On elliptic equations with superlinear nonlinearities

Miguel Ramos

University of Lisbon, CMAF - Faculty of Science
Av. Prof. Gama Pinto 2, 1649-003 Lisboa, PORTUGAL

mramos@ptmat.fc.ul.pt

Abstract

We survey some results on the solution set of equations of the form $-\Delta u = |u|^{p-2}u + f(x)$ $(p > 2)$ with Dirichlet boundary conditions on a smooth bounded domain of $\mathbb{R}^N$, from the point of view of the calculus of variations and critical point theory. We focus on the so called *perturbation from symmetry* problem.

1. Introduction

In the following Ω is a smooth bounded domain of $\mathbb{R}^N$. We shall be concerned with equations of the form

$$-\Delta u = g(u) \ \text{ in } \Omega, \quad u = 0 \text{ on } \ \partial\Omega,$$

where $g \in C^1(\mathbb{R};\mathbb{R})$ is *superlinear* in the sense that $g(s)/s \to +\infty$ as $|s| \to \infty$. The model nonlinearity is the homogeneous function

$$g(s) = |s|^{p-2}s \qquad \text{with } p > 2.$$

Classical methods based upon fixed point theorems do not apply easily to this problem because there are no a priori bounds for the solutions.

In the one-dimensional case $N = 1$, one can apply the shooting method to the ODE, and the existence of an unbounded sequence $(u_k)_k$ of solutions can be proved; moreover, the number of their nodal domains increases with k.

However, numerous open problems subsist in the case $N > 1$; among others, they concern the existence of solutions, the uniqueness in a prescribed class of functions (positive solutions, ground-state solutions, radially symmetric solutions, etc), their possible symmetry, the sign of the solutions as well as the number of their nodal domains.

In Section 2 we list a number of known results for the above problem. This list is not intended to be exhaustive but rather to provide the reader a flavor of the state of art. Since we aim mostly at the discussion in Section 4, we do not include complete references in Section 2; these can be found e.g. in [10, 31]. In Section 3 we comment briefly on the most fruitful framework that has been used so far to prove a number of such results. Section 4 is devoted to a special case where, although some basic questions still remain unsolved, new results have been obtained recently. Hereafter we will restrict our attention to the case $N \geqslant 3$; the critical Sobolev exponent $2^* := 2N/(N-2)$ will play an important role. Also, in order to keep the paper simple, we do not attempt in presenting the results in its most general form.

2. Some known results

Given $p \in \mathbb{R}$, $p > 2$, consider the problem

$$-\Delta u = |u|^{p-2}u \ \text{ in } \Omega, \quad u = 0 \text{ on } \ \partial\Omega. \tag{2.1}$$

In case $p < 2^*$, there is a compact embedding of the Sobolev space $H_0^1(\Omega)$ into $L^p(\Omega)$. Using various methods one can prove that (2.1) admits a solution different from the trivial one $u \equiv 0$; in fact, a positive solution $u > 0$.

The situation is different if $p \geqslant 2^*$. For example, if Ω is star-shaped and $p \geqslant 2^*$ then (2.1) has no solutions.

While if Ω has a nontrivial homology ($H_k(\Omega;\mathbb{Z}_2) \neq 0$ for some $k \geqslant 1$) and $p = 2^*$, then a positive solution does exist (Bahri and Coron, 1988). On the other hand, in any domain Ω the problem

$$-\Delta u = \lambda u + |u|^{p-2}u \ \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (2.2)$$

with $p = 2^*$ admits a positive solution provided $N \geqslant 4$ and $0 < \lambda < \lambda_1(\Omega)$, the first eigenvalue of $(-\Delta, H_0^1(\Omega))$ (Brezis and Nirenberg, 1983).

The problem may admit several positive solutions. For example, if $p < 2^*$ is sufficiently close to 2^* then (2.1) has at least cat(Ω)+1 positive solutions, provided Ω has a nontrivial Ljusternik-Schnirelmann category cat(Ω) > 1 (Benci, Cerami and Passaseo, 1991). While, for example, if we remove k small balls from a given ball (so that the resulting set Ω has a fixed category cat(Ω) independent of k) then, for p close to 2^*, the number of positive solutions of (2.1) increases up to $2k + 1$, provided we count them with their multiplicity in the sense of the critical groups (Benci and Cerami, 1994).

