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Contents

Editorial

Due to some logistic and budget constraints, the editorial board of the bulletin decided to publish this double issue instead of two distinct issues in 2017.

In this number of the bulletin we present three articles about recent developments in the areas of Analysis, Geometry and Mathematical Physics. We continue the cycle of historical articles honouring José Anastácio da Cunha and his work. We include an interview to Patrícia Gonçalves who won an ERC starting grant and, as usual, we publish several summaries and reports of some of the activities partially supported by CIM.

We highlight the CIM-SPM's initiative Pedro Nunes' Lectures, which brought Étienne Ghys to deliver three outstanding lectures, in Portuguese, at the Fundação Calouste Gulbenkian and at the universities of Coimbra and Porto. We feature an interview with Étienne Ghys in order to celebrate the success of his visit to Portugal.

We recall that the bulletin welcomes the submission of review, feature, outreach and research articles in Mathematics and its applications.

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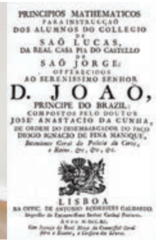
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LISBOA


ECMTB 2018
LISBON

11th European Conference on Mathematical and Theoretical Biology (ECMTB 2018)

23 to 27 July, 2018

Lisbon

The 11th European Conference on Mathematical and Theoretical Biology (ECMTB 2018) will be held in Lisbon, Portugal, from 23 to 27 July, 2018. The venue is the Faculdade de Ciências da Universidade de Lisboa (Faculty of Sciences of the University of Lisbon), FCUL, and its research centre CMAF-CIO will host the event.

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CIM-WIAS Workshop

Topics in Applied Analysis and Optimisation

(Stochastic, Partial Differential Equations and Numerical Analysis)

06 to 08 December, 2017

Lisbon

A joint CIM-WIAS Workshop co-organised at the Department of Mathematics of the Faculty of Sciences of the University of Lisbon , with the cooperation of CMAF-CIO/ULisboa and CMUC/UCoimbra. This event will present and discuss current scientific interests among the research groups of the Weierstrass Institute in Berlin and mathematics centres in Portugal. This CIM -WIAS workshop will bring together a selection of experts in Europe and is expected to launch and strengthen further collaborations. Topics of interest include PDEs with applications to Material Sciences, Thermodynamics and Laser dynamics, Scientific computing, Nonlinear optimisation and Stochastic analysis.

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On the construction of complex algebraic surfaces

by **Carlos Rito***

Quotients and *coverings* appear frequently in the construction of algebraic varieties. If a variety V has some property that makes it a quotient or a covering of some simpler variety U, then one may try to construct V starting from this simpler U. This method has proven to be very efficient in the construction of complex algebraic surfaces. This is a vast theme, here we do not intend to give a detailed survey on it. Our aim is to give a taste of the subject, for non-specialists, by presenting some examples, most of them related to the work of the author.

1 Generalities

Let S be a complex smooth algebraic surface that is projective (i.e. that has an embedding into a *projective space*). A *divisor* in S is a formal sum of curves $\sum n_i C_i$, $n_i \in \mathbb{Z}$. A divisor is *effective* if the coefficients n_i are all non-negative. Two divisors D, D' are linearly equivalent if the difference $D - D'$ is the divisor of zeros and poles of a rational function f/g on S. In this case we write $D \equiv D'$. The *complete linear system* $|D|$ is the set of all divisors linearly equivalent to D.

To give a divisor on a surface S is not trivial. If the surface is embedded in some projective space $\mathbb{P}^n$, an obvious way is to take *hyperplane sections* (i.e. intersections with hyperplanes), but this depends on the particular embedding. There is a divisor which is intrinsic to the variety (i.e. it does not depend on the embedding), the *canonical divisor* K_S: it is the divisor of a meromorphic differential 2-form $f\, dx \wedge dy$. The number of generators of the canonical linear system $|K_S|$ is the *geometric genus* $p_g(S)$. If the surface admits a holomorphic differential 1-form $fdx + gdy$, we say that it is *irregular*, and the irregularity $q(S)$ is the number of independent such forms.

The *pluricanonical* map $\phi_{|nK_S|} : S \to \mathbb{P}^d$ is the map given by the sections of the pluricanonical linear system $|nK_S|$ (hence $d = \dim(|nK_S|)$). The *Kodaira dimension* $\mathrm{Kod}(S)$ is the maximum of the dimensions of the images $\phi_{|nK_S|}(S)$, $n \in \mathbb{N}$. Obviously it is at most 2, and surfaces of Kodaira dimension 2 are said to be *of general type*.

It is known since Hironaka [Hir64] that there is always a resolution of singularities for algebraic varieties over a field of characteristic zero. He has been awarded the 1970 Fields Medal for this result. His proof consists in repeatedly *blowing-up* along non-singular subvarieties, and to show that the process ends. The idea of resolving a singularity by blowing-up is as follows. Given a smooth surface X and a point $p \in X$, there is a smooth surface Y and a map $\pi : Y \to X$ such that $E := \pi^{-1}(p) \cong \mathbb{P}^1$ and $Y \backslash E \cong X \backslash p$. Now let $C \subset Y$ be a smooth curve intersecting E at n distinct points. Then $\pi(C) \subset X$ is a curve with a singular point of multiplicity n at p, and the blow-up π resolves the singularity of the curve $\pi(C)$.

Given curves $C, C' \subset S$ without common components, with S a smooth surface, the *intersection number* CC' is the number of points in $C \cap C'$, counting multiplicities. If $C \equiv C'$, we say that $C^2 := CC'$ is the *self-intersection* of C. The definition can be extended to all curves, and some curves have negative self-intersection. For instance, consider the blow-up $\pi : Y \to \mathbb{P}^2$ at a point $p \in \mathbb{P}^2$. Let $E \subset Y$ be the *exceptional divisor* as above, $L, L' \subset \mathbb{P}^2$ be distinct lines through p and consider the strict transforms $\hat{L}, \hat{L}' \subset Y$

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of L, L'. Since $\hat{L}\hat{L}' = 0$ and $\hat{L} \equiv \hat{L}'$, then $\hat{L}^2 = 0$. We have that

$$1 = L^2 = \pi^*(L)^2 = (E + \hat{L})^2 = E^2 + 0 + 2,$$

thus $E^2 = -1$. We say that E is a (-1)-curve. Any (-1)-curve can be contracted to a smooth point of the surface. Curves with self-intersection < -1 are contracted to singular points. For example the resolution of a *node* (ordinary double point) is a (-2)-curve, i.e. a curve isomorphic to $\mathbb{P}^1$ with self-intersection -2.

We say that a smooth surface is *minimal* if it has no (-1)-curves; the *minimal model* of a smooth surface is obtained contracting all its (-1)-curves. The minimal model of a surface with non-negative Kodaira dimension is unique, i.e. different choices for the contraction of all (-1)-curves give rise to isomorphic surfaces.

The geometric genus $p_g(S)$ and the irregularity $q(S)$ of the surface S are topological invariants. Finally, the *holomorphic Euler characteristic* of S is

$$\chi(S) := 1 - q(S) + p_g(S),$$

and the *topological Euler characteristic* $\chi_{\text{top}}(S)$ satisfies

$$K_S^2 + \chi_{\text{top}}(S) = 12\chi(S).$$

Geometers want to classify surfaces according to these invariants. For surfaces of general type always $\chi \geq 1$, and naturally one wants to classify the case $\chi = 1$, but this is still far from completed.

2 Campedelli vs Godeaux

Let S be a smooth minimal surface of general type. One has $K_S^2 \geq 1$, and the Bogomolov-Miyaoka-Yau inequality $K_S^2 \leq 9\chi(S)$ holds, hence $1 \leq K_S^2 \leq 9$ for surfaces with $\chi = 1$. These surfaces satisfy $p_g = q$, and the ones with $p_g = q = 0$ have received particular attention. In the 19th century it was thought that these surfaces were *rational* (i.e. obtained from the projective plane by a sequence of blow-ups and blow-downs). Then Enriques [Enr96], in 1896, showed the existence of surfaces with $p_g = q = 0$ and Kodaira dimension 0, hence not rational. Nowadays these are called *Enriques surfaces*. In the same year, Castelnuovo [Cas96] proved his *rationality criterion*: an algebraic surface S is rational if and only if $q(S) = 0$ and the linear system $|2K_S|$ is empty.

The first examples of surfaces of general type with $p_g = q = 0$ were discovered by Godeaux [God31] ($K^2 = 1$) and Campedelli [Cam32] ($K^2 = 2$) in the 1930s.

The Godeaux construction is as follows. Let $Q \subset \mathbb{P}^3$ be the quintic surface with equation

$$x^5 + y^5 + z^5 + w^5 = 0.$$

It is invariant for the $\mathbb{Z}_5$ action

$$\sigma : (x : y : z : w) \to (x : \rho y : \rho^2 z : \rho^3 w),$$

with ρ a 5th root of unity. Since the action is free and $\chi(Q) = 5, q(Q) = 0, K_Q^2 = 5$, then the surface Q/σ is smooth and has invariants $\chi = 1, q = 0$ and $K^2 = 1$.

The Campedelli surface is obtained as (the resolution of the singularities of) a double cover of $\mathbb{P}^2$ ramified over a curve $\{f_{10} = 0\}$ of degree 10 with 6 singularities of type $(3, 3)$ (a triple point with the 3 branches sharing the same tangent line). This can be seen as the surface with equation $w^2 = f_{10}(x, y, z)$ in the *weighted projective space* $\mathbb{P}[1, 1, 1, 5]$. Its invariants are $\chi = 1, q = 0$ and $K^2 = 2$.

These are typical examples of the two most successful methods for the construction of algebraic surfaces: quotients and coverings. One can discuss which method is more efficient, but frequently the construction is given by a combination of the two. Below we give some examples.

3 Double coverings

Let S be a smooth surface with an *involution*, i.e. with a non-trivial automorphism σ such that $\sigma^2 = \mathrm{Id}$. The projection

$$\varphi : S \to X := S/\sigma$$

is a *double covering*. The *ramification set* of φ is the set of points fixed by σ; it is the union of a smooth curve with a finite number $n \geq 0$ of points $p_1, \ldots, p_n$, which correspond to nodes of S/σ. Let $S' \to S$ be the blow-up of S at these points. Then σ extends to an involution σ' on S' with ramification

$$R' := R + \sum_1^n E_i,$$

where E_i is the exceptional curve corresponding to p_i, and R is a smooth curve disjoint from E_i, $i = 1, \ldots, n$. The *branch locus* of

$$\varphi' : S' \to X' := S'/\sigma'$$

is the (smooth) curve $B := \varphi'(R')$. One can show that there exists a divisor L such that $B \equiv 2L$ (we say that B is 2-divisible). The projection $\rho : X' \to \bar{X}$ to the minimal model gives a singular curve $\bar{B} := \rho(B)$.

Conversely, from a smooth surface $\bar{X}$ and a 2-divisible (possibly singular) branch curve $\bar{B}$, we can recover the surface S: we take the double covering $\bar{S} \to \bar{X}$ ramified over $\bar{B}$; the *smooth minimal model* of $\bar{S}$, obtained by resolving the singularities of $\bar{S}$ and contracting all (-1)-curves, is a surface isomorphic to S.

$$\begin{tikzcd}
\bar{S} \arrow[d] & S' \arrow[l] \arrow[r] \arrow[d] & S \arrow[d] \\
\bar{X} & X' \arrow[l] \arrow[r] & X
\end{tikzcd}$$

Frequently the curve $\bar{B}$ is highly singular; construction methods that include the use of symmetry and computational tools have proved useful.

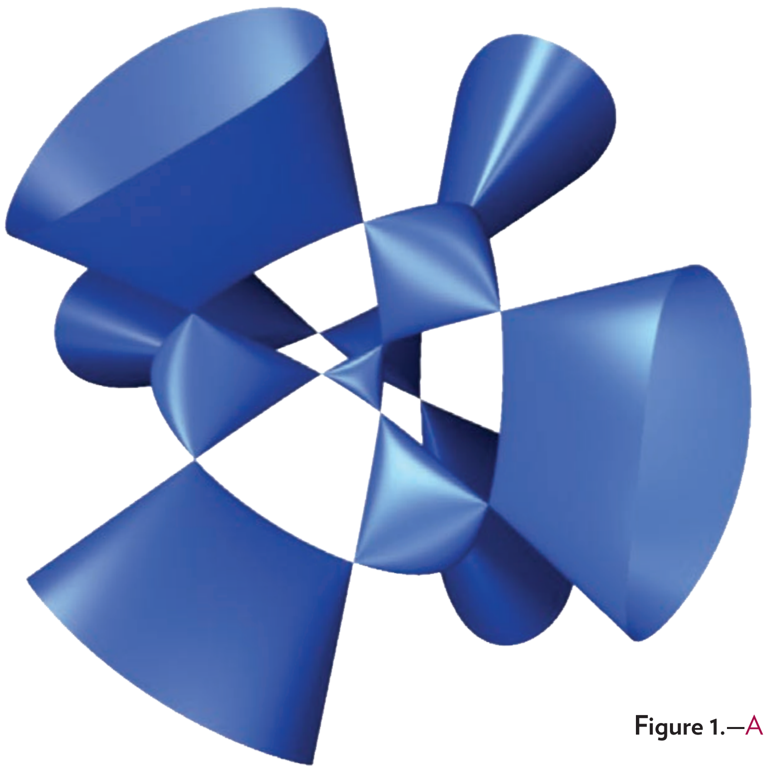

Figure 1.—A Kummer surface in $\mathbb{R}^3$

Since Godeaux and Campedelli, mathematicians have been giving examples of surfaces of general type with $\chi = 1$ for each possible value of the invariants $p_g = q$ and $3 \leq K^2 \leq 9$. The author has given the first examples for the mysterious cases $K^2 = 7$ and $p_g = q = 1, 2$ (see [Rit10], [Rit15]). Somehow surprisingly, these were obtained using double coverings.

3.1 $p_g = q = 1, K^2 = 7$

A singular point of a curve is *of type* (n, n) if it is a singular point of multiplicity n with branches sharing the same tangent line at the point. This means that we still have a singularity of multiplicity n after one blow-up; we say that these two points of multiplicity n are *infinitely near*.

A *double plane* is a surface S with an involution i such that the quotient S/i is a rational surface. It can be obtained as (the resolution of the singularities of) a double covering $\bar{S} \to \mathbb{P}^2$, ramified over a (singular) branch curve B. The problem is then how to find B, because the surface S is determined by it.

In his work on double planes, Du Val [du 52] proposed a configuration of branch curves B for the construction of some surfaces with low invariants, in particular $p_g = q = 0$. A similar configuration gives invariants $p_g = q = 1$. But these curves are highly singular (with some points of type (n, n)), and hence it is difficult to prove their existence.

Given points $p_1, \dots, p_n$, possibly infinitely near, one can use computer algebra, or more precisely the computer algebra system Magma [BCP97] and the algorithm given in [Rit10], to compute the linear system of plane curves with given degree and singularities at $p_1, \dots, p_n$. But in general, if these points are not chosen properly, this system is empty. Thus we have to compute points in a special position such that the curve exists. The idea is as follows.

Suppose we want to compute a plane curve of degree d with a singular point of multiplicity n. Its defining polynomial is a linear combination $\sum a_i m_i(x, y)$, where $m_i(x, y)$ are monomials. We want to compute the coefficients a_i. Recall that a plane curve has a point of multiplicity n if and only if the derivatives up to order $n - 1$ vanish at that point. Let M be the matrix with lines the derivatives of the m_i up to order $n - 1$. The curve exists if M is not of maximal rank. Thus we need to compute points in the variety given by the vanishing of certain minors of the matrix M. Since some points are infinitely near, the process is a combination of the above with some blow-ups. Then the success depends on the computational complexity. For the construction of the case $p_g = q = 1$ and $K^2 = 7$, see [Rit10].

3.2 $p_g = q = 2, K^2 = 7$

Recall that an *elliptic curve* is the quotient of $\mathbb{C}$ by a lattice Γ. Topologically it is a torus. In the same way, a quotient $A := \mathbb{C}^2/\Gamma$ is a complex surface, a torus of dimension 2. The ones that are algebraic are called *abelian surfaces*. Multiplication by -1 in $\mathbb{C}^2$ gives rise to an involution σ on A. The quotient A/σ is a *Kummer surface*, a complex algebraic surface with $q = 0, K \equiv 0$, and having 16 nodes, corresponding to the 16 fixed points of σ (figure 1).

Now we explain the construction of a particular abelian surface, starting from a double covering of the projective plane. Let p_0, p_1, p_2 be distinct points in the projective plane $\mathbb{P}^2$ and let T_i be the line through p_0, p_i, $i = 1, 2$. There is a 1-dimensional linear system of conics tangent to T_1, T_2 at p_1, p_2, respectively. Take C_1, C_2 distinct smooth conics in this system, and choose T_3, T_4 general lines through p_0. Let $Q' \to \mathbb{P}^2$ be the double covering with branch curve $C_1 + C_2 + T_3 + T_4$. It is well known that the smooth minimal model Q of Q' is a $K3$ surface, i.e. a smooth minimal surface with $q = 0$ and $K \equiv 0$. Notice that Q' has 8 nodes corresponding to the 8 nodes in the branch curve. These give 8 (-2)-curves in Q. We can show that Q contains 8 other disjoint (-2)-curves, contained in the pullback of the lines T_1, T_2. Hence Q is a $K3$ surface with 16 (-2)-curves $A_1, \dots, A_{16}$; it is a Kummer surface. Now consider the double covering $\psi : A' \to Q$ with branch curve $\sum_1^{16} A_i$. Since A_i is in the branch locus, then $\psi^*(A_i)$ is a double curve $2E_i$, $i = 1, \dots, 16$. From $(2E_i)^2 = 2A_i^2 = -4$, we get $E_i^2 = -1$, $i = 1, \dots, 16$. The minimal model A of A' is obtained by contracting the (-1)-curves $E_1, \dots, E_{16}$. The surface A is an abelian surface.

Choosing a certain branch curve in A, we have constructed a double covering of A that gives the first example of a surface with $p_g = q = 2$ and $K^2 = 7$, see [Rit15] for the details.

4 Triple coverings

Triple coverings are more complicated than double coverings, but they share a common nice property: both have a *canonical resolution* of singularities. Briefly this means that one can resolve the singularities of the surface via resolving the singularities of the branch locus of the covering.

Here we explain the idea of the construction of a surface of general type with $p_g = 0$ and $K^2 = 3$ which is obtained by a triple covering of a certain singular Godeaux surface.

A *cusp* singularity of a surface is a singularity with local equation $x^2 + y^2 = z^3$. Its resolution is the union of two (-2)-curves A, A' such that $AA' = 1$. It is a type of singularity that may appear when one takes the quotient of a surface by the action of a group isomorphic to $\mathbb{Z}_3$, with fixed points. Conversely, if a surface X has cusps $c_1, \dots, c_n$ satisfying a certain 3-divisibility condition, namely

$$\sum_1^n (A_i + 2A_i') \equiv 3L$$

for some divisor L, then there is a $\mathbb{Z}_3$-covering $\tau : S \to X$ with branch locus $\cup c_i$. The surface S is smooth at $\tau^{-1}(c_i)$, $i = 1, \dots, n$. The invariants of S and X are related by

$$\chi(S) = 3\chi(X) - \frac{2n}{3}, \quad K_S^2 = 3K_X^2.$$

So, if X is a Godeaux surface with a 3-divisible set of 3 cusps and no other singularities, then S is a smooth surface with $\chi = 1$ and $K^2 = 3$. As in Section 2, the surface X is a $\mathbb{Z}_5$-quotient of a quintic surface $Q \subset \mathbb{P}^3$, but now Q has 15 cusp singularities. The problem is how to find such a singular quintic.

