BULLETIN

INTERNATIONAL CENTER FOR MATHEMATICS

JULY 2010 28

Ten years after the WMY2000

In 2000, the International Mathematical Union (IMU) launched the WMY2000, the World Mathematical Year supported by UNESCO. The 1992 Rio de Janeiro declaration by the IMU, set three aims: The great challenges of 21st Century; Mathematics, a key for Development; and The image of mathematics. This last challenge, in particular, has contributed for the renewal of many worldwide initiatives on the popularisation of mathematics and, in Europe, the RPA-MATHS (Raising Public Awareness in Mathematics) project was launched by the European Mathematical Society (EMS) and was coordinated by its homonymous committee.

As a joint initiative of CIM and the EMS/RPA committee, an international workshop will be held in Óbidos, Portugal, on the 26-29 September 2010, to provide a Forum for a general reflection and an international balance by experts on the building the Image of Mathematics, ten years after the WMY2000. It aims to encourage and inspire actions directed towards raising public awareness of the importance of mathematical sciences for contemporary society in a cultural and historical perspective, as well as to provide the European mathematical societies with ideas and contributions of concerted actions with other national or international organisations and societies in matters of raising public awareness of science and technology and other important aspects of society with a strong component of mathematics.

The workshop "Raising the Public Awareness of Mathematics", organised by E. Behrends (FU Berlin), N. Crato (TU Lisbon) and J. F. Rodrigues (U Lisbon), has the support of the Portuguese Mathematical Society SPM, the town of Óbidos and the Portuguese research centers, CMAF and CEMAPRE. Interested participants are invited to write to the organisers and to visit the workshop page `http://www.cim.pt/RPAM`.

Contents

Feature Article

The solid trefoil knot as an algebraic surface

Stephan Klaus

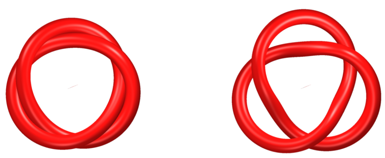
Mathematisches Forschungsinstitut
Oberwolfach
http://www.mfo.de/organisation/institute/klaus

Abstract

We give an explicit polynomial of degree 14 in three real variables x, y and z such that the zero set gives the solid trefoil knot. The polynomial depends on two further parameters which enable a deformation from an embedded torus. We use only elementary methods such that the proofs are also accessible to graduate math work groups for pupils in secondary schools. The results can be easily visualized using the free SURFER software of Oberwolfach.

Introduction

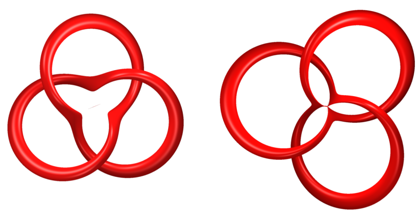
We use the elementary technique in [1] to construct an explicit polynomial of degree 14 in three real variables x, y and z such that the zero set gives the *solid trefoil knot*, i.e, the boundary surface of a tubular neighborhood of the trefoil knot. This answers a question of José Francisco Rodrigues at the Mathematisches Forschungsinstitut Oberwolfach.



Moreover, our polynomial will also depend on two real parameters a and b such that $a \mapsto 0$ describes a deformation into the shape of the standard torus.

We present some visualizations of the trefoil surface by the free SURFER software of Oberwolfach which also allows real-time deformation by changing some surface parameters. Using suitable parameter combinations there are interesting self-intersections and singularities. The pictures of this article are all created by the SURFER.

The trefoil knot is the simplest nontrivial knot and one can ask for explicit polynomials giving other solid knots. In a forthcoming paper, we will do this for general torus knots by similar techniques, whereas other types of knots seem difficult to approach. In particular, the following numerical invariant of a knot seems to be new, but difficult to approach:



Definition: *Let $K \subset \mathbb{R}^3$ be a knot. Denote by*

$$\mathrm{sad}(K) := \min \left\{ \deg(p) | p \in \mathbb{R}[x,y,z] \, \textit{and} \, p(x,y,z) = 0 \right. \\ \left. \textit{gives a tubular neighborhood around} K \right\}$$

the **solid algebraic degree** *of K.*

Our construction shows $sad(\text{trefoil}) \leq 14$. More generally, it is possible to show $sad(\mathrm{K}) \leq 8 + 2k$ for a $(2,k)$-torus knot using the method described in [1].

Keywords: trefoil knot, torus, algebraic surface
MSC: 14J25, 14Q10, 51N10, 57N05, 57N35

The idea of the construction

We denote by d the distance of a point $(x, y) \in \mathbb{R}^2$ to the origin and by $\phi \in [0, 2\pi[$ its angle to the x-axis, i.e.

$$d^2 = x^2 + y^2, \quad x = d\cos(\phi) \quad \text{and} \quad y = d\sin(\phi).$$

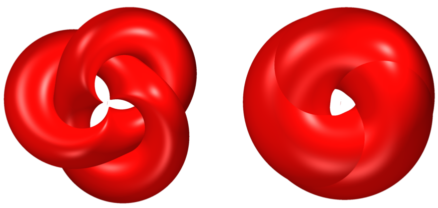
We denote $C := \cos(\phi)$ and $S := \sin(\phi)$.

Now we consider a second coordinate system $(t, z) \in \mathbb{R}^2$ and two circles of radius b and centers $(1 - a, 0)$ and $(1 + a, 0)$, which are clearly given by the equation

$$\left[(t-1-a)^2 + z^2 - b^2)((t-1+a)^2 + z^2 - b^2\right] = 0$$

where we assume a and b to be positive real numbers. We note that the two circles intersect if $a \leq b$ and that the circle around $1 - a$ includes the origin for $a + b > 1$. By expansion of $[(t-1) \pm a]^2$ and using the third binomial law, the equation reads as

$$\left[(t-1)^2 + z^2 + a^2 - b^2\right]^2 - 4a^2(t-1)^2 = 0.$$



As in [1], the idea is to rotate the pair of circles around the center $(1, 0)$ by an angle $\psi \in \mathbb{R}$ in the (t, z)-plane, while the t-axis rotates around the z-axis and spans the (x, y)-plane. If ϕ denotes the angle of the t-axis against the x-axis in the (x, y)-plane, we set up the condition

$$2\psi = 3\phi.$$

This condition yields exactly 3 twists (i.e., rotations by π in the (t, z)-plane) of the two tubes generated by the rotating circles before glueing them together after a full rotation around the z-axis. Thus, we imitate exactly the construction of the trefoil knot as a $(2, 3)$-torus knot.

We implement this idea by a coordinate rotation in the (t, z)-plane around $(1, 0)$

$$(t-1) \mapsto c(t-1) + sz \qquad z \mapsto -s(t-1) + cz,$$

where $c := \cos(\psi)$ and $s := \sin(\psi)$. This gives the equation for the rotated pair of circles

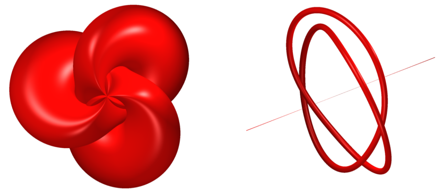
$$\left[(c(t-1) + sz)^2 + (-s(t-1) + cz)^2 + a^2 - b^2\right]^2 - 4a^2(c(t-1) + sz)^2 = 0.$$

Expansion of the two inner brackets gives equation E:

$$\left[(t-1)^2 + z^2 + a^2 - b^2\right]^2 - 4a^2\left[c^2(t-1)^2 + 2cs(t-1)z + s^2z^2\right] = 0.$$

At the same time, we have in the (x, y)-plane

$$t^2 = x^2 + y^2, \quad x = t\cos(\phi) \quad \text{and} \quad y = t\sin(\phi).$$



As a special case, we obtain the **standard torus** for $a = 0$ (and $b < 1$), as then the two circles coincide. We note that in this case $a = 0$, the SURFER has problems with the visualization as there are two surfaces at the same place which is numerically an unstable situation.

Construction of the polynomial equation

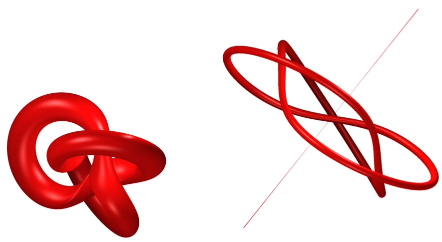
Now we will construct an implicit polynomial representation $p(x, y, z)$ for the solid trefoil knot by elimination of the variables ϕ (i.e., C and S), ψ (i.e., c and s) and t.

