Feature Article

The solid trefoil knot as an algebraic surface

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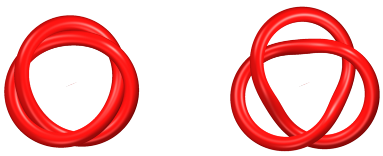
Abstract

We give an explicit polynomial of degree 14 in three real variables x, y and z such that the zero set gives the solid trefoil knot. The polynomial depends on two further parameters which enable a deformation from an embedded torus. We use only elementary methods such that the proofs are also accessible to graduate math work groups for pupils in secondary schools. The results can be easily visualized using the free SURFER software of Oberwolfach.

Introduction

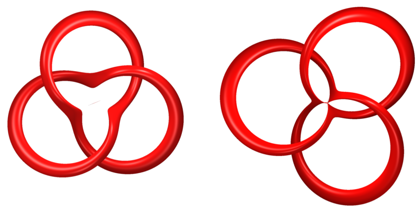
We use the elementary technique in [1] to construct an explicit polynomial of degree 14 in three real variables x, y and z such that the zero set gives the *solid trefoil knot*, i.e, the boundary surface of a tubular neighborhood of the trefoil knot. This answers a question of José Francisco Rodrigues at the Mathematisches Forschungsinstitut Oberwolfach.



Moreover, our polynomial will also depend on two real parameters a and b such that $a \mapsto 0$ describes a deformation into the shape of the standard torus.

We present some visualizations of the trefoil surface by the free SURFER software of Oberwolfach which also allows real-time deformation by changing some surface parameters. Using suitable parameter combinations there are interesting self-intersections and singularities. The pictures of this article are all created by the SURFER.

The trefoil knot is the simplest nontrivial knot and one can ask for explicit polynomials giving other solid knots. In a forthcoming paper, we will do this for general torus knots by similar techniques, whereas other types of knots seem difficult to approach. In particular, the following numerical invariant of a knot seems to be new, but difficult to approach:

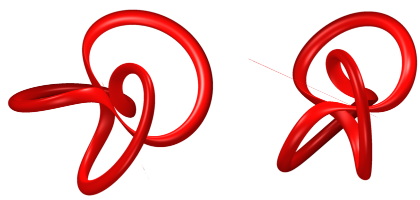


Definition: *Let $K \subset \mathbb{R}^3$ be a knot. Denote by*

$$\operatorname{sad}(K) := \min \Big\{ \deg(p) | p \in \mathbb{R}[x,y,z] \textit{ and } p(x,y,z) = 0 \\ \textit{gives a tubular neighborhood around} K \Big\}$$

the **solid algebraic degree** *of K.*

Our construction shows $sad(\text{trefoil}) \leq 14$. More generally, it is possible to show $sad(\mathrm{K}) \leq 8 + 2k$ for a $(2,k)$-torus knot using the method described in [1].



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MSC: 14J25, 14Q10, 51N10, 57N05, 57N35

The idea of the construction

We denote by d the distance of a point $(x,y) \in \mathbb{R}^2$ to the origin and by $\phi \in [0, 2\pi[$ its angle to the x-axis, i.e.

$$d^2 = x^2 + y^2, \quad x = d\cos(\phi) \quad \text{and} \quad y = d\sin(\phi).$$

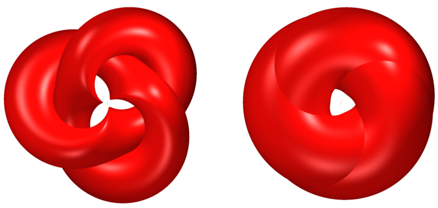
We denote $C := \cos(\phi)$ and $S := \sin(\phi)$.

Now we consider a second coordinate system $(t,z) \in \mathbb{R}^2$ and two circles of radius b and centers $(1-a, 0)$ and $(1+a, 0)$, which are clearly given by the equation

$$\left[(t-1-a)^2 + z^2 - b^2)((t-1+a)^2 + z^2 - b^2\right] = 0$$

where we assume a and b to be positive real numbers. We note that the two circles intersect if $a \leq b$ and that the circle around $1-a$ includes the origin for $a+b > 1$. By expansion of $[(t-1) \pm a]^2$ and using the third binomial law, the equation reads as

$$\left[(t-1)^2 + z^2 + a^2 - b^2\right]^2 - 4a^2(t-1)^2 = 0.$$



As in [1], the idea is to rotate the pair of circles around the center $(1,0)$ by an angle $\psi \in \mathbb{R}$ in the (t,z)-plane, while the t-axis rotates around the z-axis and spans the (x,y)-plane. If ϕ denotes the angle of the t-axis against the x-axis in the (x,y)-plane, we set up the condition

$$2\psi = 3\phi.$$

This condition yields exactly 3 twists (i.e., rotations by π in the (t,z)-plane) of the two tubes generated by the rotating circles before glueing them together after a full rotation around the z-axis. Thus, we imitate exactly the construction of the trefoil knot as a $(2,3)$-torus knot.