It is classically known that a threefold of degree 3 in $\mathbb{P}^4$ has at most 10 nodes, and there is exactly one such threefold with 10 nodes, the *Segre cubic*. The dual of the Segre cubic is the so-called *Igusa quartic*. Its singular set is an union of 15 lines. We have used computer algebra to construct a quintic threefold passing through the 15 singular lines of the Igusa quartic, with 15 cuspidal lines there. This means that a general hyperplane section of this threefold is a quintic surface with 15 cusps. The task now is to find one of these with a free action of the group $\mathbb{Z}_5$. This has been achieved by using computer algebra and some symmetry. This has produced two non-isomorphic surfaces with $p_g = 0$ and $K^2 = 3$, one is a surface implicitly constructed from results in [vdGZ77] and [Bar00], the other is new, see [Rit16].

5 Quotients of products of curves

Consider the surface $\mathbb{P}^1 \times \mathbb{P}^1$ and let f, g be the fibrations given by the projections onto the first and second factor, respectively. Let $F_1, \dots, F_4$ be fibres of f and $G_1, \dots, G_4$ be fibres of g. The curve

$$B := F_1 + \dots + F_4 + G_1 + \dots + G_4$$

has 16 nodes. There is a $\mathbb{Z}_2^2$ covering

$$E_1 \times E_2 \xrightarrow{\gamma} Q \xrightarrow{\delta} \mathbb{P}^1 \times \mathbb{P}^1,$$

with E_1, E_2 elliptic curves: the map δ is the double covering ramified over B, then Q is a Kummer surface, the double covering γ of Q ramified over its nodes is an abelian surface, and we can show that this surface is the product of two elliptic curves.

So, the surface Q, constructed as a covering of $\mathbb{P}^1 \times \mathbb{P}^1$, could be initially obtained as a quotient of $E_1 \times E_2$. This can be done more in general. Let C, D be smooth curves and G be a group acting on the product $C \times D$. The quotient $X := (C \times D)/G$ is a surface, with singularities corresponding to the points fixed by G. The invariants of the smooth minimal model of X can be easily calculated, there has been a considerable work on these type of surfaces (see e.g. [Bau12] for a survey on product-quotient surfaces). This method has proven to be very efficient.

Such quotient surfaces with $\chi = 1$ have been exhaustively studied. If the action of G is free, then $K^2 = 8$ and several examples have been obtained. A product of curves is a quotient of $\mathbb{H} \times \mathbb{H}$, where $\mathbb{H}$ is the complex upper-half plane. Thus these surfaces are covered by the *bidisk* $\mathbb{H} \times \mathbb{H}$. It was an open problem to provide an example of a surface of general type with $\chi = 1$ and $K^2 = 8$ not covered by the bidisk. Such an example was found in a collaboration with F. Polizzi and X. Roulleau. Surprisingly again, it is obtained using double coverings, see [PRR] for the details.

6 Ball quotients

So far we have talked about coverings and quotients by finite groups. But these groups can be infinite. It is known that all smooth minimal algebraic surfaces satisfying $K^2 = 9\chi$ are *ball quotients*, i.e. are obtained as a quotient $\mathbb{B}/G$, where $\mathbb{B}$ is the unit ball in $\mathbb{C}^2$ and G is some infinite group.

A surface of general type with the same invariants $p_g = q = 0$, $K^2 = 9$ as $\mathbb{P}^2$ is called a *fake projective plane*. These surfaces have been classified ([PY07], [CS10]), there are exactly 100 such surfaces, which are 50 pairs of complex-conjugated surfaces. The methods used, related to arithmetic groups, are not typical from algebraic geometry. The output of this classification is a list of the groups G, available at [Car]. Some information about the geometry of these surfaces is very hard to get. But it is interesting to play with the groups. For instance degree n coverings of the surface $\mathbb{B}/G$ correspond to index n subgroups of G, and (some of) these can be computed. Then one may wonder how to get it from the geometry...

As a by-product of the work on fake projective planes, in [CS10] the unique (up to complex-conjugation) example of a surface with $p_g = q = 1$ and $K^2 = 9$ is given. It is known as the Cartwright-Steger surface, one of the most intriguing surfaces ever found.

A geometric construction of any of the above surfaces with $\chi = 1$ and $K^2 = 9$ is a very interesting open problem on the theory of algebraic surfaces.

Acknowledgements

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DSABNS 2017
Eighth Workshop Dynamical Systems Applied to Biology and Natural Systems

by **Maíra Aguiar***

The *Workshop Dynamical Systems Applied to Biology and Natural Sciences*–DSABNS is a well established international scientific event that has been organized by the *Biomathematics and Statistics group* at *Centro de Matemática, Aplicaçõees Fundamentais e Investigacção Operacional* (CMAF-CIO) from *Lisbon University* since 2010, every year during the month of February. The scientific program includes lectures by invited speakers, contributed talks and poster presentations, covering research topics in scientific areas such as population dynamics, eco-epidemiology, epidemiology of infectious diseases, molecular and antigenic evolution and methodical topics in the natural sciences and mathematics.

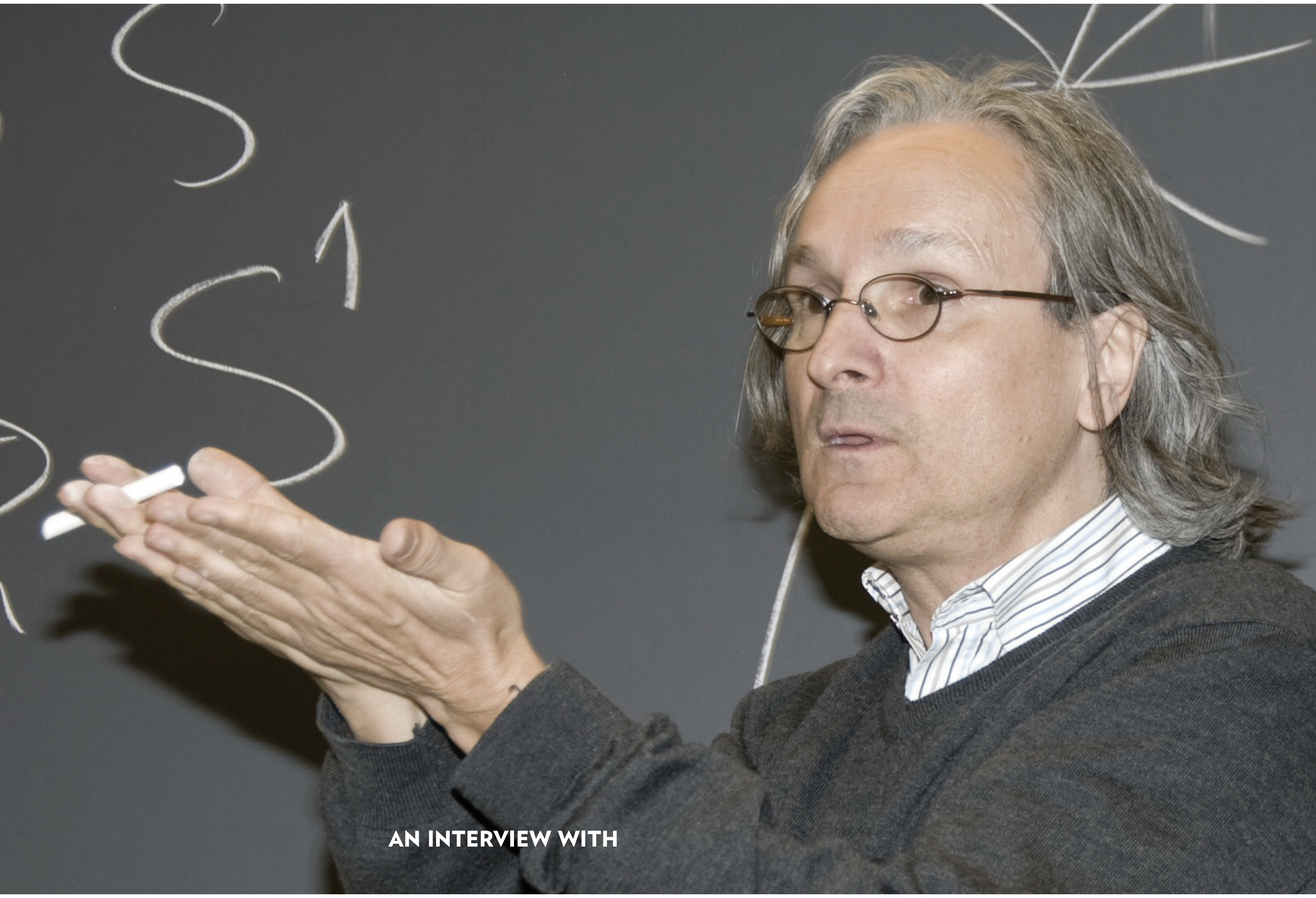
All participants are allowed to present their scientific work, which make this event an excellent opportunity for internationalization, specially for the Portuguese scientific community. The fact that there is no registration fee for the workshop encourages the participation of students, specially Portuguese, which is in fact extremely important in the area of applied mathematics.

The 8th Workshop DSABNS was held at *Colégio do Espírito Santo*, *Évora University*, hosted by its research centre CIMA and the participation of the *Department of Mathematics, Escola de Ciências e Tecnologia*, from January 31st to February 3rd, 2017. The scientific program included lectures by the 19 Plenary speakers, 10 Invited talks, 40 contributed talks and 14 posters. With more than 90 participants, from 18 different countries, the workshop had financial support from FACC-FCT and the research centers CMAF-CIO (*Universidade de Lisboa*), CMA (*Universidade Nova de Lisboa*, *NovaID*) and CIM (*Centro Internacional de Matemática*).

For more information, please visit

http : //dsabns2017.fc.ul.pt/

* CMA, FCT, Universidade Nova de Lisboa and CMAF-CIO, Universidade de Lisboa



Étienne Ghys

by **José Ferreira Alves***

Étienne Ghys (born 29 December 1954) is a French mathematician recognised for his outstanding contributions in the fields of geometry and dynamical systems, as well as for his exceptional role in the dissemination of Mathematics.

Presently, he is a CNRS Directeur de Recherche at ENS, Lyon. His impressive research has lead to many distinctions and awards, such as his elevation to the french Académie des Sciences in 2004, the title of *Chevalier de la légion d'honneur* in 2012, the invitations as a speaker at the ICM of Kyoto, as a member of the program committee for the ICM in Hyderabad, as a member of the Fields Medal committee in 2014 or the prize *Prix Servant of the Académie des Sciences*. He is an honorary member of several prestigious societies around the world and was distinguished as doctor honoris causa by the University of Geneva, in 2008. He has served as editor of several prestigious journals such as *Annals of Mathematics* and *Publications Mathématiques de l'IHÉS*.

His work in the promotion of mathematics is remarkable and was distinguished with the *Clay award for dissemination of mathematical knowledge*, in 2015, the *Prix du livre audio* 2011, the *Prix d'Alembert de la SMF*, in 2010. His series of films, produced with Aurélien Alvarez and Jos Leys and published as DVDs and online in many languages, has had a huge impact on high school students. The first, *Dimensions*, has been downloaded more than a million times.

* Centro de Matemática & Faculdade de Ciências da Universidade do Porto

Personal

How did you become a Mathematician? In particular, how did you get interested in Dynamical Systems and Geometry?

So, these are two different questions. How did I become a mathematician? It was not a decision. I was a good pupil at school and I made no positive choice. When I was a kid I loved science, any kind of science. Could be Physics or even Chemistry . . . . I remember people asking me "what do you want to do when you grow up?". My answer was "I want to be ingénieur". Because in my family, which was not a rich family, the closest to science was engineering. Then it was a progressive decision: when I was at Mathématiques Supérieures I just understood that what I loved in Physics was its mathematical aspect. So my decision was not as a yes or no question, I went to Mathematics in a progressive way.

Now the question about Dynamical Systems and Geometry. I am sorry to say, it is not related to the field but to a human contact. I was a student at École Normale Supérieure, in Saint-Cloud, I had a girlfriend—who is my wife now—, she was in my home city, Lille, I was in Paris and I was unhappy. So I said to the director of the Math Department in École Normale Supérieure, "maybe I could come back to Lille", and he asked me why. I answered "because my girlfriend is there" and he found it a good idea. So I went to Lille and I got a letter of recommendation for professor Gilbert Hector. I liked him and he was doing Geometry and Dynamical Systems: foliations. So what I did choose was the professor and not the topic. I like Mathematics, of course, but I like even more the human contact.

Besides Mathematics, which I think you do not see merely as a job, but also as a hobby, what are the other interests that you have?

To be honest, none!

No other interests?

Well, I am interested by many other things of course, but not as much as Mathematics. You know, if I would tell you that I like cinema, it would be true, but not as much as Mathematics.However, I can say something: the older I get, the more I am interested by Philosophy.

I know that you speak Portuguese very well. As you, many other top French mathematicians (Bonatti, Gambaudo, Yoccoz, who recently passed away, unfortunately) speak Portuguese fluently. This is not just a coincidence. Would you tell us the reason for that?

The reason is not quite the same for all these people. Well, officially it is the same thing, but in my case, when I was a student there was a compulsory military service in France and I was 100% anti-militarist. For example, in France we have École Polytechnique I didn't want to be a student in a

military school. So, one of the possibilities, not to have the military service, was to choose the so-called Coopération Scientifique: instead of staying one year in the military service we could go two years to a teaching position, typically in Algeria, Tunisia . . . North Africa. In my case, because my advisor had spent a year in Rio de Janeiro, at IMPA, he suggested me to go there. Fortunately I could go there and it was a great success. I spent two years there not in military service but in scientific cooperation. The motivation of Jean-Christophe (Yoccoz) was somehow different: he also came in Rio to replace military service, but I believe that Jean-Christophe was not as anti-militarist as I was. Probably, the same happened with Christian (Bonatti).

What about today?

Actually, today there is no more compulsory military service in France: good news! I think that IMPA somehow lost a natural source of young French mathematicians.

But since then IMPA has established strong relations with France . . .

Yes, but not with the very young. When I came there I was only 25.

Now French mathematicians do not go there so young, maybe not so keen on learning a new language.

Yes, I think so. Now they hire post-docs, and there are not so many French post-docs at IMPA.

Mathematics

The History of Mathematics has a lot of mathematicians that we admire. You seem to have a particular admiration for Poincaré, is it true?

Yes, it is true. Maybe one of the reasons why I like him is because his proofs are frequently incorrect, but not his ideas! So, I like his style, not because of the mistakes, of course, but because he is like a discoverer that goes in the forest without taking too much care of what is behind him.

We could say he was more concerned with the ideas than with the formalism.

Yes, his formalism is reduced to the strict minimum. When I read a paper by Poincaré I know that there is a good probability that it might be slightly incorrect and I have to stay alert and I have to question any single sentence. This is very different from other mathematicians.

You are recognized for many valuable contributions to the divulgation of Mathematics, not only with talks, but also with texts and movies. What is the importance of mathematical divulgation for you?

I want to be honest: one of the main reasons for me is that it helps me to understand. I discovered that when I try to explain things to others I understand better. There is a famous sentence by Gergonne, a French mathematician form the 19th Century, he said something like "no mathematician can claim that he understands something, unless he can explain it to the man on the street". Of course today it is more difficult, but the idea is still valid. Many times, trying to explain things to others made me understand better.

Mathematics is the art of having good ideas to solve problems, but also the art of telling other fellows the solutions. Of course not everybody has excellent skills for both. But don't you think that some of our colleagues neglect this second component of Mathematics?

Not some . . . most! [Laughs].

I was trying to be kind . . .

I think it is dangerous, because some mathematicians can get lost. As if they go along a trail in the forest and they don't take care of explaining to others. They go so far that nobody can follow them, and they don't even take care of cleaning the path behind them. They might get alone in the forest! I don't give names . . .

I have several in mind . . .

. . . but many mathematicians get isolated form the others.

It is a pity, because Mathematics is an art, sharing is important.

Poincaré was not like that. It is impressive to note that he frequently gave public lectures or wrote elementary papers in order to explain his main ideas. One good point with Dynamical Systems, Geometry and Topology is that it is somehow easier to explain.

For people on the street we have some nice sentences.

Unfortunately, always the same! [Laughs]

Since Poincaré, the field of Dynamical Systems has developed a lot and got recognition from the mathematical community with several Fields Medals awarded to dynamicists. What do you think are the main challenges for the field in the next decades?

It is a hard question. I am not sure that you will like my answer . . .

Well, I am here as a journalist . . . [Laughs]

. . . but I would say that it is important to reconnect the theory of Dynamical systems with Physics. The origin of Dynamics is clearly Physics—Celestial Mechanics in particular—and it seems to me that many experts in this field have lost contact with physicists. I think it could be a priority now. For example, one could question the relevance of these contributions on the C^1 generic Dynamical Systems for Physics. It is certainly interesting from the mathematical point of view, no doubt, but is this what we want to do? Do we want to go in the forest and get lost? So, my main comment would be, please, do not forget where we come from: Physics.

Let me say something in favor of our colleagues: it is true that mathematicians sometimes go very far and get lost in the forest, but historically it happened many times that people from other fields came to rescue the lost ones. For instance, the theory developed by Einstein used Mathematics that could possibly be considered lost in the forest.

Yes, but I do not completely agree with you. Einstein was essentially using Riemannian Geometry but Riemann's motivation was essentially Physics. Look at the story of the Lorenz attractor. We know mathematicians have been blind, not listening to Lorenz for more that 15 years. Why? Because it was Physics?

The same thing happened with Hénon.

Yes, the same thing with Hénon. Some mathematicians are proud of that: "oh, we are mathematicians, we don't need the physicists".

More generally, what do you think can be the hot spots of mathematical research in the near future?

I will give you a similar answer: to reconnect with reality. It seems to me that, specially, French Mathematics has been very abstract. I believe, too abstract. So, my opinion is that we need to come back to the concrete reality, to extract from it interesting mathematical problems.

Portugal

When was your first visit to Portugal?

My first visit to Portugal was in August 1992. I came for a conference on Dynamical Systems held in Porto.

How do you see the scientific development of Portugal in the meantime, specially in Mathematics?

Let me be honest: away from Mathematics I know almost nothing about Portugal. Regarding Portuguese Mathematics, 30 years ago it was invisible. Now it is visible and I think it plays a significant role in European mathematics. It is going in the good direction.

In Portugal, there are just a few purely research permanent positions, in contrast to the French CNRS. What do you think about that?