The relation $2\psi = 3\phi$ yields with the formulas for the double angle and for the triple angle the following relations:

$$C^3 - 3CS^2 = c^2 - s^2 \quad \text{and} \quad 3C^2S - S^3 = 2cs.$$

Because of $c^2 + s^2 = 1$ we obtain $c^2 = \frac{1}{2}(1 + C^3 - 3CS^2)$ and $s^2 = \frac{1}{2}(1 - C^3 + 3CS^2)$, hence

$$c^2 = \frac{t^3 + x^3 - 3xy^2}{2t^3}, \quad s^2 = \frac{t^3 - x^3 + 3xy^2}{2t^3} \quad \text{and}$$
$$cs = \frac{3x^2y - y^3}{2t^3}.$$



Inserting this into the equation E and multiplying with $2t^3$ in order to clear denominators gives

$$2t^3\left[(t-1)^2 + z^2 + a^2 - b^2\right]^2 + 2(3x^2y - y^3)(t-1)z - 4a^2\left[((t^3 + x^3 - 3xy^2)(t-1)^2 + (t^3 - x^3 + 3xy^2)z^2\right] = 0.$$

Separating even powers of t to the left side and odd powers to the right yields

$$-8t^4\Big[(t^2+1+z^2+a^2-b^2)+4a^2\,\big[2t^4-(x^3-3xy^2)(t^2+1)\big]+8a^2(3x^2y-y^3)z+4a^2(x^3-3xy^2)z^2\Big] = t\Big[\,\big[2t^2(t^2+1+z^2+a^2-b^2)^2+8t^4+4a^2(2(x^3-3xy^2)-\ t^2(t^2+1)\big]-8a^2(3x^2y-y^3)z-4t^2a^2z^2\Big].$$

Squaring and inserting $t^2 = x^2+y^2$ yields the **polynomial equation for the solid trefoil knot** of degree 14:

$$\Big[-8(x^2+y^2)^2(x^2+y^2+1+z^2+a^2-b^2)+4a^2\big[2(x^2+y^2)^2-(x^3-3xy^2)(x^2+y^2+1)\big]+ 8a^2(3x^2y-y^3)z+4a^2(x^3-3xy^2)z^2\Big]^2-(x^2+y^2)\Big[2(x^2+y^2)(x^2+y^2+1+z^2+a^2-b^2)^2+8(x^2+y^2)^2+ 4a^2\big[2(x^3-3xy^2)-(x^2+y^2)(x^2+y^2+1)\big]-8a^2(3x^2y-y^3)z-4(x^2+y^2)a^2z^2\Big]^2=0$$

In this article we present some visualizations for different parameter values and points of view. We remark that in some pictures, the z-axis appears as a ghost which is probably due to numerical instabilities in the SURFER software.

Bibliography

[1] Stephan Klaus: *Solid Möbius strips as algebraic surfaces*,
Preprint, 10 p. (2010, submitted), Mathematisches Forschungsinstitut Oberwolfach

---

Stephan Klaus is one of the invited speakers of MatCampus2010.

Feature Article

Symplectic topology: rigidity and flexibility of ellipsoids

J. Basto-Gonçalves

Centro de Matemática da Universidade do Porto
Portugal
jbg@fc.up.pt

Abstract

Very basic results and ideas of symplectic topology are presented in the context of symplectic embeddings of ellipsoids. A simple version of symplectic capacities is defined and used to prove rigidity results, and the “symplectic folding” construction is explained and used to prove flexibility results.

1 Classical Results

Consider the space $\mathbb{R}^{2n}$, with coordinates (p,q), and a smooth map $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$. Denote by φ_t the flow $\varphi_t : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ of Hamilton equations:

$$\begin{cases} \dot{q} &= \dfrac{\partial H}{\partial p} \\ \dot{p} &= -\dfrac{\partial H}{\partial q}. \end{cases}$$

Theorem 1.1 (Liouville). *The flow φ_t is volume preserving.*

The changes of coordinates $(P,Q) = \varphi(p,q)$ in $\mathbb{R}^{2n}$ that preserve the form of the Hamilton equations for any Hamiltonian H (called *canonical transformations* in Mechanics) form the relevant group for symplectic geometry. They can be characterized by preserving the standard 2-form ω_0:

$$\mathrm{d}P \wedge \mathrm{d}Q = \mathrm{d}p \wedge \mathrm{d}q, \quad (P,Q) = \varphi(p,q)$$

where:

$$\omega_0 = \mathrm{d}p \wedge \mathrm{d}q = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i.$$

It is an important fact that, for any fixed t:

Theorem 1.2. *The flow of Hamilton equations*

$$\varphi_t : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}, \quad (p^t, q^t) = \varphi_t(p,q)$$

is a canonical transformation:

$$\mathrm{d}p^t \wedge \mathrm{d}q^t = \mathrm{d}p^0 \wedge \mathrm{d}q^0.$$

Equivalently, $\omega = \mathrm{d}p \wedge \mathrm{d}q$ is an integral invariant of φ_t:

$$\varphi_t^* \omega = \omega.$$

All this can be generalized to a *symplectic manifold*: a pair (M,ω), where M is a $2n-$dimensional differentiable manifold and ω is a symplectic form, a 2-form satisfying:

$$\Omega = \frac{1}{n!}\omega^n \text{ is a volume form, and } \omega \text{ is closed: } \mathrm{d}\omega = 0.$$

Locally all symplectic manifolds look the same: there are no local invariants; this is in contrast to Riemannian geometry, where curvature, for instance, is a local invariant. The precise formulation is:

Theorem 1.3 (Darboux). *A symplectic manifold (M,ω) is locally symplectomorphic to:*

$$(\mathbb{R}^{2n}, \omega_0 = \mathrm{d}p \wedge \mathrm{d}q)$$

i.e. given $x \in M$, there exists a neighbourhood U of x and a diffeomorphism $\varphi : U \longrightarrow V \subset \mathbb{R}^{2n}$ such that:

$$\varphi^*(\mathrm{d}p \wedge \mathrm{d}q) = \omega.$$

Liouville theorem is valid for any canonical transformation, besides the flow of Hamilton equations. More generally, defining a *symplectic map* as a map $\varphi : (M,\omega) \longrightarrow (M',\omega')$ such that $\varphi^*\omega' = \omega$, we have:

Theorem 1.4 (Liouville). *A symplectic diffeomorphism $\varphi : (M,\omega) \longrightarrow (M',\omega')$ is volume preserving*

$$\varphi^*\omega' = \omega \quad \Longrightarrow \quad \varphi^*\Omega' = \Omega.$$

There is no interesting topology associated to volume preserving maps; in fact:

Theorem 1.5 (Moser)**.** *If $U \subset \mathbb{R}^{2n}$ is diffeomorphic to a ball B, and $vol(U) = vol(B)$, then there exists a volume preserving diffeomorphism $\Phi : B \longrightarrow U$.*

The work of Gromov in the 80's showed a completely different picture for symplectic topology. In the symplectic camel problem, the camel is represented by the closed unit ball in $\mathbb{R}^4$, and the wall with a hole by:

$$W = \left\{ x \in \mathbb{R}^4 | x_1 = 0,\ x_2^2 + x_3^2 + x_4^2 \geq 1 \right\}.$$

Then the problem, passing the camel through the wall hole, is to move the ball from one side of the wall (say, $x_1 > 0$) to the other, preserving the standard symplectic form.

That this is impossible shows a form of rigidity in symplectic geometry. We will consider other results on rigidity, but also on flexibility, in the context of embedding an ellipsoid into another one.

This type of problem has a Hamiltonian dynamics interpretation ([7]): let (p_i, q_i) be the moment-position of the ith particle; we can consider an initial ellipsoid as a representation of our knowledge of the particles, a smaller $i-$axis meaning more information, or smaller error, for particle i; it is important to know whether in future time the image of the ellipsoid by the flow can be contained in a different ellipsoid. As the flow, for fixed time, is a canonical transformation, albeit of a special type, we have an embedding problem for ellipsoids.

2 Basic definitions

A volume form on a smooth n-dimensional manifold M is a nowhere vanishing n-form Ω. On every open set $U \subset \mathbb{R}^n$ we consider the standard volume $\Omega_0 = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$; a smooth embedding $\varphi : U \hookrightarrow M$ is said to be volume preserving if:

$$\varphi^* \Omega = \Omega_0.$$

Let $\mathscr{D}(n)$ be the group of symplectic diffeomorphisms (also called symplectomorphisms or canonical transformation) of $\mathbb{R}^{2n}$, and $\mathrm{Sp}(n)$ its subgroup of linear isomorphisms.

On every open set $U \subset \mathbb{R}^{2n}$ we consider the standard symplectic form $\omega_0 = \mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \cdots + \mathrm{d}x_n \wedge \mathrm{d}y_n$; a smooth embedding $\varphi : U \hookrightarrow M$ is said to be symplectic if it is a symplectic map:

$$\varphi^* \omega = \omega_0, \text{ and therefore } \varphi^* \Omega = \Omega_0$$

where Ω and Ω_0 are the volume forms induced by the symplectic forms.

Definition 1. *An open symplectic ellipsoid of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with radii $r_i = \sqrt{a_i/\pi}$ is the set:*

$$\begin{aligned} E(a) &= E(a_1, \ldots, a_n) \\ &= \left\{ z \;\middle|\; \frac{\pi |z_1|^2}{a_1} + \cdots + \frac{\pi |z_n|^2}{a_n} < 1 \right\}, \end{aligned}$$

where we assume $a_1 \leq \ldots \leq a_n$, and $z_j = x_j + i y_j$.

Definition 2. *An open symplectic cylinder of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with radius $r = \sqrt{a/\pi}$ is the set:*

$$\begin{aligned} Z(a) &= \{(x,y) \in \mathbb{R}^{2n} :\ \pi |(x_1, y_1)|^2 < a\} \\ &= \{z \in \mathbb{C}^n :\ \pi |z_1|^2 < a\}. \end{aligned}$$

Remark 2.1. *The ball of radius r is denoted by $B(\pi r^2)$:*

$$B(a) = E(a, a, \ldots, a), \quad Z(a) = E(a, \infty, \ldots, \infty).$$

In dimension 2, an embedding is volume preserving if and only if it is symplectic; in higher dimensions there exists symplectic rigidity, as first shown in [5]:

Gromov Theorem (1985)**.** *If there is a symplectic embedding $\varphi : B(a) \longrightarrow Z(A)$ of a ball into a symplectic cylinder, then $a \leq A$.*

Remark 2.2. *It is essential for the cylinder to be symplectic; the Lagrangian cylinder:*

$$L(a) = \{(x,y) \in \mathbb{R}^{2n} :\ \pi |(x_1, x_2)|^2 < a\}$$

can be embedded into $L(A)$ for any positive A, as the map:

$$(x_1, y_1, x_2, y_2) \mapsto \left(\frac{A}{2a} x_1, \frac{2a}{A} y_1, \frac{A}{2a} x_2, \frac{2a}{A} y_2 \right)$$

is a symplectomorphism.

