Some mathematical aspects of mountain waves: work carried out at Instituto Dom Luiz

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Research in many areas of the Geosciences is pursued at Instituto Dom Luiz (IDL), an Associated Laboratory of the University of Lisbon that is a partner of CIM. In the Atmospheric and Climate Modelling group at IDL, research on basic fluid dynamics relevant to meteorology is carried out, using numerical and analytical methods. As a member of this research group with a particular interest in analytical and semi-analytical mathematical techniques, the author will try to describe in this paper some aspects of research on mountain waves, which is perhaps one of the most interesting areas from an applied mathematics point of view.

Introduction

Mountain waves are a type of internal gravity waves forced by airflow over mountains. Internal gravity waves exist, not at an interface, like ocean waves, but in the interior of the atmosphere. They require an atmosphere with stable stratification, where air parcels that are displaced vertically tend to oscillate. These waves are fairly common, but can only be visualized when the atmosphere has enough moisture for clouds to form in the regions of ascending motion. Associated with mountain waves there is a pressure distribution at the surface which causes a drag force on the mountains (Smith, 1980). To this corresponds a reaction force acting on the atmosphere, which must be represented in some way (parametrized) in global climate and weather prediction models. This is required because the dominant contributions to this force come from mountains of width ≈10km, which are typically not resolved by these models. Current research at IDL aims to understand how mountain wave drag varies with input parameters of the incoming large-scale flow, in order to contribute to the improvement of existing parametrizations of this process.

Mountain wave equations

Mountain waves, like other meteorological phenomena, are governed by a set of partial differential equations comprising the Navier-Stokes equation, the conservation of mass, a heat balance equation and an equation of state for ideal gases. In the following equation set, the rotation of the Earth is neglected, because the scale of the motions is relatively small, yet viscosity is also neglected because the scale is larger that that of viscous boundary layers.

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \vec{g}, \qquad (1)$$

$$\frac{\partial T}{\partial t} + \vec{v}\cdot\vec{\nabla}T = \frac{1}{\rho c_p}\left(\frac{\partial p}{\partial t} + \vec{v}\cdot\nabla p\right), \qquad (2)$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0, \qquad (3)$$

$$p = R_a \rho T. \qquad (4)$$

In (1)–(4), $\vec{v} = (u, v, w)$ is the velocity vector, ρ is the density, p is the pressure, T is the absolute temperature, $\vec{g}$ is the acceleration of gravity, R_a is the ideal gas constant for air and c_p is the corresponding specific heat at constant pressure.

Equation (2) results from the first law of thermodynamics for adiabatic processes, because the motions associated with mountain waves are fast enough for heat transfer to be insignificant (except when there is cloud formation).

For simplicity, 2D motion (in an $x - z$ vertical plane) is considered. The flow is also assumed to be stationary, because the waves are generated by a fixed topographic forcing. Additionally, the Boussinesq approximation is assumed. This is combined next with linearization of the equations of motion to obtain a final simplified equation set. In the Boussinesq approximation, the thermo-

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dynamic dependent variables of (1)–(4) are decomposed as $\rho = \overline{\rho}(z) + \rho'$, $p = \overline{p}(z) + p'$ and $T = \overline{T}(z) + T'$, where the overbar denotes a reference state that depends only on height and the primes denote perturbations associated with the mountain waves. The reference state is assumed to be in hydrostatic equilibrium:

$$\frac{d\overline{p}}{dz} = -\overline{\rho}g. \tag{5}$$

Additionally, constant reference values of the density and other flow variables (denoted by a zero subscript) are assumed to exist such that $\overline{\rho} \approx \rho_0$ and $\rho'/\overline{\rho} \approx \rho'/\rho_0$ and similarly for the other variables. The Boussinesq approximation also assumes that

$$\frac{\rho'}{\rho_0} \approx -\frac{\theta'}{\theta_0}, \tag{6}$$

where $\theta = \overline{\theta}(z) + \theta'$ is the potential temperature. This is defined as

$$\theta = T\left(\frac{p_0}{p}\right)^{R_a/c_p}, \tag{7}$$

where p_0 is a reference pressure (generally assumed to be $p_0 = 10^5$Pa). θ is a very important quantity in meteorology because it is conserved in adiabatic processes. Equation (6) amounts to assuming that the density is a much weaker function of pressure than of temperature, which is acceptable for motions much slower than the speed of sound.

