $\mathscr{P}$. Yet, the edges of the resulting graph are not straight. If we project directly G from a point O over a plane χ, then the image of the (straight) edges of $\mathscr{P}$ are still straight and the image of the faces are still convex. But the projected graph may cease to be plane.

To avoid this, choose a face $\mathscr{F}$ of $\mathscr{P}$, let π be the

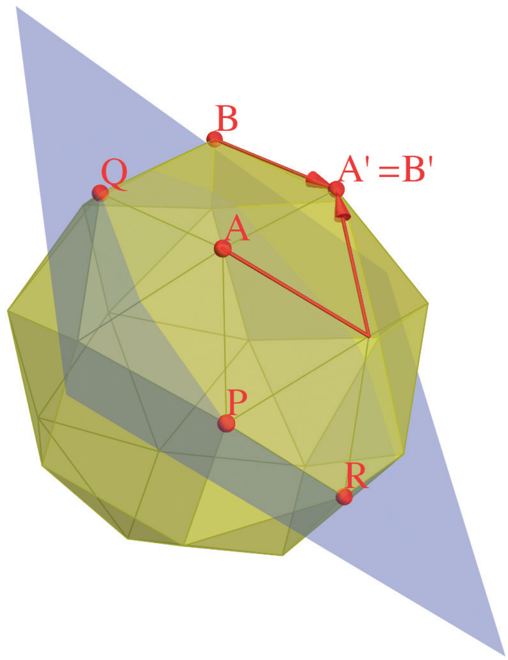

Figure 9.—Illustration of Balinki's theorem and proof

plane that contains $\mathscr{F}$ and consider O near to the centroid C of $\mathscr{F}$, on the side of π opposite to $\mathscr{P}$. Since the intersection of two convex sets is still convex, the intersection of any line with $\mathscr{P}$ is either a point or a line segment. Suppose that the projection of two different edges e and f intersect. Then the line OA, for a given point A in e, also contains a point B of f, and the intersection of line OA with $\mathscr{P}$ is the segment $\overline{AB}$. But this cannot happen if O is sufficiently close to C: neither when, say, $A \in \mathscr{F}$ nor when $A, B \notin \mathscr{F}$, since, in the latter case, OA must cross the plane containing $\mathscr{F}$ outside this face.

Before proceeding further, we state this result, that was originally obtained independently of Steinitz's theorem.

Theorem of Tutte.—Every *planar*, 3-*connected* graph can be represented in the plane with straight edges and convex faces.

Note that we consider in this theorem two different conditions. For example, it can be easily proved that the graph represented *with straight edges* in the middle of Figure 4 *cannot* be drawn in the plane with convex faces.

Let us go back to the first property. Can every planar graph be represented in the plane with straight edges? The answer, *yes*, goes back to 1936 and is due to Wagner, and new proofs were published independently in 1948, by Fáry, and in 1951, by Stein. We also consider this question here, with one more issue in mind: we want straight edges *and*, at the same time, vertices with *small integer* coordinates in a suitable coordinate system — or, in other words, with *good resolution*. The construction we describe here is due to W. Schnyder [15], and is illustrated in Figure 10 and in Figure 11.

Given any plane graph G, by possibly adding some new edges (that can be withdrawn afterwords), we obtain a new graph in which every face, including the unbounded face, is a triangle. Let us suppose that the vertices of the unbounded face are coloured with three colours, say, red, green and blue. Schnyder shows that we may orient and colour with the same three colours all the edges of such a graph, in such a way that from every vertex if we "follow" the edges of a given colour, we reach the corresponding coloured vertex.

Now, for each vertex, consider the three "coloured paths" and the partition of the bounded faces into three classes determined by these paths. In Figure 10, for example, there are 15 bounded faces. On the right-hand side, for vertex 1, the three classes have 5, 9 and 1 faces, respectively, where the class with 5 faces [resp. 9, 1] is not bounded by the "red path" [resp., "green path", "blue path"]. We obtain consecutively for all the vertices

$(5,9,1), (1,4,10), (3,10,2), (11,2,2), (9,1,5), (0,15,0),$
$(15,0,0), (7,5,3), (2,6,7)$ and $(0,0,15)$.

