Symplectic topology: rigidity and flexibility of ellipsoids

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Abstract

Very basic results and ideas of symplectic topology are presented in the context of symplectic embeddings of ellipsoids. A simple version of symplectic capacities is defined and used to prove rigidity results, and the "symplectic folding" construction is explained and used to prove flexibility results.

1 Classical Results

Consider the space $\mathbb{R}^{2n}$, with coordinates (p, q), and a smooth map $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$. Denote by φ_t the flow $\varphi_t : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ of Hamilton equations:

$$\begin{cases} \dot{q} &= \dfrac{\partial H}{\partial p} \\ \dot{p} &= -\dfrac{\partial H}{\partial q}. \end{cases}$$

Theorem 1.1 (Liouville)**.** *The flow φ_t is volume preserving.*

The changes of coordinates $(P, Q) = \varphi(p, q)$ in $\mathbb{R}^{2n}$ that preserve the form of the Hamilton equations for any Hamiltonian H (called *canonical transformations* in Mechanics) form the relevant group for symplectic geometry. They can be characterized by preserving the standard 2-form ω_0:

$$\mathrm{d}P \wedge \mathrm{d}Q = \mathrm{d}p \wedge \mathrm{d}q, \quad (P, Q) = \varphi(p, q)$$

where:

$$\omega_0 = \mathrm{d}p \wedge \mathrm{d}q = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i.$$

It is an important fact that, for any fixed t:

Theorem 1.2. *The flow of Hamilton equations*

$$\varphi_t : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}, \quad (p^t, q^t) = \varphi_t(p, q)$$

is a canonical transformation:

$$\mathrm{d}p^t \wedge \mathrm{d}q^t = \mathrm{d}p^0 \wedge \mathrm{d}q^0.$$

Equivalently, $\omega = \mathrm{d}p \wedge \mathrm{d}q$ is an integral invariant of φ_t:

$$\varphi_t^* \omega = \omega.$$

All this can be generalized to a *symplectic manifold*: a pair (M, ω), where M is a $2n-$dimensional differentiable manifold and ω is a symplectic form, a 2-form satisfying:

$$\Omega = \frac{1}{n!}\omega^n \text{ is a volume form, and } \omega \text{ is closed: } \mathrm{d}\omega = 0.$$

Locally all symplectic manifolds look the same: there are no local invariants; this is in contrast to Riemannian geometry, where curvature, for instance, is a local invariant. The precise formulation is:

Theorem 1.3 (Darboux)**.** *A symplectic manifold (M, ω) is locally symplectomorphic to:*

$$(\mathbb{R}^{2n}, \omega_0 = \mathrm{d}p \wedge \mathrm{d}q)$$

i.e. given $x \in M$, there exists a neighbourhood U of x and a diffeomorphism $\varphi : U \longrightarrow V \subset \mathbb{R}^{2n}$ such that:

$$\varphi^*(\mathrm{d}p \wedge \mathrm{d}q) = \omega.$$

Liouville theorem is valid for any canonical transformation, besides the flow of Hamilton equations. More generally, defining a *symplectic map* as a map $\varphi : (M, \omega) \longrightarrow (M', \omega')$ such that $\varphi^*\omega' = \omega$, we have:

Theorem 1.4 (Liouville)**.** *A symplectic diffeomorphism $\varphi : (M, \omega) \longrightarrow (M', \omega')$ is volume preserving*

$$\varphi^*\omega' = \omega \quad \Longrightarrow \quad \varphi^*\Omega' = \Omega.$$

There is no interesting topology associated to volume preserving maps; in fact:

Theorem 1.5 (Moser). *If $U \subset \mathbb{R}^{2n}$ is diffeomorphic to a ball B, and $vol(U) = vol(B)$, then there exists a volume preserving diffeomorphism $\Phi : B \longrightarrow U$.*

The work of Gromov in the 80's showed a completely different picture for symplectic topology. In the symplectic camel problem, the camel is represented by the closed unit ball in $\mathbb{R}^4$, and the wall with a hole by:

$$W = \left\{x \in \mathbb{R}^4 | x_1 = 0,\ x_2^2 + x_3^2 + x_4^2 \geq 1\right\}.$$

Then the problem, passing the camel through the wall hole, is to move the ball from one side of the wall (say, $x_1 > 0$) to the other, preserving the standard symplectic form.

That this is impossible shows a form of rigidity in symplectic geometry. We will consider other results on rigidity, but also on flexibility, in the context of embedding an ellipsoid into another one.