Linearization of the equations of motion goes one step further by assuming the same kind of decomposition also for the velocity vector: $\vec{v} = (U(z) + u', V(z) + v', w')$ (where the capital letters correspond to the reference wind, which is only a function of height), and neglecting all products of perturbations. With all these simplifications, which are valid for waves over relatively low mountains, the equation set (1)–(4) becomes:

$$U\frac{\partial u'}{\partial x} + w'\frac{\partial U}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}, \tag{8}$$

$$U\frac{\partial w'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + b, \tag{9}$$

$$U\frac{\partial b}{\partial x} + N^2 w' = 0, \tag{10}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0, \tag{11}$$

where $b = g(\theta'/\theta_0)$ is a buoyancy perturbation and $N^2 = (g/\theta_0)(d\overline{\theta}/dz)$ is the mean static stability.

Through differentiation and summation, these equations may be combined into one single equation for w':

$$\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2} + l^2(z)w' = 0, \tag{12}$$

where

$$l^2 = \frac{N^2}{U^2} - \frac{1}{U}\frac{d^2U}{dz^2}. \tag{13}$$

$l(z)$ is called the Scorer parameter.

Since the boundary conditions are most conveniently applied in wavenumber space and the waves are expected to be confined near an isolated topography, Fourier analysis is adopted to express all flow variables, including w':

$$w'(x,z) = \int_{-\infty}^{\infty} \widehat{w}(k,z)e^{ikx}dk, \tag{14}$$

where $\widehat{w}$ is the Fourier transform of w'. Then, (12) can be written

$$\widehat{w}'' + \left[l^2(z) - k^2\right]\widehat{w} = 0, \tag{15}$$

where the primes denote differentiation with respect to z. Despite its simplicity, in general this equation has no analytical solution. Two exceptions occur when $l(z)$ is either a slow function of z, or is piecewise constant. These two cases will be addressed next in turn.

Slowly varying Scorer parameter profile

When the Scorer parameter varies relatively slowly with height, the WKB approximation can be used to solve (15). This entails defining a new rescaled vertical coordinate as $Z = \varepsilon z$, where ε is a small parameter (Bender and Orszag, 1999), so that (15) becomes:

$$\varepsilon^2\ddot{\widehat{w}} + \left[\frac{N^2}{U^2} - \varepsilon^2\frac{\ddot{U}}{U} - k^2\right]\widehat{w} = 0, \tag{16}$$

where the dots denote differentiation with respect to Z. Additionally, a solution of the form

$$\widehat{w} = \widehat{w}(Z=0)e^{i\varepsilon^{-1}\int_0^Z [m_0(\varrho)+\varepsilon m_1(\varrho)+\varepsilon^2 m_2(\varrho)+\ldots]d\varrho} \tag{17}$$

is adopted. In this equation, the vertical wavenumber of the mountain waves is expanded as a power series of ε. By inserting (17) into (16), and considering terms only up to second-order in ε, the following set of algebraic equations for m_0, m_1 and m_2 is obtained:

$$-m_0^2 + \frac{N^2}{U^2} - k^2 = 0, \tag{18}$$

$$i\dot{m}_0 - 2m_0 m_1 = 0, \tag{19}$$

$$i\dot{m}_1 - 2m_0 m_2 - m_1^2\frac{\ddot{U}}{U} = 0. \tag{20}$$

If the wave motion itself can be considered hydrostatic, i.e.

$$\frac{1}{\rho_0}\frac{\partial p'}{\partial z} = b, \qquad (21)$$

then the k^2 term in (18) must be neglected. This corresponds to relatively wide mountains, which are those that give a dominant contribution to the drag. In this case, the definitions for m_0, m_1 and m_2 found from (18)-(20) are:

$$m_0 = \frac{N}{U}\mathrm{sgn}(k), \qquad (22)$$

$$m_1 = -\frac{1}{2}i\frac{\dot{U}}{U}, \qquad (23)$$

$$m_2 = -\frac{1}{8}\frac{U}{N}\mathrm{sgn}(k)\left(\frac{\dot{U}^2}{U^2} + 2\frac{\ddot{U}}{U}\right). \qquad (24)$$

In (22)–(24) it was assumed that N is constant, because generally the vertical variation of U is more important in the atmosphere. The sign function has been included in m_0 (and as a consequence appears also in m_2) so that the wave energy propagates upward, as is logical for waves forced topographically. It can be shown that if the vertical wavenumber of these waves (and thus m_0) has the same sign as Uk, the group velocity of the waves has a positive vertical component, as required.