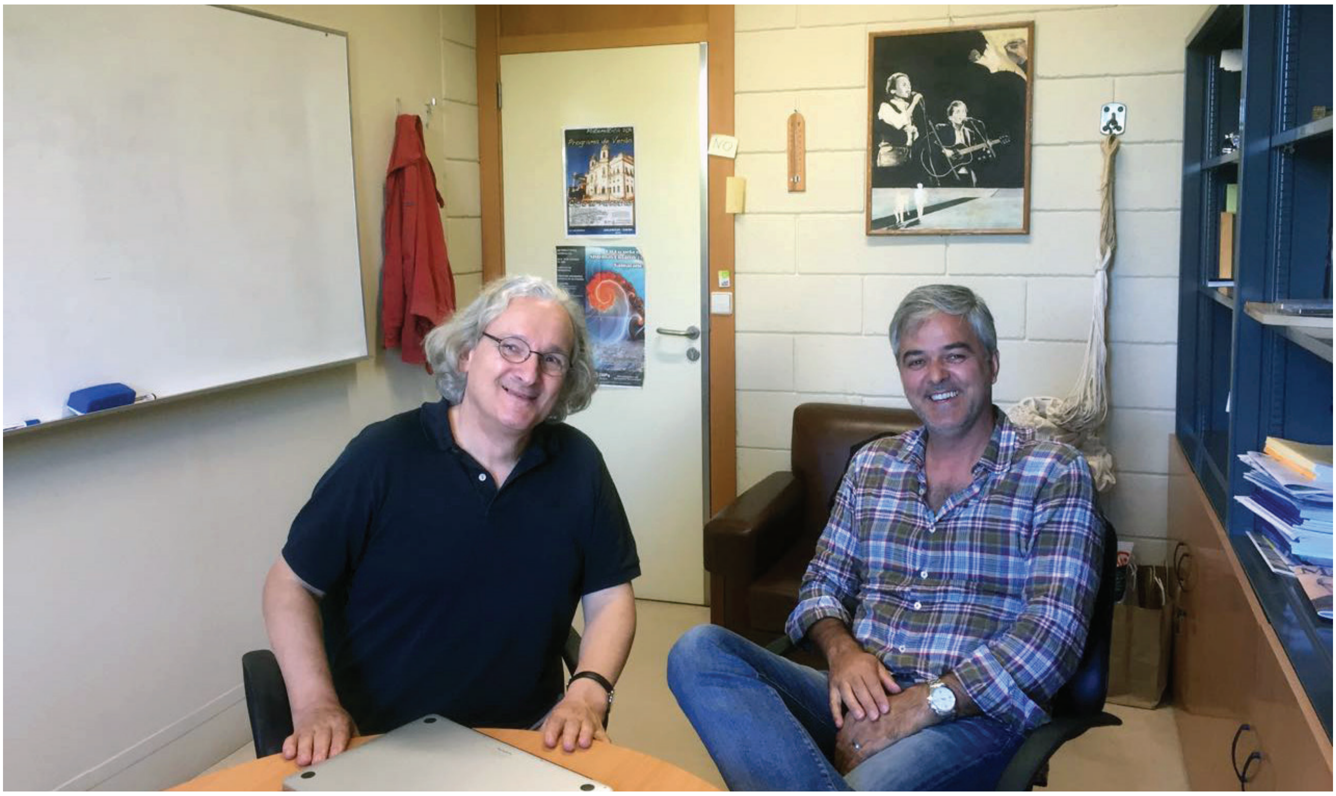
It is hard for me to answer this question. I have been a CNRS member since the beginning of my career. It is difficult for me to . . . as we say in French, cracher dans la soupe. I was hired a CNRS member before my PhD, before I wrote a single paper. I got a permanent position when I was 22 years old, can you believe it? I cannot criticize it, because I am happy with that, but I think it is too much. Let me tell you a story. I was working on my PhD thesis in Lille, working alone, and one day Dennis Sullivan came to Lille as a member of a jury of a PhD. It happens that I discussed with him and he liked what I was doing. And when he came back to Paris probably he took his telephone and I was hired. I got a permanent position because Dennis Sullivan liked my work, with no publications.

Let me say that the intuition of Dennis Sullivan with respect to that is not negligible.

Yes, but that is a good question: can a permanent position be given to a young guy at the age of 22?

That is a risk, and maybe because of running that risk France succeeds so well in Mathematics.

Yes, that is a risk, but I could give you a lot of not well succeeded cases. I remember once I was hired as the president of an evaluation committee of the Institut de Mathématiques de Jussieu, which is probably the biggest Math department in the world. The director of this department at that time was Harold Rosenberg. I was discussing with him and I said "how lucky you are, you have such a great number of excellent CNRS members". Harold answered me "by definition, you know those that you know and you do not know those that you do not know". And then he told me "look at the list of CNRS members in my department, and you will see a



lot of people that you have never heard of". It maybe be good to have such a CNRS position for 10 years, say. No obligations, no teaching for 10 years would be good. After 10 years, a committee could evaluate your work and decide if it is wise for you to continue in such a research position.

For some reasons (financial, demographic . . .) only a few positions for mathematicians have been opened in the recent years in Portuguese universities. On the other hand, the PhD programmes in Mathematics have grown and have become quite successful. Do you have any advice for the young Portuguese researchers who have just finished their PhD in terms of career opportunities?

My first answer would be that PhD is not necessarily an opening to academic. Society needs mathematicians. We do not understand enough that a PhD in Mathematics does not necessarily have to go to a university and to do research. For example in Switzerland or Germany, I think they have a different idea, most of their PhD's go to other kind of jobs. We have the same problem in France. We should try to understand that PhD is an opening to many different careers. The second answer is that the world is great. For instance, Portuguese can go to Brazil, there are many positions in Brazil. Many universities in Brazil need young PhD's and Brazil is a great country. So, go there!

Social

There are historical reasons to explain the existence of not many women doing Mathematics in the past. Though the world has changed a lot, there are still much more men than women doing research. For example, there was only one female Fields Medal. What do you think can be done to correct that?

I learnt recently that Portugal is actually the best in Europe from this point of view. I heard that among mathematicians in Portugal, 47% are female. In France it is only about 20%. If you go to Pure Mathematics, in France it is closer to 5%. If you go from the purest to the most applied you will see also a difference in the proportion of women. In Number Theory, the number of women is very small. But if you go to Applied Mathematics you have a more reasonable proportion of women. I think this is probably because we give an image of what Pure Math is if compared with Applied Math and I believe that men are the main responsible for that. It is always a big discussion, people explaining that it is not their fault and that the fault comes from the lower level. So, primary school would be the problem? I am convinced that the problem is everywhere, from primary school to the university. As an example, the 2017 promotion of the École

Normale Supérieure contains 40 students in mathematics: none of them is a female!

That is a problem of society, in general.

Now how to solve it? I do not know. Recently I was in a committee and there was an English man there. He told me something interesting: in England, professors belonging to a hiring committee have to attend a two hour class, a psychology class or something like that. Most of my English colleagues told me this is very useful. You go there, you spend two hours and the teacher helps you to understand your own stereotypes. This is compulsory, English professors—male, and probably female too—, have to participate in this kind of tutorial before going to any hiring committee. I suggested that there should be something like that in France. For example, CNRS could organize this kind of stage or formation. I think mathematicians, in particular, have to learn how to detect their own implicit stereotypes. Well, this will not be solved soon.

It takes generations some times.

Have you seen the statistics of CNRS? The number of females in the CNRS have been steadily decreasing in the last 30 years.

It is a pity that Mathematics is not profiting from a big part of the population.

Pure Mathematics is terrible from this point of view.

In the recent years, the Mathematical and the scientific community in general have been overwhelmed with the use of bibliometric data to assess and evaluate individuals and institutions. What do you think about that?

I hate them. This is ridiculous. I do not understand how it is possible to do that. Well, let me be a little bit more subtle. I think it might make sense to evaluate a department, but it is dangerous and bad for individuals. I think for evaluating a Math Department of 50 people, for example, it could make sense. But, be careful, to evaluate an individual I think it is nonsense. The good news is that, at least in France, committees do not use it.

I was recently told that the next evaluation of FCT, in Portugal, will not use it anymore. Aparently they are aware of the problems.

For example, CNRS mathematical hiring committees don't use these numbers! Using numbers is specially bad when you have to compare a mathematician, for instance, with a chemist. Chemists usually publish 500 papers!

Well, that cannot be compared.

But some people do it!

The success of Mathematics happens in two main directions: fundamental research and applied research. They are closely connected and the history of Mathematics proves it. However, in recent years there seems to be some pressure by the financing institution to direct the work of Mathematician towards fields of immediate application. What is your opinion about that?

If you want me to say that fundamental research is more important that applied research, no! One should remind our colleagues that it is not a sin to have applications. Doing Mathematics for pure pleasure of doing Mathematics with no use, is it what we want to do? I agree with you when you mention the word immediate. I agree that it is not a good idea to force immediate applications. Gauss or Poincaré were always mixing everything, pure and applied. My feeling is that we have to teach our purest colleagues that applications might be reasonable. I am in favor of applications, not in favor of forcing applications immediately.

What is the question I did not pose you that you would like to answer?

Could it be some personal question?

Yes, sure!

I am more than sixty now. I often ask it to myself what is in common between my approach to Mathematics now, what I like in Mathematics, and what I used to like when I was 15. You know, I have been in love with Mathematics for 45 years and the Étienne of today is very different from the Étienne when I was 15. And the kind of things I was liking

when I was 15 is totally different from the kind of things that I like today. What is common?

This might be significant for our earlier discussion, when you hire a young mathematician. When I was in my 20's, I was somehow competitive, I wanted to prove theorems, if possible before the others. Now I am more like contemplative and I love reading, understanding and explaining the papers of others. This is something we should take into account when we try to evaluate careers of our colleagues, that we cannot use the same kind of criterium when we evaluate a young mathematician and an older one, because they do not have the same goals in their lives. Now, maybe more from a personal point of view: how is it to become old in Mathematics? Probably, I think (i.e. I hope) it is not yet my case! But we must admit at some moment that a mathematician becomes less active, not as creative as he used to be. It is a fundamental question for a human being: how do you get old? Aging in mathematics is rarely discussed. In my case, the older I am, the more pleasure I have in reading old books from the 19th or 18th centuries.

SOERGEL BIMODULES AND 2-REPRESENTATION THEORY

by **Marco Mackaay***

1 INTRODUCTION

For almost four decades, the canonical bases of certain quantum algebras have been at the core of representation theory. Historically, the first ones were the KAZHDAN-LUSZTIG bases of Hecke algebras associated to COXETER groups [8]. For lack of space in this review, we will mostly concentrate on these, although the canonical bases of quantum groups form another interesting class of examples.

The KL bases, and the associated KL polynomials, have remarkable *positive integrality properties*, which were conjectured by KAZHDAN and LUSZTIG in [8]. For example, the multiplication constants of the Hecke algebra w.r.t. the KL basis belong to $\mathbb{N}[v, v^{-1}]$, where v is a formal parameter. (We assume that the HECKE algebras and their modules are defined over $\mathbb{C}(v)$.)

In a subsequent paper [9], KAZHDAN and LUSZTIG proved their conjectures for finite and affine WEYL groups, by interpreting the KL bases in terms of the local intersection cohomology of Schubert varieties. In that approach, the aforementioned multiplication constants become dimensions of cohomology groups and are therefore positive integral. Eventually, a geometric proof for Weyl groups of symmetrizable KAC-MOODY algebras was found [1, 2, 7]. However, the geometric arguments do not work for other COXETER groups.

Therefore, SOERGEL [17, 18] introduced an algebraic/combinatorial approach, using certain bimodules, designed to prove that the KL positive integrality properties hold for any COXETER group. This huge project, after important partial results by himself and others [17, 18, 5, 4], was eventually completed by ELIAS and WILLIAMSON [3].

Following standard terminology in this field, we say that SOERGEL's monoidal categories, resp. the indecomposable bimodules, *categorify* the Hecke algebras, resp. the KL basis elements. Alternatively, we can say that the latter *decategorify* the former.

Since the interest in HECKE algebras stems from their representation theory, it is natural to study the 2-representation theory of SOERGEL's monoidal categories, in which modules are replaced by their categorical analogue, called 2-modules.

In their systematic approach to 2-representation theory, MAZORCHUK and MIEMIETZ [16] proved a categorical version of the JORDAN-HÖLDER theorem. This led them to define the notion of a *simple transitive 2-module*, which is the correct categorical analogue of a simple module, although its decategorification is often not simple. Thus arises naturally the problem of classifying all simple transitive 2-modules of SOERGEL's monoidal category for any finite COXETER type.

In this review, we will recall what is known about this classification.

2 COXETER GROUPS AND HECKE ALGEBRAS

In this section we will briefly recall some well-known facts about COXETER groups, HECKE algebras and KAZHDAN-LUSZTIG bases. More material and proofs can be found in [6, 8, 12].

2.1 COXETER GROUPS

Let S be a finite set. A COXETER *matrix* $(m_{st})_{s,t\in S}$ is a symmetric matrix such that $m_{ss} = 1$ for all $s \in S$, and $m_{st} \in \{2, 3, \dots\} \cup \{\infty\}$ for all $s \neq t \in S$. Furthermore, let W be a group.

DEFINITION 1.— We say that (W, S) is a COXETER system if there exists a COXETER matrix $(m_{st})_{s,t\in S}$ such that $W \cong F(S)/N$, where $F(S)$ is the free group generated by S and $N \triangleleft F(S)$ the normal subgroup generated by the elements

$$(st)^{m_{st}} \tag{1}$$

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for all $s, t \in S$ with $m_{st} < \infty$.

We call W the Coxeter group and S the set of simple reflections of the Coxeter system (W, S).

By definition, the *rank* of (W, S) is the order of S, which is finite by assumption. However, this does not necessarily imply that W is of finite order.

The only Coxeter group of rank 0 is the trivial group, and the only one of rank 1 is $\mathbb{Z}/2\mathbb{Z}$. But there is an infinite family of rank 2 Coxeter groups, indexed by $m_{st} = m_{ts} = n \in \{2, 3, 4 \dots, \} \cup \{\infty\}$ with $S = \{s, t\}$. These are isomorphic to the dihedral groups of order $2n$ (which can be infinite), with st corresponding to a rotation of degree $2\pi/n$ when n is finite.

The finite Coxeter groups are classified by the finite type Coxeter diagrams [6, Sections 2.4 and 6.4], which are a generalization of the Dynkin diagrams of finite-dimensional complex semisimple Lie algebras.

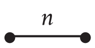
For example, for any $n \in \mathbb{N}$, the symmetric group on $n+1$ letters can be seen as a Coxeter group of type A_n, with $S = \{s_1, \dots, s_n\}$ the set of simple transpositions. Its Coxeter diagram is

Numbering the vertices of the diagram from left to right by $1, 2, \dots, n$, and writing s_i for the simple reflection associated to the vertex i, we have

$$m_{ij} = \begin{cases} 3 & \text{if } |i-j| = 1, \\ 2 & \text{if } |i-j| > 1, \\ 1 & \text{if } |i-j| = 0. \end{cases}$$

This is the general rule for obtaining the Coxeter matrix from a Coxeter diagram and vice-versa, with one exception: if $m_{st} > 3$ for two neighboring vertices in the diagram, then that number is written above the corresponding edge.

For example, for any $n > 3$, the dihedral group of order $2n$ can be seen as a Coxeter group of type $I_2(n)$ with Coxeter diagram



Note that $I_2(3) = A_2$, since they have the same Coxeter diagram.

For any $w \in W$, a *reduced expression* for w is by definition a shortest string $s_1, \dots, s_\ell \in S$ such that $w = s_1 \cdots s_\ell$. We call ℓ the *length* of the string. In general, there can be more than one reduced expression for w, but two of them can always be related by applying (3) a finite number of times, as shown by Matsumoto and Tits' theorem (see [12, Thm. 1.9]).

This allows us to define the length function $\ell : W \to \mathbb{Z}_{\geq 0}$, which associates to each $w \in W$ the length of a reduced expression for w (see [6, Sect. 1.6]).

Furthermore, it allows us to define the Bruhat *order* $\preceq$ on W, which is a partial order defined by: $u \preceq w$ iff u can be obtained as a (not necessarily reduced) subexpression of a reduced expression for w (see [6, Sect. 5.9 and 5.10]).

If W is finite, then it has a unique longest element, denoted w_0, which is also maximal w.r.t. the Bruhat order.

2.2 Hecke algebras

Let (W, S) be any Coxeter system. In the group algebra $\mathbb{C}[W]$, the relations $s^2 = e$ and $(st)^{m_{st}} = e$ can be rewritten as

$$(s+e)(s-e) = 0, \tag{2}$$

$$\underbrace{sts\cdots}_{m_{st}} = \underbrace{tst\cdots}_{m_{st}}. \tag{3}$$

The next definition is obtained by v-deforming the relation in (2).

Definition 2.— The Hecke algebra $\mathscr{H}$ associated to (W, S) is the unital associative $\mathbb{C}(v)$-algebra generated by T_s, for $s \in S$, subject to the relations

$$\begin{aligned} (T_s + 1)(T_s - v^{-2}) &= 0, \\ \underbrace{T_sT_tT_s\cdots}_{m_{st}} &= \underbrace{T_tT_sT_t\cdots}_{m_{st}}, \end{aligned} \tag{4}$$

for all $s, t \in S$. By convention, we write $T_e = 1$.

Note that $T_s^2 = (v^{-2}-1)T_s + v^{-2}$ and $T_s^{-1} = v^2T_s + v^2 - 1$.

For any $w \in W$, choose a reduced expression $w = s_1 \cdots s_{\ell(w)}$, with $s_i \in S$, and define

$$T_w := T_{s_1} \cdots T_{s_{\ell(w)}}.$$

By Matsumoto and Tits' theorem, the element T_w does not depend on the choice of reduced expression. Moreover, $\{T_w \mid w \in W\}$ is a linear basis of $\mathscr{H}$, called the *standard basis* (see [12, Prop. 3.3]). In particular, this implies that $\mathscr{H}$ is a flat deformation of the group algebra of W.

The KL *basis* $\{b_w \mid w \in W\}$ is harder to define. Let $^-$ be the *bar involution* on $\mathscr{H}$, which is the $\mathbb{C}$-linear involution given by

$$\overline{v} := v^{-1} \quad \text{and} \quad \overline{T_w} := T_{w^{-1}}^{-1}.$$

The KL *basis elements* $b_w \in \mathscr{H}$ are uniquely determined by the two properties [8, Thm 1.1]:

$$\overline{b_w} = b_w, \tag{5}$$

$$b_w = v^{\ell(w)} \sum_{y \preceq w} P_{y,w} T_y, \tag{6}$$

where $P_{y,w} \in \mathbb{Z}[v^{-2}]$ has negative v-degree strictly less than $\ell(w) - \ell(y)$ for $y \prec w$ and $P_{w,w} = 1$.

Note that the matrix $(P_{y,w})_{y,w \in W}$ is unitriangular, so the fact that the b_w form a basis follows immediately from the fact that the T_w form a basis.

In general, there is no simple formula expressing b_w in terms of the T_y. Only the KL generators are easy to compute:

$$b_e = 1 \quad \text{and} \quad b_s = v(T_s + 1) \quad \text{for all } s \in S. \tag{7}$$

However, in type $I_2(n)$ we can write down all KL basis elements explicitly [12, Ch. 7]:

$$b_w = v^{\ell(w)} \sum_{y \preceq w} T_y \quad \text{for all } w \in W.$$

A short calculation shows that, in any COXETER type, we have

$$b_s^2 = (v + v^{-1}) b_s \quad \text{for all } s \in S. \tag{8}$$

It is also easy to see that $b_s b_t = b_{st}$ for all $s \neq t \in S$. But, in general, the product of a finite number of KL basis elements is not a KL basis element, e.g. in type $I_2(3) = A_2$ we have

$$b_s b_t b_s = b_{sts} + b_s \quad \text{and} \quad b_t b_s b_t = b_{tst} + b_t,$$

where $b_{sts} = b_{tst}$ because $sts = tst$ in W. Nevertheless, if we choose a reduced expression $w = s_1 \cdots s_{\ell(w)}$ for each $w \in W$, and define $b_{\underline{w}} := b_{s_1} \cdots b_{s_{\ell(w)}}$, then $\{b_{\underline{w}} \mid w \in W\}$ is yet another basis of $H_v(W)$, called the BOTT-SAMELSON *basis*. This follows from the fact that $b_{\underline{w}} = v^{\ell(w)} T_w + \text{l.o.t.}$, where l.o.t. is a linear combination of T_y with $y \prec w$. Note, however, that $b_{\underline{w}}$ depends on the choice of reduced expression for w.