On the other hand, the uniqueness of positive solutions of (2.1) does hold if Ω is a ball and $p < 2^*$ (Gidas, Ni and Nirenberg, 1979). Similarly to the problem (2.2), if $\lambda < 0$ and $p < 2^*$ (Kwong, 1989), or if $\lambda > 0$ and $p \leqslant 2^*$ (Srikanth, 1993). In contrast, if Ω is an annulus and $p < 2^*$ is sufficiently close to 2^* then (2.1) admits one positive radial solution and a further positive nonradial solution (Brezis and Nirenberg, 1983).

As for sign-changing solutions, the number $N(u)$ of nodal domains of a solution u of (2.1) can be estimated by its Morse index $m(u)$ (see Section 3). It is an elementary fact that the inequality $N(u) \leqslant m(u)$ always holds, while if Ω is a ball or an annulus and u is radially symmetric then $N(u) \leqslant 1 + \frac{m(u)}{N+1}$ (Aftalion and Pacella, 2004).

Other, more specialized questions were studied in the past decades. It is also of great interest to consider more general nonlinearities, and in fact some of the results above hold for more general equations than (2.1). In the sequel, by a *superlinear and subcritical* nonlinearity g we will mean a function $g \in C^1(\mathbb{R};\mathbb{R})$ such that:

(i) $g'(s)s^2 \geqslant g(s)s > 0 \ \forall s \neq 0$;

(ii) $g(s)s \geqslant \mu G(s)$ for large $|s|$, where $\mu > 2$;

(iii) $|g(s)| \leqslant C(1 + |s|^{p-1}) \ \forall s$, with $2 < p < 2^*$.

We have used the notation $G(s) := \int_0^s g(\xi)\,d\xi$ (so $G(s) = |s|^p/p$ if $g(s) = |s|^{p-2}s$). We stress that, in strong contrast with the one-dimensional problem, the mere existence of solutions for a more general equation

$$-\Delta u = g(u) \ \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (2.3)$$

with g superlinear and subcritical, is not settled. In spite of the numerical evidence suggesting the existence of many solutions for (2.3) (Ding, Costa and Chen, 1999), only a three-solutions theorem is established so far (Wang, 1991): (2.3) admits a positive solution, a negative solution, and a further sign-changing solution.

However, in case g is superlinear, subcritical and *odd symmetric* ($g(-s) = -g(s)\ \forall s$) then the existence of an infinite number of solutions can be proved. In particular,

Theorem 1. [3] *For any $p < 2^*$ and any domain Ω, problem (2.1) admits an unbounded sequence of solutions.*

It is not known whether the number of nodal domains of these solutions is arbitrary large, neither whether a solution having at least three nodal domains does exist. In Section 3 we develop the content of Theorem 1 from a general point of view, while in Section 4 we present some new results in this direction.

3. Minimax theorems

In the rest of the paper g is a superlinear and subcritical nonlinearity. Solutions of (2.3) can be seen as *critical points* of the energy functional

$$I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(u), \qquad u \in H_0^1(\Omega), \quad (3.1)$$

where $G(s) := \int_0^s g(\xi)\,d\xi$, that is, points u such that $I'(u) = 0$. A number $c \in \mathbb{R}$ is a *critical value* of I if $I(u) = c$ for some critical point u. Since the functional is unbounded both from below and from above, global minimization/maximization is excluded. One can find critical points of I by either using constrained minimization or a minimax theorem.

The most celebrated minimax theorems in critical point theory are the Mountain Pass Theorem and the Saddle Point Theorem, due to Ambrosetti and Rabinowitz [3] and Rabinowitz [22] respectively, in the 70s. This followed earlier work which can be traced back to Birkhoff (1917), Ljusternik and Schnirelmann (1934) and M.A. Krasnosel'skiĭ (1964), among others.

