On the construction of complex algebraic surfaces

by **Carlos Rito***

Quotients and *coverings* appear frequently in the construction of algebraic varieties. If a variety V has some property that makes it a quotient or a covering of some simpler variety U, then one may try to construct V starting from this simpler U. This method has proven to be very efficient in the construction of complex algebraic surfaces. This is a vast theme, here we do not intend to give a detailed survey on it. Our aim is to give a taste of the subject, for non-specialists, by presenting some examples, most of them related to the work of the author.

1 Generalities

Let S be a complex smooth algebraic surface that is projective (i.e. that has an embedding into a *projective space*). A *divisor* in S is a formal sum of curves $\sum n_i C_i$, $n_i \in \mathbb{Z}$. A divisor is *effective* if the coefficients n_i are all non-negative. Two divisors D, D' are linearly equivalent if the difference $D-D'$ is the divisor of zeros and poles of a rational function f/g on S. In this case we write $D \equiv D'$. The *complete linear system* $|D|$ is the set of all divisors linearly equivalent to D.

To give a divisor on a surface S is not trivial. If the surface is embedded in some projective space $\mathbb{P}^n$, an obvious way is to take *hyperplane sections* (i.e. intersections with hyperplanes), but this depends on the particular embedding. There is a divisor which is intrinsic to the variety (i.e. it does not depend on the embedding), the *canonical divisor* K_S: it is the divisor of a meromorphic differential 2-form $f\,dx \wedge dy$. The number of generators of the canonical linear system $|K_S|$ is the *geometric genus* $p_g(S)$. If the surface admits a holomorphic differential 1-form $fdx + gdy$, we say that it is *irregular*, and the irregularity $q(S)$ is the number of independent such forms.

The *pluricanonical* map $\phi_{|nK_S|} : S \to \mathbb{P}^d$ is the map given by the sections of the pluricanonical linear system $|nK_S|$ (hence $d = \dim(|nK_S|)$). The *Kodaira dimension* $\mathrm{Kod}(S)$ is the maximum of the dimensions of the images $\phi_{|nK_S|}(S)$, $n \in \mathbb{N}$. Obviously it is at most 2, and surfaces of Kodaira dimension 2 are said to be *of general type*.

It is known since Hironaka [Hir64] that there is always a resolution of singularities for algebraic varieties over a field of characteristic zero. He has been awarded the 1970 Fields Medal for this result. His proof consists in repeatedly *blowing-up* along non-singular subvarieties, and to show that the process ends. The idea of resolving a singularity by blowing-up is as follows. Given a smooth surface X and a point $p \in X$, there is a smooth surface Y and a map $\pi : Y \to X$ such that $E := \pi^{-1}(p) \cong \mathbb{P}^1$ and $Y \backslash E \cong X \backslash p$. Now let $C \subset Y$ be a smooth curve intersecting E at n distinct points. Then $\pi(C) \subset X$ is a curve with a singular point of multiplicity n at p, and the blow-up π resolves the singularity of the curve $\pi(C)$.

Given curves $C, C' \subset S$ without common components, with S a smooth surface, the *intersection number* CC' is the number of points in $C \cap C'$, counting multiplicities. If $C \equiv C'$, we say that $C^2 := CC'$ is the *self-intersection* of C. The definition can be extended to all curves, and some curves have negative self-intersection. For instance, consider the blow-up $\pi : Y \to \mathbb{P}^2$ at a point $p \in \mathbb{P}^2$. Let $E \subset Y$ be the *exceptional divisor* as above, $L, L' \subset \mathbb{P}^2$ be distinct lines through p and consider the strict transforms $\hat{L}, \hat{L}' \subset Y$

* CMUP, Faculdade de Ciências da Universidade do Porto · crito@fc.up.pt
On leave from: Universidade de Trás-os-Montes e Alto Douro, UTAD · crito@utad.pt

of L, L'. Since $\hat{L}\hat{L}' = 0$ and $\hat{L} \equiv \hat{L}'$, then $\hat{L}^2 = 0$. We have that

$$1 = L^2 = \pi^*(L)^2 = (E + \hat{L})^2 = E^2 + 0 + 2,$$

thus $E^2 = -1$. We say that E is a (-1)-curve. Any (-1)-curve can be contracted to a smooth point of the surface. Curves with self-intersection < -1 are contracted to singular points. For example the resolution of a *node* (ordinary double point) is a (-2)-curve, i.e. a curve isomorphic to $\mathbb{P}^1$ with self-intersection -2.