3 SOERGEL BIMODULES

For any COXETER group W, take $\mathfrak{h}$ to be the complexification of SOERGEL's finite-dimensional real W-module in [18, Prop. 2.1], which generalizes the usual representation of an affine WEYL group on the CARTAN subalgebra of an affine KAC-MOODY algebra.

Let R be the complex algebra of regular functions on $\mathfrak{h}$, equipped with a $\mathbb{Z}$-grading such that $\deg(\mathfrak{h}^\star) = 2$. The action of W on $\mathfrak{h}$ extends naturally to an action on R by degree-preserving automorphisms.

Let R–fmod–R be the monoidal category of all finitely generated graded R–R bimodules, where the monoidal product is given by the tensor product over R. By definition, the morphisms are the degree-preserving bimodule maps. Note that R–fmod–R is additive, because we can also take the direct sum of two bimodules. Furthermore, the homogeneous direct summands of the hom-spaces are all finite-dimensional complex vector spaces and composition is bilinear, so R–fmod–R is $\mathbb{C}$-linear.

For any $s \in S$, let R^s be the graded subalgebra of s-invariant polynomials and define

$$B_s := R \otimes_{R^s} R\{1\},$$

where $\{1\}$ indicates a downward grading shift of 1. This is a graded $R - R$ bimodule with left and right actions given by $a \cdot (x \otimes y) \cdot b := (ax) \otimes (yb)$, for any $a, b, x, y \in R$.

We have $R \cong R^s \oplus R^s\{-2\}$ as R^s-bimodules, so

$$B_s \otimes B_s \cong B_s\{+1\} \oplus B_s\{-1\} \quad \text{for all } s \in S.$$

This isomorphism categorifies the equality in (8).

More generally, for any finite number of simple reflections $s_1, \ldots, s_m \in S$, the corresponding BOTT-SAMELSON *bimodule* is defined as

$$B_{s_1} \otimes_R B_{s_2} \otimes_r \cdots \otimes_R B_{s_m}.$$

Let $w \in W$ and suppose $w = s_1 \cdots s_{\ell(w)}$ is a reduced expression. Then we denote the corresponding BOTT-SAMELSON bimodule by $B_{\underline{w}}$.

DEFINITION 3.— The monoidal category of SOERGEL bimodules $\mathcal{S}$ is the full subcategory of R–fmod–R containing all direct sums of direct summands of BOTT-SAMELSON bimodules with grading shifts.

The additive category $\mathcal{S}$ is idempotent complete and KRULL-SCHMIDT [18, Rem. 1.3].

Before we state the categorification theorem for SOERGEL bimodules, recall that the *split* GROTHENDIECK *algebra* of $\mathcal{S}$, denoted $[\mathcal{S}]$, is by definition the $\mathbb{C}(v)$-vector space spanned by the isoclasses of the SOERGEL bimodules, subject to the relations:

$$[U \oplus V] = [U] + [V] \quad \text{and} \quad [U\{t\}] = v^t [U]$$

for all SOERGEL bimodules U, V and $t \in \mathbb{Z}$. It becomes an algebra after putting

$$[U \otimes_R V] := [U][V]$$

for all SOERGEL bimodules U, V. By the above, it follows that $\left\{[B_w] \mid w \in W\right\}$ is a basis of $[\mathcal{S}]$.

The first three points in the following theorem are due to SOERGEL [18, Thm. 1.10 and Satz 6.16]. The fourth point is due to ELIAS and WILLIAMSON [3, Thm. 1.1].

THEOREM 4.— Let (W, S) be an arbitrary COXETER system. Then

1. there is a well-defined isomorphism of $\mathbb{C}(v)$-algebras $\rho_S : \mathcal{H} \to [\mathcal{S}]$ uniquely determined by
$$b_s \mapsto [B_s] \quad \text{for all } s \in S;$$
2. for every $w \in W$, there exists an indecomposable B_w in $\mathcal{S}$, unique up to degree-preserving isomorphism, that is a direct summand of the BOTT-SAMELSON bimodule $B_{\underline{w}}$, for any reduced expression for w, and is not a direct summand of $B_{\underline{u}}\{t\}$ for any $u \prec w$ and $t \in \mathbb{Z}$; in particular, the isoclass of B_w does not depend on the choice of reduced expression for w;

3. every indecomposable Soergel bimodule is isomorphic to $B_w\{t\}$ for some $w \in W$ and $t \in \mathbb{Z}$;

4. for every $w \in W$, we have $\rho_S(b_w) = [B_w]$.

Note that this theorem immediately implies that the KL basis of $\mathscr{H}$ is positive integral: for any $u, v \in W$, we have

$$b_u b_v := \sum_{w \in W} \gamma_{u,v}^{w} b_w$$

such that $\gamma_{u,v}^{w} \in \mathbb{N}[v, v^{-1}]$, because these multiplication constants are equal to the graded decomposition numbers of $[B_u \otimes B_v]$ in terms of the $[B_w]$

4 2-Representation theory

From now on, let (W, S) be an finite type Coxeter system, i.e. we assume that W is a finite group.

Recall that a category is *graded finitary* if it is additive, $\mathbb{C}$-linear, idempotent complete and Krull-Schmidt, such that the homogeneous direct summands of its hom-spaces are finite-dimensional and it has finitely many isoclasses of indecomposable objects up to grading shifts, e.g. the category of finitely generated graded projective modules over a non-negatively graded algebra which is finite-dimensional in each degree.

Let $\mathscr{S}$ be the monoidal category of Soergel bimodules for (W, S). A 2*-module* of $\mathscr{S}$ is by definition a graded finitary category $\mathscr{M}$ on which the Soergel bimodules act as linear endofunctors and the bimodule maps as natural transformations, such that all structures (including the grading) are preserved. In general, the 2-action is allowed to be weak in a restricted sense, but this is not the right place to explain such technical details.

A 1*-intertwiner* between two 2-modules of $\mathscr{S}$ is by definition a degree-preserving $\mathbb{C}$-linear functor between the underlying categories which commutes with the 2-action. Again, we suppress all technical conditions which control the level of weakness that is allowed. Two 2-modules are called equivalent if there is a fully faithful and essentially surjective 1-intertwiner between them.

Finally, there is a next layer of structure, formed by natural transformations between 1-intertwiners which satisfy additional conditions. We call these 2-intertwiners.

Together, the 2-modules of $\mathscr{S}$ and the 1 and 2-intertwiners between them form a 2-category, which we denote by $\mathscr{S}$-2fmod.

Note that we only consider additive 2-modules. In this review, we do not discuss abelian or triangulated 2-modules.

4.1 Cell modules

The decategorified story of cell modules of Hecke algebras is due to Kazhdan and Lusztig [8].

Definition 5.— We define the *left pre-order* $\geq_L$ on W by putting $w \geq_L v$ if $\gamma_{u,v}^{w} \neq 0$ for some $u \in W$.

We set $w \sim_L u$ provided that $u \geq_L w$ and $w \geq_L u$. The equivalence classes of this equivalence relation are called the *left cells* of W.

The right and two-sided pre-orders $\geq_R$ and $\geq_J$, and the right and two-sided cells for the corresponding equivalence relations $\sim_R$ and $\sim_J$ are defined similarly, using multiplication from the right and from both sides respectively.

Note that each left (resp. right) cell is contained in a two-sided cell, that each two-sided cell is the disjoint union of the left (resp. right) cells it contains, and that W is the disjoint union of all two-sided cells.

In general, it is not so easy to compute cells explicitly. In type A_n, Kazhdan and Lusztig [8] proved that $u \sim_L w$ iff $Q(u) = Q(w)$, where Q is the recording tableau in the Robinson-Schensted correspondence. Similarly, $u \sim_R v$ iff $P(u) = P(w)$, where P is the insertion tableau.

In type $I_2(n)$, the computation of the cells is straightforward and gives:

$$\mathscr{J}_e = \mathscr{L}_e = \mathscr{R}_e = \{e\}$$

$$\mathscr{J}_s = \mathscr{J}_t = \begin{array}{c|c|c|c} & \mathscr{R}_s & \mathscr{R}_t & \\ \hline & s, sts, \ldots & ts, tsts, \ldots & \mathscr{L}_s \\ \hline & st, stst, \ldots & t, tst, \ldots & \mathscr{L}_t \\ \hline \end{array}$$

$$\mathscr{J}_{w_0} = \mathscr{L}_{w_0} = \mathscr{R}_{w_0} = \{w_0\}.$$

If $\mathscr{L}$ is a left cell of W, we write $w \geq_L \mathscr{L}$ if $w \geq_L u$ for all $u \in \mathscr{L}$, and we write $w >_L \mathscr{L}$ if $w \geq_L \mathscr{L}$ and $w \notin \mathscr{L}$. Let $M_{\geq_L \mathscr{L}}$ and $M_{>_L \mathscr{L}}$ be the subvector-spaces of $\mathscr{H}$ spanned by all b_w satisfying $w \geq_L \mathscr{L}$, and $w >_L \mathscr{L}$ respectively. Both are left ideals of $\mathscr{H}$ and $M_{>_L \mathscr{L}} \subset M_{\geq_L \mathscr{L}}$.

Definition 6.— The *left cell module* $C_{\mathscr{L}}$ is defined as

$$C_{\mathscr{L}} := M_{\geq_L \mathscr{L}} \big/ M_{>_L \mathscr{L}}$$

with the natural left $\mathscr{H}$ action.

Note that $C_{\mathscr{L}}$ inherits a KL-basis, consisting of all b_w with $w \in \mathscr{L}$. Clearly, this provides the cell module with a positive integral basis, i.e. on the KL-bases of $\mathscr{H}$ and $C_{\mathscr{L}}$, the *action constants* all belong to $\mathbb{N}[v, v^{-1}]$.

As we already remarked, the left cells in type A_n are parametrized by standard tableaux. As a matter of fact, every left cell-module of $\mathscr{H}$ is simple and its isomorphism class is determined by the partition underlying the corresponding standard tableau. This establishes a bijection between the isoclasses of left cell-modules and the isoclasses of simple modules, which is atypical: in other Coxeter types most simple modules do not have a positive integral basis and most cell modules are not simple.

For example, consider type $I_2(n)$. Any one-dimensional module is completely determined by its character $\chi\colon \mathscr{H} \to \mathbb{C}(q)$. By the quadratic relations in $\mathscr{H}$, we must have $\chi(T_s) = \epsilon_1$ and $\chi(T_t) = \epsilon_2$ with $\epsilon_1, \epsilon_2 \in \{v^{-2}, -1\}$. If n is even, there is no extra condition, so there are four different characters. If n is odd, then $\epsilon_1 = \epsilon_2$ is required to hold, so there are only two different characters. We denote the corresponding one-dimensional modules by $V_{\epsilon_1,\epsilon_2}$.

We have $C_{\mathscr{L}_e} \cong V_{-1,-1}$, because $b_s = v(T_s + 1)$ and $b_t = v(T_t + 1)$ act as zero. Similarly, we have $C_{\mathscr{L}_{w_0}} \cong V_{v^{-2},v^{-2}}$, because b_s and b_t both act as multiplication by $v+v^{-1}$. When n is even and at least 4, the modules $V_{v^{-2},-1}$ and $V_{-1,v^{-2}}$ are not equivalent to cell modules, because there are no more one-element left cells.

All other simple modules are known to be of dimension two. Since $C_{\mathscr{L}_s}$ and $C_{\mathscr{L}_t}$ have dimension $n-1$, they cannot be simple for $n \geq 4$.

Furthermore, in type $I_2(n)$ there are other interesting modules of $\mathscr{H}$ with a positive integral basis, as we will explain below.

4.2 Cell 2-modules

There is a natural categorification of the (left) cell-modules, due to Mazorchuk and Stroppel [13] in the case of finite Weyl groups, and Mazorchuk and Miemietz [15] in general (see also [16, Sec. 3.3]). Let $\mathscr{L}$ be a left cell and take $\mathscr{M}_{\geq_L \mathscr{L}}$ to be the full subcategory of $\mathcal{S}$ generated by the B_w for $w \geq_L \mathscr{L}$. This subcategory contains a unique ideal $\mathcal{I}_{\mathscr{L}}$ which is maximal in the set of all $\mathcal{S}$-stable ideals.

Definition 7.— The left cell 2-module associated to $\mathscr{L}$ is defined as

$$\mathscr{C}_{\mathscr{L}} := \mathscr{M}_{\geq_L \mathscr{L}} \big/ \mathcal{I}_{\mathscr{L}}$$

with the natural 2-action of $\mathcal{S}$.

By construction, we have $C_{\mathscr{L}} \cong [C_{\mathscr{L}}]$ as $\mathscr{H}$-modules.

4.3 Simple transitive 2-modules

Mazorchuk and Miemietz [16] found that the correct categorification of the notion of simple module, is that of *simple transitive 2-module*. A 2-module $\mathscr{M}$ of $\mathcal{S}$ is *transitive* if for any two indecomposable objects X, Y in $\mathscr{M}$, there exists a Soergel bimodule B in $\mathcal{S}$ such that X is a direct summand of BY. A transitive 2-module $\mathscr{M}$ is simple transitive if it has no non-zero proper $\mathcal{S}$-stable ideals. Any transitive 2-module has a simple transitive quotient [16, Lem. 4]. By construction, any cell 2-module is simple transitive.

In type A_n the converse is also true: any simple transitive 2-module is equivalent to a cell 2-module [15, Sec. 7.1].

However, in type $I_2(n)$ there are simple transitive 2-modules that are not equivalent to cell 2-modules. There is an ADE-classification for the simple transitive 2-modules in type $I_2(n)$, and only the ones of type A are equivalent to cell 2-modules.

To explain this, we first note that any simple transitive 2-module $\mathscr{M}$ has an underlying quiver, which can be graded, so that $\mathscr{M}$ becomes equivalent to the category of graded finitely generated projective modules of the quiver algebra after modding out by a virtually nilpotent ideal. As it turns out, for type $I_2(n)$ Soergel bimodules, the quiver underlying a simple transitive 2-module can always be obtained from a simply laced Dynking diagram of finite type. The main part of the following theorem can be found in [10, Thm. 1 and Sec. 6], with only a construction of the simple transitive 2-modules of Dynkin type E missing, which can be found in [14].

Theorem 8.— Let $\mathcal{S}$ be the monoidal category of Soergel bimodules of type $I_2(n)$. For any $n > 2$, $\mathcal{S}$ has two inequivalent cell 2-modules of rank one, namely $\mathscr{C}_{\mathscr{L}_e}$ and $\mathscr{C}_{\mathscr{L}_{w_0}}$.

Furthermore, there are two cell 2-modules of rank $n-1$, namely $\mathscr{C}_{\mathscr{L}_s}$ and $\mathscr{C}_{\mathscr{L}_t}$, whose underlying graph is of Dynkin type A_{n-1}. They are equivalent iff n is odd.

1. If $n = 2k+1 > 2$ or $n = 4$, then all simple transitive 2-modules are equivalent to the above cell 2-modules.
2. If $n = 2k > 4$, there are two additional inequivalent simple transtive 2-modules, whose underlying graph is of Dynkin type D_{k+1}.
3. If $n = 12$, 18 or 30, there are also two inequivalent exceptional simple transitive 2-modules, whose underlying graph is of Dynkin type E_6, E_7 and E_8 respectively.

The above gives a total classification of the simple transitive 2-modules of $\mathcal{S}$.

It is interesting to note that the two inequivalent simple transitive 2-modules of Dynking type E_6 decategorify to isomorphic $\mathscr{H}$-modules. The same happens for Dynkin type E_8, but the two inequivalent simple transitive 2-modules of Dynkin type E_7 have non-isomorphic decategorifications.

We also note that the decategorified story was already known to Lusztig [11, Prop. 3.8].

The classification of the simple transitive 2-modules of $\mathcal{S}$ in other finite Coxeter types is very incomplete. In [10] the following (very) partial result was proved. For every finite type COXETER system (W, S), there is a unique lowest order two-sided cell $\mathcal{J}_S$ which does not contain e. One simple description of $\mathcal{J}_S$ is that it consists of all $w \neq e \in W$ with a unique reduced expression. Now assume that (W, S) has rank > 2. Then any simple transitive 2-module of $\mathcal{S}$ that is annihilated by all B_w with $w >_J \mathcal{J}_S$, is equivalent to a cell 2-module [10, Thm. 1].

The rest of the classification is unknown and forms an interesting but difficult open problem, except for COXETER type A_n where the cell 2-modules exhaust the simple transitive 2-modules.

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REPORT

Axiomatic Thinking

by **Fernando Ferreira*** and **Reinhard Kahle****

On October 11-14, 2017, a meeting on Axiomatic Thinking took place at the *Academia das Ciências de Lisboa* and the *Faculdade de Ciência e Tecnologia da Universidade Nova de Lisboa*. It was jointly organized by the *Académie Internationale de Philosophie des Sciences* and the *Academia das Ciências de Lisboa*, with support of CIM, SPM, FCT-MCTES,Nova.ID.FCT, TAP and the research centers CMA (FCT,UNL), CMAF-CIO (FC,UL) and CFCUL (FC,UL). The programme committee consisted of Fernando Ferreira (Lisbon), Gerhard Heinzmann (Nancy), and Reinhard Kahle (Lisbon).

100 years ago, in 1917, David Hilbert gave his seminal talk *Axiomatisches Denken* at a meeting of the Swiss Mathematical Society in Zurich. It marks the beginning of Hilbert's return to his foundational studies, which ultimately resulted in the establishment of proof theory as a new branch in the emerging field of mathematical logic. Hilbert also used the opportunity to bring Paul Bernays back to Göttingen as his main collaborator in foundational studies in the years to come.

At the conference the success of Hilbert's axiomatic method was reevaluated. It did not only lay the foundations for the understanding of modern mathematics, but it also found its way in many applications, first of all—and as vividly advocated by Hilbert—in Physics. Even for the seemingly negative results, as in the case of Gödel's incompleteness theorems, it was Hilbert's sharply posed questions which opened up completely new and unexpected perspectives for our understanding of Mathematics.

On Wednesday, at the Academia das Ciências de Lisboa, the speakers were Michael Detlefsen, Wolfram Pohlers, Steve Simpson, Peter Koellner, Michael Rathjen, and Mark van Atten. The day finished with a panel discussion on the current state of foundational reasearch in Mathematics by Koellner, Rathjen and van Atten, moderated by Fernando Ferreira. Thursday to Saturday, at FCT-UNL, the speakers were: Evandro Agazzi, José Ferreirós, Wilfried Sieg, Marco Buzzoni, Gregor Schiemann, Daniel Vanderveken, Fabio Minazzi, Michel Ghins, Dennis Dieks, Mario Alai, Lorenzo Magnani, Olga Pombo, Valentin Bazhanov, Paul Weingartner, Itala D'Ottaviano, and Alberto Cordero.

The organizers plan to publish a volume with contributions of the meeting and further papers by experts in the field.

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AN INTERVIEW WITH

Patrícia Gonçalves

by **Lisa Santos***

Let us start with what was behind this interview, the millionaire starting grant you won from the European Research Council (ERC). Tell me a bit about the preparation of your application . . .