We specialize the underlying idea to our problem. For every $k \in \mathbb{N}$, let us denote by E_k the finite dimensional space spanned by the first k eigenfunctions of

$(-\Delta, H_0^1(\Omega))$. For a given number $R_k > 0$, let $Q_k := B_{R_k}(0) \cap E_k$ be the ball of radius R_k in E_k, centered at the origin, and consider the class of (continuous) maps,

$$\Gamma_k := \{\gamma : Q_k \to H_0^1(\Omega) : \ \gamma \text{ odd}, \ \gamma|_{\partial Q_k} = Id\}.$$

Then we define the number

$$b_k := \inf_{\gamma \in \Gamma_k} \sup_{u \in Q_k} I(\gamma(u)). \tag{3.2}$$

It can be proved that $b_k > 0$ if R_k is sufficiently large, in particular $b_k \in \mathbb{R}$. Moreover,

$$b_k \leqslant b_{k+1} \ \forall k \quad \text{ and } \quad \lim_{k \to \infty} b_k = +\infty.$$

The numbers b_k are natural candidates for being critical values of I. However, this will not be the case unless I is an *even* functional (i.e. $I(-u) = I(u) \ \forall u$). This amounts to ask that $g(s)$ is an odd nonlinearity.

So, in this case I admits indeed an infinite number of critical values; the corresponding critical points $(u_k)_k$ constitute a sequence of solutions to problem (2.3) whose $H_0^1(\Omega)$-norms tend to infinity as $k \to \infty$, and this settles Theorem 1 above.

Moreover, regardless of its symmetry, if g is asymptotically dominated by a pure-power nonlinearity $|s|^{p-2}s$, then one has the following estimates on the growth of b_k.

Proposition 2. [6, 30] *If* $g(s)s - |s|^p = \mathrm{o}(|s|^p)$ *as* $|s| \to \infty$ *then*

$$c_1 k^{2p/N(p-2)} \leqslant b_k \leqslant c_2 k^{2p/N(p-2)}$$

for some $c_1, c_2 > 0$ *independent of* k.

The second inequality is related to the asymptotic behavior of the eigenvalues of $(-\Delta, H_0^1(\Omega))$ and follows from the very definition of b_k. As for the first inequality, it arises from a semiclassical inequality of Cwikel, Lieb and Rosenbljum [16, 21, 28], which is used here in the context of *Morse index estimates.*

If u is a solution of (2.3), its Morse index $m(u)$ is defined as the number of negative eigenvalues of the linearized problem

$$-\Delta v = g'(u)v + \lambda v, \qquad v \in H_0^1(\Omega). \tag{3.3}$$

In an equivalent way, $m(u)$ is the supremum of the dimensions of the subspaces Z of $H_0^1(\Omega)$ where the quadratic form $I''(u)$ is definite negative (i.e. $I''(u)(\varphi, \varphi) < 0 \ \forall \varphi \in Z, \ \varphi \neq 0$).

Loosely speaking, for a general functional I, Morse theory is concerned with relating the structure of the critical point set of I in $\{a \leqslant I \leqslant b\}$ $(a, b \in \mathbb{R})$ with the homology, homotopy, homeomorphism, and diffeomorphism type of the pair $(\{I \leqslant b\}, \{a \leqslant I \leqslant b\})$.

The pioneering work of M. Morse on compact manifolds goes back to the 30s, followed by later developments and extensions to the infinite dimensional case by Palais, Rothe, Sard and Smale among others, in the 60s. As mentioned before, a decade later critical point theory in the framework of PDEs was giving its first steps. In the 80s these two methods were put aside; we quote the following paragraph from [6]:

"The two main methods in critical point theory are probably Morse theory (including Morse inequalities) and minimax variational approaches (as initiated by Ljusternik and Schnirelman). Morse theory usually provides (in some cases) critical points with a local information (i.e., the Morse index) but requires nondegenerate functionals and does not give precise indications on the energy levels. On the other hand, minimax critical point theory usually yields critical values by explicit formulas but lacks real local understanding of the structure of associated critical points. [...] More recently, attempts to understand the local nature of minimax critical points have been made."

This led to a huge literature on the subject, mostly in the 90s (see e.g. [15, 19, 24]). This is the context which relates the definition of the minimax levels b_k with the first estimate in Proposition 2. In our final section we explain the relevance of such type of estimates for problem (2.3).

4. Perturbation from symmetry

As mentioned above, (2.3) admits an unbounded sequence of solutions in case g is (superlinear, subcritical and) odd symmetric. In the one-dimensional case $(N = 1, \ \Omega = (-1, 1))$, under these assumptions the picture is rather clear. It is known ([12, 30]) that the solution set of the corresponding ODE consists precisely of a sequence $u_0, \pm u_1, \pm u_2, \dots$ where $u_0 = 0$ and, up to the sign of $u_k'(0)$, u_k is completely determined by the condition: u_k possesses exactly $k-1$ zeros in $(-1, 1)$. Moreover, going back to the numbers b_k defined in (3.2), we have that $b_k = I(u_k) < I(u_{k+1}) = b_{k+1}$ for every k. Finally, each solution u_k has Morse index k and is non-degenerate, in the sense that 0 is not an eigenvalue of the linearized (ordinary differential) equation in (3.3).

