Stratified Models in First-Order Logic

by **José Roquette***

The various nature of the mathematical objects in what concerns their complexity, our knowledge of them or the possibility to make them explicit (for example, infinitesimal or ilimited real numbers) is a strong motivation to consider their distribution into *levels* or *strata*. The *stratification* depends on the selected property (or properties) of the mathematical objects that are the subject-matter of our study.

Stratified Models

Along this article $\mathscr{L}$ is a first-order language with equality, no constant symbols, no function symbols and the logical symbols:

parentheses	")" and "("
variables	$v_0, v_1, \dots, v_n, \dots$
0-ary connective	$\perp$ (falsity, *falsum*, *absurdum*)
binary connective	$\Rightarrow$ (implication)
universal quantifier	$\forall$

The basic definitions and conventions of $\mathscr{L}$ are as usual (see [2]); in particular, $\neg\phi$ ("not" ϕ), $\phi \vee \psi$, (ϕ "or" ψ), $\phi \wedge \psi$ (ϕ "and" ψ) and $(\exists v_i)\phi(v_i)$ ("there is a" v_i "such that" $\phi(v_i)$) abbreviate, respectively, $\phi \Rightarrow \perp$, $(\neg\phi) \Rightarrow \psi$, $\neg(\neg\phi) \vee (\neg\psi))$ and $\neg(\forall v_i)\neg\phi(v_i)$.

The expressions $\mathrm{Term}(\mathscr{L}')$, $\mathrm{Term}_C(\mathscr{L}')$, $\mathrm{Atom}(\mathscr{L}')$, $\mathrm{Form}(\mathscr{L}')$, $\mathrm{Sent}(\mathscr{L}')$, $\mathrm{At}(\mathscr{L}')$ denote, respectively, the classes of the *terms*, the *closed terms*, the *atomic formulae*, *the formulae*, the *sentences* and the *atomic sentences* of whatever first-order language $\mathscr{L}'$ we are using.

Definition.—Let P be a set and $\leq$ a total, dense, preordering relation on P. The expressions "$p < q$" and "$p =_{\leq} q$" abbreviate, respectively: "$p \leq q$ and $q \nleq p$" and "$p \leq q$ and $q \leq p$", for each $p, q \in P$.

We will be interested on total dense preorderings $\mathbb{P} = (P, \leq)$ having a $\leq$-minimal element $\mathbb{0}_{\leq}$ (which we denote simply by $\mathbb{0}$ when no confusion arises) and no $\leq$-maximal element; more explicitely: $\mathbb{0} \leq p$, for every $p \in P$; and given $q \in P$ there is a $p \in P$ such that $q \leq p$ and $p \nleq q$.

Fix $\mathbb{P} = (P, \leq, \mathbb{0})$ as above and consider a class valued function D defined on P such that, for each $p \in P$, the image of p under D is a non-empty class. For $\mathbb{P}$, D as before, a sequence $\mathscr{F} := (P, \leq, \mathbb{0}, D) = (\mathbb{P}, D)$ is called a *stratifying frame*. The elements of P are the *nodes* of $\mathscr{F}$ and for each $p \in P$, the set $D(p)$ is the *domain* of $\mathscr{F}$ at the node p.

To each $a \in D(p)$ we associate a constant symbol $\bar{a}$ (using different constant symbols for different elements of $D(p)$). If $a \in D(p) \cap D(q)$, then the constant symbol associated with a is the same.

At this point it is convenient to introduce some extensions of the original first-order language $\mathscr{L}$.
By $\mathscr{L}_*$, we understand the first-order extension of $\mathscr{L}$ defined as $\mathscr{L}_* := \mathscr{L} \cup \{\mathbf{0}, \sqsubseteq\}$, where $\mathbf{0}$ is a *constant symbol* and $\sqsubseteq$ is a new *binary relation symbol* called *precedence of level*.