The application was a complex process and it was very difficult for me to finish it. During this time I went to India, where I spent two weeks. Before leaving, I went to the traveler's medical consultation in Portugal, I did the malaria prevention, but three days after arriving back to Portugal, I picked up a very high fever whose cause no one could explain, my face was full of blest. I remember finishing the application, on the last day, which coincided with my son's birthday, with a high fever.

I submitted the application on November 17. I knew that I was approved for the interview phase on Women's Day, March 8. The interview took place on June 1, I had plenty of time to prepare myself. It took me a lot of time to make the slides, I made millions of versions, I asked for the opinion of many colleagues who had already competed for ERC scholarships. Several of my co-authors have read the project. I knew that if they did not understand what I wrote, the panel would not understand either.

The application has two parts: the part B1, which describes a generalist mathematics, trying to catch the attention of the two or three referees who will read this part; the part B2 is a technical part that I know (now)

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that is sent to eight referees. In the end, I received all the reports. The referees assign a level to the candidate, a level to the project and a level to its suitability to the institution where the candidate is inserted. Even the time of dedication of the PI to the project is classified. In this ranking, I was told that I was in the first third of the candidates. But there have been cases of candidates who stayed in the second third and who, in the end, got the scholarship.

Is the requirement of the application, compared to an FCT project, much bigger?

Yes, it's really much bigger. The application is very demanding, and has many rules, from letter size, to file names, exact values of hiring amounts of team members, . . .

What about the rules for spending money, the process is less complicated than the Portuguese?

No, the difficulties are the same. I started to spend the money in early December but the problem is that we are in Portugal, subject to a general law that corsets us. For example, I cannot buy a computer. Although ERC financed the purchase, I have to make an exception request which is currently waiting for approval from the Ministry of Finance, because the money is considered public money. In the next semester I am going to spend three months in Paris and I'm having a hard time to pay for an apartment. Portuguese legislation only allows you to pay hotels and not apartments, . . . it is absurd, and much more expensive, to stay in a hotel for three months!

Let us talk now about what was the genesis of your application.

I knew very little about ERC grants. When I was finishing my PhD at IMPA, in Brazil, a colleague from my area visited IMPA and had just applied for a starting grant. While I was in Brazil, it made no sense to compete, but I wished to return to Portugal.

In the edition in which I applied there were more then eighty applications in Mathematics, from all over the world, and only ten were selected. These applications are very different from the applications for projects to which we are used to compete. It is necessary to present a fantastic idea supported by arguments that we are able to put this idea into practice.

Out of all the applications, a short list is chosen, with those who pass to the interview. Being part of this short list is already very good. For example, candidates in France who go through this phase and are not funded are called by the French Ministry of Science and Technology and receive funding, because the mere fact of being selected for the interview means that the project is excellent. In Portugal, I asked to the Office of Promotion of R&D Framework Program (http://www.gppq.fct.pt/) if there was something like this and the answer was no.

In Germany, France and England training is given to people who are applying for ERC grants. When I applied I had no help, except in the financial component of the project, where I received some support from the Project Support Office of the University of Minho.

In France, people who have won an ERC grant give training to anyone who wants to compete. There is a kind of open day where they tell their experience, which helps those who want to compete. If they asked me that, I would do it with pleasure. In Portugal, in the area of Mathematics, I was the 13th candidate to apply since the launch of the ERC grants contest but the first one to get it.

The year 2015 was the penultimate year in which I could run for a starting grant. In fact, I could not have competed for a strating grant at this point, if I had not children, because for each child the deadline extends for a year and a half. It turned out that 2015 was the perfect timing. I have a work with Milton on the KPZ equation (Kardar-Parisi-Zhang equation). There is a conjecture that says this equation is a universal law that describes certain patterns of growth in nature. The theme of the project is the universality of the physical systems and how to prove this universality starting from stochastic discrete systems. Milton and I submitted a result, which was considered very good, we were able to prove the existence of a weak solution of this equation, and the definition of weak solution in this context has been introduced by us. We have also proved that all particle systems satisfying certain conditions fall into that equation, but there was one small problem, we had not been able to prove the uniqueness of that solution. Martin Hairer proved the uniqueness in a more general context and for a different notion. A while later he won the Fields Medal! Up to now, none has been able to apply his notion to the particle systems setting. We could not verify that our particle systems satisfy the hypotheses that Martin Hairer poses. In September 2015, some colleagues also proved the uniqueness, but in the context that interested us, thus sustaining that our definition of solution was not a crazy thing. The conditions that susteined my application to an ERC grant were met. I could move on to the proof that more particle systems can fall into this equation.

The application is very ambitious, no doubt. Our main objective in the area of particle systems is to obtain the hydrodynamic limit, a partial differential equation (PDE) or a stochastic partial differential equation that describes how is the evolution, in space and time, of a quantity which the particle system conserves. We usually pick up a particle system, which is a sort of discretization of a PDE (the best way to discretize a PDE) and establish the link between the discrete problem and the continuous problem. I have at hand in this moment several problems that have nothing to do with probability, involving fractional laplacians, fractional derivatives, regularity of weak solutions of PDEs, . . . We have sometimes questions concerning the part of analysis, but the results that we find are in such a generelized setting that they are not useful to our concrete questions. In short, the project consists of

identifying the PDEs that are obtained in the hydrodynamic limit of certain particle systems, understanding if they are universal, in what sense, how will we get them from the particle systems and what peculiarities we have in a certain particle system that allow us to arrive at these equations. For example, changing locally the dynamics of a particle system, does this change the PDE we have reached?

My interview in Brussels should have started with a video, which I will describe shortly, but there was a problem. They tell us in the guidelines for not using videos or simulations but I used a pdf file that had the video and several simulations because I thought I would explain much better what I wanted to say. They make exceptions and I asked the secretary to test my file, which had a video, on her computer. We soon saw that it did not work but she let me use my laptop and called the technician.

The interview took place in a very small room with U-distribution. I already knew this because I was installed in a hotel just opposite to the place of the interview, and from the window of my room, I saw the interviews of the others. The interviews are scheduled, but we have to get there one hour in advance. I knew there was a person in my area in the competition, but when I got there I saw another person from my area, in the waiting room, whom I had no idea he was competing and I thought things were complicated (we ended up with the grant, the three of us!). But at that moment I became more relaxed. I thought, I'm here to show to these mathematicians, who are excellent researchers, a very nice subject and I hope they enjoy it and feel the pleasure I have in doing Mathematics.

At that U-table were about sixteen persons and there were two screens, one where the time runs and other where our presentation is placed, ready to start. I had twelve minutes to make my point. The technician took my computer (and I could not have a quick last look of my slides) and told me that everything was working. When I got to the interview room, my computer was hibernated, I was not able to open it, I shut it down and openned it again and nothing happened, there was some problem of incompatibility of the projector with my Mac . . . and the whole panel was waiting for me! I had another presentation without the video—the plan B! I believe that the panel saw the video while the technician was doing the verifications. The video describes growth patterns. It starts with someone sitting inside a car that sees ice particles falling, which begin to gather in the windshield of the car, and do a very nice pattern. The geometry of ice particles is quite funny, they leave holes. And this pattern of growth is described by the KPZ equation. This pattern also exists in the growth of tumors, in the growth of bacteria, and if we burn a paper sheet, the way the fire moves has the same behavior. The objective of the project is to study to what extent this equation is universal. It is universal for certain types of physical systems, which we characterize as a certain class of universality (it is not unique, there are others). The questions are: how to model these systems

microscopically; how does the growth of the particles evolves over time; whether it is this equation or others which model this growth . . . and we also ask what kind of universality classes exist. In the interview they asked me several times if the topic was not too ambitious, if I thought I could do everything.

Talking to you, it is obvious you love what you do!

Yes, I never say, “I’m going to work”, I say “I am going to have fun!”

I feel it too. I often say: happy is the one who is paid to do what he loves! Did you trust the success of your application?

I always had some hope of success, otherwise I would not even try to compete, because it’s a lot of work. I had faith that if someone read the project carefully and if I tried to convince that panel, I had a chance. I look back and despite the mishap of the video, I left the interview very satisfied, I thought it went very well. The interview was on June 1, in Portugal it was almost summer, in Brussels it was raining and it was cold. I remember thinking about how lucky we are with our weather . . .

How did it feel when you learned you had won?

I went on vacations on the 28th or 29th of July, I was on the beach and I thought I had to see the mail (the internet on my cell phone was very recent) and I saw that I had a mail saying “Project has been retained for funding”. If I had not read all those blogs of other colleagues, I would not know that this was the magic phrase. They did not say amounts and asked not to advertise the prize at the institution because I would receive a formal letter with the amounts and all the details. The formal letter was only received on August 22, and the disclosure only occurred on September 8.

At a meeting of the Pedagogical Council of the University of Minho, a colleague from the Department of Chemistry congratulated the Department of Mathematics and Applications and I did not even understand why! Only later I did realize that the newspapers announced your scholarship indicating the University of Minho as the host institution.

Yes, the host institution was the University of Minho. In the meantime, I won a contest for Associate Professor at IST and my host institution has changed.

You said, somewhere, that you would like to create a *school* in Portugal. Do you want to detail a little more?

They asked me in the interview why I was asking for five masters students. I explained that in Portugal there are no people working in my area and that it is possible for a master’s student to start being introduced to these subjects slowly. My idea was to start forming a team in my research area. They asked me what kind of thesis I could give to a master’s student. I replied that I already

had a student who did a master's thesis with me in this area, and that this is possible if the student has a good background in Probability. This student is now doing his PhD with me and with Cédric Bernardin, based at the University of Nice. In fact, they only gave me two one-year scholarships for master's students.

I'm sure I could attract some established researcher from PDEs or from Physics. The interaction of people from different areas is very good and very important. I could get people who already have a solid career . . . but it would be harder to make them leave their research to get into a new thing, it would be simpler to grab younger students and train them at first. They also gave me two PhD fellowships, each lasting four years and three postdoc fellowships, each lasting two years. My main concern is: how will I find MSc students who want to make a career in this field? The path I need, in stochastic processes, is note very developed in Portugal. The Probability part that is given is, in large part, more towards Statistics. I really want to create a group that bridges the gap between Probability and PDEs.

My collaborators who work on this topic are all on the project, but as external collaborators. The team is me and the MSc, PhD and posdocs students that I will hire.

MSc scholarships are things that do not exist at the moment, in Portugal, which makes them appealing.

Yes. And they are of the same level as those of PhD. They are 980 euros per month.

I had prepared a question to ask you: "Is there any challenging result that you have not yet been able to prove?" But you have already told me that you and Milton could not prove the uniqueness of the KPZ equation and that the appearance of this proof was the touchstone to submit the application to the ERC grant.

Yes. From the results published so far, the one I'm most proud of (it was published in ARMA) was the existence of solution of the KPZ equation and it was relevant in my decision to move forward for this application. There is another article I like very much, but this one is the result that has the most impact. When I finished the exams and qualifying exam at IMPA, Milton finished his PhD, also at IMPA, having stayed there for one or two years as a postdoc and that's when we began to interact. Since 2008 I've been telling Milton, "we have to do this" but we knew it was too hard and it was going to take time. The other article I like a lot was published in CPAM, it is a work easier to understand, which was done while I was at the Courant Institute.

How did you go to the Courant?

I really wanted to have a contact with Varadhan because he is fantastic. Almost every particle systems problem has a little bit of him. He developed several methods, extended the techniques to various contexts . . . I told my PhD advisor at IMPA that I would enjoy spending three months

in the Courant, I had money from the Gulbenkian's prize *Estímulo à investigação* that I won. I wrote to Varadhan and I asked Claudio to write him, telling him who I was, and he agreed to invite me. It was spectacular, I was very well received, I shared an office with Martin Hairer, it was very cute. At that time, I was pregnant, from home to the Courant there were about four blocks, but there were pizzas to sell on the street and I would come to the Institute very sick . . . and I used to say, "I hate New York, a person is sick all day!" I presented at the Courant a seminar on the work that I later submitted with Milton at CPAM.

Publish in the CPAM helps the prestige of the authors.

Yes, it is true. But the publication of this article took more than a year. This article and the one I mentioned above are my two most beautiful works. I also have one with Cédric Bernardin, Mariele Simon and Milton Jara, which I like very much, which is in the context of other classes of universality.

The KPZ equation is a universal law, but it is not the only one, there are many others. The question that the project also addresses is the following: we define any dynamics of a particle system, which has a probabilistic law. For systems that conserve an amount that has some physical interest, this system is related to a stochastic PDE. But for systems that hold two, or three, or four, there are more equations. So these equations are part of different classes of universality. And the question is, what laws do we have to arrive at those kind of equations?

Imagine the following example, we have particles that move on coupled oscillators. We may think that we have an Hamiltonian dynamics given by a certain potential, and the particles begin to move and their dynamics conserve several quantities as, for example, the energy and volume of the particles. Each conserved quantity "lives in its world", the energy is described by a certain stochastic PDE and the volume by one other. And the question is, depending on the type of dynamics we choose, what kind of PDE do we get? And how are these equations related, are they coupled or not? And by slightly changing the parameters of a model, with a conserved quantity, as in the case of the KPZ equation, how to go from a certain equation to other equations? They are connected, in fact. I usually draw a picture with clouds, to represent this. One cloud is one class, another cloud is another class, and there is a line connecting them. And in the middle of this line, there is something that can be considered as a fixed point of a more general application. We are making changes in the microscopic system and getting a link from one class of equations to another. The KPZ equation is a bit out of date, because with my work with Milton the subject is now quite well understood. What is now in fashion is the study of other universality classes, for systems with more than one conservation law, which lead us to coupled equations, fractional Laplacian equations, which are very trendy. It appears fractional Laplacians in which the exponents are given by the sequence of Fibonacci! Moving below, in the microscopic scenario, in the dynamics between particles, how do I go up to the macroscopic scenario, for example, to arrive at a fractional heat equation with an exponent that is related to the sequence of Fibonacci?

Tell me a little about your academic path: University of Porto, IMPA (versus a permanent place).

I first made the fourth year of the educational branch of Mathematics at the Faculty of Sciences of the University of Porto.

But why, were you looking for a safe job?

No, I would have been the only student in Pure Mathematics that year, and I did not want to be alone on the course, I thought it was awful to have teachers teaching just for me, but I quickly came to the conclusion that I did not want the teaching path. The following year, I did the fourth year of Pure Mathematics, we were four students. And at the time of first semester exams, I decided to go to IMPA, to take a summer course, to try to understand if I adapted. That was when I met Claudio Landim, my future advisor. I attended a course, taught by him, of Measure and Integration, which did not exist in my time in Porto.

The Licenciatura in Porto, in my time, was fantastic. When I arrived at IMPA, to do my PhD, I did Functional Analysis without any difficulty, I found it easy.

How did you end up in particle systems?

My teachers of the 4th year told us that IMPA was spectacular, specially in the area of Dynamical Systems. I had no idea IMPA had someone working on particle systems. The subjects that gave me pleasure to study were ODEs and Probability. My visit to IMPA at the end of 1st semester of the 4th year of Pure Mathematics was disastrous. When I arrived from IMPA, the 2nd semester classes had already begun and I had to do the 1st semester exams. It was very hard! But I came back with the idea that I needed to return to IMPA. I was hesitating between Analysis and Probability. I never thought of Dynamic Systems. I loved the disciplines of Functional Analysis, Spectral Theory and Stochastic Processes. I spoke with Claudio and he accepted me as his student. I really liked Claudio, his group, the exceptional environment of IMPA. It was a shot in the dark that went very well. The qualifying exams are very hard, I did one in Probability and one in Analysis (my secondary area), but my doctorate went very well, IMPA has this ability to motivate the students. The trick is to do our work with passion. I really love what I do!

Your first prize was the *estímulo à investigação*, from the Gulbenkian Foundation. Tell me how it happened.

I saw the announcement near the Assis' office and I thought, "Why not?" Of course it was great, it gave me money to do a lot of things, like going three months to the Courant, for example. Of course the money makes this possible, contacts and funding are fundamental.

Finishing your doctorate, you returned to Portugal, you had a posdoc grant with Alberto Pinto, a scholarship from *Programa Ciência*, and then a development grant of *Programa Jovens Investigadores*. Am I right?

No. Before returning to Portugal I still did a six-month postdoc in São Paulo, with Pablo Ferrari and that was also a shot in the dark. I did not like São Paulo as much as Rio, but I loved working with him. In six months we made an article. At that time I had a four-year postdoc fellowship in Brazil, but I decided to come back to Portugal for personal reasons. I was welcomed by Cecília Azevedo, who had money from a project with Salvatore, which allowed me to stay at the University of Minho for three months. This was followed by a postdoc with Alberto Pinto, also in Minho, for a year and a half. I was a visiting assistant professor during a semester at the Universidade Nova de Lisboa, maintaining the postdoc in Minho, which gave me a small salary supplement. At the end of the semester, I returned to Minho, where I was advised by Luis Pinto, who was then the Director of the Centro de Matemática, to apply for a scholarship from FCT *Programa Ciência*, with the University of Minho as host institution, and my application was successful . At that time I was a little tired of making successive applications. I later canceled this scholarship and went to teach at PUC in Rio de Janeiro, but meanwhile I applied for another postdoc fellowship from the FCT, the *Programa Jovens Investigadores*.

The relevant question, when describing your academic path, is to make visible the hard life of a young researcher in Portugal, even in the case of one as successful as yours.

Yes, it is true, our life is really hard! My path had several other applications in the middle (involving many hours of work), some to positions in Portuguese universities and it did not end yet. In the meantime, I ran for a place at the Getúlio Vargas Foundation, and got a contract, with a salary of associate professor, fantastic. This foundation is a kind of IMPA, very well known in the area of Economics and has a Department of Mathematics that is growing. I was considering dropping the *Jovem Investigador* scholarship and returning to Brazil for this Foundation, but in the meantime, I gained the position of Associate Professor at IST.

Yes, it is easy to understand, by your description, that the pursuit of stability in Portugal for a doctorate is a very painful thing.

Yes, we apply and wait for the results, and sometimes we see aberrant things . . . , that deceive us. When I applied for the ERC scholarship, my contract was as a *Jovem Investigadora* at the University of Minho. I had tried, several times and without success, to obtain a place of teaching in a Portuguese university.

Tell me a little of your daily life as a mother. You travel a lot, right? How do you manage your life with two small children?

I travel, in fact, a lot, I make many trips a year. My oldest son is five, when I am out, he does not talk to me on skype, he says “you’re not here”. It is the power he thinks he has over me . . .

But when you take long term visits, you take them with you?

Yes of course! In the 2nd semester of this year I will go three months to Paris and they also go. More than fifteen days without them and I fell a tightening heart, more than a week already cost me . . . When my son enters the school, I do not know how to solve these situations, maybe I can get a school that accepts him for a few months.

Yes, the life of a woman researcher (still) is not equivalent to that of a man. Even admitting that there is total equality in the tasks with the children and at home, the truth is that when the children are small, they need much more of the mother than of the father. Later that changes.

When my son was little, he hated my computer. The computer was his enemy, who stole my presence from him. I recognize that these five years of my son were very complicated. With my daughter I am already more relaxed, I already have a stable job. When I heard about my job at IST, I was at a dinner party with friends, I looked at my cell phone, I saw that I had won the contest, and tears were streaming down my face. My friends were worried and I told them I had good news. It was hard, my life. I have, at the moment, 38 years old and there was a lot of fight to get here, but there it is, as I love what I do, looking back I can only say it was great!