The detection of embedding obstructions and the proof of the corresponding rigidity results will be based on symplectic capacities:

Definition 3. *An extrinsic symplectic capacity c on $(\mathbb{R}^{2n}, \omega_0)$ is a map c such that, for every $A \subset \mathbb{R}^{2n}$, $c(A) \in [0, +\infty]$, satisfying the following properties:*

Monotonicity: $c(A) \leq c(A')$ *if there exists $\varphi \in \mathscr{D}(n)$ such that $\varphi(A) \subset A'$.*

Conformality: $c(\alpha A) = \alpha^2 c(A)$*, for any $\alpha \in \mathbb{R}^*$.*

Nontriviality: $0 < c(B(\pi)), \quad c(Z(\pi)) < \infty.$

3 Rigidity

When considering linear symplectic embeddings, there exists symplectic rigidity:

Theorem 3.1 ([8]). *Given two ellipsoids $E(a)$ and $E(a')$, there exists a linear symplectic map $S \in Sp(n)$ such that $S(E(a)) \subset E(a')$ if and only if $a_i \leq a'_i$, for all $i = 1, \ldots, n$.*

Even when allowing nonlinear symplectomorphisms, symplectic rigidity can still be present:

Theorem 3.2 ([4]). *Given two ellipsoids $E(a_1, a_2)$ and $E(a'_1, a'_2)$ with:*

$$\nu \leq a_1, a_2, a'_1, a'_2 \leq 1, \quad \frac{1}{2} < \nu < 1$$

there exists a symplectic embedding φ such that $\varphi(E(a)) \subset E(a')$ if and ony if $a_i \leq a'_i$, for $i = 1, 2$.

Gromov theorem can also be seen as a rigidity result for embeddings of ellipsoids and it follows immediatly from it that, if $E(a)$ embeds symplectically into $E(a')$, then:

$$a_1 \leq a'_1.$$

Going back to the Hamiltonian dynamics interpretation, this means that we cannot improve our knowledge of the best known particle, but (flexibility results) if we allow a loss in information for that particle, the error in the others can become smaller.

In $\mathbb{C}^2 \cong \mathbb{R}^4$ it is natural to characterize the shape of a symplectic ellipsoid by:

Definition 4. *Two ellipsoids $E(a_1, a_2)$ and $E(a'_1, a'_2)$ in $\mathbb{C}^2 \cong \mathbb{R}^4$ have the same shape type if:*

$$\exists k \in \mathbb{N}: \quad k \leq \frac{a_2}{a_1} < k+1, \quad k \leq \frac{a'_2}{a'_1} < k+1.$$

In higher dimensions the definition will be more general:

Definition 5. *Given an ellipsoid $E(a_1, \ldots, a_n)$, let $\{\mu_i\}$ be the sequence of the numbers $\{ka_j\}$, with $k \in \mathbb{N}$ and $j = 1, \ldots, n$, written (maybe with repetitions) in increasing order. The* Ekeland-Hofer i-capacity *for $E(a)$ is given by:*

$$c_i(E(a)) = \mu_i.$$

Definition 6. *Two ellipsoids $E(a)$ and $E(a')$ in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ have the same shape type if:*

$$\exists \alpha_1 = 1 < \cdots < \alpha_n: \quad \mu_{\alpha_i}(a) = a_i, \quad \mu_{\alpha_i}(a') = a'_i.$$

Example 1. *An ellipsoid $E(a) \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ has the shape type of a ball whenever $a_n \leq 2a_1$; then the associated sequences are:*

$$\begin{aligned} \mu &= \{\overbrace{A, A \ldots, A}^{n}, \overbrace{2A, \ldots, 2A}^{n}, 3A, \ldots\} && \text{for } B(A) \\ \mu' &= \{a_1, a_2, \ldots, a_n, 2a_1, \ldots, 2a_n, 3a_1, \ldots\} && \text{for } E(a) \end{aligned}$$

and we can choose $\alpha_i = i$, $i = 1, \ldots, n$.

Example 2. *$E(1,2,3)$ and $E(1,3,4)$ have the same shape type, their associated sequences being respectively:*

$$\begin{aligned} \mu &= \{1, 2, 2, 3, 3, 4, 4, 5, 6, 6, 6, 7 \ldots\} \\ \mu' &= \{1, 2, 3, 3, 4, 4, 5, 6, 6, 7, \ldots\} \end{aligned}$$

We can choose $\alpha_1 = 1$, $\alpha_2 = 3$ and $\alpha_3 = 5$.

Having the same shape type is an equivalence relation if we exclude resonant ellipsoids, for which the sequence $\{\mu_i\}$ is not strictly increasing; it is easy to see that then the two definitions agree for $n = 2$.

Example 3. *$B(a)$ and $E(a, 2a)$ have the same shape type using the definition 6: their associated sequences are respectively:*

$$\begin{aligned} \mu &= \{a, a, 2a, 2a, 3a, 3a, 4a, 4a, \ldots\} \\ \mu' &= \{a, 2a, 2a, 3a, 4a, 4a, 5a, 6a, \ldots\} \end{aligned}$$

and we can choose $\alpha_1 = 1$ and $\alpha_2 = 2$. On the other hand, they have different shape types using the first definition (def. 4).

Theorem 3.2 considers ellipsoids with the shape type of a ball ($k = 1$), but the result can be extended to ellipsoids having the same shape type:

Theorem 3.3 ([1]). *If the two ellipsoids $E(a)$ and $E(a')$ in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ have the same shape type, there exists a symplectic embedding φ such that $\varphi(E(a)) \subset E(a')$ if and only if:*

$$a_i \leq a'_i, \quad i = 1, \ldots, n.$$

Proof. If $E(a)$ embeds in $E(a')$ then it follows from the definition of capacity that:

$$c_j(E(a)) \leq c_j(E(a'))$$

for all Ekeland-Hofer capacities, in particular if they have the same shape type:

$$a_i = c_{\alpha_i}(E(a)) \leq c_{\alpha_i}(E(a')) = a'_i, \quad i = 1, \ldots, n.$$

□

This is a generalization of a result of F. Schlenk [12, 13]: If $a_n \leq 2a_1$, there exists no symplectic embedding of the ellipsoid $E(a) = E(a_1, \ldots, a_n)$ into a ball $B(A)$ with $A < a_n$ (the shape type of the ellipsoid is that of a ball).

4 Flexibility

The following result shows that, if the shape type of the ellipsoids is sufficiently different, there is flexibility:

Theorem 4.1 ([6, 4])**.** *For any $a > 0$, and for a sufficiently small $\varepsilon > 0$, there exists a symplectic embedding φ such that:*

$$\varphi\left(E(\varepsilon, \ldots, \varepsilon, a)\right) \subset B(\pi).$$

There are no estimates on the size of ε, but F. Schlenk, using symplectic folding, proved:

Theorem 4.2 ([12, 13])**.** *If $\beta > 2\alpha$, there exists a symplectic embedding φ of the ellipsoid $E(r) = E(\alpha, \ldots, \alpha, \beta) \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$ into a ball $B(A)$ with:*

$$E(\alpha, \ldots, \alpha, \beta) \hookrightarrow B(A), \quad A > \frac{\beta}{2} + \alpha.$$

Remark 4.1. *This theorem has been much improved in (complex) dimension 2 ([11]). But the methods used to obtain the best embedding results do not have a straightforward generalization to higher dimensions.*

Definition 7. *An open polydisk is the set:*

$$\begin{aligned} P(a) = P(a_1, \ldots, a_n) &= B(a_1) \times \cdots \times B(a_n) \\ &= \left\{ z \,\middle|\, \pi \frac{|z_1|^2}{a_1} < 1, \ldots, \pi \frac{|z_n|^2}{a_n} < 1 \right\}, \end{aligned}$$

where we assume $a_1 \leq \cdots \leq a_n$.

A very impressive result concerning flexibility of polydisks is due to L. Guth:

Theorem 4.3 ([7])**.** *There is a dimensional constant C_n such that, given two polydisks $P(r)$ and $P(r')$, if:*

$$C_n a_1 < r'_1, \quad C_n a_1 \ldots a_n < a'_1 \ldots a'_n$$

there exists a symplectic embedding of $P(a)$ into $P(a')$.

This result has an obvious application to ellipsoids:

Example 4. *In $\mathbb{C}^3 \cong \mathbb{R}^6$, there exists a constant $K > C_3\pi$ such that:*

$$E(\pi, a, a) \hookrightarrow E\left(3K, 3K, \frac{4}{K}a^2\right) \quad a > 3K.$$

This follows from the embedding:

$$P(\pi, a, a) \hookrightarrow P\left(K, K, \frac{a^2}{C_3\pi}\right)$$

and the inclusions $E(\pi, a, a) \subset P(\pi, a, a)$ and:

$$P\left(K, K, \frac{a^2}{C_3\pi}\right) \subset E\left(3K, 3K, \frac{4}{K}a^2\right).$$

A similar result is valid in any dimension; it shows that if the shape type of the ellipsoid is sufficiently different from that of a ball ($a > 3K$ above) then there exists considerable flexibility and the relevant obstructions are (derived from) just the first capacity and the volume.