For each $p \in D(p)$ we denote by $\mathscr{L}_*^p$ the first-order extension of $\mathscr{L}_*$ given by $\mathscr{L}_*^p := \mathscr{L}_* \cup \{\bar{a} \mid a \in D(p)\}$.

Finally, $\mathscr{L}_*^+$ is the first-order extension of $\mathscr{L}_*$ defined

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as: $\mathscr{L}_*^+ := \cup_{p\in P}\mathscr{L}_*^p$. The language $\mathscr{L}_*$ is the *stratifying language* associated with $\mathscr{L}$.

So, the class of all closed terms of $\mathscr{L}_*^+$ is:

$$\mathrm{Term}_C(\mathscr{L}_*^+) = \{\bar{a} \mid (\exists p \in P) a \in D(p)\} \cup \{\mathbf{0}\}.$$

For any terms $t_1, t_2 \in \mathrm{Term}(\mathscr{L}_*^+)$, the expression $t_1 \sqsubset t_2$ abbreviates "$t_1 \sqsubseteq t_2$ and $t_2 \not\sqsubseteq t_1$". (The relations $t_1 \sqsubseteq t_2$ and $t_1 \sqsubset t_2$ must be read, respectively, as "t_1 *precedes* t_2" and "t_1 *strictly precedes* t_2".)

Consider a function V defined on $\mathrm{Term}_C(\mathscr{L}_*^+)$ and with values in P such that $V(\mathbf{0}) = \mathbb{0}$, $a \in D(V(\bar{a}))$ and for each $p \in P$ if $a \in D(p)$, then $V(\bar{a}) \leq p$ and $a \notin D(q)$, for arbitrary $q < V(\bar{a})$ in P. ($V(\bar{a})$ must be thought as "the"[1] first level of interpretation of $\bar{a}$.)

Having described how closed terms are interpreted we will now make the necessary preparatory steps towards the description of the semantics in *stratified models* (a concept to be introduced later). Consider a function Σ defined on P such that, for each $p \in P$, the value of p under Σ is a set of atomic sentences of $\mathscr{L}_*^p$. (The set $\Sigma(p)$ establish, for each $p \in P$ the "basic truths" at p.)

The functions D, Σ and V satisfy the following conditions:

1. If $p \leq q$, then $D(p) \subseteq D(q)$.
2. $\perp \notin \Sigma(p)$, for every p.
3. If $p \leq q$, then $\Sigma(p) \subseteq \Sigma(q)$.
4. The formula $\bar{a}_1 = \bar{a}_2$ is in $\Sigma(p)$ iff $V(\bar{a}_1) \leq p$, $V(\bar{a}_2) \leq p$ and $a_1 = a_2$.
5. the formula $\bar{a}_1 \sqsubseteq \bar{a}_2$ is in $\Sigma(p)$ iff $V(a_1) \leq V(\bar{a}_2) \leq p$.
6. If R_i is a n_i-ary relation symbol of $\mathscr{L}$ and $a_1, \dots, a_{n_i}, b_1, \dots, b_{n_i} \in D(p)$ with $a_1 = b_1, \dots, a_{n_i} = b_{n_i}$, then: $R_i(\bar{a}_1, \dots, \bar{a}_{n_i})$ in $\Sigma(p)$ implies that $R(\bar{b}_1, \dots, \bar{b}_{n_i})$ is in $\Sigma(p)$. (Evidently, if R_i is a n_i-ary relation symbol of $\mathscr{L}$ and $R_i(\bar{a}_1, \dots, \bar{a}_{n_i})$ is in $\Sigma(p)$, then: $V(a_j) \leq p$ [for $1 \leq j \leq n_i$] since $\Sigma(p)$ consists of atomic formulas of $\mathscr{L}_*^p$.