INEQUALITIES IN NONLINEAR FOURIER ANALYSIS

by **Diogo Oliveira e Silva***

This note is meant as a gentle introduction to nonlinear Fourier analysis. In particular, we highlight the parallel with the classical linear theory, describe some recent developments, and mention some open problems in the field.

1 THE FOURIER TRANSFORM

The FOURIER transform is an ubiquitous tool in mathematical analysis. Its power stems from the fact that it reveals certain properties about the function which are not readily apparent by inspection. One of the first questions that arises concerns its mapping properties on the scale of LEBESGUE spaces L^p. Given a sufficiently nice function $f : \mathbb{R} \to \mathbb{C}$, we shall define its FOURIER transform as follows:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, \mathrm{d}x.$$

One easily checks that the FOURIER transform of an integrable function is a uniformly continuous function which decays at infinity, an observation which is historically attributed to early works of RIEMANN and LEBESGUE. With the above normalization, a straightforward application of the triangle inequality shows that the FOURIER transform is a contraction from L^1 to L^∞. It also extends to an isometry on the HILBERT space L^2, as a consequence of PLANCHEREL's theorem. In other words, the following hold:

$$\|\hat{f}\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}, \tag{1}$$

$$\|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}. \tag{2}$$

Estimates (1) and (2) can be interpolated with the classical convexity theorem of RIESZ-THÖRIN. As a consequence, we obtain the HAUSDORFF-YOUNG inequality, which in turn asserts the following: Given an exponent $1 \leq p \leq 2$, the inequality

$$\|\hat{f}\|_{L^{p'}(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})} \tag{3}$$

holds for every $f \in L^p$. Here p' denotes the exponent conjugate to p, given by $\frac{1}{p} + \frac{1}{p'} = 1$.

The concept of L^p convergence differs substantially from that of pointwise convergence. A powerful link between the two is provided by *maximal functions*, which are in themselves central objects of study in FOURIER analysis. In general terms, one expects L^p-bounds for a maximal function to imply pointwise almost everywhere convergence of the original operator. In our setting, we are led to define the maximally truncated FOURIER transform,

$$\mathscr{F}_* f(\xi) = \sup_{y \in \mathbb{R}} \left| \int_{-\infty}^{y} f(x) e^{-2\pi i x \xi}\, \mathrm{d}x \right|.$$

The classical MENSHOV-PALEY-ZYGMUND inequality states that, for every $1 \leq p < 2$, there exists a constant $M_p < \infty$ such that

$$\|\mathscr{F}_* f\|_{L^{p'}(\mathbb{R})} \leq M_p \|f\|_{L^p(\mathbb{R})}, \tag{4}$$

for every $f \in L^p$. The case $p = 2$ of inequality (4) is considerably more subtle, and follows from the celebrated result of

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Carleson [3] on the pointwise convergence of Fourier series of square integrable functions. A powerful *variational* refinement of these results has been recently proved by Oberlin, Seeger, Tao, Thiele, and Wright [13]: Given $1 \leq p \leq 2$ and $r > p$, there exists a constant $C_{p,r} < \infty$ such that

$$\Big\| \int_{-\infty}^{y} f(x) e^{-2\pi i x\xi} \, \mathrm{d}x \Big\|_{L_\xi^{p'}(\mathbb{R};\mathscr{V}_y^r(\mathbb{R}))} \leq C_{p,r} \|f\|_{L^p(\mathbb{R})}, \qquad (5)$$

for every $f \in L^p$. In the case $1 \leq p < 2$ and $r = \infty$, inequality (5) reduces to (4). When $p = 2$ and $2 < r < \infty$, inequality (5) strengthens Carleson's theorem by establishing L^2-estimates for the r-variation[1] of the partial sum operator for the Fourier transform.

We conclude this succinct account of linear Fourier analysis by asking the following natural questions: What is the optimal constant in inequality (3)? What are the corresponding *extremizers*? By this we mean functions which saturate the sharp inequality, turning it into an equality. Beckner [2] proved that the inequality

$$\|\hat{f}\|_{L^{p'}(\mathbb{R})} \leq \mathbf{B}_p \|f\|_{L^p(\mathbb{R})} \qquad (6)$$

holds with constant $\mathbf{B}_p := p^{\frac{1}{2p}} p'^{-\frac{1}{2p'}}$, which is strictly less than 1 if $1 < p < 2$, and is sharp. Moreover, equality is attained by Gaussians. Lieb [9] later proved that there exist no other extremizers besides the Gaussian functions. More recently, Christ [4] further refined inequality (6), establishing the following stable version: There exists a constant $c_p > 0$ such that

$$\|\hat{f}\|_{L^{p'}(\mathbb{R})} \leq \Big(\mathbf{B}_p - c_p \frac{\mathrm{dist}_p^2(f, \mathfrak{G})}{\|f\|_{L^p(\mathbb{R})}^2}\Big) \|f\|_{L^p(\mathbb{R})}, \qquad (7)$$

for every nonzero $f \in L^p$. Here $\mathrm{dist}_p(f, \mathfrak{G})$ denotes the L^p-distance from f to the set of all Gaussians, denoted $\mathfrak{G}$. Sharp inequalities and stable versions thereof, together with a characterization of the corresponding sets of extremizers, have a rich history in mathematical analysis. The brief description given here only scratches the surface of this fascinating topic for the very particular case of the Hausdorff-Young inequality, which will nonetheless be of interest to us further along the discussion.

2 The nonlinear Fourier transform

One of the many useful features of the Fourier transform is that it maps a *linear* partial differential equation into an algebraic equation, which can be explicitly solved, and then pulled back to a solution of the original problem via Fourier inversion. There have been many attempts to find suitable replacements for this mechanism in the world of *nonlinear* partial differential equations.

[1] See §3.2 below for a discussion of variation norms.

For the remainder of this note, we shall focus on a simple nonlinear model of the Fourier transform, also known as the Dirac *scattering transform*, or the $\mathrm{SU}(1,1)$-*scattering transform*. To describe it precisely, let us take a measurable, bounded and compactly supported function $f : \mathbb{R} \to \mathbb{C}$, which will generally be referred to as a *potential*. Given an arbitrary number $\xi \in \mathbb{R}$, consider the initial value problem

$$\begin{gathered} \frac{\partial}{\partial x} \begin{bmatrix} a(x,\xi) \\ b(x,\xi) \end{bmatrix} = \begin{bmatrix} 0 & \overline{f(x)} e^{2\pi i x\xi} \\ f(x) e^{-2\pi i x\xi} & 0 \end{bmatrix} \begin{bmatrix} a(x,\xi) \\ b(x,\xi) \end{bmatrix}, \\ \begin{bmatrix} a(-\infty,\xi) \\ b(-\infty,\xi) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{gathered} \qquad (8)$$

This system is well known to have unique absolutely continuous solutions $a(\cdot,\xi)$ and $b(\cdot,\xi)$. Defining functions $a, b : \mathbb{R} \to \mathbb{C}$ via $a(\xi) := a(+\infty,\xi)$ and $b(\xi) := b(+\infty,\xi)$, one is led to study properties of the *forward transform* $f \mapsto (a,b)$. The differential equation (8) forces $|a(\xi)|^2 - |b(\xi)|^2 = 1$, which in particular means that a certain size of the above vector is controlled by the quantity $|a(\xi)|$ alone. It is sometimes convenient to add an extra column and turn this vector into a 2×2 matrix belonging to the classical Lie group

$$\mathrm{SU}(1,1) = \Big\{ \begin{pmatrix} w & z \\ \bar{z} & \bar{w} \end{pmatrix} : w, z \in \mathbb{C} \text{ and } |w|^2 - |z|^2 = 1 \Big\},$$

which is isomorphic to $\mathrm{SL}(2,\mathbb{R})$. It should be emphasized that we are *not* considering the linear Fourier transform on the group $\mathrm{SU}(1,1)$, as the map $f \mapsto (a,b)$ is highly nonlinear. This is at the root of several foundational and technical issues, and among other issues prevents any sort of interpolation scheme from holding in general. On the other hand, the nonlinear Fourier transform enjoys several symmetries which are shared by its linear counterpart, e.g. with respect to L^1-normalized dilations, translations, and modulations.

Sources of motivation for considering this precise instance of the nonlinear Fourier transform include the eigenvalue problem for the Dirac operator, the study of completely integrable systems and scattering theory, and the Riemann-Hilbert problem; see the expository paper [15] for further information. The Dirac scattering transform is the simplest nonlinear model that cannot be solved explicitly of a more general transform, the AKNS-ZS nonlinear Fourier transform; see [1, 17] for details.

We now describe some nonlinear analogues of the classical inequalities for the linear Fourier transform from the first section. First of all, Grönwall's inequality from ODE theory implies the following analogue of the Riemann-Lebesgue estimate (1):

$$\big\| (\log |a(\xi)|^2)^{\frac{1}{2}} \big\|_{L_\xi^\infty(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}, \qquad (9)$$

for every potential f.

Secondly, the nonlinear PLANCHEREL's theorem is a well-known scattering identity, a variant of which goes back at least to work of VERBLUNSKY from the 1930s. It states that

$$\big\|(\log|a(\xi)|^2)^{\frac{1}{2}}\big\|_{L^2_\xi(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}, \qquad (10)$$

for every potential f. Identity (10) can be established via a contour integration argument, see e.g. [11, §6], and it is curious to note that no other proof seems available in the literature. The reader might wonder about the role of the square root of the logarithm on the left-hand sides of inequalities (9) and (10). It helps to notice that both inequalities reduce to their linear analogues in first order approximation. In particular, the linear FOURIER transform coincides with the linearization of the nonlinear Fourier transform at the origin.

Similarly to the linear case, one would like to use interpolation in order to obtain a nonlinear HAUSDORFF-YOUNG inequality but, as previously mentioned, this is not available in the current nonlinear setting. However, the seminal work of CHRIST and KISELEV [5, 6] on the spectral theory of one-dimensional SCHRÖDINGER operators implies the following result: If $1 \le p < 2$, then there exists a constant $C_p < \infty$ such that

$$\big\|(\log|a(\xi)|^2)^{\frac{1}{2}}\big\|_{L^{p'}_\xi(\mathbb{R})} \le C_p\|f\|_{L^p(\mathbb{R})}, \qquad (11)$$

for every potential f. The proof produces a family of constants C_p which, contrary to the linear case, blows up as $p \to 2^-$. Thus it is natural to ask:

QUESTION 1.— Do the constants C_p from inequality (11) remain uniformly bounded, as p tends to 2?

Question (1) was originally asked by MUSCALU, TAO and THIELE [11], and was solved in a particular toy model by KOVAČ [7], but remains open in its full generality. In §3.1 we shall describe some recent investigations around this circle of problems. On the other hand, a variational refinement generalizing the nonlinear analogue of the MENSHOV-PALEY-ZYGMUND inequality was established by OBERLIN et al. [13]. This has recently been extended to the discrete setting, and we present some details in §3.2 below.

We close this section by mentioning a nonlinear analogue of CARLESON's theorem on the pointwise convergence of FOURIER series. It was originally formulated in [11, Conjecture 1.2], and we record it here.

QUESTION 2.— Does the following inequality hold, for every square integrable function f?

$$\Big\|\sup_{x\in\mathbb{R}}(\log|a(x,\xi)|^2)^{\frac{1}{2}}\Big\|_{L^2_\xi(\mathbb{R})} \le C\|f\|_{L^2(\mathbb{R})}$$

Question 2 was solved in a particular toy model by MUSCALU, TAO and THIELE [11], but remains open in its full generality. It is known [12] that this fundamental question cannot be settled by estimating the terms in the natural multilinear expansion of the scattering transform.

3 SOME RECENT PROGRESS

3.1 TOWARDS QUESTION 1

By considering truncated Gaussian potentials and linearizing, one may check that the constant C_p from (11) dominates BECKNER's constant from (6), $C_p \ge \mathbf{B}_p$. It may be tempting to conjecture that $C_p = \mathbf{B}_p$. While this is still an open problem, which would immediately provide an affirmative answer to Question 1, the main result from [8] hints at some supporting evidence in this direction. To describe it precisely, fix an exponent $1 < p < 2$, a *height* $H > 0$, and a *width* $W > 0$. We only consider potentials $f : \mathbb{R} \to \mathbb{C}$ of controlled height and width, i.e. such that $|f| \le H$ and f is supported on an interval of length at most W.

THEOREM 1 ([8]).— There exist $\delta, \varepsilon > 0$, depending on p, H, W, such that

$$\big\|(\log|a(\xi)|^2)^{\frac{1}{2}}\big\|_{L^{p'}_\xi(\mathbb{R})} \le \left(\mathbf{B}_p - \varepsilon\|f\|^2_{L^1(\mathbb{R})}\right)\|f\|_{L^p(\mathbb{R})}, \qquad (12)$$

for every potential f satisfying the above hypotheses and $\|f\|_{L^1} \le \delta$.

Note that inequality (12) implies (11) with $C_p = \mathbf{B}_p$, but only for the restricted class of potentials considered in the theorem. Since this class is allowed to depend on the exponent p, no uniformity is claimed. The emphasis is rather on the perhaps surprising fact that the nonlinear HAUSDORFF-YOUNG ratio beats the linear one for sufficiently small values of $\|f\|_{L^1}$.

We briefly describe the main idea behind the proof of Theorem 1. The strategy is to split the analysis into two cases, depending on whether or not the potential f is far from the set of Gaussians in the relative L^p-distance. In the former case, one invokes CHRIST's sharpened HAUSDORFF-YOUNG inequality (7) in order to absorb the error terms coming from linearization. In the latter case, one calculates a few terms of the multilinear expansion of $(\log|a|^2)^{1/2}$, and approximates f by a suitable Gaussian. The error terms that appear are controlled by successive applications of the MENSHOV-PALEY-ZYGMUND inequality (4).

3.2 DISCRETE ANALOGUES

There is a close and fruitful connection between the continuous FOURIER transform and discrete FOURIER series. In a similar vein, TAO and THIELE [16] introduced a discrete model for the solution curves of the nonlinear FOURIER transform. To define it, consider a compactly supported, complex-valued sequence F satisfying $|F_n| < 1$, for every n, and transfer matrices $\{T_n\}$ given by

$$T_n(z) = (1 - |F_n|^2)^{-\frac{1}{2}} \begin{pmatrix} 1 & F_n z^n \\ \overline{F_n} z^{-n} & 1 \end{pmatrix},$$

where $z \in \mathbb{T}$ is a unimodular complex number. Note that $T_n(z) \in \mathrm{SU}(1,1)$. The nonlinear FOURIER transform of the sequence F is defined as an $\mathrm{SU}(1,1)$-valued function on the unit circle given by the expression

$$(a,b)(z) = \lim_{N\to\infty} \prod_{n=-N}^{N} T_n(z),$$

where the ordered product is seen to converge in an appropriate sense provided $F \in \ell^2$. Discrete analogues of the nonlinear RIEMANN-LEBESGUE, PLANCHEREL and HAUSDORFF-YOUNG inequalities are available, see [16, §1–3]. To describe a variational refinement of latter, consider the following truncated versions of the linear and the nonlinear FOURIER transforms of F, respectively denoted by $\sigma = \sigma(F)$ and $\gamma = \gamma[F]$, and given at level N by

$$\begin{aligned} \sigma(F)(N;z) &= \textstyle\sum_{n=-\infty}^{N} \begin{pmatrix} 0 & F_n z^n \\ \overline{F_n} z^{-n} & 0 \end{pmatrix}, \quad \text{and} \\ \gamma[F](N;z) &= \textstyle\prod_{n=-\infty}^{N} T_n(z). \end{aligned} \tag{13}$$

For fixed $z \in \mathbb{T}$, we shall think of the maps $N \mapsto \gamma[F](N;z)$ and $N \mapsto \sigma(F)(N;z)$ as discrete curves taking values on the LIE group $\mathrm{SU}(1,1)$ and its LIE algebra $\mathfrak{su}(1,1)$, respectively. Endow the LIE algebra with the operator norm $\|\cdot\|_{\mathrm{op}}$, and the LIE group with the distance

$$d(X,Y) = \log(1 + \|X^{-1}Y - I\|_{\mathrm{op}}).$$

Given an exponent $r \geq 1$, we are interested in measuring the r-variation in the variable N of the curves σ and γ. The variation is defined as

$$\mathscr{V}_r(\gamma)(z) = \sup_K \sup_{N_0<\ldots<N_K} \left(\sum_{j=0}^{K-1} d(\gamma_{N_j}(z), \gamma_{N_{j+1}}(z))^r \right)^{\frac{1}{r}},$$

and similarly for $\mathscr{V}_r(\sigma)$. Here the supremum is taken over all strictly increasing finite sequences of integers $N_0 < N_1 < \ldots < N_K$ and over all integers K. We are finally in a position to state the following discrete, variational, nonlinear HAUSDORFF-YOUNG inequality.

THEOREM 2 ([14]).— Let $1 \leq p < 2$ and $r > p$. Then there exists a constant $D_{p,r} < \infty$ such that

$$\begin{aligned} \|\mathscr{V}_r(\gamma[F])\|_{L^{p'}(\mathbb{S})} + \|\mathscr{V}_r(\gamma[F])\|^{1/r}_{L^{p'/r}(\mathbb{T}\setminus\mathbb{S})} &\leq \\ \leq D_{p,r} \left\| \log\left(\tfrac{1+|F_n|}{1-|F_n|}\right)\right\|_{\ell^p(\mathbb{Z})}&, \end{aligned} \tag{14}$$

for every $F \in \ell^p$ satisfying $|F_n| < 1$, for every n. Here, $\mathbb{S} := \{z \in \mathbb{T} : \mathscr{V}_s(\gamma[F])(z) \leq 1\}$, where $s = r$ if $p < r < 2$, and $s = (p+2)/2$ if $r \geq 2$.

We briefly describe the proof of Theorem 2, which comprises two parts. The first part is inspired by the adaption of LYONS' theory of rough paths [10] by OBERLIN et al. [13] to study variation norms on $\mathrm{SU}(1,1)$. In particular, given a potential F, one shows that the r-variation of the discrete curve $\gamma[F]$ can be controlled by the r-variation of the linearized curve $\sigma(F)$, plus an extra term that accounts for the possible presence of large jumps. The second part amounts to a discrete variational version of the MENSHOV-PALEY-ZYGMUND inequality (4). This step requires $r > p$, and is accomplished via an adaptation of the original argument of CHRIST-KISELEV [5] to the variational setting.

We finish by noting that the range of exponents promised by Theorem 2 is almost sharp. Indeed, given $p > 1$, one easily checks that inequality (14) can only hold if $r > p$. On the other hand, extending this inequality to $p = 2$, already in the simplest case $r = \infty$, would provide an affirmative answer to a discrete version of Question 2. This remains a central open problem in the area.

ACKNOWLEDGEMENTS

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REPORT

FOLIO 2017
Mathematics and Literature II

An International Workshop and an Exhibition held in Óbidos, Portugal, the 21st October 2017

by **Carlota Simões***

The *Centro Internacional de Matemática* (http://www.cim.pt/) in partnership with the *Science Museum of the University of Coimbra* (http://www.museudaciencia.org), and the *Portuguese Mathematical Society* (https://www.spm.pt/), with the support of the *Center for Mathematics, Fundamental Applications and Operational Research of the University of Lisbon* (http://cmafcio.campus.ciencias.ulisboa.pt) and the *FCT Doctoral Program in Materialities of Literature of the Faculty of Arts and Humanities of the University of Coimbra* (https://apps.uc.pt/courses/en/course/2341), organized MATHEMATICS AND LITERATURE II, a one day international workshop (October 21) and the exhibition ET SIC IN INFINITUM (October 21–29), taking part of the program of FOLIO–*International Festival of Literature of Óbidos*, organized by the Municipality of Óbidos (http://www.obidos.pt/) from 19 to 29 of October of 2017.