Schnyder proves that if we take a triangle in the plane, consider the vertices with these triplets as suitable multiples of the barycentric coordinates and draw straight edges, then we obtain a plane representation of the initial

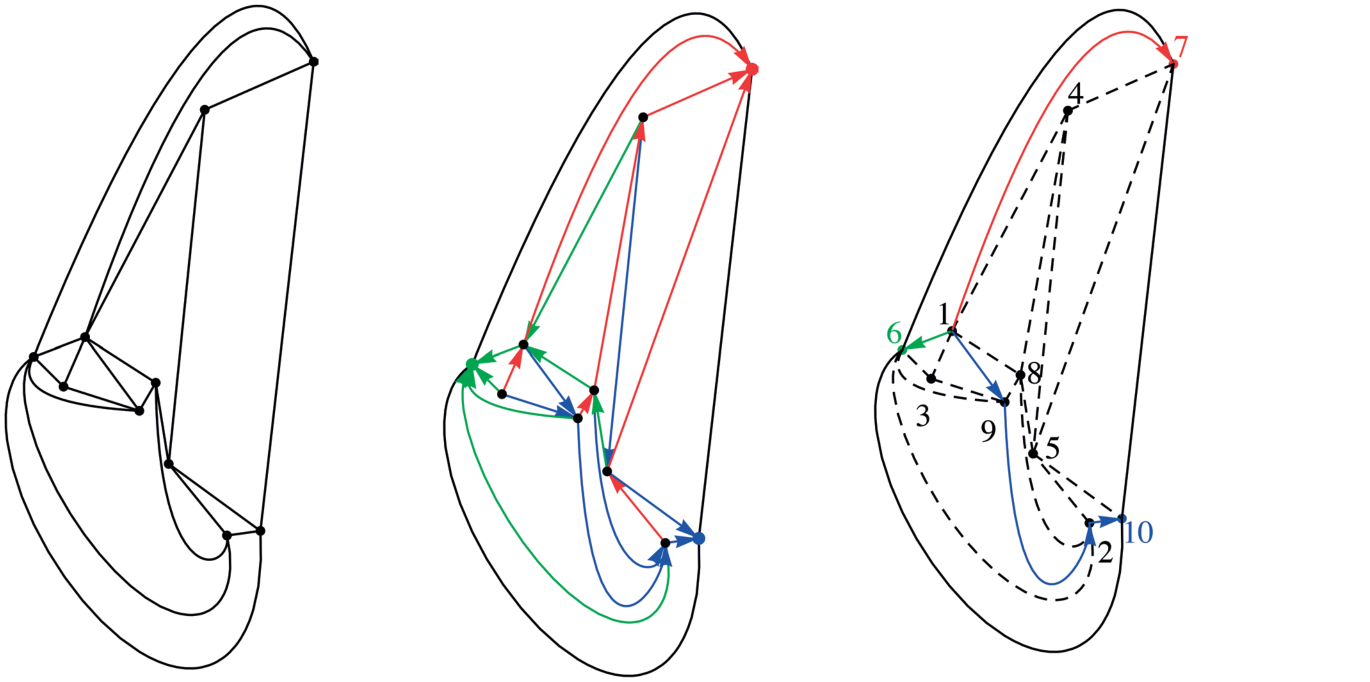

Figure 10.—How to draw a graph with straight edges and good resolution (but without convex faces) I

graph with integer (barycentric) coordinates not greater than the number of faces. In fact, with a slight modification of this method, Schnyder proves that the coordinates can be limited to integers between 0 and $v-2$, where v is the number of vertices of the graph. In Figure 11, we use these coordinates on the left-hand side, and on the right-hand side we show without any further explanation a *geodesic embedding* of the graph, that is also based in Schnyder's construction (see the book of S. Felsner [8] for more information on this subject). Note that, as in our example, by Schnyder's method we may end up with non-convex faces, after deletion of the edges added at the beginning for obtaining triangular faces. But this can be circumvented [8].

