Math in the Media

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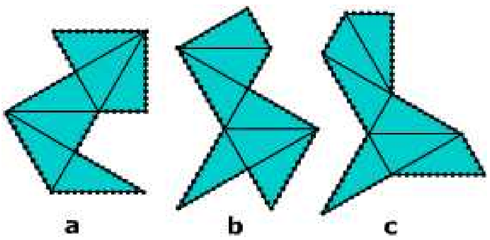
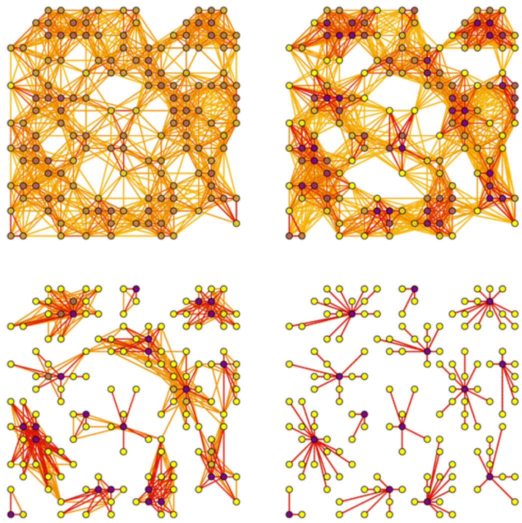
Hearing molecular drums. *Science* for February 8, 2008 ran a Report by a 6-member Stanford team entitled "Quantum Phase Extraction in Isospectral Electronic Nanostructures." The team, led by Hari Manoharan, took advantage of the discovery (Carolyn Gordon, David Webb, Scott Wolpert, 1992) of pairs of distinct polygonal shapes *isospectral* in the sense that they had exactly the same vibrational profile: identical responses at every frequency. This discovery was the long-awaited answer to Mark Kac's 1966 question "Can you hear the shape of a drum?" The *Science* authors use carbon monoxide molecules to draw a pair of different but geometrically isospectral shapes on the surface of a copper crystal. Each has area about 57 square nanometers, and encloses about 30 of "the 2D Fermi sea of electrons" that inhabit the surface; this pond of electrons will function as a "vibrating medium."



Three quantum nano-resonators assembled from carbon monoxide molecules. **a** "Bilby," **b** "Hawk," **c** "Broken Hawk." Each is assembled on a copper crystal by placing 90 CO molecules (black dots) around a polygonal contour. The polygon is geometrically the union of seven identical 30-60-90-degree triangles. In polygons **a** and **b**, any two adjacent triangles are related by reflection across their common border; this does not hold for **c**. The polygonal shapes **a** and **b** are known to be mathematically isospectral (but different in this respect from **c** even though **c** matches, for example, their area and perimeter); the authors exploit this feature to directly access the phase of the quantum-mechanical system formed by the surface electrons trapped inside the CO walls. Image after Manoharan *et al.*

The authors remark that "the time-independent Schrödinger equation is also a wave equation defined by the Laplacian and boundary conditions," i.e. the same equation that governs the sound of a drum, and that therefore the electronic resonances of the set of captured electrons will be the same for the two structures. Their main result is showing that "the complete phase information of wave functions in both structures can be experimentally determined" by "harnessing the topological property of isospectrality as the additional degree of freedom." This is physically significant because the spatial variation of the phase of the wave functions is measured without the usual reliance on interference phenomena. The supplementary information for this report includes a movie with soundtrack where the Schrödinger vibrations of the Bilby, Hawk and Broken Hawk nano-structures can be "heard" (at the rate of 100 THz $\sim$ 1 KHz).

Markov Clusters in the tree of life. "Long-held ideas regarding the evolutionary relationships among animals have recently been upended by sometimes controversial hypotheses based largely on insights from molecular data." So begins the abstract of a paper in the April 10 2008 *Nature*. The authors, an 18-member international team led by Casey Dunn (Brown), present in "Broad phylogenomic sampling improves resolution of the animal tree of life" a new method for selecting the genes to analyze in order to more accurately understand the relative position of species on the evolutionary tree.