We say that a smooth surface is *minimal* if it has no (-1)-curves; the *minimal model* of a smooth surface is obtained contracting all its (-1)-curves. The minimal model of a surface with non-negative Kodaira dimension is unique, i.e. different choices for the contraction of all (-1)-curves give rise to isomorphic surfaces.

The geometric genus $p_g(S)$ and the irregularity $q(S)$ of the surface S are topological invariants. Finally, the *holomorphic Euler characteristic* of S is

$$\chi(S) := 1 - q(S) + p_g(S),$$

and the *topological Euler characteristic* $\chi_{\text{top}}(S)$ satisfies

$$K_S^2 + \chi_{\text{top}}(S) = 12\chi(S).$$

Geometers want to classify surfaces according to these invariants. For surfaces of general type always $\chi \geq 1$, and naturally one wants to classify the case $\chi = 1$, but this is still far from completed.

2 Campedelli vs Godeaux

Let S be a smooth minimal surface of general type. One has $K_S^2 \geq 1$, and the Bogomolov-Miyaoka-Yau inequality $K_S^2 \leq 9\chi(S)$ holds, hence $1 \leq K_S^2 \leq 9$ for surfaces with $\chi = 1$. These surfaces satisfy $p_g = q$, and the ones with $p_g = q = 0$ have received particular attention. In the 19th century it was thought that these surfaces were *rational* (i.e. obtained from the projective plane by a sequence of blow-ups and blow-downs). Then Enriques [Enr96], in 1896, showed the existence of surfaces with $p_g = q = 0$ and Kodaira dimension 0, hence not rational. Nowadays these are called *Enriques surfaces*. In the same year, Castelnuovo [Cas96] proved his *rationality criterion*: an algebraic surface S is rational if and only if $q(S) = 0$ and the linear system $|2K_S|$ is empty.

The first examples of surfaces of general type with $p_g = q = 0$ were discovered by Godeaux [God31] ($K^2 = 1$) and Campedelli [Cam32] ($K^2 = 2$) in the 1930s.

The Godeaux construction is as follows. Let $Q \subset \mathbb{P}^3$ be the quintic surface with equation

$$x^5 + y^5 + z^5 + w^5 = 0.$$

It is invariant for the $\mathbb{Z}_5$ action

$$\sigma : (x : y : z : w) \to (x : \rho y : \rho^2 z : \rho^3 w),$$

with ρ a 5th root of unity. Since the action is free and $\chi(Q) = 5$, $q(Q) = 0$, $K_Q^2 = 5$, then the surface Q/σ is smooth and has invariants $\chi = 1$, $q = 0$ and $K^2 = 1$.

The Campedelli surface is obtained as (the resolution of the singularities of) a double cover of $\mathbb{P}^2$ ramified over a curve $\{f_{10} = 0\}$ of degree 10 with 6 singularities of type $(3, 3)$ (a triple point with the 3 branches sharing the same tangent line). This can be seen as the surface with equation $w^2 = f_{10}(x, y, z)$ in the *weighted projective space* $\mathbb{P}[1, 1, 1, 5]$. Its invariants are $\chi = 1$, $q = 0$ and $K^2 = 2$.

These are typical examples of the two most successful methods for the construction of algebraic surfaces: quotients and coverings. One can discuss which method is more efficient, but frequently the construction is given by a combination of the two. Below we give some examples.

3 Double coverings

Let S be a smooth surface with an *involution*, i.e. with a non-trivial automorphism σ such that $\sigma^2 = \text{Id}$. The projection

$$\varphi : S \to X := S/\sigma$$

is a *double covering*. The *ramification set* of φ is the set of points fixed by σ; it is the union of a smooth curve with a finite number $n \geq 0$ of points $p_1, \ldots, p_n$, which correspond to nodes of S/σ. Let $S' \to S$ be the blow-up of S at these points. Then σ extends to an involution σ' on S' with ramification

$$R' := R + \sum_1^n E_i,$$

where E_i is the exceptional curve corresponding to p_i, and R is a smooth curve disjoint from E_i, $i = 1, \ldots, n$. The *branch locus* of

$$\varphi' : S' \to X' := S'/\sigma'$$

is the (smooth) curve $B := \varphi'(R')$. One can show that there exists a divisor L such that $B \equiv 2L$ (we say that B is 2-divisible). The projection $\rho : X' \to \bar{X}$ to the minimal model gives a singular curve $\bar{B} := \rho(B)$.