The lower boundary condition, which requires that, in inviscid conditions, the flow is tangential to the topography, can be expressed, in the linearized approximation, as

$$w'(z=0) = U_0\frac{\partial h}{\partial x} \Rightarrow \widehat{w}(z=0) = iU_0k\widehat{h}, \qquad (25)$$

where $U_0 = U(z=0)$, $h(x)$ is the surface elevation and $\widehat{h}(k)$ is its Fourier transform. This completely specifies the solution to the problem. If an explicit solution is required, the integrals in the exponent of (17) must be calculated. This is possible analytically for the term involving m_1, but not in general for those involving m_0 and m_2. Nevertheless, for calculating mountain wave drag this is not necessary, since, the drag per unit spanwise length of the mountain is defined as

$$D = \int_{-\infty}^{+\infty} p'(z=0)\frac{\partial h}{\partial x}dx$$
$$= 2\pi i\int_{-\infty}^{+\infty} k\widehat{p}^*(z=0)\widehat{h}\,dk, \qquad (26)$$

where the asterisk denotes complex conjugate and $\widehat{p}$ is the Fourier transform of the pressure perturbation. Using (8) and (11), this quantity can be expressed as

$$\widehat{p} = i\frac{\rho_0}{k}(U'\widehat{w} - U\widehat{w}'), \qquad (27)$$

which means that at the surface, using (17), it becomes

$$\widehat{p}(z=0) = i\rho_0U_0^2\Big[m_0(z=0) + \varepsilon m_1(z=0) + i\frac{U_0'}{U_0} + \varepsilon^2 m_2(z=0)\Big]\widehat{h}, \qquad (28)$$

where $U_0' = U'(z=0)$. If (28) is used in (26), and (22)–(24) are also employed, the drag normalized by its value D_0 for a constant mean wind U_0 is given by (Teixeira and Miranda, 2004)

$$\frac{D}{D_0} = 1 - \frac{1}{8}\frac{U_0'^2}{N^2} - \frac{1}{4}\frac{U_0''U_0}{N^2}, \qquad (29)$$

correct to second-order in ε, where

$$D_0 = 4\pi\rho_0NU_0\int_0^{+\infty} k|\widehat{h}|^2\,dk. \qquad (30)$$

Thus the WKB approximation allows one to obtain a closed-form analytical expression for the correction to the drag due to the variation of the wind with height. Something analogous could be done if N was assumed to be a function of height as well.

Two-layer atmosphere

Consider now that the atmosphere has a two-layer structure, with different (constant) values of l in each layer: l_1 near the surface ($0 < z < H$) and l_2 aloft ($z > H$). It will be assumed that $l_1 > l_2$, since unlike the opposite possibility, this allows wave trapping near the surface, which affects mountain wave drag in an interesting way (Scorer, 1949). In this situation, (15) has solutions of the form:

$$\widehat{w} = a_1e^{im_1z} + b_1e^{-im_1z} \text{ if } k^2 < l_1^2, \qquad (31)$$

$$\widehat{w} = c_1e^{-n_1z} + d_1e^{n_1z} \text{ if } k^2 > l_1^2, \qquad (32)$$

in the lower layer, where $m_1^2 = l_1^2 - k^2$ and $n_1^2 = k^2 - l_1^2$. In the upper layer, on the other hand,

$$\widehat{w} = a_2e^{im_2z} \text{ if } k^2 < l_2^2, \qquad (33)$$

$$\widehat{w} = c_2e^{-n_2z} \text{ if } k^2 > l_2^2, \qquad (34)$$

where $m_2^2 = l_2^2 - k^2$ and $n_2^2 = k^2 - l_2^2$. The first solutions in (31)–(32) and (33)–(34) correspond to waves whose energy propagates vertically, while the second solutions are evanescent. In the upper layer (33) corresponds to an upward propagating solution, whereas (34) corresponds to a wave that decays with height. This makes physical sense for topographically generated waves. a_1, b_1, c_1, d_1, a_2 and c_2 are functions of k which are determined by the boundary conditions. These prescribe that the waves either propagate upward or decay as $z \to +\infty$ (this is already included in (33)–(34), as mentioned above), that the flow is tangential to the topography at the surface, (25), and that the streamline slope and pressure perturbation are continuous at $z = H$. For simplicity, it is assumed next that the discontinuity of l is due to N and not to U, which is taken as constant. This slightly simplifies the boundary conditions, but other possibilities could be accommodated without too much effort, if required.