Definition.—For $P, \leq, D, \Sigma, V, \mathbb{0}$ as described previously let $\mathscr{F} := (P, \leq, D, \mathbb{0})$ be a stratifying frame. A *stratified model* for $\mathscr{L}_*$ is a sequence $\mathscr{S}_* := (\mathscr{F}, \Sigma, V, o)$ such that,

1. $o \in D(\mathbb{0})$;
2. $\bar{o}$ is identified with $\mathbf{0}$.

The *nodes* of $\mathscr{S}_*$ and the *domain* of $\mathscr{S}_*$ at each $p \in P$ are those of $\mathscr{F}$.

Remark.—If $\mathscr{S}_* = (\mathscr{F}, \Sigma, V, o)$ is a stratified model for $\mathscr{L}_*$, then it is easy to prove that P is an infinite set and, at each $p \in P$, D and Σ determine a classical structure (see [1]) $\mathfrak{A}_p$ whose domain (which we denote by $|\mathfrak{A}_p|$) is $D(p)$ and:

1. if $p \leq q$, then $|\mathfrak{A}_p| \subseteq |\mathfrak{A}_q|$.
2. The interpretations $R_i^{\mathfrak{A}_p}$ of a n_i-ary relation symbol R_i of $\mathscr{L}$ and $\sqsubseteq^{\mathfrak{A}_p}$ of $\sqsubseteq$ are: $R_i^{\mathfrak{A}_p}(a_1, \dots, a_{n_i})$ iff the formula $R_i(\bar{a}_1, \dots, \bar{a}_{n_i})$ belongs to $\Sigma(p)$, and $a_1 \sqsubseteq^{\mathfrak{A}_p} a_2$ iff the formula $\bar{a}_1 \sqsubseteq \bar{a}_2$ belongs to $\Sigma(p)$.

   So, for $p \leq q$ we have that $R_i^{\mathfrak{A}_p} \subseteq R_i^{\mathfrak{A}_q}$ (i.e. $R_i^{\mathfrak{A}_q}$ extends $R_i^{\mathfrak{A}_p}$) and $\sqsubseteq^{\mathfrak{A}_p} \subseteq \sqsubseteq^{\mathfrak{A}_q}$ (i.e. $\sqsubseteq^{\mathfrak{A}_q}$ extends $\sqsubseteq^{\mathfrak{A}_p}$).
3. $\bar{a}^{\mathfrak{A}_p} := a$ for every $p \in P$ and for every $a \in D(p)$;
4. in particular, since $\mathbf{0}$ is $\bar{o}$ we obtain, $\mathbf{0}^{\mathfrak{A}_p} := o$ for every $p \in P$.

Proposition.—Let $\mathscr{S}_* := (\mathscr{F}, \Sigma, V, o)$ be a stratified model for $\mathscr{L}_*$. Then,

1. if $\bar{a}_1, \dots, \bar{a}_n$ are closed terms of $\mathscr{L}_*^+$, then there is a $p \in P$ such that:
   $a_1, \dots, a_n \in D(p)$ and $a_1, \dots, a_n \in D(q)$ implies $p \leq q$, for every $q \in P$
   (we refere to this last proposition as "(*)").
2. If $\bar{a}_1, \dots, \bar{a}_n$ are closed terms of $\mathscr{L}_*^+$, then there is a $p \in P$ such that: $V(\bar{a}_j) \leq p$ (for $1 \leq j \leq n$) and if $V(\bar{a}_j) \leq q$ (for $1 \leq j \leq n$), then $p \leq q$, for every $q \in P$.
3. If $p_1, p_2 \in P$ satisfy proposition (*), then $p_1 =_{\leq} p_2$.

If $\mathscr{S}_* := (\mathscr{F}, \Sigma, V, o)$ is a stratified model for $\mathscr{L}_*$ and $\bar{a}_1, \dots, \bar{a}_n$ are closed terms of $\mathscr{L}_*^+$, we define $V(\bar{a}_1, \dots, \bar{a}_n)$ as "the" p (unique modulo $=_{\leq}$) satisfying the proposition (*). (The level $V(\bar{a}_1, \dots, \bar{a}_n)$ is "the" *first level of interpretation of all the* $\bar{a}_1, \dots, \bar{a}_n$.)