When the nonlinear term of the equation is no longer odd symmetric but rather behaves asymptotically like one, there seems to be no reason why a great number of solutions should cease to exist. Consider for example the following basic perturbed problem

$$-\Delta u = |u|^{p-2}u + f(x), \qquad u \in H_0^1(\Omega), \tag{4.1}$$

where $2 < p < 2^*$ and, say, $f \in L^2(\Omega)$. This problem was first studied in [5, 6, 23, 29, 30]. In particular, the following holds.

Theorem 3. [6, 30] *If*

$$p < \frac{2N-2}{N-2} \tag{4.2}$$

then (4.1) *admits an unbounded sequence of solutions.*

We mention that in the case where non-homogeneous Dirichlet boundary conditions are considered, a similar conclusion holds provided $p < 2N/(N-1)$, cf. [13].

It remains an open and challenging problem to know if the full range $p < 2^* = 2N/(N-2)$ can be allowed in Theorem 3. The following two results somehow suggest that this is the case.

Theorem 4. [2] *Given* $f \in L^2(\Omega)$ *and* $k \in \mathbb{N}$ *there exists* $\varepsilon_0 = \varepsilon_0(k)$ *such that for* $|\varepsilon| < \varepsilon_0$ *the problem*

$$-\Delta u = |u|^{p-2}u + \varepsilon f(x), \qquad u \in H_0^1(\Omega),$$

with $p < 2^*$ *admits at least* k *solutions.*

Theorem 5. [4] *If* $p < 2^*$ *then the set of* $f \in H^{-1}(\Omega)$ *such that the problem* (4.1) *has infinitely many weak solutions is a dense residual set in* $H^{-1}(\Omega)$.

Let us describe roughly the underlying idea in the proof of Theorem 3. As in (3.1), let

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p - \int_\Omega f(x)u, \qquad u \in H_0^1(\Omega).$$

We denote by S the unit sphere in $H_0^1(\Omega)$ and by J the functional

$$J(u) := \max_{t>0} I(tu), \qquad u \in S.$$

It can be proved that there is a one-to-one correspondence between critical points of I and critical points of J. Moreover, the numbers b_k constructed above can also be defined as

$$b_k = \inf_{\gamma \in \mathscr{A}_k} \sup_{u \in S_k} J(\gamma(u)),$$

where $S_k = S \cap E_k$ and

$$\mathscr{A}_k := \{\gamma : S_k \to S,\ \gamma \text{ continuous and odd}\}.$$

Since J is not an even functional, b_k is not expected to be a critical value of J. However, let

$$I^*(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p, \qquad u \in H_0^1(\Omega),$$

and consider the corresponding functional J^* and minimax levels b_k^*. Then b_k^* is a critical value for J^*. Moreover it can be proved that given $a > 0$ then J has a critical value $c > a$ provided

$$\{J^* \leqslant b_k^* + \varepsilon\} \subset \{J \leqslant a\} \subset \{J \leqslant a + \varepsilon\} \subset \{J^* \leqslant b_{k+1}^* - \varepsilon\}$$

for some $\varepsilon > 0$.

So we see that the existence of infinitely many critical values for J will be a consequence of proving that the intervals (b_k^*, b_{k+1}^*) are large enough with respect to the difference $|J - J^*|$. Here is where Proposition 2 and the condition (4.2) come into play.

Going back to the symmetric problem (2.1), a further natural question concerns the sign of these solutions. The following theorem complements Theorem 1.