The workshop consisted of presentations by F. J. Craveiro de Carvalho (Universidade de Coimbra), Samuel Lopes (Universidade do Porto), Carlos Santos (Universidade de Lisboa), José Francisco Rodrigues (Universidade de Lisboa), Michèle Audin (Université de Strasbourg), António Cordoba (Universidad Autónoma de Madrid) and

* Museu da Ciência e Departamento de Matemática da Faculdade de Ciências e Tecnologia da Universidade de Coimbra

Figure 1.—A glimpse on the participants of the international workshop.

Figure 2.—F.J. Craveiro de Carvalho presenting several authors such as Adam Zagajewski, Katharine O'Brien or Miroslav Holub.

Manuel Portela (Universidade de Coimbra).

The talks covered literature with mathematical influence, like the concept of dimension in Edwin Abbott Abbott's Flatland by Samuel Lopes, works of literature by authors who are also mathematicians, as in the talk by José Francisco Rodrigues on *literary mathematicians, from Galileo to Hausdorff*, or in the presentation by Michèle Audin, a mathematician and member of the OuLiPo (Ouvroir de Littérature Potentielle), who talked about her own works, as well as computational literature, created by computer programs, as in the talk by Manuel Portela.

On poetry and mathematics, Carlos Santos recovered a study on *The Lusíadas*, proving that the main astronomical source of Camões was the *Tratado da Esfera* from the great

Figure 3.—Carlos Santos talked about the astronomy contained in the Portuguese epic poem *Os Lusíadas*, from Luís de Camões.

Figure 4.—Samuel Lopes's talk was on the 1884 novel *Flatland: A Romance of Many Dimensions*, from Edwin Abbott Abbott.

Figure 5.—José Francisco Rodrigues talked about literary mathematicians, from Galileo to Hausdorff.

Figure 6.—Michèle Audin, a member of the OuLiPo (*Ouvroir de Littérature Potentielle*).

Figure 7.—António Cordoba, presenting *Poetry among Theorems*.

Figure 8.—Manuel Portela, presenting *Permutation and Generation in Computational Literature*.

Figure 9.—The exhibition ET SIC IN INFINITUM: *an intermedial and transliterary installation.*

mathematician Pedro Nunes. In his talk *Poetry among theorems*, António Cordoba presented poetry with mathematical influence by several authors, referring in particular, the Spanish José Echegaray, professor of mathematics and Nobel Prize of Literature in 1904. Craveiro de Carvalho presently dedicates himself to the translation into Portuguese of poems related to mathematics, originally written in English language. In this talk he presented several authors such as Adam Zagajewski, Katharine O'Brien or Miroslav Holub.

The exhibition ET SIC IN INFINITUM: *an intermedial and transliterary installation*, from Carolina Martins and Diogo Marques, is the result of a partnership between the artists association wreading-digits (wreading-digits.com), and the Doctoral Program in Literature Materialities of the Faculty of Arts and Humanities of the University of Coimbra and completed the program of FOLIO 2017–*Mathematics and Literature II* (http://www.cim.pt/agenda/event/182).

Acknowledgement: The author thanks Graça Brites for the photos of the Workshop, J.F. Rodrigues for the picture of Óbidos and Ana Calçada and her collaborators from FOLIO for their hospitality in Óbidos.

JOSÉ ANASTÁCIO DA CUNHA (1744–1787)

by **Luis Saraiva***

Introduction

José Anastácio da Cunha is one of the few Portuguese mathematicians that can be said to have made significant contributions to the world of mathematics. He was born in Lisbon, in 1744. On his life see [Ferro, 1987]. Earlier in his youth (some sources mention the period 1760–1763) he was educated by the priests of the *Oratorian Congregation*. In 1764 he was appointed first lieutenant of the Company of Artillerymen of the *Porto Artillery Regiment*. Although the regiment is said to be from Porto, in fact it was stationed at Valença, where it remained during the time da Cunha was a member. The majority of the officers of the regiment were foreigners, many from protestant countries, and so da Cunha contacted authors, ideas and books that normally a Portuguese officer of that time would not have access. While in this Regiment, he wrote the *Essay on the Mines* (*Ensaio sobre as Minas*") [da Cunha, 1994], which is said to have been decisive for the Marquis of Pombal noticing him, and also, in 1769, the *Physical-Mathematical Letter, on the Theory of Powder in general, etc* (*Carta Fisico-Mathematica, sobre a Theorica da Polvora em Geral, etc*). None was published during his lifetime.

In 1772 took place the *Reform of Coimbra University*, the first reform of the University since 1612, aiming at bringing Portugal up to the level of the best European Universities. In particular, it was created the first Faculty of Mathematics in Portugal. In September of that year three teachers for the new faculty were appointed: Miguel Franzini, Miguel António Ciera and José Monteiro da Rocha. In October of the following year Anastácio da Cunha was appointed lecturer of the Faculty. He taught there during four academic years: 1774/75 to 1777/78.

The King D. José died in 1777, the *Marquis of Pombal* was dismissed, there was a political change, the backward forces that Pombal had subdued came back to power, and the arrests by the Inquisition started. In January 1778 da Cunha was arrested on the grounds of religious heterodoxy, following a denunciation. He was taken to the Inquisition tribunal and condemned in October. He was expelled from the university, and was sentenced to stay three years with the *Oratorian Congregation*, to be followed by four years of exile in Évora. He was said to never again return either to the University or to Valença. In January of 1781, the last year of the stay at the *Oratorian Congregation* and the four years of Exile in Évora were pardoned. On his Inquisition Trial, see [Ferro, 1987]. It is not known the exact date he was appointed to teach mathematics at the Colégio de S. Lucas of Casa Pia de Lisboa. It is probable that the year he started to teach there was 1782. We know for sure that in 1783 he was already teaching there, The Casa Pia is an institution founded in 1780 by Queen D. Maria I and initially organized by the Police Superintendent Pina Manique, which aimed to protect and educate young children either orphans or with very poor parents, who had not enough money to be able to offer their children the minimum for surviving and having an education. In this way da Cunha avoided to be unemployed, but his salary at the S. Lucas college was one quarter of the one he had at the University. We do not know exactly when he left the S. Lucas College, we know by the time he died, on January 1, 1787, he was no longer a teacher there. He completed his masterwork, *Principios Mathematicos* while he was teaching at the Casa Pia, although the book was only published in 1790. three years after his death. In 1807 Domingos de Sousa Coutinho (1760–1833)[1] published in

[1] Brother of Rodrigo de Sousa Coutinho. He obtained a degree in Law awarded by the University of Coimbra. He was a professional diplomat, and started in Denmark in 1788. He was in Turim from 1796 to 1803. He was at the Portuguese Embassy in London from 1803 to 1814.

* CIUHCT• CMAF-CIO Universidade de Lisboa

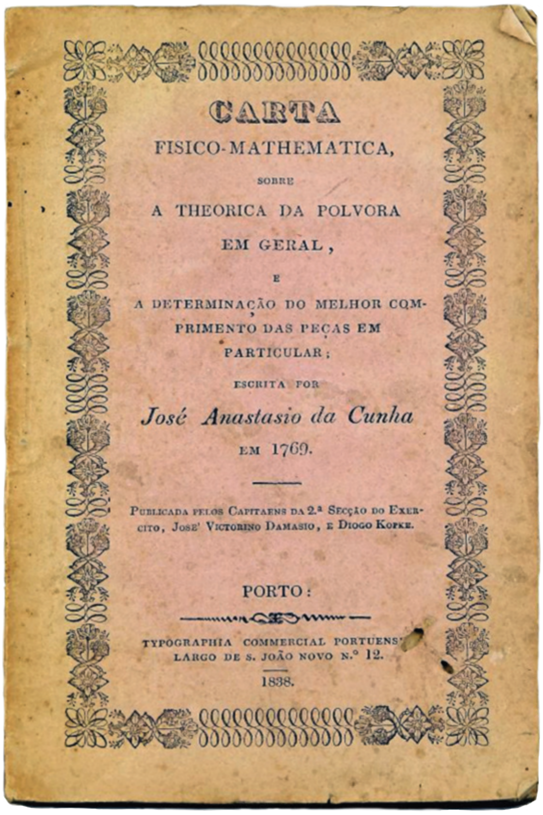


Figure 1. Physical-Mathematical Letter

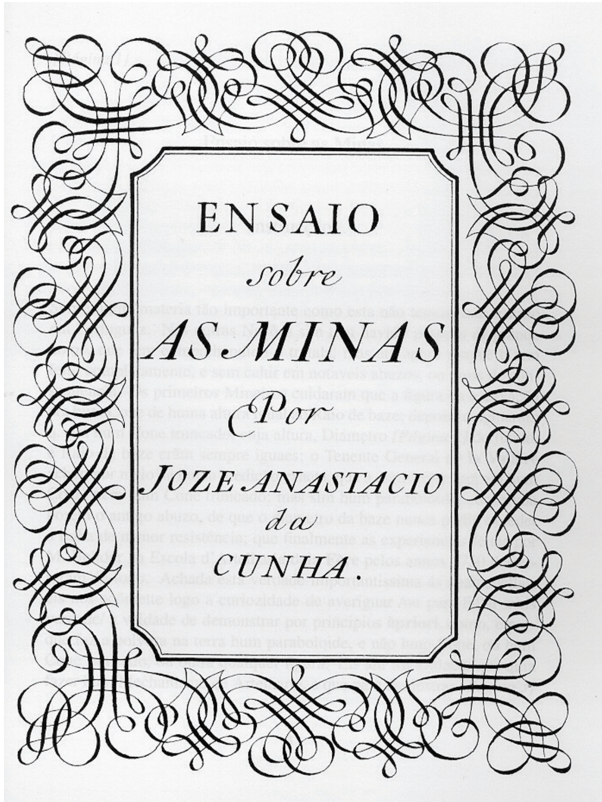


Figure 2. Essay on the Mines

London his *Essay on the Principles of Mechanics* (*Ensaio sobre os Princípio da Mecânica*). In 1811 it is the turn of the French translation of his *Mathematical Principles* by his student and friend João Manuel de Abreu (1757–1815)[2] to be published in Bordeaux, with a so called second edition, but probably just the leftovers of the first one with a different front cover, appearing in Paris in 1816.

The Mathematical Principles[3]

The main work of da Cunha is his *Mathematical Principles*, a 302-page book divided in 21 chapters, which he calls "books". It covers a wide range of subjects, and there are different levels of both rigour and depth in this work, from the elementary to the highly specialized and innovative. Subjects go from Euclidean to analytic geometry, from differential calculus to differential equations, from algebraic equations to the theory of number series.

There are no references in his book, but that might be because he died before completing it. Nevertheless there are clear influences of Newton, D' Alembert, Lagrange and Euler. Da Cunha uses the Greek method of unfolding his subjects, so he uses systematically a sequence of axioms-definitions-propositions-proofs, he tries to be the most concise possible. This was most unusual in his time, and so it had the consequence of becoming a difficult book for the University students of his time, as its understanding implied the reader to be able to reason mathematically.

Internationally, the first historian who had an impact in calling the attention to da Cunha's work in the 20th century was the Soviet historian Adolph Pavlovich Youschkevitch (1906–1993) who wrote an important paper in *Revue d'Histoire des Sciences* [Youschkevitch, 1973]. Youschkevitch concludes his paper saying [Youschkevitch, 1973; p. 22]

> nous avons le droit et le devoir de ranger J. A. da Cunha parmi les éminents prédécesseurs de la reforme du calcul infinitesimal réalisée peu après sa mort prématurée par Bolzano, Gauss, Cauchy, Abel et d'autres géomètres du XIXème siècle

[2] He was at the *Porto Artillery Regiment* with da Cunha. He was first a student, then a friend of da Cunha. He graduated in mathematics at the Coimbra University. He was also arrested by the Inquisition in 1778, and condemned to three years of confinement

[3] In this chapter I will follow my paper [Saraiva, 2012]

We will go into more detail in books IX and XV, respectively on the theory of series and on differential calculus, where DA CUNHA was as innovator.[4]

Book XV is a 15 page chapter [da Cunha, 1790; pp. 196–120] where the author provides a rigorous study on the main properties of number series. His definition of a convergent series is

> Mathematicians call convergent series a series whose terms are initially determined, each one by the preceding terms, such that the series can be continued, and finally it makes no difference whether it is continued or not, because we can disregard without any noticeable error the sum of any number of terms that we wish to add to those already written or designated; and these last terms are denoted by writing &c. after the first two or three terms, or any number of terms we want; however it is necessary that either the written terms show how we could continue the series, or that this is known in some other way.[5]

This definition can be said to be formally equivalent to CAUCHY's although there are some subtle differences in attitude, which do not affect the practical use of both definitions, as it is expressed in [Giusti, 1990; p. 45]:

> ce qui pour Cauchy sera choisi arbitrairement, pour da Cunha est arbitrairement fixé

For CAUCHY we chose an arbitrary quantity, and we prove that from a certain order onwards the sum of any number of consecutive terms is less than the arbitrary quantity chosen. But for DA CUNHA, he assumes an agreement has been reached on what can be considered negligible, that is, this must be established beforehand, although this quantity must be arbitrary.

Then using his definition, DA CUNHA proves correctly that when we have $r > 0$, then the series $\sum_{n\geq 1} r^n$ is convergent if $r < 1$. From this he obtains that the series $\sum_{n\geq 1} a^n/n!$ is convergent for any [positive] a. He then goes on to define a^b: let c be such that $\sum_{n\geq 1} c^n/n! = a$, that is, $e^c = a$. Then a^b means $\sum_{n\geq 1}(bc)^n/n!$, that is, in today's terms, $e^{bc} = (e^b)^c = a^b$. After putting this definition, DA CUNHA proves that c always exists, giving its explicit value as a sum of a series.

He goes on and proves some properties of exponentials, and ends the chapter by defining logarithm as the inverse of the exponential, and using the definition and properties of the exponential he has just proved, he shows that $\ln(a^n) = n(\ln a)$.

The subject on Book XV is differential calculus. It is a short chapter, it has only 12 pages. DA CUNHA uses Leibnizian calculus with Newtonian notations. [Mawhin, 1990] says he is the first mathematician to formulate a modern analytic definition of the differential of a real function of real variable, anticipating CAUCHY's definition. He starts the chapter stating some basic definitions: *variable* (an expression which can have more than one value), *constant* (an expression which only has a single value), *infinite variable* (a variable that admits values greater than any given magnitude), *infinitesimal variable* (a [positive] variable that can have smaller values than any given magnitude) and *function* (if the value of a certain expression A depends on the value of another expression B, then A is said to be a function of B, and B is said to be the root of A) [da Cunha, 1790; p. 193]. He defines differential as follows:

> Let dx be a quantity homogeneous [that is, of the same kind] to the root [that is , the argument] x, which we call the fluxion of the root; we call the fluxion of Γx, and we write $d\Gamma x$, the quantity that will make $d\Gamma x/dx$ constant and
> $$\frac{\Gamma(x+dx)-\Gamma x}{dx} - \frac{d\Gamma x}{dx}$$
> an infinitesimal or zero, if dx is an infinitesimal and all that does not depend on x remains constant.[6]

[Mawhin, 1990: pp. 99–100] immediately concludes that this definition implies that $d\Gamma x$, is linear with dx, that is, there is a constant k such that $d\Gamma x = k \cdot dx$, and that

$$\Gamma(x+dx) - \Gamma(x) - d\Gamma x = dx \cdot A(dx),$$

with $A(dx) \to 0$ when $dx \to 0$ that is,

$$\Gamma(x+dx) - \Gamma(x) = k \cdot dx + dx \cdot A(dx).$$

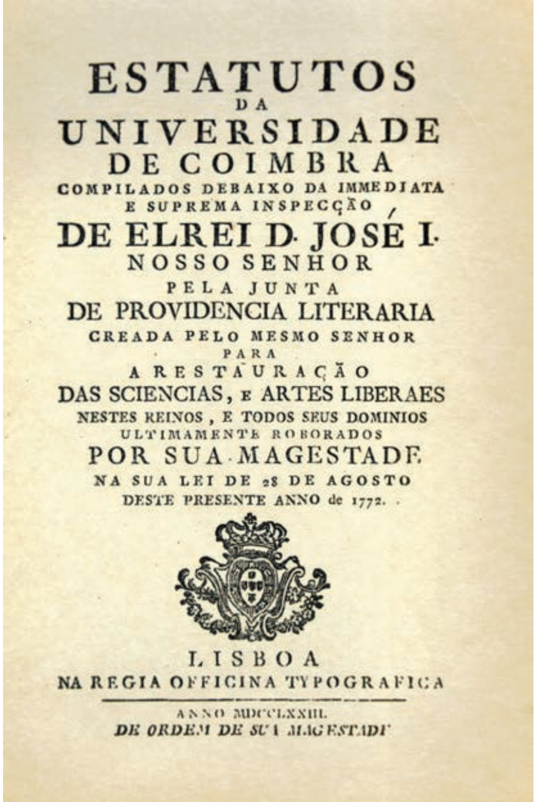
This corresponds to the modern way of defining differential of a function.[Grattan-Guinness, 1990; p. 57] expresses doubts concerning the clarity of the definitions of dx and of the fluxion of x, but this does not detract from the innovation of this definition.

DA CUNHA defines differentials of higher order and proves correctly that if A, B, C, D, &c., are constants, and if x is an infinitesimal, then $Ax + Bx^2 + Cx^3 + Dx^4 +$ &c.

[4] There are many papers on da Cunha's *Mathematical Principles*. In English see [Giusti, 1990], [Queiró, 1988] and [Oliveira, 1988] ; in Portuguese, among others, see [Rodrigues, 1999; mainly pp. 78–83]

[5] "Serie convergente chamam os Mathematicos àquella, cujos termos saõ semelhantemente determinados, cada hum pelo numero dos termos precedente[s], de sorte que sempre a serie se possa continuar, e finalmente venha a ser indiferente o continua-la ou naõ, por se poder desprezar sem erro notável a somma de quantos termos se quisesse ajuntar aos já escritos ou indicados: e estes ultimos indicam-se escrevendo &c. depois dos primeiros dois, ou três, ou quantos se quiser: he porem necessário que os termos escritos mostrem como se poderia continuar a serie, ou que isso se saiba por outra via" [da Cunha, 1790; p. 106]

[6] "Escolhida qualquer grandeza, homogenea a uma raiz *x*, para se chamar fluxaõ dessa raiz, e denotada assim *dx*; chamar-se há fluxaõ de Γ*x*, e se denotará assim *d*Γ*x*, a grandeza que faria constante, e infinitessimo ou cifra, se dx fosse infenitessimo [sic] e constante tudo o que naõ depende de *dx*."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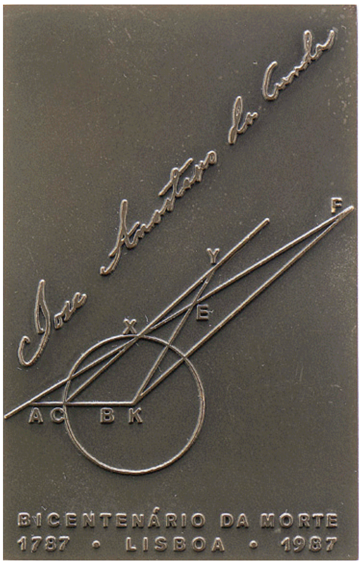


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OBRA POSTHUMA
DE
JOZÉ ANASTACIO DA CUNHA.
DADA A LUZ
Por D. D. A. de S. C.
POSSUIDOR DO MS. AUTOGRAPHO.
London:
PRINTED BY COX, SON, AND BAYLIS,
GREAT QUEEN-STREET;
FOR J. BUDD, BOOKSELLER TO H. R. H. THE PRINCE OF WALES,
AT THE CROWN AND MITRE, PALL-MALL.
1807.