This type of problem has a Hamiltonian dynamics interpretation ([7]): let (p_i, q_i) be the moment-position of the ith particle; we can consider an initial ellipsoid as a representation of our knowledge of the particles, a smaller $i-$axis meaning more information, or smaller error, for particle i; it is important to know whether in future time the image of the ellipsoid by the flow can be contained in a different ellipsoid. As the flow, for fixed time, is a canonical transformation, albeit of a special type, we have an embedding problem for ellipsoids.

2 Basic definitions

A volume form on a smooth n-dimensional manifold M is a nowhere vanishing n-form Ω. On every open set $U \subset \mathbb{R}^n$ we consider the standard volume $\Omega_0 = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$; a smooth embedding $\varphi : U \hookrightarrow M$ is said to be volume preserving if:

$$\varphi^*\Omega = \Omega_0.$$

Let $\mathscr{D}(n)$ be the group of symplectic diffeomorphisms (also called symplectomorphisms or canonical transformation) of $\mathbb{R}^{2n}$, and $\mathrm{Sp}(n)$ its subgroup of linear isomorphisms.

On every open set $U \subset \mathbb{R}^{2n}$ we consider the standard symplectic form $\omega_0 = \mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \cdots + \mathrm{d}x_n \wedge \mathrm{d}y_n$; a smooth embedding $\varphi : U \hookrightarrow M$ is said to be symplectic if it is a symplectic map:

$$\varphi^*\omega = \omega_0, \text{ and therefore } \varphi^*\Omega = \Omega_0$$

where Ω and Ω_0 are the volume forms induced by the symplectic forms.

Definition 1. *An open symplectic ellipsoid of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with radii $r_i = \sqrt{a_i/\pi}$ is the set:*

$$\begin{aligned} E(a) &= E(a_1, \ldots, a_n) \\ &= \left\{ z \;\middle|\; \frac{\pi|z_1|^2}{a_1} + \cdots + \frac{\pi|z_n|^2}{a_n} < 1 \right\}, \end{aligned}$$

where we assume $a_1 \leq \ldots \leq a_n$, and $z_j = x_j + iy_j$.

Definition 2. *An open symplectic cylinder of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with radius $r = \sqrt{a/\pi}$ is the set:*

$$\begin{aligned} Z(a) &= \{(x, y) \in \mathbb{R}^{2n} :\ \pi|(x_1, y_1)|^2 < a\} \\ &= \{z \in \mathbb{C}^n :\ \pi|z_1|^2 < a\}. \end{aligned}$$

Remark 2.1. *The ball of radius r is denoted by $B(\pi r^2)$:*

$$B(a) = E(a, a, \ldots, a), \quad Z(a) = E(a, \infty, \ldots, \infty).$$

In dimension 2, an embedding is volume preserving if and only if it is symplectic; in higher dimensions there exists symplectic rigidity, as first shown in [5]:

Gromov Theorem (1985). *If there is a symplectic embedding $\varphi : B(a) \longrightarrow Z(A)$ of a ball into a symplectic cylinder, then $a \leq A$.*

Remark 2.2. *It is essential for the cylinder to be symplectic; the Lagrangian cylinder:*

$$L(a) = \{(x, y) \in \mathbb{R}^{2n} :\ \pi|(x_1, x_2)|^2 < a\}$$

can be embedded into $L(A)$ for any positive A, as the map:

$$(x_1, y_1, x_2, y_2) \mapsto \left(\frac{A}{2a}x_1, \frac{2a}{A}y_1, \frac{A}{2a}x_2, \frac{2a}{A}y_2\right)$$

is a symplectomorphism.

The detection of embedding obstructions and the proof of the corresponding rigidity results will be based on symplectic capacities:

Definition 3. *An extrinsic symplectic capacity c on $(\mathbb{R}^{2n}, \omega_0)$ is a map c such that, for every $A \subset \mathbb{R}^{2n}$, $c(A) \in [0, +\infty]$, satisfying the following properties:*

Monotonicity: *$c(A) \leq c(A')$ if there exists $\varphi \in \mathscr{D}(n)$ such that $\varphi(A) \subset A'$.*

Conformality: *$c(\alpha A) = \alpha^2 c(A)$, for any $\alpha \in \mathbb{R}^*$.*

Nontriviality: $0 < c(B(\pi))$, $\quad c(Z(\pi)) < \infty$.