Capacities (in general) involve the 2-dimensional area of some object; volume can considered a generalized capacity and is $2n$-dimensional. It is natural to search for intermediate capacities that involve $2k$-dimensional volumes; it follows from the results of [7] that there are no reasonably continuous intermediate capacities.

Symplectic folding is described in [9, 10, 12, 13]; we shall use a slightly different version [1], but the very careful and detailed presentation in [12, 13] should be considered for all technical aspects.

We define $T(a, b)$ as the set:

$$\begin{aligned} T(a, b) = \Big\{ & (z_1, z_2) = (u_1, v_1, u_2, v_2) \in \mathbb{R}^4 \,\Big| \\ & (u_1, v_1) \in]0, a[\times]0, 1[, \quad (u_2, v_2) \in]0, b[\times]0, 1[\\ & \frac{u_1}{a} + \frac{u_2}{b} < 1 \Big\} \end{aligned}$$

and $T(a) = T(a, a)$. The projection of $T(a, b)$ on the (u_1, u_2) plane is a triangle and the fibres are the unit square.

Lemma 4.4 ([12, 13])**.** *Assume $\varepsilon > 0$. Then:*

1. *$E(a, b)$ symplectically embeds into $T(a + \varepsilon, b + \varepsilon)$.*
2. *$T(a, b)$ symplectically embeds into $E(a + \varepsilon, b + \varepsilon)$.*

Sketch of the proof. The main fact involved in the proof is the existence of an area preserving map $(u, v) = \sigma(z)$ in the plane [12, 13] that, outside an arbitrarily small neighbourhood of the origin, where it is a translation, essentially takes open circles of area a into open rectangles $]0, a[\times]0, 1[$ (figure 1).

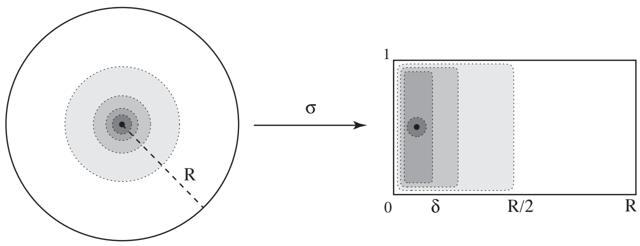


Figure 1: *Area preserving map in the plane*

Let $D(a)$ be the disk of area a; then:

$$E(a, b) = \left\{ z \,\middle|\, z_1 \in D(a), \; z_2 \in D\left(b(1 - \pi|z_1|^2/a)\right) \right\}$$

The symplectic embedding of E into T is then:

$$(z_1, z_2) \mapsto ((u_1, v_1), (u_2, v_2)) = (\sigma(z_1), \sigma(z_2))$$

The inverse of this map is used to embed T into E. $\square$

Here and subsequently we ignore everything 'small': an arbitrary small δ is involved in the construction of σ, we should therefore consider maps σ_δ with sufficiently small δ, but it is easier to proceed as if δ could be zero.

It follows from lemma 4.4 that embedding results for ellipsoids can be obtained from the corresponding results for sets of the form $T(a,b)$, and we describe symplectic folding for these sets in section 5. Figure 2 summarises the process (cf. figure 3.13 in [12]).

Since U embedding symplectically into V is equivalent to λU embedding symplectically into λV for $\lambda \neq 0$, we normalize the ellipsoids $E(a)$, and therefore the sets T, so that $a_1 = \pi$. In the figures we really represent $T(a,\pi)$ instead of $T(\pi,a)$, as in [12].

Theorem 4.5 ([1])**.** *If the ellipsoid $E(r) = E(r_1, r_2)$ in $\mathbb{C}^2 \cong \mathbb{R}^4$ has shape type $k \geq 3$ with:*

$$3 \leq k < r_2/r_1 < k+1$$

there exists a symplectic embedding φ such that $\varphi(E(r)) \subset E(r')$ with:

$$r_2 > r_2' \quad \textit{and} \quad n \leq \frac{r_2'}{r_1'} < n+1$$

for all shape types $n = 1, \ldots, \left[\dfrac{2k}{3}\right]$.

Proof. We consider the normalised ellipsoid $E(\pi, a)$, with $k\pi < a < (k+1)\pi$ and $k \geq 3$. Symplectic folding gives an embedding (figure 2):

$$T(\pi,a) \hookrightarrow T\left(\frac{a}{2} + \pi + \varepsilon\right)$$

and lines above the image of $T(\pi,a)$ in the $(u_1', u_2')-$plane correspond to sets $T(\alpha,\beta)$ into which $T(\pi,a)$ embeds; $(\alpha,0)$ and $(0,\beta)$ are the intersections of the line with the coordinate axes.

Going from T-type sets to ellipsoids:

$$E(\pi,a) \hookrightarrow E\left(\frac{3}{2}\pi + \varepsilon, \frac{3}{4}(a+\pi) + \varepsilon\right),$$

with:

$$\frac{3}{4}(a+\pi) < a \Longleftrightarrow k \geq 3.$$

The same construction also gives an embedding:

$$E(\pi,a) \hookrightarrow B\left(\frac{a}{2} + \pi + \varepsilon\right)$$

and clearly embeddings for all in between shape types. For any b such that:

$$\frac{3}{4}(a+\pi) < b < a$$

there is a trivial embedding (again see figure 2):

$$E\left(\frac{3}{2}\pi + \varepsilon, \frac{3}{4}(a+\pi) + \varepsilon\right) \hookrightarrow E\left(\frac{3}{2}\pi + \varepsilon, b\right)$$

and the shape type can thus be extended up to $\left[\dfrac{2k}{3}\right]$. □

Open Question ([12, 13])**.** *Does the ellipsoid $E(a, 2a, 3a)$ symplectically embed into $B(A)$ for some $A < 3a$?*

Ekeland-Hofer capacities show that:

- $E(a, 3a, \ldots, 3a)$ does not symplectically embed into a ball $B(A)$ with $A < 3a$.
- $E(a, 2a, \ldots, 2a, 3a)$ does not symplectically embed into a ball $B(A)$ with $A < 2a$.

On the other hand, there is also some flexibility, as it follows from theorem 4.2 that:

$$E(a,3a) \hookrightarrow B\left(\frac{5}{2}a + \varepsilon\right)$$

The change introduced in the symplectic folding process allows estimates (lemma 4.7) that are decisive in the proof of:

Theorem 4.6 ([1])**.** *For any positive ε, there exists a symplectic embedding:*

$$E(\pi, b_1, \ldots, b_{n-2} = b, a) \hookrightarrow B(A+\varepsilon), \quad A < a$$

when $a > b + \pi$, with A given by:

$$A = \frac{a+b+\pi}{2}.$$

Remark 4.2. *For $n = 3$, $b = 2\pi$, $a = 3\pi$:*

$$E(\pi, 2\pi, 3\pi) \hookrightarrow B(A+\varepsilon), \quad A = \frac{3\pi + 2\pi}{2} + \frac{\pi}{2} = 3\pi$$

and thus $E(\pi, 2\pi, 3\pi)$ is in the boundary of (known) flexibility.

Remark 4.3. *$b = \pi$ gives theorem 3.1.1 in [12] (or theorem 4.2): for all $\varepsilon > 0$,*

$$E(\pi, \ldots, \pi, a) \textit{ symplectically embeds into } B\left(\frac{a}{2} + \pi + \varepsilon\right)$$

Lemma 4.7 ([1])**.** *For any $\varepsilon > 0$, symplectic folding gives an embedding $\psi : T(\pi,a) \hookrightarrow \mathbb{C}^2 \cong \mathbb{R}^4$:*

$$\psi((u_1,v_1),(u_2,v_2)) = ((u_1',v_1'),(u_2',v_2'))$$

such that

$$u_1' + u_2' < A - b + \frac{b}{\pi}u_1 + \frac{b}{a}u_2 + \varepsilon, \quad A = \frac{a+b+\pi}{2}.$$

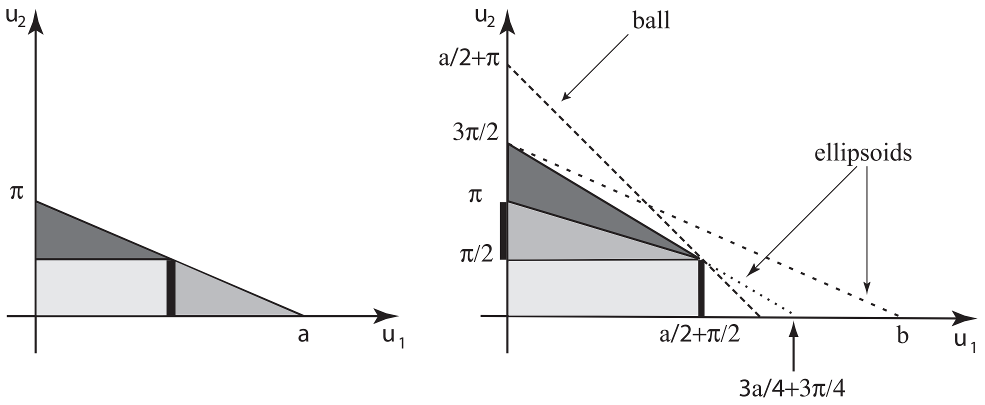


Figure 2: *Scheme of symplectic folding in the (u_1, u_2) plane*

Theorem 4.6 follows from lemmas 4.7 and 4.8:

Lemma 4.8 ([1]). *If for any positive ε there exists a symplectic embedding $\psi : T(\pi, a) \hookrightarrow \mathbb{C}^2 \cong \mathbb{R}^4$:*

$$\psi((u_1, v_1), (u_2, v_2)) = ((u_1', v_1'), (u_2', v_2'))$$

such that:

$$u_1' + u_2' < A - b + \frac{b}{\pi} u_1 + \frac{b}{a} u_2 + \varepsilon$$

then there exists a a symplectic embedding Φ:

$$E(\pi, b_1, \dots, b_{n-2} = b, a) \hookrightarrow B(A + \varepsilon)$$

Proof. It follows from lemma 4.4 and the estimate on ψ that there exists a symplectic embedding σ:

$$\sigma : E(\pi, a) \hookrightarrow \mathbb{C}^2 \cong \mathbb{R}^4, \qquad \sigma(z_1, z_2) = (z_1', z_2')$$

such that:

$$\pi |z_1'|^2 + \pi |z_2'|^2 < A - b + \frac{b}{\pi} \pi |z_1|^2 + \frac{b}{a} \pi |z_2|^2 + \varepsilon$$

with:

$$A = \frac{a + b + \pi}{2}.$$

Then $\sigma \times \mathrm{id}_{n-2}$, after a suitable permutation τ, defined by $\tau(z_1, z_2, \dots) = (z_1, z_n, z_2, \dots)$, gives the desired symplectic embedding:

$$\Phi = (\sigma \times \mathrm{id}_{n-2}) \circ \tau : E(\pi, b_1, \dots, b_{n-2}, a) \hookrightarrow \mathbb{C}^n \cong \mathbb{R}^{2n}$$

□

5 Symplectic folding

Step 1: We separate the region $u_2 > \pi$ from the region $u_2 < \pi$, the large fibres from the small ones: here the fibres are related to the projection on the (u_1, v_1) plane, and the symplectic map is the product $\varphi \times \mathrm{id}$ of an area preserving map φ in the (u_1, v_1) plane (figure 3) and the identity on the (u_2, v_2) plane.

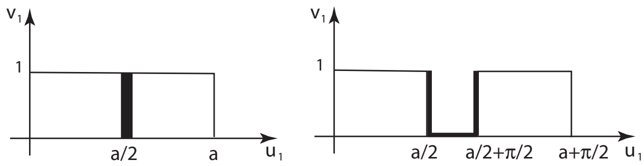


Figure 3: *Separating the fibres: black regions have the same area*

Remark 5.1. *Again we should consider the regions $u_2 > b/2 + \delta$ and $u_2 < b/2 - \delta$ and deform $b/2 - \delta < u_2 < b/2 + \delta$ for a conveniently small δ (the black region); the map outside that region is the identity on the left and a translation on the right.*

The result can also be seen in the (u_1, u_2) plane:

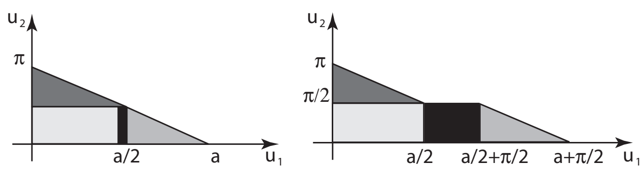


Figure 4: *Separating the fibres, (u_1, u_2) plane*

Remark 5.2. *The (u_1, v_1) and (u_2, v_2) planes are symplectic, the symplectic form on them is an area form, and it is convenient to preserve the area on them; but the plane (u_1, u_2) is Lagrangean, the symplectic form vanishes on it and therefore no area preserving on that plane is involved.*

Step 2: We rearrange the fibres: the symplectic map is the product of an area preserving map σ_1 in the (u_2, v_2) plane (figure 5), and the identity on the (u_1, v_1) plane.

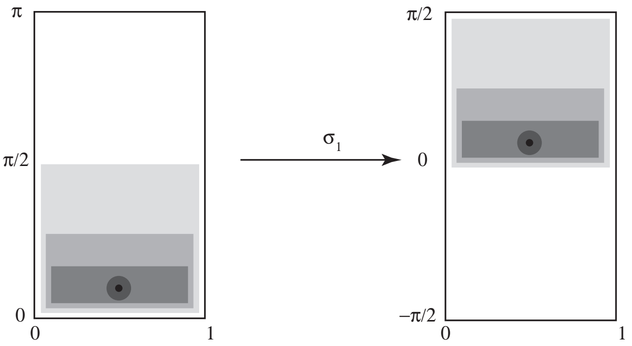


Figure 5: *Rearranging the fibres in the (u_2, v_2) plane*

The result can again be seen in the (u_1, u_2) plane, the top triangle goes to the bottom:

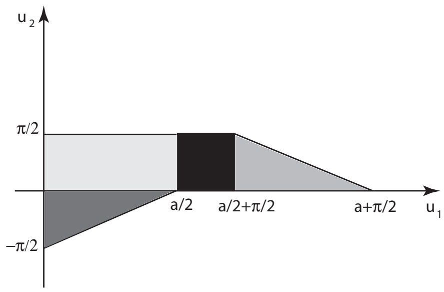


Figure 6: *Rearranging the fibres, in the (u_1, u_2) plane*

Step 3: We lift the region $a/2 + \pi/2 < u_1 < a + \pi/2$ by $\pi/2$ along the u_2 direction. Now the symplectic map is not a product of area preserving maps: its action can be seen in the (u_1, u_2) and (u_1, v_1) planes (figure 7), but we refer to [12] for the construction of the lift map.

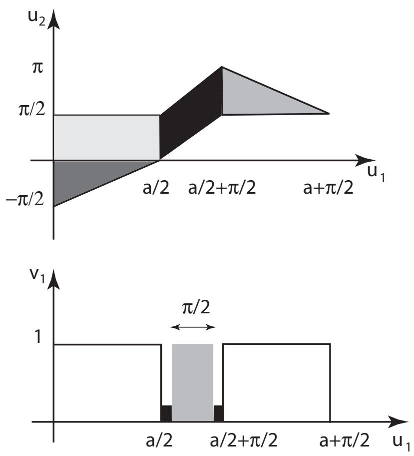


Figure 7: *Lifting*

The grey region in the plane (u_1, v_1) is the projection on that plane of points lifted less than $\pi/2$, and more than 0, and has area bigger than $\pi/2$.

Step 4: We contract along the v_1 direction, and extend along the u_1 direction, by $a/(a+\pi)$, keeping (u_2, v_2) unchanged (figure 8); again this is the product of an area preserving map on the (u_1, v_1) plane and the identity on the (u_2, v_2) plane.

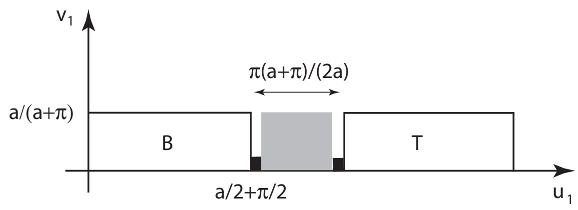


Figure 8: *Rearranging in the (u_1, v_1) plane*

Step 5: We now turn T over B: we extend the grey area, then we fold twice in the base (figure 9).

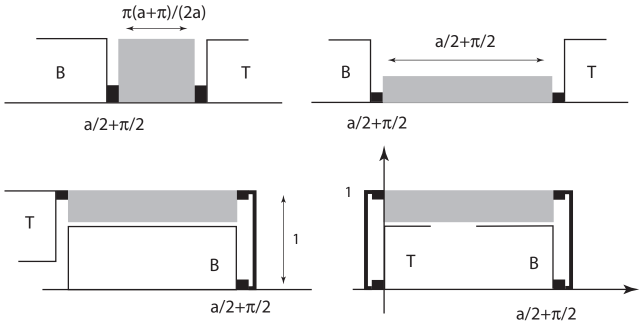


Figure 9: *Folding in the (u_1, v_1) plane*

The transformation of the grey area (in the (u_1, v_1) plane) is as in the previous step, with a factor of π/a now, but using the identity outside that area on the left and a translation on the right. The end result in the (u_1, u_2) plane is:

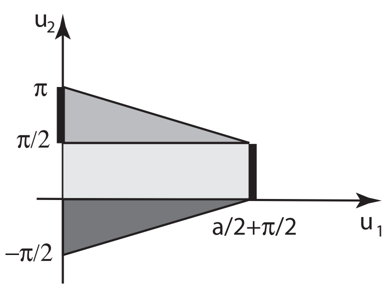


Figure 10: *Folding in the (u_1, u_2) plane*

Step 6: We rearrange the fibres in the (u_2, v_2) plane again:

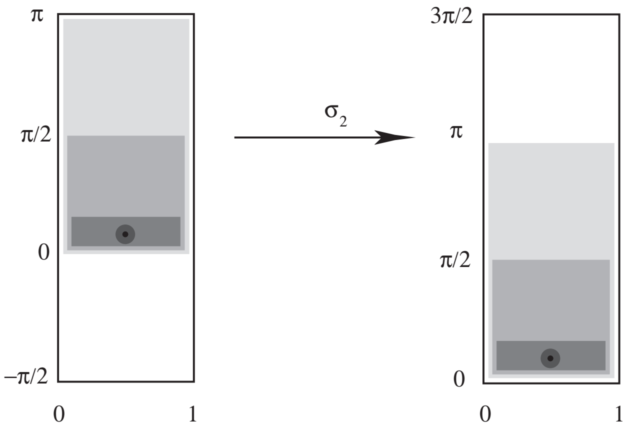


Figure 11: *Rearranging the fibres in the* (u_2, v_2) *plane*

The symplectic map is the product of an area preserving map σ_2 in the (u_2, v_2) plane (figure 11), and the identity on the (u_1, v_1) plane. Seen in the (u_1, u_2) plane, the bottom triangle goes to the top (figure 2).