Conversely, from a smooth surface $\bar{X}$ and a 2-divisible (possibly singular) branch curve $\bar{B}$, we can recover the surface S: we take the double covering $\bar{S} \to \bar{X}$ ramified over $\bar{B}$; the *smooth minimal model* of $\bar{S}$, obtained by resolving the singularities of $\bar{S}$ and contracting all (-1)-curves, is a surface isomorphic to S.

$$\begin{array}{ccccc} \bar{S} & \longleftarrow & S' & \longrightarrow & S \\ \downarrow & & \downarrow & & \downarrow \\ \bar{X} & \longleftarrow & X' & \longrightarrow & X \end{array}$$

Frequently the curve $\bar{B}$ is highly singular; construction methods that include the use of symmetry and computational tools have proved useful.

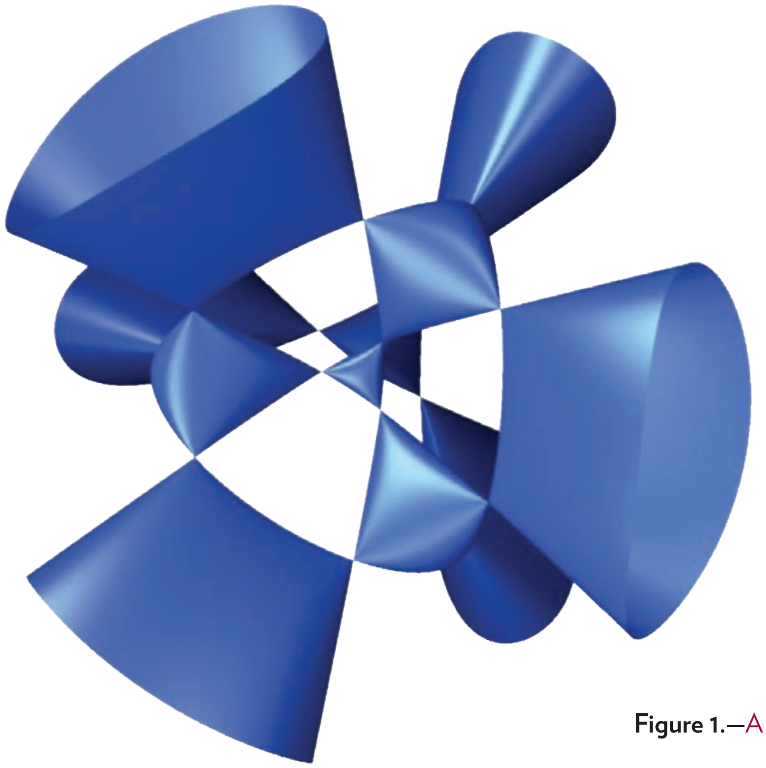

Figure 1.—A Kummer surface in $\mathbb{R}^3$

Since Godeaux and Campedelli, mathematicians have been giving examples of surfaces of general type with $\chi = 1$ for each possible value of the invariants $p_g = q$ and $3 \leq K^2 \leq 9$. The author has given the first examples for the mysterious cases $K^2 = 7$ and $p_g = q = 1, 2$ (see [Rit10], [Rit15]). Somehow surprisingly, these were obtained using double coverings.

3.1 $p_g = q = 1, K^2 = 7$

A singular point of a curve is *of type* (n, n) if it is a singular point of multiplicity n with branches sharing the same tangent line at the point. This means that we still have a singularity of multiplicity n after one blow-up; we say that these two points of multiplicity n are *infinitely near*.

A *double plane* is a surface S with an involution i such that the quotient S/i is a rational surface. It can be obtained as (the resolution of the singularities of) a double covering $\bar{S} \to \mathbb{P}^2$, ramified over a (singular) branch curve B. The problem is then how to find B, because the surface S is determined by it.

In his work on double planes, Du Val [du 52] proposed a configuration of branch curves B for the construction of some surfaces with low invariants, in particular $p_g = q = 0$. A similar configuration gives invariants $p_g = q = 1$. But these curves are highly singular (with some points of type (n, n)), and hence it is difficult to prove their existence.

Given points $p_1, \ldots, p_n$, possibly infinitely near, one can use computer algebra, or more precisely the computer algebra system Magma [BCP97] and the algorithm given in [Rit10], to compute the linear system of plane curves with given degree and singularities at $p_1, \ldots, p_n$. But in general, if these points are not chosen properly, this system is empty. Thus we have to compute points in a special position such that the curve exists. The idea is as follows.