3 Rigidity

When considering linear symplectic embeddings, there exists symplectic rigidity:

Theorem 3.1 ([8]). *Given two ellipsoids $E(a)$ and $E(a')$, there exists a linear symplectic map $S \in Sp(n)$ such that $S(E(a)) \subset E(a')$ if and only if $a_i \leq a_i'$, for all $i = 1, \ldots, n$.*

Even when allowing nonlinear symplectomorphisms, symplectic rigidity can still be present:

Theorem 3.2 ([4]). *Given two ellipsoids $E(a_1, a_2)$ and $E(a_1', a_2')$ with:*

$$\nu \leq a_1, a_2, a_1', a_2' \leq 1, \quad \frac{1}{2} < \nu < 1$$

there exists a symplectic embedding φ such that $\varphi(E(a)) \subset E(a')$ if and ony if $a_i \leq a_i'$, for $i = 1, 2$.

Gromov theorem can also be seen as a rigidity result for embeddings of ellipsoids and it follows immediatly from it that, if $E(a)$ embeds symplectically into $E(a')$, then:

$$a_1 \leq a_1'.$$

Going back to the Hamiltonian dynamics interpretation, this means that we cannot improve our knowledge of the best known particle, but (flexibility results) if we allow a loss in information for that particle, the error in the others can become smaller.

In $\mathbb{C}^2 \cong \mathbb{R}^4$ it is natural to characterize the shape of a symplectic ellipsoid by:

Definition 4. *Two ellipsoids $E(a_1, a_2)$ and $E(a_1', a_2')$ in $\mathbb{C}^2 \cong \mathbb{R}^4$ have the same shape type if:*

$$\exists k \in \mathbb{N} : \quad k \leq \frac{a_2}{a_1} < k+1, \quad k \leq \frac{a_2'}{a_1'} < k+1.$$

In higher dimensions the definition will be more general:

Definition 5. *Given an ellipsoid $E(a_1, \ldots, a_n)$, let $\{\mu_i\}$ be the sequence of the numbers $\{ka_j\}$, with $k \in \mathbb{N}$ and $j = 1, \ldots, n$, written (maybe with repetitions) in increasing order. The* Ekeland-Hofer i-capacity *for $E(a)$ is given by:*

$$c_i(E(a)) = \mu_i.$$

Definition 6. *Two ellipsoids $E(a)$ and $E(a')$ in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ have the same shape type if:*

$$\exists \alpha_1 = 1 < \cdots < \alpha_n : \quad \mu_{\alpha_i}(a) = a_i, \quad \mu_{\alpha_i}(a') = a_i'.$$

Example 1. *An ellipsoid $E(a) \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ has the shape type of a ball whenever $a_n \leq 2a_1$; then the associated sequences are:*

$$\mu = \{\overbrace{A, A \ldots, A}^{n}, \overbrace{2A, \ldots, 2A}^{n}, 3A, \ldots\} \qquad \text{for } B(A)$$
$$\mu' = \{a_1, a_2, \ldots, a_n, 2a_1, \ldots, 2a_n, 3a_1, \ldots\} \qquad \text{for } E(a)$$

and we can choose $\alpha_i = i$, $i = 1, \ldots, n$.

Example 2. *$E(1, 2, 3)$ and $E(1, 3, 4)$ have the same shape type, their associated sequences being respectively:*

$$\mu = \{1, 2, 2, 3, 3, 4, 4, 5, 6, 6, 6, 7 \ldots\}$$
$$\mu' = \{1, 2, 3, 3, 4, 4, 5, 6, 6, 7, \ldots\}$$

We can choose $\alpha_1 = 1$, $\alpha_2 = 3$ and $\alpha_3 = 5$.

Having the same shape type is an equivalence relation if we exclude resonant ellipsoids, for which the sequence $\{\mu_i\}$ is not strictly increasing; it is easy to see that then the two definitions agree for $n = 2$.

Example 3. *$B(a)$ and $E(a, 2a)$ have the same shape type using the definition 6: their associated sequences are respectively:*

$$\mu = \{a, a, 2a, 2a, 3a, 3a, 4a, 4a, \ldots\}$$
$$\mu' = \{a, 2a, 2a, 3a, 4a, 4a, 5a, 6a, \ldots\}$$

and we can choose $\alpha_1 = 1$ and $\alpha_2 = 2$. On the other hand, they have different shape types using the first definition (def. 4).

Theorem 3.2 considers ellipsoids with the shape type of a ball ($k = 1$), but the result can be extended to ellipsoids having the same shape type:

Theorem 3.3 ([1]). *If the two ellipsoids $E(a)$ and $E(a')$ in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ have the same shape type, there exists a symplectic embedding φ such that $\varphi(E(a)) \subset E(a')$ if and only if:*

$$a_i \leq a_i', \quad i = 1, \ldots, n.$$

Proof. If $E(a)$ embeds in $E(a')$ then it follows from the definition of capacity that:

$$c_j(E(a)) \leq c_j(E(a'))$$

for all Ekeland-Hofer capacities, in particular if they have the same shape type:

$$a_i = c_{\alpha_i}(E(a)) \leq c_{\alpha_i}(E(a')) = a_i', \quad i = 1, \ldots, n.$$

□

This is a generalization of a result of F. Schlenk [12, 13]: If $a_n \leq 2a_1$, there exists no symplectic embedding of the ellipsoid $E(a) = E(a_1, \ldots, a_n)$ into a ball $B(A)$ with $A < a_n$ (the shape type of the ellipsoid is that of a ball).