Stratified Semantics

In order to completely characterize the semantics in stratified models, we need to extend the considerations made in the previous section to arbitrary formulae. The next proposition fully describes the situation. In fact, by induction on the complexity of formulae we can prove the following,

Proposition.—Let $\mathscr{S}_* := (\mathscr{F}, \Sigma, V, o)$ be a stratified model for $\mathscr{L}_*$. Then there exists a unique function Σ^*, defined on P, such that for each $p \in P$, $\Sigma(p)$ is a subset of $\Sigma^*(p)$ which consists of sentences of $\mathscr{L}_*^p$ and

1. if ϕ is an atomic formula of $\mathscr{L}_*^p$ and $\phi \notin \Sigma(p)$, then $\phi \notin \Sigma^*(p)$.

{1} The definite article refers to the binary relation $=_{\leq}$.

2. if $\phi \Rightarrow \psi$ is a formula of $\mathscr{L}_*^+$ then $\phi \Rightarrow \psi$ belongs to $\Sigma^*(p)$ iff $\phi \notin \Sigma^*(p)$ or $\psi \in \Sigma^*(p)$ and both ϕ and ψ are sentences of $\mathscr{L}_*^p$.
3. if $(\forall v_i)\phi(v_i)$ is a formula of $\mathscr{L}_*^+$ then $(\forall v_i)\phi(v_i)$ belongs to $\Sigma^*(p)$ iff $\phi(\bar{a})$ belongs to $\Sigma^*(p)$, for every $a \in D(p)$.

Notation.—[2] We write $p \Vdash \phi$ for $\phi \in \Sigma^*(p)$ (read "p *forces* ϕ").

So, we have, for each $p \in P$:

1. $p \Vdash \bar{a}_1 = \bar{a}_2$ iff $V(\bar{a}_1) \leq p$, $V(\bar{a}_2) \leq p$ and $a_1 = a_2$.
2. $p \Vdash \bar{a}_1 \sqsubseteq \bar{a}_2$ iff $V(\bar{a}_1) \leq V(\bar{a}_2) \leq p$.
3. $p \nVdash \bot$.
4. $p \Vdash \phi \Rightarrow \psi$ iff $p \nVdash \phi$ or $p \Vdash \psi$, for all sentences ϕ and ψ of $\mathscr{L}_*^p$.
5. $p \Vdash (\forall v_i)\phi(v_i)$ iff $p \Vdash \phi(\bar{a})$, for every $a \in D(p)$.

As a direct consequence of the proposition above we can derive a few more properties of the forcing relation:

6. $p \Vdash \neg\phi$ iff $p \nVdash \phi$.
7. $p \Vdash \neg\neg\phi$ iff $p \Vdash \phi$, for every sentence ϕ of $\mathscr{L}_*^p$.
8. $p \Vdash \phi \vee \psi$ iff $p \Vdash \phi$ or $p \Vdash \psi$.
9. $p \Vdash \phi \wedge \psi$ iff $p \Vdash \phi$ and $p \Vdash \psi$, for all sentences ϕ and ψ of $\mathscr{L}_*^p$.
10. $p \Vdash (\exists v_i)\phi(v_i)$ iff there is an $a \in D(p)$ such that $p \Vdash \phi(\bar{a})$.
11. $V(\bar{a}_1) \Vdash \bar{a}_1 \sqsubseteq \bar{a}_2$ iff $V(\bar{a}_1) =_{\leq} V(\bar{a}_2)$.
11. $V(\bar{a}_2) \Vdash \bar{a}_1 \sqsubseteq \bar{a}_2$ iff $V(\bar{a}_1) \leq V(\bar{a}_2)$.
12. $p \Vdash \bar{a}_1 \sqsubset \bar{a}_2$ iff $V(\bar{a}_1) < V(\bar{a}_2) \leq p$.