We implement this idea by a coordinate rotation in the (t,z)-plane around $(1,0)$

$$(t-1) \mapsto c(t-1) + sz \qquad z \mapsto -s(t-1) + cz,$$

where $c := \cos(\psi)$ and $s := \sin(\psi)$. This gives the equation for the rotated pair of circles

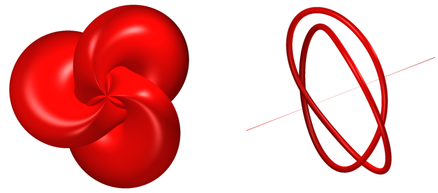
$$\left[(c(t-1)+sz)^2 + (-s(t-1)+cz)^2 + a^2 - b^2\right]^2 - 4a^2(c(t-1)+sz)^2 = 0.$$

Expansion of the two inner brackets gives equation E:

$$\left[(t-1)^2 + z^2 + a^2 - b^2\right]^2 - 4a^2\left[c^2(t-1)^2 + 2cs(t-1)z + s^2z^2\right] = 0.$$

At the same time, we have in the (x,y)-plane

$$t^2 = x^2 + y^2, \quad x = t\cos(\phi) \quad \text{and} \quad y = t\sin(\phi).$$



As a special case, we obtain the **standard torus** for $a = 0$ (and $b < 1$), as then the two circles coincide. We note that in this case $a = 0$, the SURFER has problems with the visualization as there are two surfaces at the same place which is numerically an unstable situation.

Construction of the polynomial equation

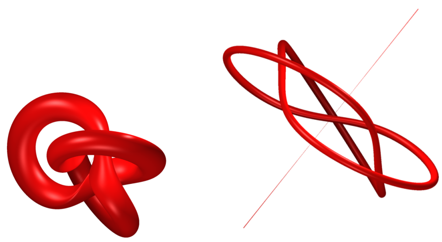
Now we will construct an implicit polynomial representation $p(x,y,z)$ for the solid trefoil knot by elimination of the variables ϕ (i.e., C and S), ψ (i.e., c and s) and t.

The relation $2\psi = 3\phi$ yields with the formulas for the double angle and for the triple angle the following relations:

$$C^3 - 3CS^2 = c^2 - s^2 \quad \text{and} \quad 3C^2S - S^3 = 2cs.$$

Because of $c^2 + s^2 = 1$ we obtain $c^2 = \frac{1}{2}(1 + C^3 - 3CS^2)$ and $s^2 = \frac{1}{2}(1 - C^3 + 3CS^2)$, hence

$$c^2 = \frac{t^3 + x^3 - 3xy^2}{2t^3}, \quad s^2 = \frac{t^3 - x^3 + 3xy^2}{2t^3} \quad \text{and}$$
$$cs = \frac{3x^2y - y^3}{2t^3}.$$



Inserting this into the equation E and multiplying with $2t^3$ in order to clear denominators gives

$$2t^3\left[(t-1)^2 + z^2 + a^2 - b^2\right]^2 + 2(3x^2y - y^3)(t-1)z - 4a^2\left[((t^3 + x^3 - 3xy^2)(t-1)^2 + (t^3 - x^3 + 3xy^2)z^2\right] = 0.$$

Separating even powers of t to the left side and odd powers to the right side yields

$$-8t^4\Big[(t^2+1+z^2+a^2-b^2)+4a^2\,[2t^4-(x^3-3xy^2)(t^2+1)]+8a^2(3x^2y-y^3)z+4a^2(x^3-3xy^2)z^2\Big] = t\Big[\,[2t^2(t^2+1+z^2+a^2-b^2)^2+8t^4+4a^2(2(x^3-3xy^2)-\,t^2(t^2+1)]-8a^2(3x^2y-y^3)z-4t^2a^2z^2\Big].$$

Squaring and inserting $t^2 = x^2 + y^2$ yields the **polynomial equation for the solid trefoil knot** of degree 14:

$$\Big[-8(x^2+y^2)^2(x^2+y^2+1+z^2+a^2-b^2)+4a^2\,[2(x^2+y^2)^2-(x^3-3xy^2)(x^2+y^2+1)]+ 8a^2(3x^2y-y^3)z+4a^2(x^3-3xy^2)z^2\Big]^2-(x^2+y^2)\Big[2(x^2+y^2)(x^2+y^2+1+z^2+a^2-b^2)^2+8(x^2+y^2)^2+ 4a^2\,[2(x^3-3xy^2)-(x^2+y^2)(x^2+y^2+1)]-8a^2(3x^2y-y^3)z-4(x^2+y^2)a^2z^2\Big]^2=0$$

In this article we present some visualizations for different parameter values and points of view. We remark that in some pictures, the z-axis appears as a ghost which is probably due to numerical instabilities in the SURFER software.

Bibliography

[1] Stephan Klaus: *Solid Möbius strips as algebraic surfaces*,
Preprint, 10 p. (2010, submitted), Mathematisches Forschungsinstitut Oberwolfach

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Stephan Klaus is one of the invited speakers of MatCampus2010.