4 Flexibility

The following result shows that, if the shape type of the ellipsoids is sufficiently different, there is flexibility:

Theorem 4.1 ([6, 4])**.** *For any $a > 0$, and for a sufficiently small $\varepsilon > 0$, there exists a symplectic embedding φ such that:*

$$\varphi\left(E(\varepsilon, \ldots, \varepsilon, a)\right) \subset B(\pi).$$

There are no estimates on the size of ε, but F. Schlenk, using symplectic folding, proved:

Theorem 4.2 ([12, 13])**.** *If $\beta > 2\alpha$, there exists a symplectic embedding φ of the ellipsoid $E(r) = E(\alpha, \ldots, \alpha, \beta) \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ into a ball $B(A)$ with:*

$$E(\alpha, \ldots, \alpha, \beta) \hookrightarrow B(A), \quad A > \frac{\beta}{2} + \alpha.$$

Remark 4.1. *This theorem has been much improved in (complex) dimension 2 ([11]). But the methods used to obtain the best embedding results do not have a straightforward generalization to higher dimensions.*

Definition 7. *An open polydisk is the set:*

$$\begin{aligned} P(a) = P(a_1, \ldots, a_n) &= B(a_1) \times \cdots \times B(a_n) \\ &= \left\{ z \,\middle|\, \pi\frac{|z_1|^2}{a_1} < 1, \ldots, \pi\frac{|z_n|^2}{a_n} < 1 \right\}, \end{aligned}$$

where we assume $a_1 \leq \cdots \leq a_n$.

A very impressive result concerning flexibility of polydisks is due to L. Guth:

Theorem 4.3 ([7])**.** *There is a dimensional constant C_n such that, given two polydisks $P(r)$ and $P(r')$, if:*

$$C_n a_1 < r'_1, \quad C_n a_1 \ldots a_n < a'_1 \ldots a'_n$$

there exists a symplectic embedding of $P(a)$ into $P(a')$.

This result has an obvious application to ellipsoids:

Example 4. *In $\mathbb{C}^3 \cong \mathbb{R}^6$, there exists a constant $K > C_3\pi$ such that:*

$$E(\pi, a, a) \hookrightarrow E\left(3K, 3K, \frac{4}{K}a^2\right) \quad a > 3K.$$

This follows from the embedding:

$$P(\pi, a, a) \hookrightarrow P\left(K, K, \frac{a^2}{C_3\pi}\right)$$

and the inclusions $E(\pi, a, a) \subset P(\pi, a, a)$ and:

$$P\left(K, K, \frac{a^2}{C_3\pi}\right) \subset E\left(3K, 3K, \frac{4}{K}a^2\right).$$

A similar result is valid in any dimension; it shows that if the shape type of the ellipsoid is sufficiently different from that of a ball ($a > 3K$ above) then there exists considerable flexibility and the relevant obstructions are (derived from) just the first capacity and the volume.

Capacities (in general) involve the 2-dimensional area of some object; volume can considered a generalized capacity and is $2n$-dimensional. It is natural to search for intermediate capacities that involve $2k$-dimensional volumes; it follows from the results of [7] that there are no reasonably continuous intermediate capacities.

Symplectic folding is described in [9, 10, 12, 13]; we shall use a slightly different version [1], but the very careful and detailed presentation in [12, 13] should be considered for all technical aspects.

We define $T(a, b)$ as the set:

$$\begin{aligned} T(a, b) = \Big\{ &(z_1, z_2) = (u_1, v_1, u_2, v_2) \in \mathbb{R}^4 \,\Big| \\ &(u_1, v_1) \in]0, a[\times]0, 1[, \quad (u_2, v_2) \in]0, b[\times]0, 1[\\ &\frac{u_1}{a} + \frac{u_2}{b} < 1 \Big\} \end{aligned}$$

and $T(a) = T(a, a)$. The projection of $T(a, b)$ on the (u_1, u_2) plane is a triangle and the fibres are the unit square.