Theorem 6. [8, 11, 20] *If* $p < 2^*$ *then* (2.1) *admits a sequence of unbounded sign-changing solutions.*

The proof of Theorem 6 uses an homological description of the minimax levels. In order to deal with the perturbed symmetric problem, in [27] an elementary approach, based upon the ideas described in the previous section, was proposed. Now we deal with perturbations such as

$$-\Delta u = |u|^{p-2}u + f(x,u), \qquad u \in H_0^1(\Omega). \tag{4.3}$$

Theorem 7. [27] *If* $p < (2N-2q)/(N-2)$ *and* $f(x,s)$ *is a continuous function such that* $f(x,s)/ \to s \to 0$ *as* $s \to 0$ *uniformly in* x *and* $0 \leqslant f(x,s)s \leqslant C(1+|s|^q)$, $0 < q < p$, *then* (4.3) *admits a sequence of unbounded sign-changing solutions.*

A related problem concerns the case where

$$-\Delta u = V(x)|u|^{p-2}u + f(x), \qquad u \in H_0^1(\Omega). \tag{4.4}$$

and $V \in C^1(\overline{\Omega})$ changes sign in Ω. This new feature of the nonlinearity causes a lack of compactness (in fact if we deal with the homogeneous function $g(s) = |s|^{p-1}s$ this is not a serious problem, but it becomes a serious one whenever, say, $g(s) = |s|^{p-1}s + |s|^{r-1}r$ with $p \neq r$). The following was proved by means of a variational technique which relates a priori bounds of the solutions with a priori bounds of their Morse indices, an idea which goes back to [7].

Theorem 8. *Assume that $V(x)$ has only nondegenerate zero points in Ω. Then problem (4.4) has an unbounded sequence of solutions in the following two situations:*

(a) [26] $f \equiv 0$ *and* $p < 2^*$*;*

(b) [25] $f \in C(\overline{\Omega})$ *and* $p < (2N-2)/(N-2)$*.*

Concerning the symmetry of the solutions, we state the following deep result. A corresponding one for the perturbed problem (with, say, $f(x)$ radially symmetric) is not known.

Theorem 9. [17] *If Ω is a ball or an annulus and $p < 2^*$ then problem (2.1) admits an unbounded sequence of radially symmetric solutions and a further unbounded sequence of nonradially symmetric solutions.*

We also mention that the above problems have a natural extension to systems of the form

$$-\Delta u = |v|^{q-2}v + f_1(x), \quad -\Delta v = |u|^{p-2}u + f_2(x), \quad (4.5)$$

with $u, v \in H_0^1(\Omega)$, $p, q > 2$ and, say, $p \leqslant q$ (this reduces to (4.1) if $p = q$ and $f_1 = f_2$).

Theorem 10. *Problem (4.5) has an unbounded sequence of solutions $u, v \in H_0^1(\Omega)$ in the following two situations:*

(a) [1] $f_2 \equiv f_2 \equiv 0$ *and* $\frac{1}{p} + \frac{1}{q} > \frac{N-2}{N}$*;*

(b) [14] $f_1, f_2 \in L^2(\Omega)$ *and* $\frac{N}{2}(1 - \frac{1}{p} - \frac{1}{q}) < \frac{p-1}{p}$.

We observe that if $p = q$ then case (a) reduces to the assumption that $p < 2^*$, while (b) is precisely (4.2). No "generic" results in the spirit of Theorems 4 and 5 are known for (4.5).

Finally, we mention two directions of research on this type of problems: the case where Ω is the entire space $\mathbb{R}^N$ (see [9] for sign-changing solutions in the symmetric case; the non-symmetric case seems to be open); the case where the problem is sublinear rather than superlinear, i.e. $g(s) = |s|^{p-2}s$ with $p < 2$ in the model equation (see [18] for the non-symmetric case).

References

[1] S. Angenent, R. van der Vorst, A superquadratic indefinite elliptic system and its Morse-Conley-Floer homology, *Math. Z.* 231 (1999), 203–248.

[2] A. Ambrosetti, A perturbation theorem for superlinear boundary value problems, Math. Res. Center, Univ. Wisconsin-Madison, Tech. Sum. Report 1442 (1974).

[3] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* 14 (1973) 349–381.

[4] A. Bahri, Topological results on a certain class of functionals and application, *J. Funct. Anal.* 41 (1981) 397–427.

[5] A. Bahri, H. Berestycki, A perturbation method in critical point theory and applications, *Trans. Amer. Math. Soc.* 267 (1981) 1–32.

[6] A. Bahri, P.L. Lions, Morse-index of some min-max critical points. I. Application to multiplicity results, *Comm. Pure Appl. Math.* 41 (1988) 1027–1037.