Suppose we want to compute a plane curve of degree d with a singular point of multiplicity n. Its defining polynomial is a linear combination $\sum a_i m_i(x, y)$, where $m_i(x, y)$ are monomials. We want to compute the coefficients a_i. Recall that a plane curve has a point of multiplicity n if and only if the derivatives up to order $n - 1$ vanish at that point. Let M be the matrix with lines the derivatives of the m_i up to order $n - 1$. The curve exists if M is not of maximal rank. Thus we need to compute points in the variety given by the vanishing of certain minors of the matrix M. Since some points are infinitely near, the process is a combination of the above with some blow-ups. Then the success depends on the computational complexity. For the construction of the case $p_g = q = 1$ and $K^2 = 7$, see [Rit10].

3.2 $p_g = q = 2, K^2 = 7$

Recall that an *elliptic curve* is the quotient of $\mathbb{C}$ by a lattice Γ. Topologically it is a torus. In the same way, a quotient $A := \mathbb{C}^2/\Gamma$ is a complex surface, a torus of dimension 2. The ones that are algebraic are called *abelian surfaces*. Multiplication by -1 in $\mathbb{C}^2$ gives rise to an involution σ on A. The quotient A/σ is a *Kummer surface*, a complex algebraic surface with $q = 0, K \equiv 0$, and having 16 nodes, corresponding to the 16 fixed points of σ (figure 1).

Now we explain the construction of a particular abelian surface, starting from a double covering of the projective plane. Let p_0, p_1, p_2 be distinct points in the projective plane $\mathbb{P}^2$ and let T_i be the line through p_0, p_i, $i = 1, 2$. There is a 1-dimensional linear system of conics tangent to T_1, T_2 at p_1, p_2, respectively. Take C_1, C_2 distinct smooth conics in this system, and choose T_3, T_4 general lines through p_0. Let $Q' \to \mathbb{P}^2$ be the double covering with branch curve $C_1 + C_2 + T_3 + T_4$. It is well known that the smooth minimal model Q of Q' is a $K3$ surface, i.e. a smooth minimal surface with $q = 0$ and $K \equiv 0$. Notice that Q' has 8 nodes corresponding to the 8 nodes in the branch curve. These give 8 (-2)-curves in Q. We can show that Q contains 8 other disjoint (-2)-curves, contained in the pullback of the lines T_1, T_2. Hence Q is a $K3$ surface with 16 (-2)-curves $A_1, \ldots, A_{16}$; it is a Kummer surface. Now consider the double covering $\psi : A' \to Q$ with branch curve $\sum_1^{16} A_i$. Since A_i is in the branch locus, then $\psi^*(A_i)$ is a double curve $2E_i$, $i = 1, \ldots, 16$. From $(2E_i)^2 = 2A_i^2 = -4$, we get $E_i^2 = -1$, $i = 1, \ldots, 16$. The minimal model A of A' is obtained by contracting the (-1)-curves $E_1, \ldots, E_{16}$. The surface A is an abelian surface.

Choosing a certain branch curve in A, we have constructed a double covering of A that gives the first example of a surface with $p_g = q = 2$ and $K^2 = 7$, see [Rit15] for the details.

4 Triple coverings

Triple coverings are more complicated than double coverings, but they share a common nice property: both have a *canonical resolution* of singularities. Briefly this means that one can resolve the singularities of the surface via resolving the singularities of the branch locus of the covering.

Here we explain the idea of the construction of a surface of general type with $p_g = 0$ and $K^2 = 3$ which is obtained by a triple covering of a certain singular Godeaux surface.

A *cusp* singularity of a surface is a singularity with local equation $x^2 + y^2 = z^3$. Its resolution is the union of two (-2)-curves A, A' such that $AA' = 1$. It is a type of singularity that may appear when one takes the quotient of a surface by the action of a group isomorphic to $\mathbb{Z}_3$, with fixed points. Conversely, if a surface X has cusps $c_1, \ldots, c_n$ satisfying a certain 3-divisibility condition, namely

$$\sum_1^n (A_i + 2A_i') \equiv 3L$$

for some divisor L, then there is a $\mathbb{Z}_3$-covering $\tau : S \to X$ with branch locus $\cup c_i$. The surface S is smooth at $\tau^{-1}(c_i)$, $i = 1, \ldots, n$. The invariants of S and X are related by

$$\chi(S) = 3\chi(X) - \frac{2n}{3}, \quad K_S^2 = 3K_X^2.$$

So, if X is a Godeaux surface with a 3-divisible set of 3 cusps and no other singularities, then S is a smooth surface with $\chi = 1$ and $K^2 = 3$. As in Section 2, the surface X is a $\mathbb{Z}_5$-quotient of a quintic surface $Q \subset \mathbb{P}^3$, but now Q has 15 cusp singularities. The problem is how to find such a singular quintic.