Tutte's original proof is different, and the ideas behind it are still used nowadays [12]. They correspond to the following "physical" idea: suppose that, in a board, we fix nails corresponding to the vertices of the unbounded face of the graph, and that we connect with rubber bands the vertices that are incident with any edge, by tying up the bands on points corresponding to vertices as indicated by the graph. When we leave such a system to itself, if in equilibrium there is some tension in all the rubber bands, then the edges will be straight and the faces will be convex.

More precisely, Tutte proves the following. Consider, for each vertex v not belonging to the unbounded face, with neighbours[2] say, $w_1, w_2, \ldots, w_k$, the (vectorial) equation

$$\sum_{i=1}^{k} c\,(\mathbf{p}_{\mathbf{w}_i} - \mathbf{p}_{\mathbf{v}}) = 0.$$

In this equation, c is an elasticity coefficient that we can neglect by now, by considering it constant, and $\mathbf{p}_\mathbf{x}$ represents the constant coordinate vector of the point x if x is a "nailed" vertex of the unbounded face, or a pair of variables, otherwise. Then, the system of equations has a unique solution, that represents the coordinates of the vertices of a plane graph associated with the initial graph, for which the faces are convex provided the edges are straight.

4. On the "difficult part" of the theorem of Steinitz

All the known proofs of the fact that every planar 3-connected graph with more than 3 vertices is the graph of a polyhedron present (naturally …) some difficulties. We will mention here some of these proofs, but in quite a vague way. For more precision and even for a correct attribution of results to authors, please see Ziegler [18,19] and Richter [13] and the bibliography therein.

We may say that there are three kinds of known proofs of this theorem [19, p. 8]. For each of them, new

[2] That is, the vertices w such that vw is an edge of G.

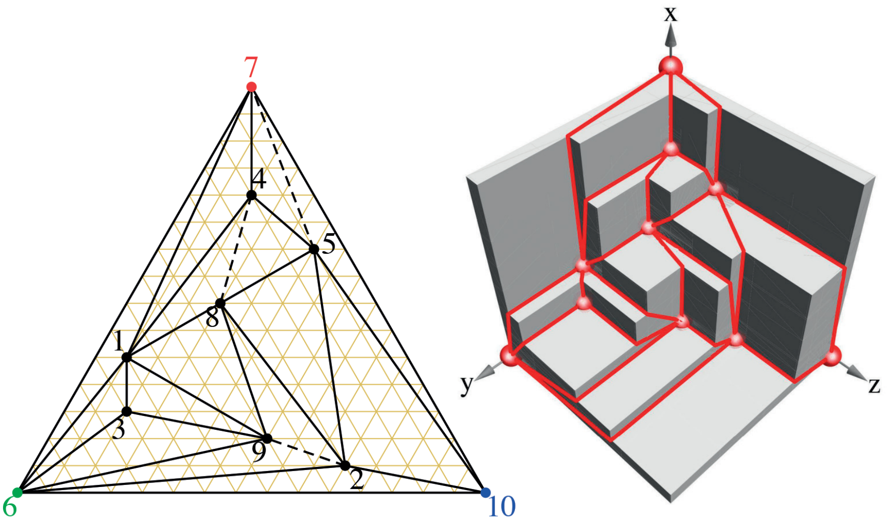

Figure 11.—How to draw a graph with straight edges and good resolution (but without convex faces) II