[7] A. Bahri, P.L. Lions, Solutions of superlinear elliptic equations and their Morse indices, *Comm. Pure Appl. Math.* 45 (1992) 1205–1215.

[8] T. Bartsch, Critical point theory on partially ordered Hilbert spaces, *J. Funct. Anal.* 186 (2001) 117–152.

[9] T. Bartsch, Z. Liu, T. Weth, Sign changing solutions of superlinear Schrödinger equations, *Comm. Partial Differential Equations* 29 (2004) 25–42.

[10] T. Bartsch, Z.Q. Wang, M. Willem, The Dirichlet problem for superlinear elliptic equations, in: *Handbook of Differential Equations*, Stationary Partial Differential Equations, vol. 2, M. Chipot and P. Quittner Eds., Elsevier, 2005, 1–55.

[11] T. Bartsch, T. Weth, A note on additional properties of sign changing solutions to superlinear elliptic equations, *Topol. Methods Nonlinear Anal.* 22 (2003) 1–14.

[12] H. Berestychi, Le nombre de solutions de certains problèmes semi-linéaires elliptiques, *J. Funct. Anal.* 40 (1981) 1–29.

[13] Ph. Bolle, N. Ghoussoub, H. Tehrani, The multiplicity of solutions to non-homogeneous boundary value problems, *Manuscripta Math.* 101 (2000), 325–350.

[14] D. Bonheure, M. Ramos, Multiple critical points of perturbed symmetric strongly indefinite functionals, Preprint CMAF, 2007.

[15] K.C. Chang, *Infinite dimensional Morse theory and multiple solution problems*, Progress in Nonlinear Differential Equations and their Applications, vol. 6, Birkhäuser, Boston, Mass., 1993.

[16] M. Cwikel, Weak type estimates for singular values and the number of bound states of Schršdinger operators, *Ann. Math.* 106 (1977) 93–100.

[17] R. Kajikiya, Orthogonal group invariant solutions of the Emden-Fowler equation, *Nonlinear Anal. TMA* 44 (2001) 845–896.

[18] R. Kajikiya, Multiple solutions of sublinear Lane-Emden elliptic equations, *Calc. Var. Partial Differential Equations* 26 (2006) 29–48.

[19] A.C. Lazer, S. Solimini, Nontrivial solutions of operator equations and Morse indices of critical points of min-max type, *Nonlinear Anal. TMA* 12 (1988) 761–775.

[20] S. Li, Z.Q. Wang, Ljusternik-Schnirelmann theory in partially ordered Hilbert spaces, *Trans. Amer. Math. Soc.* 354 (2002) 3207–3227.

[21] E.H. Lieb, Bounds on the eigenvalues of the Laplace and Schrödinger operators, *Bull. Amer. Math. Soc.* 82 (1976) 751–753.

[22] P. Rabinowitz, Some critical point theorems and applications to semilinear elliptic partial differential equations, *Ann. Scuola Norm. Sup. Pisa* 5 (1978) 215–223.

[23] P. Rabinowitz, Multiple critical points of perturbed symmetric functionals, *Trans. Amer. Math. Soc.* 272 (1982) 753–770.

[24] M. Ramos, L. Sanchez, Homotopical linking and Morse index estimates in min-max theorems, *Manuscripta Math.* 87 (1995) 269–284.

[25] M. Ramos, H. Tehrani, Perturbation from symmetry for indefinite semilinear elliptic equations, Preprint CMAF, 2008.

[26] M. Ramos, S. Terracini, C. Troestler, Superlinear indefinite elliptic problems and Pohožaev type identities, *J. Funct. Anal.* 159 (1998) 596–628.

[27] M. Ramos, W. Zou, Nodal solutions for perturbed symmetric elliptic equations via Morse index estimates, Preprint CMAF, 2007.

[28] G. Rosenbljum, The distribution of the discrete spectrum for singular differential operators, *J. Sov. Math. Dokl.* 13 (1972), 245–249 ; translation from Dokl. Akad. Nauk SSSR 202 (1972) 1012–1015.

[29] M. Struwe, Infinitely many critical points for functionals which are not even and applications to superlinear boundary value problems, *Manuscripta Math.* 32 (1980) 335-364.

[30] K. Tanaka, Morse indices at critical points related to the symmetric mountain pass theorem and applications, *Comm. Partial Differential Equations* 14 (1989) 99–128.

[31] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.