Figure 3. Statutes of the University of Coimbra, 1772

Figure 4. Signature of DA CUNHA

Figure 5. Essay on the Principles of Mechanics

is also an infinitesimal. Then some of the properties of differentials are proved, including the differentials of the sum, of the product, of the power and of the logarithm.

DA CUNHA defines differential of any order of a function of more than one variable [da Cunha, 1790; p.199]:

$$\frac{d^n\Gamma(u, x, z, \&c.)}{du\ dx\ dz\ \&c.}$$

denotes what results from dividing the fluxion of $\Gamma(u, x, z, \&c.)$ in order to the root u by du; then we divide the fluxion of this new quotient in order to the root x by dx; and so on.[7]

He proves incorrectly TAYLOR's theorem, as he assumes that all functions are equal to the sum of their power series.

Then he proceeds to prove that the mixed differentials of a function of two variables are equal. For his proof he uses TAYLOR's expansion of the function. Nevertheless, as is remarked in [Grattan-Guiness, 1990; p. 57] this is worthy of note, as this is a result that seemed obvious for many of DA CUNHA's contemporaries, so they did not think that a proof was required.

RECEPTION OF MATHEMATICAL PRINCIPLES

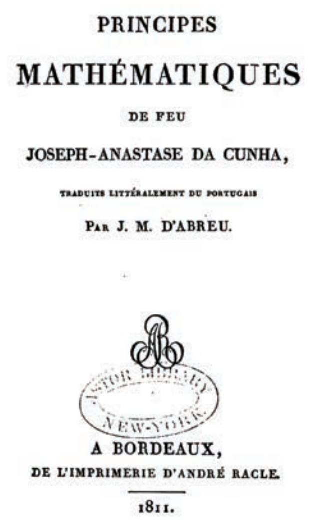
During his life DA CUNHA created and maintained a dedicated circle of students and friends that would after his death continue to defend their teacher and friend, whenever they felt he had an unfair critic. The Scottish mathematician JOHN PLAYFAIR (1748–1719) published in the July-November 1812 issue of the *Edinburgh Review* (volume XX) a review of DA CUNHA's *Mathematical Principles*, after the publication of ABREU's French translation. This is included in [Proceedings, 1990; pp. 415–423]. A Portuguese translation of PLAYFAIR's review appeared in 1813 in volume VII of the Portuguese journal published in London *O Investigador Portuguêz em Inglaterra* (*The Portuguese Researcher in England*). The review was considered among the DA CUNHA circle to be completely missing the importance of the book: therefore both JOÃO MANUEL DE ABREU and ANASTÁCIO JOAQUIM RODRIGUES (?–1818), another student and friend of DA CUNHA, wrote answers to PLAYFAIR. Both were published in volume VIII of *Investigador*

[7] "denota o que resulta de dividir por *du* a fluxaõ de Γ(*u*,*x*,*z*,&c.) tomada relativamente á raiz *u*; e por *dx* a fluxaõ do novo quociente, tomada relativamente á raiz *x*; e assim por diante." I corrected a couple of printing mistakes in the definition. But there is no doubt in DA CUNHA's definition, as the practical example that is solved after the definition clearly shows.

PRINCIPIOS MATHEMATICOS
PARA INSTRUCÇAÕ
DOS ALUMNOS DO COLLEGIO
DE
SAÕ LUCAS,
DA REAL CASA PIA DO CASTELLO
DE
SAÕ JORGE:
OFFERECIDOS
AO SERENISSIMO SENHOR
D. JOAÕ,
PRINCIPE DO BRAZIL:
COMPOSTOS PELO DOUTOR
JOSE' ANASTACIO DA CUNHA,
DE ORDEM DO DESEMBARGADOR DO PAÇO
DIOGO IGNACIO DE PINA MANIQUE,
Intendente Geral da Policia da Corte,
e Reino, &c, &c, &c.

LISBOA
NA OFFIC. DE ANTONIO RODRIGUES GALHARDO.
Impreſſor do Eminentiſſimo Senhor Cardeal Patriarca.
ANNO M.DCC.XC.
Com licença da Real Meza da Commiſſaõ Geral
ſobre o Exame, e Cenſura dos Livros.

Figure 6. Mathematical Principles by José Anastácio da Cunha (1790)



PRINCIPES
MATHÉMATIQUES
DE FEU
JOSEPH-ANASTASE DA CUNHA,
TRADUITS LITTÉRALEMENT DU PORTUGAIS
PAR J. M. D'ABREU.

A BORDEAUX,
DE L'IMPRIMERIE D'ANDRÉ RACLE.
1811.

Figure 7. The French translation of João Manuel de Abreu (1811)

Portuguez and both are included in [Proceedings, 1990: pp. 449–486 and pp. 425–448]. But of course, as the two answers were written in Portuguese and published in a journal published in England, but whose public was essentially for the emigrants of the Portuguese cultured community in England, their impact was minimal.

Rodrigues also wrote a review of da Cunha's *Principios Mathematicos* in the August 8, 1811 issue of *Moniteur Universel*, transcribed in [Proceedings, 1990; pp. 399–404].

There were other reviews: besides the above mentioned Playfair review, there was an anonymous review in the November 14, 1911 of *Göttingishche gelehrte Anzeigen*, but thought to be by Johann Tobias Mayer. Gauss, in a letter to Bessel, criticized this review. This was analysed in [Youschkevitch, 1978]. We have already mentioned the review of the *Mathematical Principles* that appeared in *Moniteur Universel*. Recently João Caramalho Domingues found an anonymous review (but thought to be of Vincenzo Brunacci), in the March/April 1816 issue of the Italian journal, *Giornale di Fisica, Chimica, Storia Naturale, Medicina ed Arti* [Domingues, 2011].

Da Cunha also influenced mathematicians in Portugal. On this matter see [Duarte and Silva, 1990]

Final Notes

In this brief paper we just give some information on da Cunha, and mainly on his *Mathematical Principles*, but all his other works are worthy of analysis. And in the last twenty five years more works by da Cunha were found: first it was the *Essay on the Mines* (*Ensaio sobre as Minas*), a work already mentioned by da Cunha in his 1769 work *Physical-Mathematical Letter*. It was found in the Archives of the Braga District by Maria Fernanda Estrada, and published in book form in 1994 [da Cunha, 1994]. In 2005, in the same Archive of the Braga District, two other researchers, Maria do Céu Silva and Maria Elfrida Ralha found another seven da Cunha manuscripts. A team of researchers was formed to study those manuscripts, and an year later a Conference took place in Braga, where the results of their research was displayed, and two books were published, one with the da Cunha papers [da Cunha, 2006a], the other one with the papers written about those manuscripts, together with others on different aspects of da Cunha's life and work and his time [da Cunha, 2006b]. Today the research continues to be done on other manuscripts that were recently found.

Anastácio da Cunha has been a much celebrated mathematician, first by his contemporaries, like João Manuel de Abreu, Anastácio Joaquim Rodrigues and other close friends, but also by people that in some way at a certain moment opposed him, as Francisco de Borja Garção Stockler[8] (1759–1829), a mathematician who wrote the first history of mathematics in Portugal and in fact it is the first history of mathematics in a single country that was published: *Historical Essay on the Origins and Developments of Mathematics in Portugal* (*Ensaio Histórico sobre as origens e progressos das matemáticas em Portugal*) published in London in 1819, thirty two years after da Cunha's death. Stocker limits his research up to 1779, the year of the foundation of Lisbon's Academy of Scienc-

[8] On Stockler see [Saraiva, 1993].

es. By that time DA CUNHA had no works published. However STOCKLER felt obliged to write a 6-page note on DA CUNHA, only second in length to what he wrote on PEDRO NUNES (1502–1578), the Portuguese leading mathematician of the 16th century.

FRANCISCO GOMES TEIXEIRA (1851–1933) the leading Portuguese mathematician of the second half of the 19th century, and the founder of the first Portuguese international journal, the *Journal of Mathematical and Astronomical Sciences* (*Jornal de Sciencias Mathematicas e Astronomicas*),[9] wrote about DA CUNHA in his *History of Mathematics in Portugal* something similar to what YOUSCHKEVITCH would write forty years later [Teixeira, 1934; p. 260]:

> In the XVII$^{\text{th}}$ century, Anastácio da Cunha is one of the forerunners of the geometers who in the XIX$^{\text{th}}$ century did this considerable work of organizing logically the new domains that were started in the world of numbers, and his works and his name must be included in the splendid history of this organization.[10]

We hope that this introduction to JOSÉ ANASTÁCIO DA CUNHA makes some of its readers interested in exploring more in depth the mathematician and free thinker that DA CUNHA was.

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[9] On Gomes Teixeira and his journal, see [Saraiva, 2014]

[10] "Anastácio da Cunha é no século XVIII um dos precursores dos geómetras que no século XIX realizaram esta obra considerável da organização lógica dos novos domínios que se tinham aberto no Mundo dos números e os seus trabalhos e o seu nome devem figurar na história brilhante desta organização"

ESGI119
The 119th European Study Group with Industry

by **Eliana Costa e Silva***, **Isabel Cristina Lopes****, and **Aldina Correia*****

The 119th European Study Group with Industry—ESGI119 was held at *Porto Design Factory*, from June 27th to July 1st, 2016, organized by ESTGF, ESEIG, and CIICESI-*Center for Research and Innovation in Business Sciences and Informations Systems* of the Polytechnic of Porto.

This was the tenth edition in Portugal[1] of a series of events that have annually brought together academics from the most diverse areas of Mathematics with experience in solving industrial problems, for R&D transfer. Suggesting methodologies from their area of expertise, participants worked collaboratively for a week to address industrial challenges that companies face every day. About 60 researchers, from different European higher education institutions, participated in solving the following five challenges proposed by Portuguese companies.

SAVANA SHOES (footwear company with more than 27 years of existence and specialized in children's footwear) proposed the study of its packaging process. The aim was to reduce the variety of carton sizes, the waste of empty space in the interior of the cartons, and to eliminate the need to carry out the experimental procedure used by the company up to now, thus reducing packing time and increasing process efficiency.

THE LUXURY HOTEL GROUP *Douro Palace Hotel Resort & Spa* and *Douro Royal Valley Hotel & Spa* wanted to study the best pricing policy, by finding an algorithm that could manage the optimal price with respect to some characteristics that the company considers important.

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*** aic@estg.ipp.pt • CIICESI/ESTG–Politécnico do Porto

[1] http://www.estgf.ipp.pt/esgi/

Figure 1.—Representatives of Savana company meeting with researchers during the ESGI119

Figure 2.— Representative of Primavera company discussing with researchers during the ESGI119

The third problem was presented by Aveleda S.A., the world's leading producer of *vinho verde*, exporting more than half of its annual production to more than 70 countries around the world. This company was concerned that the processing capacity of the wine house is far from its maximum, due to the irregularity of the flow of grapes throughout the day, and needed to improve the existing process (Fig. 3).

Figure 3.— The group working on the challenge proposed by Aveleda

The company Primavera BSS requested an efficient scheduling algorithm in order to add new features to its production management software that could be sufficiently generic and adaptable to be used by different industries, such as metal, furniture, wood, textile and food industry (Fig. 2).

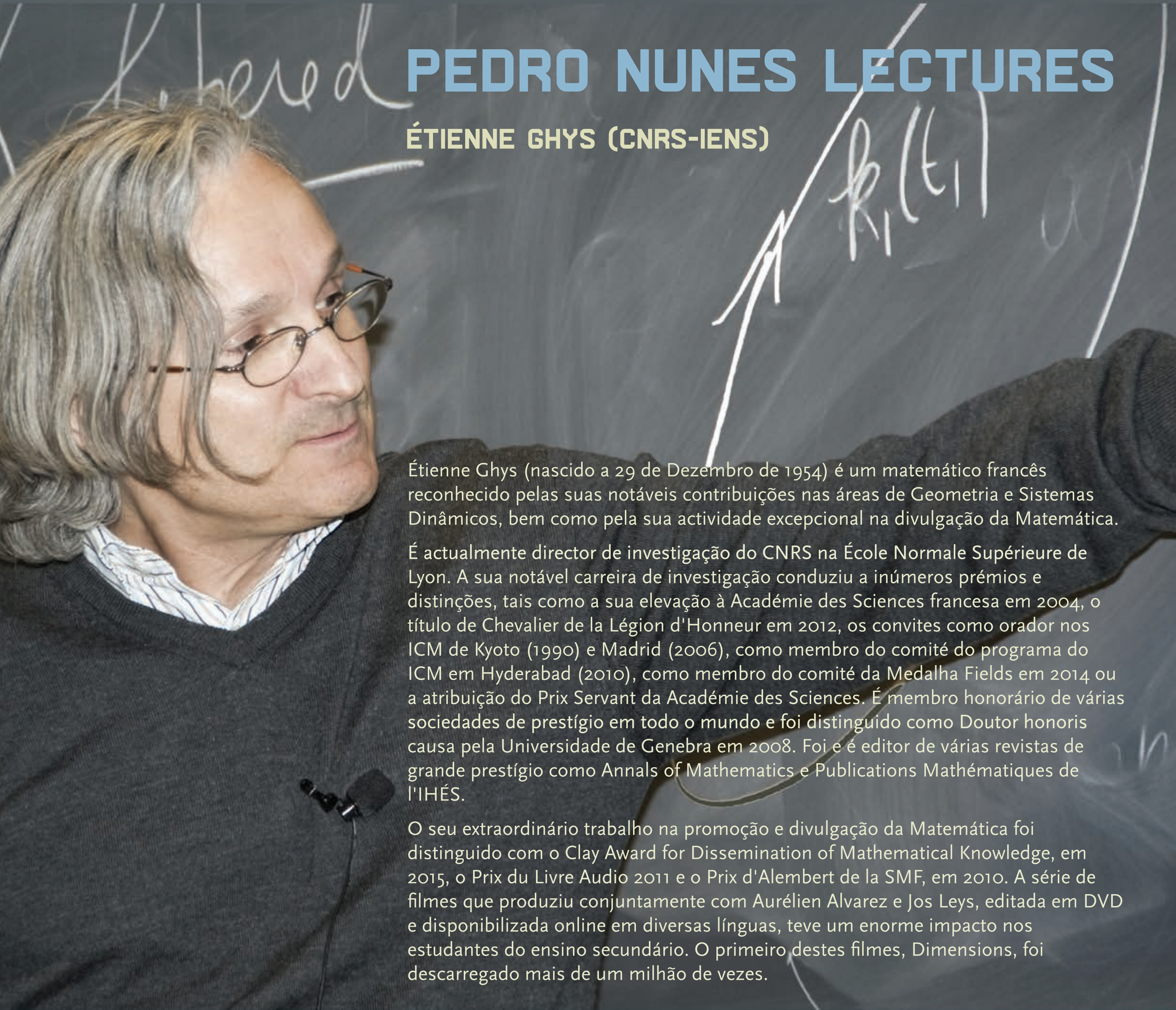
EDP–Energias de Portugal, one of the largest energy operators in the Iberian peninsula and the largest Portuguese industrial group, presented a challenge that consisted in simulating electricity prices, not only for the purpose of calculating risk measures, but also for scenario analysis in terms of prices and strategy.

The shoe company Savana was also very satisfied with the solution achieved by the ESGI119 working group (Fig. 1). The representative of this company, Jorge Fernandes, said: "We think that the group that was involved in these processes presented solutions that really can make the difference in the near future. We are sure that the ESGI is a problem solver booster, using Mathematics applied to industry. We hope that this kind of event will show not only to companies, but also to the general public, that the science of logical and abstract reasoning is not a science that is reserved for schools but a science that is very useful in the lives of people and companies."

The ESGI in Portugal have been financed by the participating companies and the research centres most directly involved in the organization. ESGI119 was funded by COST Action TD1409, *Mathematics for Industry Network* (MI-NET).[2] The COST Program—*European Cooperation in Science and Technology*—is one of the oldest European instruments for cooperation between researchers, engineers and academics from across Europe.

[2] https://mi-network.org

International Center for Mathematics
SOCIEDADE PORTUGUESA DE MATEMÁTICA
FCT Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E ENSINO SUPERIOR



PEDRO NUNES LECTURES

ÉTIENNE GHYS (CNRS-IENS)

Étienne Ghys (nascido a 29 de Dezembro de 1954) é um matemático francês reconhecido pelas suas notáveis contribuições nas áreas de Geometria e Sistemas Dinâmicos, bem como pela sua actividade excepcional na divulgação da Matemática.

É actualmente director de investigação do CNRS na École Normale Supérieure de Lyon. A sua notável carreira de investigação conduziu a inúmeros prémios e distinções, tais como a sua elevação à Académie des Sciences francesa em 2004, o título de Chevalier de la Légion d'Honneur em 2012, os convites como orador nos ICM de Kyoto (1990) e Madrid (2006), como membro do comité do programa do ICM em Hyderabad (2010), como membro do comité da Medalha Fields em 2014 ou a atribuição do Prix Servant da Académie des Sciences. É membro honorário de várias sociedades de prestígio em todo o mundo e foi distinguido como Doutor honoris causa pela Universidade de Genebra em 2008. Foi e é editor de várias revistas de grande prestígio como Annals of Mathematics e Publications Mathématiques de l'IHÉS.

O seu extraordinário trabalho na promoção e divulgação da Matemática foi distinguido com o Clay Award for Dissemination of Mathematical Knowledge, em 2015, o Prix du Livre Audio 2011 e o Prix d'Alembert de la SMF, em 2010. A série de filmes que produziu conjuntamente com Aurélien Alvarez e Jos Leys, editada em DVD e disponibilizada online em diversas línguas, teve um enorme impacto nos estudantes do ensino secundário. O primeiro destes filmes, Dimensions, foi descarregado mais de um milhão de vezes.

25 DE SETEMBRO DE 2017
FUNDAÇÃO CALOUSTE GULBENKIAN

18:00 — AUDITÓRIO 3

ENGARRAFAMENTOS NAS GRANDES CIDADES: O PREÇO DA ANARQUIA*

* LIVESTREAM EM: HTTPS://LIVESTREAM.COM/FCGLIVE/20170925CONFERENCIAETIENNEGHYS

27 DE SETEMBRO DE 2017
UNIVERSIDADE DE COIMBRA
DEPARTAMENTO DE MATEMÁTICA

15:00 — SALA PEDRO NUNES

TRÊS BOLAS MATEMÁTICAS

29 DE SETEMBRO DE 2017
UNIVERSIDADE DO PORTO
DEPARTAMENTO DE MATEMÁTICA

14:30 — SALA 0.07

SINGULARIDADES DE CURVAS ALGÉBRICAS REAIS: UM PONTO DE VISTA TOPOLÓGICO

FUNDAÇÃO CALOUSTE GULBENKIAN
CMUC Centro de Matemática Universidade de Coimbra
CENTRO DE MATEMÁTICA UNIVERSIDADE DO PORTO