The symplectic folding construction is summarised in figure 2 (it should be compared to figure 3.13 in [12]): the advantage of the change relative to [12, 13] is that we can get embeddings into ellipsoids, keeping the same estimates obtained for embeddings into balls.

Bibliography

[1] J. Basto-Gonçalves, *Symplectic rigidity and flexibility of ellipsoids*, preprint 31, CMUP 2009

[2] I. Ekeland, H. Hofer, *Symplectic topology and Hamiltonian dynamics*, Math. Z. 200 (1989), 355-378.

[3] I. Ekeland, H. Hofer, *Symplectic topology and Hamiltonian dynamics II*, Math. Z. 203 (1990), 553-567.

[4] A. Floer, H. Hofer, K. Wysocki, *Applications of symplectic homology I*, Math. Z. 217 (1994), 577-606.

[5] M. Gromov, *Pseudo-holomorphic curves in symplectic manifolds*, Inv. Math., 82 (1985), 307-347.

[6] M. Gromov, *Partial Differential Relations*, Ergebniss der Math. 9, Springer, 1986.

[7] L. Guth, *Symplectic embeddings of polydisks*, Inv. Math., 172 (2008), 477-489.

[8] H. Hofer, E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhauser Advanced Texts, Birkhauser, 1994.

[9] F. Lalonde, D. McDuff, *The geometry of symplectic energy*, Ann. of Math., 141 (1995), 349-371.

[10] F. Lalonde, D. McDuff, *Hofer's L^∞-geometry: energy and stability of Hamiltonian flows II*, Inv. Math. 122 (1995), 35-69.

[11] D. McDuff, *Symplectic Embeddings and Continued fractions: a survey*, Japanese J. Math. 4 (2009), 121-129

[12] F. Schlenk, *Embedding Problems in Symplectic Geometry*, De Gruyter Expositions in Mathematics 40, Walter de Gruyter, 2005.

[13] F. Schlenk, *Symplectic embedding of ellipsoids*, Israel J. Math. 138 (2003), 215-252.

On GAeL: Géometrie Algébrique en Liberté XVIII

Margarida Melo

Department of Mathematics
University of Coimbra, Portugal
mmelo@mat.uc.pt

Abstract

Géometrie Algébrique en Liberté is a school and conference organized by and for algebraic geometers in the beginning of their scientific careers. The 18th edition of GAeL took place in the Mathematics Department of the University of Coimbra, Portugal, on June 7-11 2010. It gathered together about 70 participants coming from whole parts of the world who got the opportunity to learn and discuss together and "en Liberté" the most recent developments in this area of research. The senior speakers for this year were the Professors Olivier Debarre (C.N.R.S.-Paris), Gerard van der Geer (Amsterdam) and Bernd Sturmfels (Berkeley).

Algebraic Geometry

Algebraic Geometry is the branch of mathematics that consists of the study of algebraic varieties: geometric incarnations of solutions of systems of polynomial equations. The simplicity and generality of this idea, that involves a big number of mathematical objects, allowed the concept to grow in many different directions developing an amazingly rich theory. It is nowadays a wide area of mathematics that combines tools from many different disciplines as Abstract Commutative Algebra, Number Theory, Complex Analysis, Differential and Complex Geometry, Algebraic Topology, Category Theory, Homological Algebra, Algebraic Combinatorics and Representation Theory to study problems arising from Geometry. The following words by the fields medalist David Mumford reflect how rich and complex are the ideas and problems that appear in this area of pure mathematics.

Algebraic geometry seems to have acquired the reputation of being esoteric, exclusive, and very abstract, with adherents who are secretly plotting to take over all the rest of mathematics. In one respect this last point is accurate. David Mumford

Even if it is one of the most classical areas of pure mathematics, it is certainly one of the most active as well. The importance of this area of research in the global context of mathematical sciences is clear, for instance, by the strong presence of algebraic geometers among the speakers of the next ICM conference (http://www.icm2010.org.in/about-icm-2010/) that will take place in Hyderabad, India on August 19-27 2010.

The great developments of Algebraic Geometry in the last decades are also clear by the list of the field medalist winners: about a quarter of the total number of field medals so far was delivered to algebraic geometers. The field medalists Kodaira, Serre, Atiyah, Grothendieck, Hironaka, Bombieri, Mumford, Deligne, Yau, Donaldson, Faltings, Drinfel'd, Mori, Witten, Kontsevich, Lafforgue and Okounkov got their award by their contributions in algebraic geometry or closely related areas. To these we should also add the name of Andrew Wiles, who got a "Special Tribute" of the Fields Institute for his proof of the famous "Fermat's last theorem" which relies upon methods from Algebraic Geometry, namely elliptic curves and modular forms.

One can say that the roots of Algebraic Geometry date back to the arabian and greek mathematicians, who often used the geometry of plane curves and their intersection properties to solve algebraic equations. The same kind of ideas were also used much later by some mathematicians in the renaissance period like Cardano and Tartaglia while studying the cubic equation. However, after a long period of great development of analytical methods in Geometry, the systematic use of algebraic methods in Geometry was established only at the end of the 19th century by the italian school of Algebraic Geometry composed by mathematicians like Enriques, Chisini, Castelnuovo and Segre. The foundations of Algebraic Geometry using notions from Commutative Algebra like the theory of ideals were established in the beginning of the 20th century by mathematicians like van der Waerden, Oscar Zariski and André Weil. In the 1950's and 1960's Serre and

Grothendieck used sheaf theory and techniques from Homological Algebra to introduce in Algebraic Geometry the notion of scheme. This led to an enormous transformation in the whole theory: classical Projective Geometry was more concerned with the geometric notion of point while the later emphasizes the concepts of regular function and regular map. Moreover, this new point of view provided Algebraic Geometry with tools to treat a wide number of problems from other areas of mathematics like Commutative Algebra and Algebraic Number Theory (recall for instance elliptic curve cryptography and Wiles' celebrated proof of Fermat's last theorem). It also allowed to solve classical problems on algebraic varieties like moduli problems or resolution of singularities. Nowadays, Algebraic Geometry is again in great transformation after the introduction of stacks by Grotendieck, Deligne, Mumford and Artin and, more recently, by the development of the theory of Derived Algebraic Geometry.

Even if Algebraic Geometry is fundamentally a rather abstract discipline, recently it has been heavily used in other rather applied areas of mathematics like statistics, control theory, robotics and also in other sciences. For instance, it is fundamental in the development of the physics' theory of strings and it has deep connections with the Phylogenetics theory of biology.

GAeL origins

The origins of "GAeL: Géometrie Algébrique en Liberté" date back to France, as one can guess by its name. In fact, the first 13th editions of GAeL took place at the "CIRM: Centre International des Rencontres Mathématiques" (Marseille, France). However, nowadays all communication and talks are in English. Most recently, GAeL has taken place in in Bedlewo (Poland), Istanbul (Turkey), Madrid (Spain), and Leiden (The Netherlands). This year edition was the 18th GAeL conference and it took place in the Mathematics Department of the University of Coimbra, Portugal. GAeL has been, since its beginning, a reference meeting for young algebraic geometers specially from European countries.

As the name indicates, the aim of GAeL is to give young algebraic geometers the opportunity to learn and discuss the most recent developments of Algebraic Geometry in a relaxed atmosphere with no fear to ask questions of any type. Young participants have the opportunity to lecture, often for the first time, in front of an international audience. At the same time, senior experts deliver courses in selected topics at the cutting edge of modern and classical Algebraic Geometry. Among the speakers of previous editions of GAeL are Batyrev, Beauville, Bridgeland, Campana, Ciliberto, Colliot-Thélène, Corti, Demailly, Faber, Fantechi, Farkas, Hassett, Huybrechts, Itenberg, Izvoskiy, Lehn, Manivel, Mukai, Muller-Stäch, Mustaţa, Okonek, Oort, Oxbury, Peskine, Reid, Siebert, Sottile, Teissier, Thomas, Tyurin, Vakil, Van Straten, Vistoli, Voisin, and Zak.

Between the senior speakers' mini-courses and the junior speakers talks there is also time for discussions and for exercise sessions. The junior participants who do not give a talk must present their work on the poster session at the beginning of the conference. Posters remain available for the rest of the week so that at the end everybody gets the opportunity to learn about each others own research. For that reason the number of participants of GAeL is quite limited: each person can participate at most in two editions of GAeL.

The organizing committee is also made of young researchers in Algebraic Geometry. Each organizer will organize GAeL for two years: in each edition organizers with some experience help the beginning ones. This year organizing committee was Víctor González Alonso (Universitat Politècnica de Catalunya, Nathan Ilten (Freie Universität, Berlin), Pedro Macias Marques (Universidade de Évora), Margarida Melo (Universidade de Coimbra), Kaisa Taipale (University of Minnesota) and Filippo Viviani (Università Roma Tre and Universidade de Coimbra).