Lemma 4.4 ([12, 13])**.** *Assume $\varepsilon > 0$. Then:*

1. *$E(a, b)$ symplectically embeds into $T(a + \varepsilon, b + \varepsilon)$.*
2. *$T(a, b)$ symplectically embeds into $E(a + \varepsilon, b + \varepsilon)$.*

Sketch of the proof. The main fact involved in the proof is the existence of an area preserving map $(u, v) = \sigma(z)$ in the plane [12, 13] that, outside an arbitrarily small neighbourhood of the origin, where it is a translation, essentially takes open circles of area a into open rectangles $]0, a[\times]0, 1[$ (figure 1).

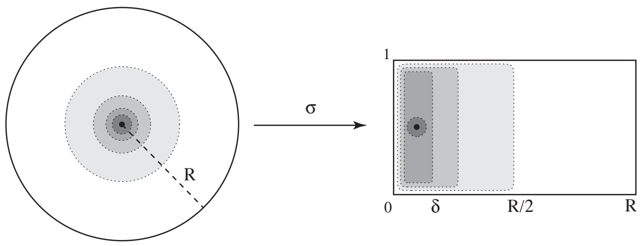

Figure 1: *Area preserving map in the plane*

Let $D(a)$ be the disk of area a; then:

$$E(a, b) = \left\{ z \,\middle|\, z_1 \in D(a), \; z_2 \in D\left(b(1 - \pi|z_1|^2/a)\right) \right\}$$

The symplectic embedding of E into T is then:

$$(z_1, z_2) \mapsto ((u_1, v_1), (u_2, v_2)) = (\sigma(z_1), \sigma(z_2))$$

The inverse of this map is used to embed T into E. $\square$

Here and subsequently we ignore everything 'small': an arbitrary small δ is involved in the construction of σ, we should therefore consider maps σ_δ with sufficiently small δ, but it is easier to proceed as if δ could be zero.

It follows from lemma 4.4 that embedding results for ellipsoids can be obtained from the corresponding results for sets of the form $T(a, b)$, and we describe symplectic folding for these sets in section 5. Figure 2 summarises the process (cf. figure 3.13 in [12]).

Since U embedding symplectically into V is equivalent to λU embedding symplectically into λV for $\lambda \neq 0$, we normalize the ellipsoids $E(a)$, and therefore the sets T, so that $a_1 = \pi$. In the figures we really represent $T(a, \pi)$ instead of $T(\pi, a)$, as in [12].

Theorem 4.5 ([1]). *If the ellipsoid $E(r) = E(r_1, r_2)$ in $\mathbb{C}^2 \cong \mathbb{R}^4$ has shape type $k \geq 3$ with:*

$$3 \leq k < r_2/r_1 < k+1$$

there exists a symplectic embedding φ such that $\varphi(E(r)) \subset E(r')$ with:

$$r_2 > r_2' \quad \text{and} \quad n \leq \frac{r_2'}{r_1'} < n+1$$

for all shape types $n = 1, \ldots, \left[\dfrac{2k}{3}\right]$.

Proof. We consider the normalised ellipsoid $E(\pi, a)$, with $k\pi < a < (k+1)\pi$ and $k \geq 3$. Symplectic folding gives an embedding (figure 2):

$$T(\pi, a) \hookrightarrow T\left(\frac{a}{2} + \pi + \varepsilon\right)$$

and lines above the image of $T(\pi, a)$ in the $(u_1', u_2')-$plane correspond to sets $T(\alpha, \beta)$ into which $T(\pi, a)$ embeds; $(\alpha, 0)$ and $(0, \beta)$ are the intersections of the line with the coordinate axes.

Going from T-type sets to ellipsoids:

$$E(\pi, a) \hookrightarrow E\left(\frac{3}{2}\pi + \varepsilon, \frac{3}{4}(a+\pi) + \varepsilon\right),$$

with:

$$\frac{3}{4}(a+\pi) < a \Longleftrightarrow k \geq 3.$$

The same construction also gives an embedding:

$$E(\pi, a) \hookrightarrow B\left(\frac{a}{2} + \pi + \varepsilon\right)$$

and clearly embeddings for all in between shape types. For any b such that:

$$\frac{3}{4}(a+\pi) < b < a$$

there is a trivial embedding (again see figure 2):

$$E\left(\frac{3}{2}\pi + \varepsilon, \frac{3}{4}(a+\pi) + \varepsilon\right) \hookrightarrow E\left(\frac{3}{2}\pi + \varepsilon, b\right)$$

and the shape type can thus be extended up to $\left[\dfrac{2k}{3}\right]$. □

Open Question ([12, 13]). *Does the ellipsoid $E(a, 2a, 3a)$ symplectically embed into $B(A)$ for some $A < 3a$?*

Ekeland-Hofer capacities show that:

- $E(a, 3a, \ldots, 3a)$ does not symplectically embed into a ball $B(A)$ with $A < 3a$.
- $E(a, 2a, \ldots, 2a, 3a)$ does not symplectically embed into a ball $B(A)$ with $A < 2a$.