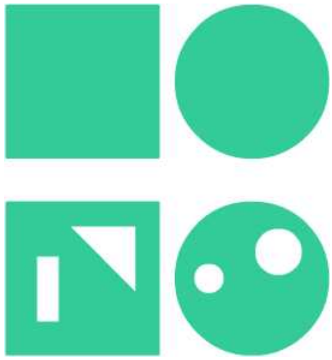
The Markov Cluster Algorithm at work. Red represents intensity. In Nieland's words: "... flow between different dense regions of nodes which are sparsely connected eventually evaporates, showing cluster structure present in the original input graph." Image courtesy of Stijn van Dongen.

"We present a new approach to identification of orthologous genes in animal phylogenomic studies that relies on a Markov cluster algorithm to analyse the structure of BLAST hits to a subset of the NCBI HomoloGene Database." BLAST (basic local alignment search tool) is a powerful algorithm, invented in 1990, for locating occurrences of a piece of genetic code in the NCBI (National Center for Biotechnology Information) database. The Markov Cluster Algorithm was devised in 2000 by Stijn van Dongen. It uses a stochastic, dynamic procedure to pinpoint the most significant part of a graph. In Henk Nieland's words: "Simulate many random walks (or flow) within the whole graph, and strengthen flow where it is already strong, and weaken it where it is weak. By repeating the process an underlying cluster structure will gradually become visible." (animated MCL algorithm simulation available in the main MCL website at `http://micans.org/mcl`).

The path to algebra: fractions. "News of the Week" in *Science* (March 21, 2008) was a story by Jeffrey Mervis about the National Mathematics Advisory Panel's release the week before of "a 120-page report on the importance of preparing students for algebra ... and its role as a gateway course for later success in high school, college, and the workplace." The report is available online (www.ed.gov/about/bdscomm/list/mathpanel). Mervis spoke with Larry Faulkner, the chair of the panel, and reports that the panel "avoided taking sides in a debilitating 2-decade-long debate on the appropriate balance between drilling students on the material and making sure they understand what they are doing." The recommendations are that "students should memorize basic arithmetic facts and spend more time on fractions and their meaning." But, as Mervis explains, "how teachers achieve those goals is up to them."

- Why do so many students have trouble with fractions? Faulkner: "Fractions have been downplayed." He mentions the perception that decimals and spreadsheets have eliminated the need for fractions. "But it's important to have an instinctual sense of what a third of a pie is, or what 20% of something is, to understand the ratio of numbers involved and what happens as you manipulate it."
- Q: Was the panel disappointed by the overall quality of existing education research? A: " ... We found a serious lack of studies with adequate scale and design for us to reach conclusions about their applicability for implementation."
- Q: Should the government be spending more money on this research? A:" ... If you want to get the value, you probably need to pay for it."
- Professional development programs? A: "There's tremendous variation in in-service programs. And the evidence is that many are not very effective."
- Calculators? "We feel strongly that they should not get in the way of acquiring automaticity [memorization of basic facts]. But the larger issue is the effectiveness of pedagogical software. At this stage, there's no evidence of substantial benefit or damage, but we wouldn't rule out products that could show a benefit."

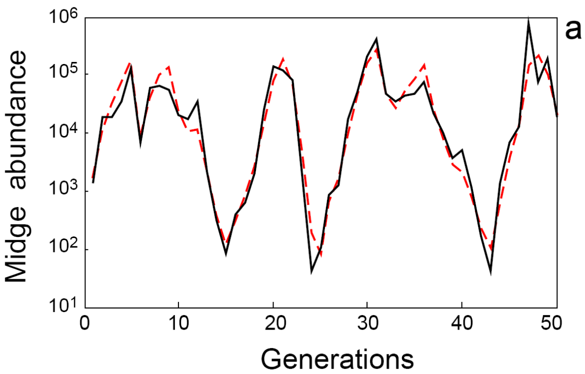
The hole truth?



The Schwarz-Christoffel formula (top) gives a conformal map from an arbitrary polygon to the unit disc. The generalization published in March 2007 by Darren Crowdy (below) applies to polygonally bounded domains of arbitrary topology. Image after Crowdy.