It is classically known that a threefold of degree 3 in $\mathbb{P}^4$ has at most 10 nodes, and there is exactly one such threefold with 10 nodes, the *Segre cubic*. The dual of the Segre cubic is the so-called *Igusa quartic*. Its singular set is an union of 15 lines. We have used computer algebra to construct a quintic threefold passing through the 15 singular lines of the Igusa quartic, with 15 cuspidal lines there. This means that a general hyperplane section of this threefold is a quintic surface with 15 cusps. The task now is to find one of these with a free action of the group $\mathbb{Z}_5$. This has been achieved by using computer algebra and some symmetry. This has produced two non-isomorphic surfaces with $p_g = 0$ and $K^2 = 3$, one is a surface implicitly constructed from results in [vdGZ77] and [Bar00], the other is new, see [Rit16].

5 Quotients of products of curves

Consider the surface $\mathbb{P}^1 \times \mathbb{P}^1$ and let f, g be the fibrations given by the projections onto the first and second factor, respectively. Let $F_1, \ldots, F_4$ be fibres of f and $G_1, \ldots, G_4$ be fibres of g. The curve

$$B := F_1 + \ldots + F_4 + G_1 + \cdots + G_4$$

has 16 nodes. There is a $\mathbb{Z}_2^2$ covering

$$E_1 \times E_2 \xrightarrow{\gamma} Q \xrightarrow{\delta} \mathbb{P}^1 \times \mathbb{P}^1,$$

with E_1, E_2 elliptic curves: the map δ is the double covering ramified over B, then Q is a Kummer surface, the double covering γ of Q ramified over its nodes is an abelian surface, and we can show that this surface is the product of two elliptic curves.

So, the surface Q, constructed as a covering of $\mathbb{P}^1 \times \mathbb{P}^1$, could be initially obtained as a quotient of $E_1 \times E_2$. This can be done more in general. Let C, D be smooth curves and G be a group acting on the product $C \times D$. The quotient $X := (C \times D)/G$ is a surface, with singularities corresponding to the points fixed by G. The invariants of the smooth minimal model of X can be easily calculated, there has been a considerable work on these type of surfaces (see e.g. [Bau12] for a survey on product-quotient surfaces). This method has proven to be very efficient.

Such quotient surfaces with $\chi = 1$ have been exhaustively studied. If the action of G is free, then $K^2 = 8$ and several examples have been obtained. A product of curves is a quotient of $\mathbb{H} \times \mathbb{H}$, where $\mathbb{H}$ is the complex upper-half plane. Thus these surfaces are covered by the *bidisk* $\mathbb{H} \times \mathbb{H}$. It was an open problem to provide an example of a surface of general type with $\chi = 1$ and $K^2 = 8$ not covered by the bidisk. Such an example was found in a collaboration with F. Polizzi and X. Roulleau. Surprisingly again, it is obtained using double coverings, see [PRR] for the details.

6 Ball quotients

So far we have talked about coverings and quotients by finite groups. But these groups can be infinite. It is known that all smooth minimal algebraic surfaces satisfying $K^2 = 9\chi$ are *ball quotients*, i.e. are obtained as a quotient $\mathbb{B}/G$, where $\mathbb{B}$ is the unit ball in $\mathbb{C}^2$ and G is some infinite group.

A surface of general type with the same invariants $p_g = q = 0$, $K^2 = 9$ as $\mathbb{P}^2$ is called a *fake projective plane*. These surfaces have been classified ([PY07], [CS10]), there are exactly 100 such surfaces, which are 50 pairs of complex-conjugated surfaces. The methods used, related to arithmetic groups, are not typical from algebraic geometry. The output of this classification is a list of the groups G, available at [Car]. Some information about the geometry of these surfaces is very hard to get. But it is interesting to play with the groups. For instance degree n coverings of the surface $\mathbb{B}/G$ correspond to index n subgroups of G, and (some of) these can be computed. Then one may wonder how to get it from the geometry...

As a by-product of the work on fake projective planes, in [CS10] the unique (up to complex-conjugation) example of a surface with $p_g = q = 1$ and $K^2 = 9$ is given. It is known as the Cartwright-Steger surface, one of the most intriguing surfaces ever found.

A geometric construction of any of the above surfaces with $\chi = 1$ and $K^2 = 9$ is a very interesting open problem on the theory of algebraic surfaces.

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