On the other hand, there is also some flexibility, as it follows from theorem 4.2 that:

$$E(a, 3a) \hookrightarrow B\left(\frac{5}{2}a + \varepsilon\right)$$

The change introduced in the symplectic folding process allows estimates (lemma 4.7) that are decisive in the proof of:

Theorem 4.6 ([1]). *For any positive ε, there exists a symplectic embedding:*

$$E(\pi, b_1, \ldots, b_{n-2} = b, a) \hookrightarrow B(A + \varepsilon), \quad A < a$$

when $a > b + \pi$, with A given by:

$$A = \frac{a + b + \pi}{2}.$$

Remark 4.2. *For $n = 3$, $b = 2\pi$, $a = 3\pi$:*

$$E(\pi, 2\pi, 3\pi) \hookrightarrow B(A + \varepsilon), \quad A = \frac{3\pi + 2\pi}{2} + \frac{\pi}{2} = 3\pi$$

and thus $E(\pi, 2\pi, 3\pi)$ is in the boundary of (known) flexibility.

Remark 4.3. *$b = \pi$ gives theorem 3.1.1 in [12] (or theorem 4.2): for all $\varepsilon > 0$,*

$$E(\pi, \ldots, \pi, a) \text{ symplectically embeds into } B\left(\frac{a}{2} + \pi + \varepsilon\right)$$

Lemma 4.7 ([1]). *For any $\varepsilon > 0$, symplectic folding gives an embedding $\psi : T(\pi, a) \hookrightarrow \mathbb{C}^2 \cong \mathbb{R}^4$:*

$$\psi((u_1, v_1), (u_2, v_2)) = ((u_1', v_1'), (u_2', v_2'))$$

such that

$$u_1' + u_2' < A - b + \frac{b}{\pi}u_1 + \frac{b}{a}u_2 + \varepsilon, \quad A = \frac{a + b + \pi}{2}.$$

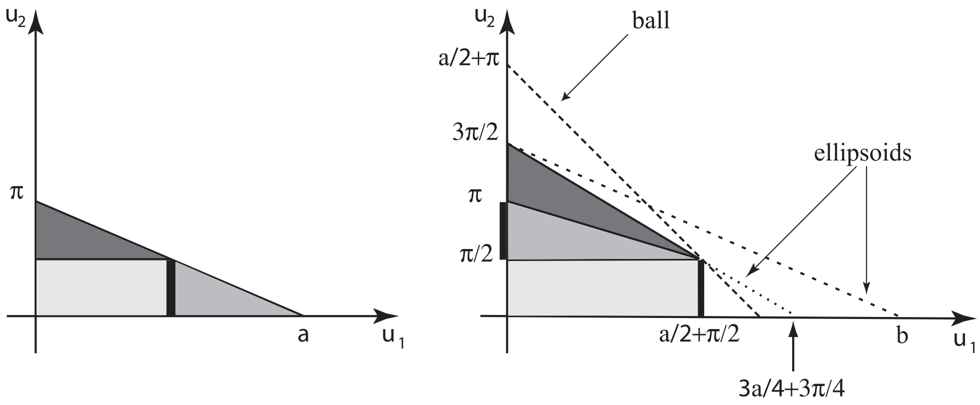

Figure 2: *Scheme of symplectic folding in the (u_1, u_2) plane*

Theorem 4.6 follows from lemmas 4.7 and 4.8:

Lemma 4.8 ([1]). *If for any positive ε there exists a symplectic embedding $\psi : T(\pi, a) \hookrightarrow \mathbb{C}^2 \cong \mathbb{R}^4$:*

$$\psi((u_1, v_1), (u_2, v_2)) = ((u_1', v_1'), (u_2', v_2'))$$

such that:

$$u_1' + u_2' < A - b + \frac{b}{\pi} u_1 + \frac{b}{a} u_2 + \varepsilon$$

then there exists a a symplectic embedding Φ:

$$E(\pi, b_1, \ldots, b_{n-2} = b, a) \hookrightarrow B(A + \varepsilon)$$

Proof. It follows from lemma 4.4 and the estimate on ψ that there exists a symplectic embedding σ:

$$\sigma : E(\pi, a) \hookrightarrow \mathbb{C}^2 \cong \mathbb{R}^4, \qquad \sigma(z_1, z_2) = (z_1', z_2')$$

such that:

$$\pi|z_1'|^2 + \pi|z_2'|^2 < A - b + \frac{b}{\pi}\pi|z_1|^2 + \frac{b}{a}\pi|z_2|^2 + \varepsilon$$

with:

$$A = \frac{a + b + \pi}{2}.$$

Then $\sigma \times \mathrm{id}_{n-2}$, after a suitable permutation τ, defined by $\tau(z_1, z_2, \ldots) = (z_1, z_n, z_2, \ldots)$, gives the desired symplectic embedding:

$$\Phi = (\sigma \times \mathrm{id}_{n-2}) \circ \tau : E(\pi, b_1, \ldots, b_{n-2}, a) \hookrightarrow \mathbb{C}^n \cong \mathbb{R}^{2n}$$

□

5 Symplectic folding

Step 1: We separate the region $u_2 > \pi$ from the region $u_2 < \pi$, the large fibres from the small ones: here the fibres are related to the projection on the (u_1, v_1) plane, and the symplectic map is the product $\varphi \times \mathrm{id}$ of an area preserving map φ in the (u_1, v_1) plane (figure 3) and the identity on the (u_2, v_2) plane.

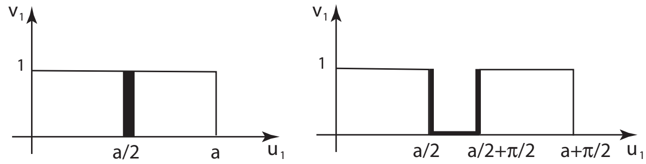

Figure 3: *Separating the fibres: black regions have the same area*

Remark 5.1. *Again we should consider the regions $u_2 > b/2 + \delta$ and $u_2 < b/2 - \delta$ and deform $b/2 - \delta < u_2 < b/2 + \delta$ for a conveniently small δ (the black region); the map outside that region is the identity on the left and a translation on the right.*

The result can also be seen in the (u_1, u_2) plane:

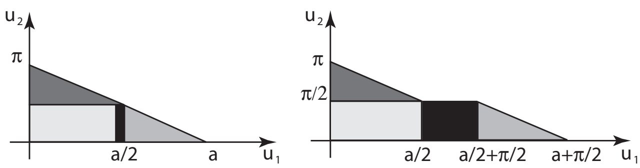

Figure 4: *Separating the fibres, (u_1, u_2) plane*

Remark 5.2. *The (u_1, v_1) and (u_2, v_2) planes are symplectic, the symplectic form on them is an area form, and it is convenient to preserve the area on them; but the plane (u_1, u_2) is Lagrangean, the symplectic form vanishes on it and therefore no area preserving on that plane is involved.*

Step 2: We rearrange the fibres: the symplectic map is the product of an area preserving map σ_1 in the (u_2, v_2) plane (figure 5), and the identity on the (u_1, v_1) plane.

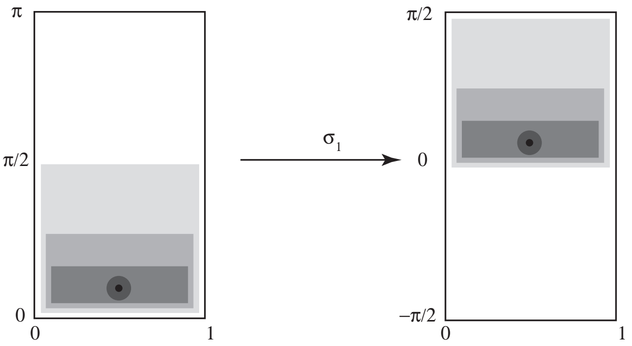

Figure 5: *Rearranging the fibres in the* (u_2, v_2) *plane*

The result can again be seen in the (u_1, u_2) plane, the top triangle goes to the bottom:

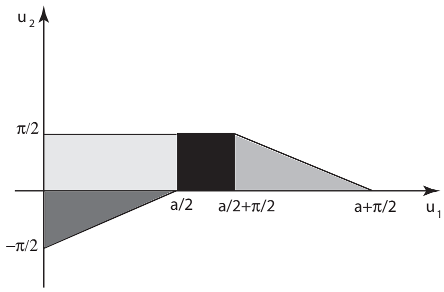

Figure 6: *Rearranging the fibres, in the* (u_1, u_2) *plane*

Step 3: We lift the region $a/2+\pi/2 < u_1 < a+\pi/2$ by $\pi/2$ along the u_2 direction. Now the symplectic map is not a product of area preserving maps: its action can be seen in the (u_1, u_2) and (u_1, v_1) planes (figure 7), but we refer to [12] for the construction of the lift map.