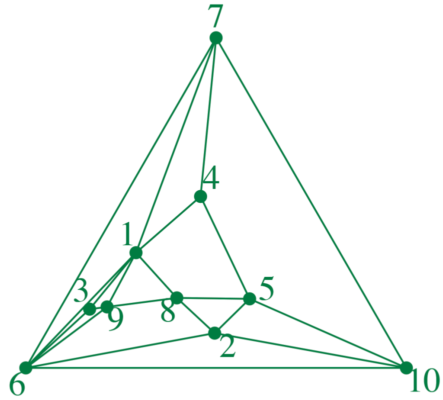

Figure 12.—Rubber band representation of the graph of $\mathcal{P}$

proofs lead to new results. Steinitz gave three different proofs, all starting from a tetrahedron and showing that vertices can be added or moved so as to fit to the graph. For other proofs of the same kind, and for the variety of results that we can obtain from them, see e.g. [9]. For example, from a modification of a proof of the same kind, it has be shown that one can "prescribe" the shape of any face of a polyhedron with a given graph. It can also be shown that the (combinatorial) symmetry of the graph can be carried over to a (geometric) symmetry of the polyhedron. These properties do not hold in dimension 4: for example, there exists a 4-polytope with 8 vertices for which one particular face, an octahedron, cannot be regular. In fact, there exists a 4-polytope for which we cannot prescribe freely the shape of a 2-face, an hexagon. From the first example, it was possible to construct a 4-polytope with 2 new vertices with "hidden" symmetries, that is, combinatorial symmetries without geometric counterpart [4]. The "realization space" (the euclidean space of coordinates of the vertices) of this polytope is not connected [5]; it is the smallest known polytope with this property.

Another kind of proofs exploit Tutte's "rubber band" idea, by "lifting" the rubber band diagram, similarly to 13, obtained from the graph of the polyhedron of Figure 2 as indicated by Richter [13] by using a constant elasticity coefficient, c. It can be proved that all the polyhedra with the same graph can be obtained this way, but with variable values of c.

J. Richter [13] bases on this method a proof for the fact that every polyhedron has the graph of another polyhedron with vertices with rational (and hence also with integral) coordinates. The best *resolution* of these integer coordinates is a new research problem, called the *quantitative Steinitz theorem*, with very recent developments [12]. Note that we know of an 8-polytope with twelve vertices that *cannot* be constructed with rational coordinates. The most important consequence of Richter's proof (and of some other proofs of the same kind) is that, for polyhedrons, the realization space is topologically very simple, in the (very imprecise) sense that we can deform continuously any polytope into any other one, up to a mirror image, provided they have the same graph.

This is not the case in dimension 4. On the contrary, the realization space of 4-polytopes is "as rich as possible", in a precise sense that we will not consider here. For details see J. Richter's book [13], which is centred exactly on this very important issue. The study of graphs of 4-polytopes and general d-polytopes for $d \geq 4$ is a rich field of growing research [19].

As an example of active research, consider Ziegler's question [20,11], regarding the polytopality of the Cartesian product of two Petersen graphs.

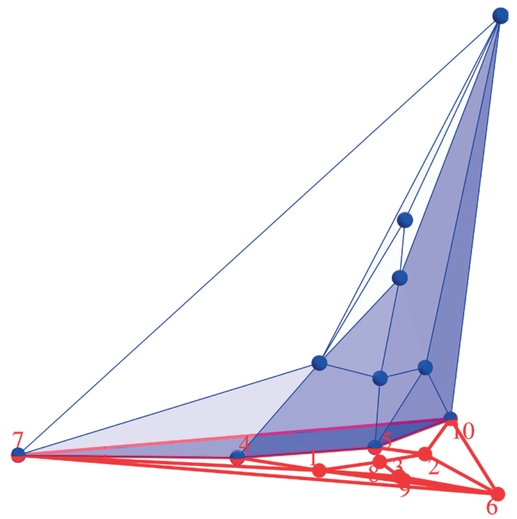

Figure. 13—"Lifting" of the rubber band representation of the graph of $\mathcal{P}$