Definition.—If $\mathscr{S}_* := (\mathscr{F}, \Sigma, V, o)$ is a stratified model for $\mathscr{L}_*$ and ϕ is a formula of $\mathscr{L}_*$,[3] we define: $p \Vdash \phi$ iff $p \Vdash \mathrm{cl}(\phi)$, where $\mathrm{cl}(\phi)$ is the *universal closure* of ϕ .

Proposition.—Let $\mathscr{S}_* := (\mathscr{F}, \Sigma, V, o)$ be a stratified model for $\mathscr{L}_*$. Then, if v_i and v_j are different variables of $\mathscr{L}$:

1. $p \Vdash v_i \sqsubseteq v_i$, for each $p \in P$.
2. $p \Vdash v_i \sqsubseteq v_j$ iff $V(\bar{a}) =_{\leq} V(\bar{b})$, for every $a, b \in D(p)$.
3. $p \Vdash v_i \sqsubseteq \mathbf{0}$ iff $V(\bar{a}) =_{\leq} \mathbb{0}$, for every $a \in D(p)$.
4. $p \Vdash \mathbf{0} \sqsubseteq v_i$, for each $p \in P$.

In certain circumstances truth is preserved when moving to an upper strata. The next definition isolates classes of formulae for which this is indeed the case.

Let $\mathscr{F} := (P, \leq, D, \mathbb{0})$ be a stratifying frame. The classes of the *elementary progressive* and the *elementary regressive* sentences of $\mathscr{L}_*^+$, denoted, respectively, by $\mathrm{Prg}_0(\mathscr{L}_*^+)$ and $\mathrm{Rgr}_0(\mathscr{L}_*^+)$, are defined inductively as follows:

P1 If ϕ is an atomic formula of $\mathscr{L}_*^+$, then ϕ is elementary progressive.

P2 If ϕ_1 and ϕ_2 are elementary progressive, then $\phi_1 \wedge \phi_2$ and $\phi_1 \vee \phi_2$ are elementary progressive.

R1 $\bot$ is elementary regressive and if ϕ is an atomic sentence of $\mathscr{L}_*^+$, different from $\bot$, then $\neg\phi$ is elementary regressive.

R2 If ϕ_1 and ϕ_2 are elementary regressive, then $\phi_1 \wedge \phi_2$ and $\phi_1 \vee \phi_2$ are elementary regressive.

PR If ϕ_1 is elementary progressive and ϕ_2 is elementary regressive, then $\phi_1 \Rightarrow \phi_2$ is elementary regressive.

RP If ϕ_1 is elementary regressive and ϕ_2 is elementary progressive, then $\phi_1 \Rightarrow \phi_2$ is elementary progressive.

We may now define the classes of the *extended elementary progressive* and the *extended elementary regressive* sentences of $\mathscr{L}_*^+$, denoted, respectively, by $\mathrm{Prg}(\mathscr{L}_*^+)$ and $\mathrm{Rgr}(\mathscr{L}_*^+)$:

Pi If ϕ is elementary progressive, then ϕ is extended elementary progressive.

Pii If $\phi(v_i)$ is a formula of $\mathscr{L}_*^+$ such that for each $p \in P$ and $a \in D(p)$, the formula $\phi(a)$ is elementary progressive, then $(\exists v_i)\phi(v_i)$ is extended elementary progressive.

Ri If ϕ is elementary regressive, then it is extended elementary regressive.

Rii If $\phi(v_i)$ is a formula of $\mathscr{L}_*^+$ such that for each $p \in P$ and $a \in D(p)$, the formula $\phi(\bar{a})$ is elementary regressive, then $(\forall v_i)\phi(v_i)$ is extended elementary regressive.