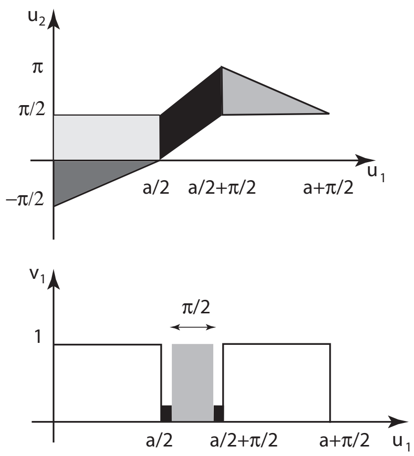

Figure 7: *Lifting*

The grey region in the plane (u_1, v_1) is the projection on that plane of points lifted less than $\pi/2$, and more than 0, and has area bigger than $\pi/2$.

Step 4: We contract along the v_1 direction, and extend along the u_1 direction, by $a/(a+\pi)$, keeping (u_2, v_2) unchanged (figure 8); again this is the product of an area preserving map on the (u_1, v_1) plane and the identity on the (u_2, v_2) plane.

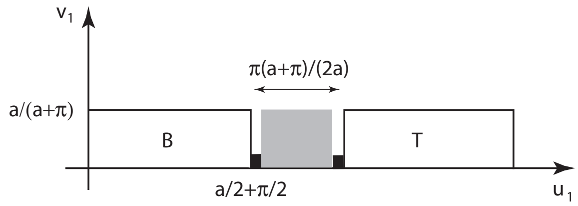

Figure 8: *Rearranging in the* (u_1, v_1) *plane*

Step 5: We now turn T over B: we extend the grey area, then we fold twice in the base (figure 9).

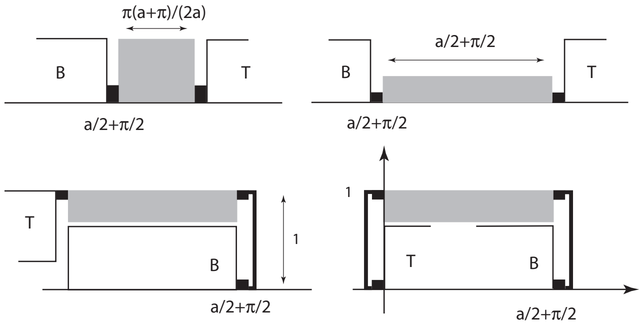

Figure 9: *Folding in the* (u_1, v_1) *plane*

The transformation of the grey area (in the (u_1, v_1) plane) is as in the previous step, with a factor of π/a now, but using the identity outside that area on the left and a translation on the right. The end result in the (u_1, u_2) plane is:

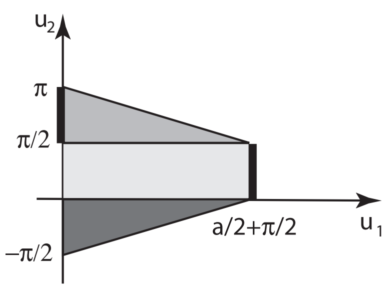

Figure 10: *Folding in the* (u_1, u_2) *plane*

Step 6: We rearrange the fibres in the (u_2, v_2) plane again:

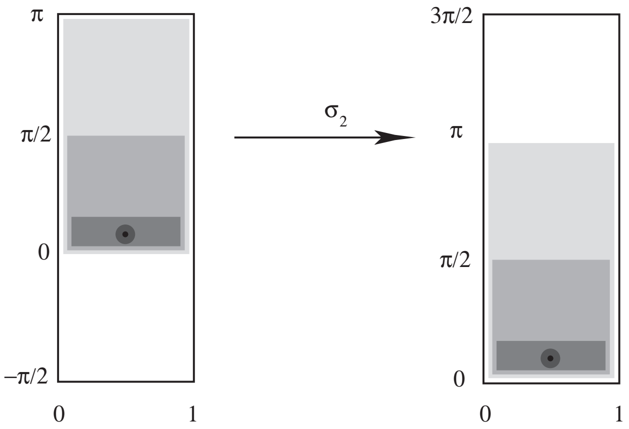

Figure 11: *Rearranging the fibres in the* (u_2, v_2) *plane*

The symplectic map is the product of an area preserving map σ_2 in the (u_2, v_2) plane (figure 11), and the identity on the (u_1, v_1) plane. Seen in the (u_1, u_2) plane, the bottom triangle goes to the top (figure 2).

The symplectic folding construction is summarised in figure 2 (it should be compared to figure 3.13 in [12]): the advantage of the change relative to [12, 13] is that we can get embeddings into ellipsoids, keeping the same estimates obtained for embeddings into balls.

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