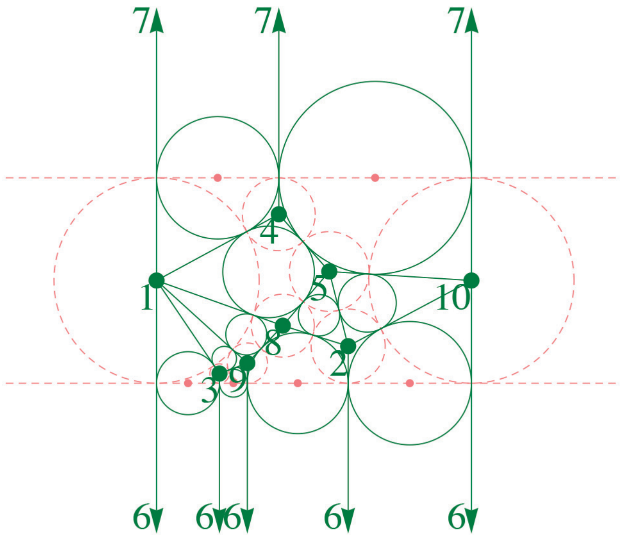

Figure 14.—Illustration of the theorem of Koebe-Andreev-Thurston I

Before considering here the third and last kind of proof of Steinitz theorem, note that this theorem claims, in particular, that a spherical triangular graph can be realized in space with straight edges. Recently, it was proved that the same happens with toroidal graphs [2]. But we know that the same does not happen in a quintuple torus — or sphere with five handles [6,14]. It is not clear what happens between the simple torus and the quintuple torus.

Finally, there is a third kind of proof of the theorem of Steinitz, that we may follow thoroughly in the work of Ziegler [19], for example.

Starting with the graph of the polytope $\mathscr{P}$ of Figure 14 and following [19], we obtained the graph of Figure 14, which has the following properties (we view a straight line as a circle of infinite radius and two parallel lines as tangent circles):

- Any vertex of the graph is the centre of a circle, and two circles are tangent if and only if the vertices are incident with an edge of the graph; these circles are in (dotted) pink[3] in Figure 14.
- Each face contains also a (green, in Figure 14) circle and the circles are tangent if and only if the faces are adjacent.
- Finally, the (pink) circles centred in the vertices and the (green) circles contained in the faces are pairwise mutually orthogonal.

It can be proved that this construction may be made for the graph of any polyhedron, and from this it follows the following remarkable theorem:

Theorem of Koebe-Andreev-Thurston.—Every graph with more than three vertices, planar and 3-connected is the graph of a polyhedron of edges tangent to a given sphere.

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[3] Note that the horizontal line above has centre in the vertex 7 and the one below in vertex 6.

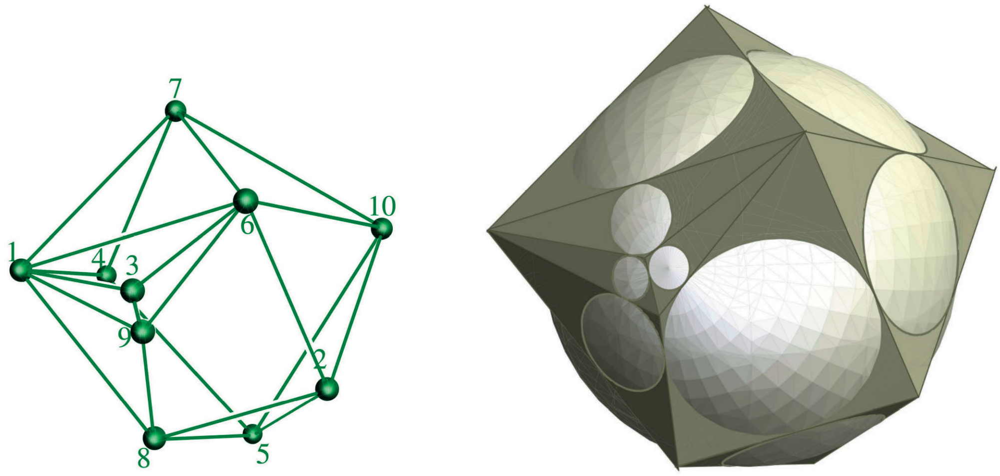

Figure 15.—Illustration of the theorem of Koebe-Andreev-Thurston II

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