Then it can be shown that

$$a_1 = \frac{iUk\hat{h}(m_1 + m_2)e^{-im_1 H}}{2m_1\cos(m_1 H) - 2im_2\sin(m_1 H)}, \tag{35}$$

$$b_1 = \frac{iUk\hat{h}(m_1 - m_2)e^{im_1 H}}{2m_1\cos(m_1 H) - 2im_2\sin(m_1 H)}, \tag{36}$$

if $k^2 < l_2^2$. If $l_2^2 < k^2 < l_1^2$ instead, then:

$$a_1 = \frac{Uk\hat{h}(im_1 - n_2)e^{-im_1 z}}{2m_1\cos(m_1 H) + 2n_2\sin(m_1 H)}, \tag{37}$$

$$b_1 = \frac{Uk\hat{h}(im_1 + n_2)e^{im_1 z}}{2m_1\cos(m_1 H) + 2n_2\sin(m_1 H)}. \tag{38}$$

When the waves are evanescent in both layers, i.e. $k^2 > l_1^2$, it can be shown that no mountain wave drag is produced, since the pressure perturbation is symmetric with respect to the orography. When $l_1^2 > l_2^2$, two possibilities exist: either the waves propagate vertically in both layers (when $k^2 < l_2^2$), or they propagate in the first layer but not in the second, i.e. are trapped ($l_2^2 < k^2 < l_1^2$). In both cases, the drag is given by (26) with

$$\hat{p}(z = 0) = \frac{\rho_0 U m_1}{k}(a_1 - b_1) \tag{39}$$

which results from (27) and (31). If the necessary calculations are performed, the drag is found to be given by two contributions: one from wavenumbers between 0 and l_2 (D_1) and the other from wavenumbers between l_2 and l_1 (D_2). These two contributions can be written:

$$D_1 = 4\pi\rho_0 U^2 \int_0^{l_2} k|\hat{h}|^2 \frac{m_1^2 m_2}{m_1^2\cos^2(m_1 H) + m_2^2\sin^2(m_1 H)}dk, \tag{40}$$

$$D_2 = \mathrm{Re}\left[4\pi i\rho_0 U^2 \int_{l_2}^{l_1} k|\hat{h}|^2 m_1 \times \frac{m_1\sin(m_1 H) - n_2\cos(m_1 H)}{m_1\cos(m_1 H) + n_2\sin(m_1 H)}dk\right]. \tag{41}$$

The integral in (40), which gives the drag due to mountain waves that propagate in the two atmospheric layers, must be evaluated numerically. Since the integrand in (41) is real, contributions to the drag from this integral only come from singularities along the real axis. These correspond to the modes of the trapped lee waves. These modes are given by the condition that the denominator in the integrand of (41) vanishes, that is

$$\tan(m_1 H) = -\frac{m_1 H}{n_2 H}. \tag{42}$$

The wavenumber of each lee wave mode, say k_i (with $i = 1, 2, \ldots$), can be found by solving (42) numerically. Then, the lee wave drag, which is produced by waves which are trapped in the lower atmospheric layer, can be calculated by finding the imaginary part of the integral in (40). In this calculation, which only receives contributions from the singularities on the real axis, the integration path must be indented above each singularity. This is because, with the addition of Rayleigh damping frictional terms to the governing equations, the singularities move to the negative imaginary semi-plane. Then, it can be shown that the lee wave drag takes the form:

$$D_2 = 4\pi^2\rho_0 U^2 \sum_i |\hat{h}|^2(k_i)\frac{m_1^2(k_i)n_2(k_i)}{1 + n_2(k_i)H}. \tag{43}$$

where the sum is performed over all lee wave modes.

Concluding remarks

Two examples of mathematical methods that can be profitably employed in the study of mountain waves have been described. Additional asymptotic techniques, such as the method of multiple scales, or matched asymptotic expansions, to give only two examples, are routinely used in fluid mechanics, and can be applied to appropriate problems in meteorology, or in the geosciences in general, as long as small parameters exist, of which the researcher may take advantage.

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