There is also a scientific committee that helps the organizing committee in aspects like the choice of topics or funding advice. The actual scientific committee for GAeL is composed by Professor Frances Kirwan (Oxford), Professor Yuri Manin (Max-Planck Institute für Mathematik) and by Professor Farns Oort (Utrecht).

This year's GAeL

This year edition of GAeL, the XVIII, took place in the Mathematics Department of the University of Coimbra, Portugal, on June 7-11. It was possible thanks to the financial support of the Foundation Compositio Mathematica, the Center for Mathematics of the University of Coimbra (CMUC), the International Center for Mathematics (CIM), the Foundation for Science and Technology of Portugal (FCT), the FCT project "Geometria Algébrica em Portugal" and of the Mathematics Department of the University of Coimbra.

As in the previous editions of GAeL, the program of the conference consisted of 3 mini-courses of 4 hours each delivered by selected experts in different areas of Algebraic Geometry and of 25 contributed talks by the junior participants. The poster session was organized in the first afternoon of the program: we had this year almost 30 posters presented by participants at different stages of their careers. The talks and posters presented by the junior participants where quite various: there were talks on moduli spaces of curves and sheaves, moduli spaces of surfaces, moduli spaces of abelian varieties, toric geometry, tropical geometry, derived algebraic geometry, arithmetic geometry, deformation theory, singularity theory, minimal resolutions among others. The program of the conference as well as a list of participants and abstracts can be found in http://severian.mit.edu/gael/files/quickschedule.pdf. This diversity of subjects allowed the young participants to learn from their colleagues several recent achievements and ongoing projects in this vast discipline and also favored discussions and questions in a GAeL flavor: always "en Liberté"!

The senior speakers of this edition of GAeL were the Professors Olivier Debarre (http://www.math.ens.fr/~debarre/) from the C.N.R.S. -École Normale Supérieure of Paris, Gerard van der Geer (http://www.science.uva.nl/~geer/) from the University of Amsterdam, The Netherlands, and Bernd Sturmfels (http://math.berkeley.edu/~bernd/) from the University of Berkeley, California.

Professor Olivier Debarre lectured on "Rational curves on algebraic varieties", a very classical argument that had important developments on the last few years yielding important contributions to the recent spectacular advances towards the proof of the minimal model program (see [1]). One of the most classical arguments of algebraic geometry concerns the classification of algebraic varieties. The most classical case is the classification of algebraic curves, which was already understood by Riemann in the 19th century. The classification of algebraic surfaces is more intricate and was one of the biggest achievements of the italian school of the beginning of the 20th century. After several decades without significant progress towards the higher dimensional case, S. Mori proposed a program, the so-called "minimal model program", which would lead to the classification of algebraic varieties in any dimension. The presence of rational curves on algebraic varieties detects several important information on birational invariants of the same varieties (see [4]). Professor Debarre lectures were mainly on the contributions of S. Mori himself and later of J. Kollár on this part of the program. Professor Debarre notes for this course are available at http://www.math.ens.fr/~debarre/NotesGAEL.pdf.

Professor van der Geer delivered a course on "Algebraic cycles on abelian varieties". Abelian varieties are among the most studied objects in Algebraic Geometry due to their extremely interesting and rich properties: they combine the structure of a projective variety with the structure of a compact algebraic group (see, for instance, [2] or [3]). The study of algebraic cycles on varieties is closely related to the famous "Hodge conjecture", which is one of the millennium problems. Moreover, the case of abelian varieties is commonly considered to be a crucial test case towards the validity or disproval of this famous conjecture. Among abelian varieties the so called jacobian varieties are of particular interest. Recent conjectures relate classical aspects of the geometry of algebraic curves, namely the study of their linear series which is the subject of Brill-Noether theory, with the existence of cycles on their associated jacobian varieties. Professor van der Geer lectures culminated in this important aspect of this theory.

Professor Sturmfels' lectures were on a new branch of algebraic geometry that lies on the border line between Algebraic Geometry and Optimization Theory: "Convex algebraic geometry". This new area of mathematics arises from the necessity of studying certain convex objects arising from linear and/or semidefinite programing. These objects seem particularly featured to be studied with tools from algebraic geometry. Professor Sturmfels lectures were centered on several examples that are crucial to understand the foundations of this new discipline in an attempt to systematically study such convex objects. References as well as the slides of Professor Sturmfels lectures are available at http://severian.mit.edu/gael/sturmfels.html.

To relax a bit of such an intense program there were also some social activities scheduled. Monday June 7th there was a visit to "Pátio das Escolas" and the historical buildings of the University of Coimbra. There was also a small reception while the poster session was held. On June 10th afternoon a visit to "Conimbriga" was organized and, after that, the participants were transported to Figueira da Foz were the social dinner was held.

Even if algebraic geometry is a very classical subject of mathematics there were not many portuguese mathematicians working on it until recently. However things are changing rapidly and a proof of this is the fact that there were more then 10 portuguese junior participants. Most of these obtained their PhD in Algebraic Geometry either in Portugal or abroad in the last 3 years. Most algebraic geometers in Portugal participate in the project "Algebraic Geometry in Portugal", run by Margarida Mendes Lopes, that aims to promote a strong portuguese Algebraic Geometry community. Already in this century, there have been several other events in Portugal related to algebraic geometry. among which we recall the "IST courses on algebraic geometry" that are regularly organized by Margarida Mendes Lopes: there were already 5 editions, the "Oporto meeting on Geometry, Topology and Physics", the "Geometry in Lisbon Summer school", the 2010 edition of "VBAC: Vector bundles on algebraic curves", the "Lisbon Summer Lectures in Geometry" in IST (Lisbon) and the "Coimbra-Salamanca Algebraic Geometry seminar", whose first edition was held in Coimbra in February 2010. On the "GA-P: Geometria Algébrica em Portugal" webpage, http://home.utad.pt/~ga-p/index.html, run by Carlos Rito from UTAD, is possible to find updated information on people working in Algebraic Geometry in Portugal, on events in Algebraic Geometry either in Portugal and abroad as well as several useful links and other information concerning this beautiful discipline, that is creating solid roots in Portugal as well.

Bibliography

[1] C. Birkar, P. Cascini, C. D. Hacon, J. McKernan: *Existence of minimal models for varieties of log general type*. J. Amer. Math. Soc. 23 (2010), 405-468.

[2] G. van der Geer, B. Moonen: Abelian varieties, Book in preparation, http://staff.science.uva.nl/~bmoonen/boek/BookAV.html

[3] C. Birkenhake, H. Lange: Complex abelian varieties. Second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 302. Springer-Verlag, Berlin, 2004.

[4] J. Kollár: Rational curves on algebraic varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 32. Springer-Verlag, Berlin, 1996.

Figure 12: *Groupe Picture,* On GAeL: Géometrie Algébrique en Liberté XVIII.

Summer School and Workshop on Imaging Sciences and Medical Applications

Isabel M. Narra Figueiredo

Department of Mathematics,
University of Coimbra, Portugal
http://www.mat.uc.pt/~isabelf

The *Summer School and Workshop on Imaging Sciences and Medical Applications* was an initiative of the UTAustin|Portugal Program, for Mathematics, in partnership with CIM (Center for International Mathematics). It took place at the Department of Mathematics at the University of Coimbra Faculty of Sciences and Technology, in Coimbra, Portugal, on June 15-23, 2010. This event had also the scientific support of CMUC (Centre for Mathematics, University of Coimbra), and two Portuguese medical associations, the *Brain Imaging Network* and the *Society of Digestive Endoscopy.*

The choice of the topic (and, *a posteriori*, its location) was motivated by the fact that, currently, we have a research project (reference UTAustin/MAT/0009/2008), in the framework of the UTAustin|Portugal Program (for Mathematics), and one of the project main subjects is precisely image processing of medical images, more exactly, endoscopic images in gastroenterology. Moreover, this *Summer School and Workshop on Imaging Sciences and Medical Applications* was also, in some sense, a natural consequence (and a continuation) of the *Workshop on Mathematical Aspects of Imaging, Modeling and Visualization in Multiscale Biology*, in which we were directly and strongly involved, and that took place at the ICES (Institute for Computational Engineering and Sciences), of the University of Texas, at Austin, USA, from March 31st to April 4th 2009.

The main goal of the *Summer School and Workshop on Imaging Sciences and Medical Applications* was, obviously, to promote new collaborations, to exchange and share new ideas and scientific results, and simultaneously, to give an opportunity to PhD students and young researchers for improving their scientific knowledge, in such a complex area as imaging sciences, which has strong interdisciplinary features.

The *Summer School* featured five excellent short courses, each one with the duration of five hours, presented by brilliant speakers, experts in imaging sciences: *Image segmentation*, by Sung Ha Kang (Georgia Institute of Technology, Atlanta, USA), *Flexible algorithms for image registration*, by Jan Modersitzki (McMaster University, Canada), *Image reconstruction in tomography*, by Alfred K. Louis (Saarland University, Germany), *Highly accurate image restoration and matching*, by Andrés Almansa (Télécom Paris Tech, France), and *Variational models in image inpainting*, by Selim Esedöglu (University of Michigan, USA).