For these classes of sentences the weak monotonicity of the forcing relation holds, i.e. if $\mathscr{S}_* = (\mathscr{F}, \Sigma, V, o)$ is a stratified model for $\mathscr{L}_*$, then $p \leq q$ implies that if $p \Vdash \phi$,

{2} For a *modal view* of *forcing*, see [6].
{3} Every sentence of $\mathscr{L}_*$ is also a sentence of $\mathscr{L}_*^p$ and every formula of $\mathscr{L}_*$ is also a formula of $\mathscr{L}_*^p$.

$$(\text{PLR}_1)\frac{}{v_i \sqsubseteq v_i} \qquad (\text{PLR}_2)\frac{}{v_i \sqsubseteq v_j \wedge v_j \sqsubseteq v_k \Rightarrow v_i \sqsubseteq v_k} \qquad (\text{PLR}_3)\frac{}{v_i \sqsubseteq v_j \vee v_j \sqsubseteq v_i} \qquad (\text{PLR}_4)\frac{}{\mathbf{0} \sqsubseteq v_i}$$

Figure 1. Precedence of Level Rules

then $q \Vdash \phi$ for every extended elementary progressive sentence ϕ of $\mathscr{L}_*^p$ and $p \leq q$ implies that if $q \Vdash \phi$, then $p \Vdash \phi$, for every extended elementary regressive sentence ϕ of $\mathscr{L}_*^p$.

The following notions are in part borrowed, in part adapted from first order-logic:

1. If ϕ is a sentence of $\mathscr{L}_*$ and $\mathscr{S}_* := (\mathscr{F}, \Sigma, V, 0)$ is a stratified model for $\mathscr{L}_*$ we define $\mathscr{S}_* \Vdash \phi$ (read "*$\mathscr{S}_*$ forces ϕ*" or "*$\mathscr{S}_*$ is a stratified model of ϕ*") as: $\mathscr{S}_* \Vdash \phi$ iff $p \Vdash \phi$, for every $p \in P$.

   We also define $\Vdash \phi$ (read "*ϕ is universally valid*" or "*ϕ is valid*") as: $\Vdash \phi$ iff $\mathscr{S}_* \Vdash \phi$, for every stratified model $\mathscr{S}_*$ for $\mathscr{L}_*$.

   If Δ is a set of sentences of $\mathscr{L}_*$, we define $\mathscr{S}_* \Vdash \Delta$ (read "*$\mathscr{S}_*$ forces Δ*" or "*$\mathscr{S}_*$ is a stratified model of Δ*") as: $\mathscr{S}_* \Vdash \Delta$ iff $\mathscr{S}_* \Vdash \phi$, for every $\phi \in \Delta$.

2. If $\Gamma \cup \{\phi\}$ is a set of sentences of $\mathscr{L}_*$, we define $\Gamma \Vdash \phi$ (read "ϕ is a stratified logical consequence of Γ") as: $\Gamma \Vdash \phi$ iff for every stratified model $\mathscr{S}_*$, if $\mathscr{S}_* \Vdash \Gamma$, then $\mathscr{S}_* \Vdash \phi$.

   If $\Gamma \cup \{\phi\}$ is a set of fomulas of $\mathscr{L}_*$ and the free variabes of the formulas in $\Gamma \cup \{\phi\}$ are among $v_{i_1}, \dots, v_{i_n}$, we define $\Gamma \Vdash \phi$ (read "*ϕ is a stratified logical consequence of Γ*") as: $\Gamma \Vdash \phi$ iff for every stratified model $\mathscr{S}_*$, for every $p \in P$ and for every $a_1, \dots, a_n \in D(p)$, if $p \Vdash \Gamma(\bar{a}_1, \dots, \bar{a}_n)$, then $p \Vdash \phi(\bar{a}_1, \dots, \bar{a}_n)$; where $p \Vdash \Gamma(\bar{a}_1, \dots, \bar{a}_n)$ abbreviates: "$p \Vdash \psi(\bar{a}_1, \dots, \bar{a}_n)$, for every $\psi \in \Gamma$.

Soundness, Conservativeness and Completeness

Along the two preceding sections we described models (and the corresponding semantics) for stratified first-order logic. But a logical system is not completely described before we introduce a notion of formal proof—a syntactical device conceived to capture truth. Soundness and completeness in a certain extent measure the adequacy of this formal device to its purpose. In this case the notion of proof generalizes the usual one in the case of first-order logic with equality (see [2]). We will skip the details here being enough to know that a formal proof of a formula ϕ is a tree, each node of it is a formula obtained from nodes that are immediate successors of it by applying some basic rule of inference, the bottom node of that tree being the formula ϕ. As a matter of fact proofs are well-founded trees, a fact that allows a form of induction on "proof complexity".

As we said before our basic rules of inference are those of first-order logic and four more rules that deal with the novelty relatively to first-order logic—the binary *precedence of level* predicate (see Figure 1). We present them here just for the sake of completeness. The non-specialist can safely ignore them since the understanding of their meaning is of no importance for the sequel.

The soundness theorem establishes precisely the fact that if a formula ϕ can be formaly proved or derived from hypothesis on a set of $\mathscr{L}_*$-formulas Γ, then ϕ is true in every stratified model of Γ. Using the notation $\Gamma \vdash \phi$ to indicate the fact that there is a derivation of ϕ with hypothesis in Γ the soundness theorem is usually restated as

$$\text{if } \Gamma \vdash \phi \text{ then } \Gamma \Vdash \phi.$$

(The soundness theorem can be proved using induction on the derivation of ϕ from Γ.)

It is typical of mathematical reasoning to adopt different frameworks to represent the same objects just for the sake of making these objects more understandable or making easier to establish relations between them. It is well known that Hilbert tought that this was the case of the use of infinitary notions. Hilbert was correct only to a certain extent. But this attitude revealed fruitful in fields of mathematics such as non-standard analysis or more generaly in the field of mathematical logic via non-standard models.

The stratified first-order logic which we have been describing is relatively to first-order logic in this exact relation. In fact we can prove a result of conservativeness, more precisely: if $\Gamma \cup \{\phi\}$ are formulas of $\mathscr{L}$, then $\Gamma \vdash \phi$ in the context of first-order logic iff $\Gamma \vdash \phi$ in the context of stratified first-order logic.

The semantic analog of this relation (the semantic extension property) can be obtained from this, the completeness theorem for first-order logic and stratified soundness. The semantic extension property establishes that if ϕ is a logical consequence of Γ, then it is a stratified consequence of Γ. In fact, by completeness of first-order logic if ϕ is a logical consequence of Γ then there is a proof of ϕ from Γ. By conservativeness there is also a stratified proof of ϕ from Γ. And using soundness, we can conclude that ϕ is a stratified consequence of Γ.

Definition.—We denote by $\mathscr{P}_c(\mathrm{Sent}(\mathscr{L}_*))$ the set of all $\Gamma \subseteq \mathrm{Sent}(\mathscr{L}_*)$ such that whenever Γ as a first-order model, then Γ has a stratified model.

Using the previous definition and the completeness theorem for first-order logic we can easily prove the following result.

Theorem [Completeness].—If Γ is a consistent{4} subset of $\mathrm{Sent}(\mathscr{L}_*)$ and $\Gamma \cup \{\neg\phi\} \in \mathscr{P}_c(\mathrm{Sent}(\mathscr{L}_*))$, then: if ϕ is a stratified consequence of Γ, then there is a proof of ϕ with hypothesis in Γ.

Conclusion

The *stratification* presented in this work may be applied to any theory (in the usual, informal sense, of this word) formalizable in a first-order language like $\mathscr{L}$. So, we may stratify ZFC{5} or even such theories as *Nelson internal set theory*, IST, or *Hrbăcek set theory*, HST, that are largely used in *nonstandard analysis* (see[5]).

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{4} Here as elsewhere in this work, "consistent" has the usual meaning in first-order logic.

{5} For a different approach, see [3], [4].

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