In the *Workshop* there were nine plenary lectures, with a predominance of Portuguese guest speakers : *Interest point detection and matching for 3D reconstruction in medical endoscopy*, by João Pedro Barreto (University of Coimbra, Portugal), *Unmixing of positive sources in hyperspectral imaging*, by José Bioucas (Instituto Superior Técnico, Lisbon, Portugal), *From models of brain function to clinical applications: new challenges in neuroimaging*, by Miguel Castelo-Branco (University of Coimbra, Portugal), *CAGE - Computer assisted gastroenterology examination*, by Miguel Coimbra (University of Porto, Portugal), *A combinatorial point of view for non-linear evolutions*, by Jérôme Darbon (Ecole Normale Supérieure de Cachan, France), *Removing non-additive noise using variable splitting and augmented lagrangian optimization*, by Mário Figueiredo (Instituto Superior Técnico, Lisbon, Portugal), *Spatially adapted regularization in total variation based image restoration*, by Michael Hintermüller (Humboldt-University of Berlin, Germany), *New trends in photogrammetry and computer vision applied to 3D city modeling and culturage heritage*, by Marc Pierrot-Deseilligny (Laboratoire MATIS, IGN, France), and *Tracking moving objects in image sequences*, by João Manuel R. S. Tavares (University of Porto, Portugal). The *Workshop* also included four sessions of contributed talks and one poster session, which gave the possibility to young researchers to report their on going work and results.

A broad audience of sixty participants attended this event. It included mathematicians, electrical and computer engineers, mechanical engineers, biomedical engineers, geographical engineers, computer scientists and a neuroscientist.

This was a remarkable event, with distinguished guest speakers, who have strongly contributed to a top level scientific atmosphere, promoting and encouraging interactions and collaborative research among all the participants.

[Note - For more information visit the event website http://www.mat.uc.pt/~isma2010/?menu=home]

Coming Events

July, 05 and 07, 2010: Pedro Nunes Lectures, by Maxim Kontsevich. Please see last page and

http://www.cim.pt/?q=glocos-pedronunes

July, 07, 2010: Jornada Matemática SPM/CIM on "Trends in Quantum Geometry"

Rio Tejo, Portugal.

Organizer

Ricardo Schiappa (CAMGSD/IST)

Aims

This Jornada SPM/CIM takes place on the occasion of the Pedro Nunes Lectures in Portugal by Professor Maxim Kontsevich (1997- Henri Poincaré Prize, 1998 - Fields Medal, 2008 - Crafoord Prize) and aims to gather young researchers, in an informal setting, to discuss recent advancements in topics related to Quantum Geometry and other aspects of Kontsevich's work (which concentrates on geometric aspects of mathematical physics, most notably on knot theory, deformation quantization and mirror symmetry). At the end of these talks a small discussion session will be held, led by Maxim Kontsevich.

Invited Speakers

J. Baptista (University of Amsterdam)

M. Cirafici (Instituto Superior Técnico)

C. Rossi (Instituto Superior Técnico)

G. Tabuada (Universidade Nova de Lisboa)

For more information about the event, see

http://www.cim.pt/?q=spm_cim_jornada_Quantum_Geometry_2010

October, 11-15, 2010: Educational Interfaces between Mathematics and Industry. This conference, first scheduled to April, 2010, was postponed due to volcano Eyjafjllajokull.

Fundação Calouste Gulbenkian and Universidade de Lisboa.

Organizers

José Francisco Rodrigues (Universidade de Lisboa)

Assis Azevedo (Universidade do Minho)

António Fernandes (Instituto Superior Técnico)

Adérito Araújo (Universidade de Coimbra)

For more information about the event, see

http://www.cim.pt/eimi

July, 9-10, 2010: 8th EUROPT Workshop "Advances in Continuous Optimization"

Aveiro

Organizers

Domingos M Cardoso (Universidade de Aveiro)

Tatiana Tchemisova (Universidade de Aveiro)

Miguel Anjos (University of Waterloo)

Edite Fernandes (Universidade do Minho)

Vicente Novo (Univ. Nac. Educación a Distancia)

Juan Parra (Universidad Miguel Hernández de Elche)

Gerhard-Wilhelm Weber (Middle East Tech. Univ.)

Aims

This meeting continues in the line of the EUROPT workshops, the first held in 2000 in Budapest, followed by the workshops in Rotterdam in 2001, Istanbul in 2003, Rhodes in 2004, Reykjavik in 2006, Prague in 2007, and Remagen in 2009.

The workshop aims to provide a forum for researchers and practitioners in continuous optimization and related fields to discuss and exchange their latest works.

Invited Speakers

Immanuel M. Bomze (University of Vienna)

Mirjam Dür (University of Groningen)

Alexander Shapiro (Georgia Tech)

Tamás Terlaky (Lehigh University)

Luís Nunes Vicente (University of Coimbra)

Henry Wolkowicz (University of Waterloo)

For more information about the event, see

http://www.europt2010.com/

July, 18-31, 2010: MatCampus2010

Braga & Santiago de Compostela

Organizers

María Victoria Otero Espinar, María Elena Vázquez Abal, Rosa María Crujeiras Casais, Pilar García Agra, Rafael Fernández Casado, Alexandre Cortés Ayaso (Santiago de Compostela)

Paula Mendes Martins, Assis Azevedo, Cláudia Mendes Araújo, Suzana Mendes Gonçalves, Isabel Leite (Braga)

For more information about the event, see

http://www.matcampus2010.org/pt

September 26-29, 2010: Raising European Public Awareness in Mathematics

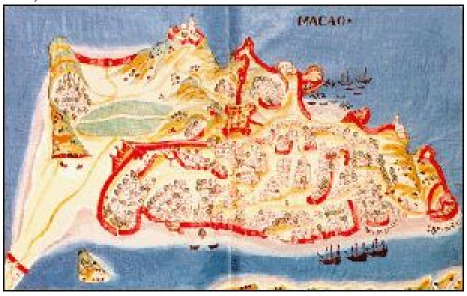
Please see first page.

For updated information on these events, see http://www.cim.pt/?q=events

---

Ten years ago CIM sponsored MACAO 2000

The international conference on MATHEMATICS AND ITS ROLE IN CIVILIZATION that took place in Macau, China on January 11-14 2000, was one of the first initiatives of the World Mathematical Year 2000 (http://www.emis.de/misc/cdrom/WMY2000/Macau/mo2000.html#about). The conference was a joint Portuguese-Chinese organisation, and was held under the advice of an international Scientific Committee, chaired by the former IMU president J.-L. Lions. It covered several communications and round tables on topics such as: Comparison of the role of Mathematics in different cultures, Exchanges and interactions of the East-West mathematical cultures, The role of Mathematics as a driving force in human progress, Contributions of Mathematics for sustainable economical, industrial and social development, Mathematical research, education and popularization in diverse cultures, and Mathematics in the future of civilization and its role in the Information Society (more details can be read in the conference report (http://wmy2000.math.jussieu.fr/9_macao.htm).



Editors: Assis Azevedo (assis@math.uminho.pt)
António Fernandes (amfern@math.ist.utl.pt).

Address: Departamento de Matemática, Universidade de Coimbra, 3001-454 Coimbra, Portugal.

The CIM Bulletin is published twice a year. Material intended for publication should be sent to one of the editors. The bulletin is available at www.cim.pt.

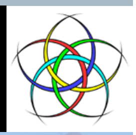
The CIM acknowledges the financial support of FCT - Fundação para a Ciência e a Tecnologia.

Centro Internacional de Matemática

PEDRO NUNES LECTURES

SOCIEDADE PORTUGUESA DE MATEMÁTICA

Maxim Kontsevich



Review of homological mirror symmetry

5th July / 15.00 h / Porto

www.fc.up.pt/cmup

Towards non-commutative integrability

7th July / 14.30 h / Lisboa

www.ciul.ul.pt

ABOUT PEDRO NUNES LECTURES

Pedro Nunes Lectures *is an initiative of* ***Centro Internacional de Matemática (CIM)*** *in cooperation with* ***Sociedade Portuguesa de Matemática (SPM)****, with the support of the* ***Fundação Calouste Gulbenkian****, to promote visits of notable mathematicians to Portugal. Each visitor is invited to give two or three lectures at Portuguese Universities on the recent developments in mathematics, their applications and cultural impact.* ***Pedro Nunes Lectures*** *are aimed to a vast audience, with wide mathematical interests, especially PhD students and youth researchers.*

Webstreaming available at:
www.cim.pt/?q=glocos-pedronunes

Professor Maxim Kontsevich work concentrates on geometric aspects of mathematical physics, most notably on knot theory, quantization, and mirror symmetry. His most famous result is a formal deformation quantization that holds for any Poisson manifold. He also introduced knot invariants defined by complicated integrals analogous to Feynman integrals.
In topological field theory, he introduced the moduli space of stable maps, which may be considered a mathematically rigorous formulation of the Feynman integral for topological string theory.
He received a Fields Medal in 1998, at the 23rd International Congress of Mathematicians in Berlin. He also received the Henri Poincaré Prize in 1997 and a Crafoord Prize in 2008.

For further information:
CENTRO INTERNACIONAL DE MATEMÁTICA
www.cim.pt

SOCIEDADE PORTUGUESA DE MATEMÁTICA
www.spm.pt

SPONSORS:

FUNDAÇÃO CALOUSTE GULBENKIAN

CMAF UL

Grupo de Física Matemática da Universidade de Lisboa

CMUP Centro de Matemática Universidade do Porto

CAMGSD