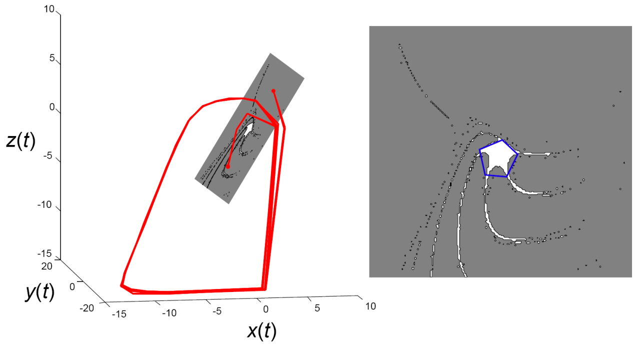
The Riemann mapping theorem guarantees a conformal map between any proper simply-connected planar domain and the open unit disc. In general, the map is constructed as the limit of an infinite process; but in case the domain is a polygon, an explicit, finite formula was found in the 1860s by Schwarz and Christoffel. An item by Adrian Cho, published March 6, 2008 on *Science*'s "Science Now" website, covers the publication a year ago (*Math. Proc. Camb. Phil. Soc.* 142 (2007) 319) of a generalization of the Schwarz-Christoffel formula to multiply-connected polygonal domains. The author was Darren Crowdy (Imperial College London); Cho quotes him: "If you give me any polygon with any number of polygonal holes, I can map it to a circle with the same number of circular holes." Crowdy's discovery "has been creating a buzz this week with coverage in several newspapers in the United Kingdom." For example, "140 Year-Old Schwarz-Christoffel Math Problem Solved" on `scientificblogging.com`. The point of Cho's piece, however, is not the mathematics but a priority controversy. Thomas DeLillo and Alan Elcrat (Wichita State), together with John Pfaltzgraff (Chapel Hill) published "Schwarz-Christoffel Mapping of Multiply Connected Domains" in the *Journal d'Analyse* (94 (2004) 17-47), and claim their share of the glory. According to Cho, "The Americans' formula ... involves the multiplication of an infinite number of terms, which goes haywire if the holes are too close together." Crowdy asserts that his method, which "replaces that product with an obscure beast known as Schottky-Klein prime function" (in Cho's words) is more reliable. Pfalzgraff is "very skeptical." Cho ends on a conciliatory note by quoting Michael Siegel (NJIT Newark) "It's a breakthrough, and all these people contributed." Cho's title: "Mathematicians Debate the Hole Truth."

Midge dynamics in Lake Myvatn.



50 generations of midge population in Lake Myvatn. The solid line represents observations, the dashed line output from the mathematical model with nine tuned parameters. Image courtesy of Anthony Ives.

"Mathematics Explains Mysterious Midge Behavior" is the title of an article by Kenneth Chang in the March 7 2008 *New York Times*. At Myrvatn ("Midge Lake") in northern Iceland, during mating season, the air can be thick with male midges (*Tanytarsus gracilentus*), billions of them. Chang quotes Anthony Ives (Wisconsin) "It's like a fog, a brown dense fog that just rises around the lake." And yet in other years, at the same time, there are almost none. Ives was the lead author on a report in *Nature* (March 6 2008) that gave an explanation for this boom-and-bust behavior in which, as Chang describes it, "the density of midges can rise or fall by a factor of a million within a few years." In the *Nature* report ("High-amplitude fluctuations and alternative dynamical states of midges in Lake Myvatn"), Ives and his co-authors characterize the midge ecology as one "driven by consumer-resource interactions, with midges being the consumers and algae/detritus the resources" and they set up a system of three coupled nonlinear difference equations, one each for midges, algae and detritus, to model it. The dynamics of this system include a stable state as well as a stable high-amplitude cycle; small variations in parameters can drive the system from one of those attractors to the other.



Alternative stable states of the midge-algae-detritus model. In the panel on the left, the plane is tangent to the manifold containing the cyclic component of the dynamics around the stationary point. The white region in the plane shows the domain of attraction to the invariant closed set, whereas the region in grey gives the domain of attraction to the outer stable cycle. The red lines give two examples of trajectories that converge to the outer stable cycle. The panel on the right shows the plane in more detail to illustrate the fine structure of the domain of attraction to the invariant closed set. The blue pentagon shows the unstable period 5 cycle that makes up part of the boundary between domains of attraction to the inner invariant closed set and the outer stable cycle. Image courtesy of Anthony Ives.

Parallel transport for qubits. *Science* for December 21, 2007 ran "Observation of Berry's Phase in a Solid-State Qubit." The authors are a team of ten from ETH, Sherbrooke and Yale, headed by Peter Leek and Andreas Wallraff. Their work falls under the rubric of what Seth Loyd called "holonomic quantum computation" (*Science* 292 5222): they use Berry phase changes produced by motion along paths (here the paths are in parameter space) to systematically manipulate the state of a qubit.

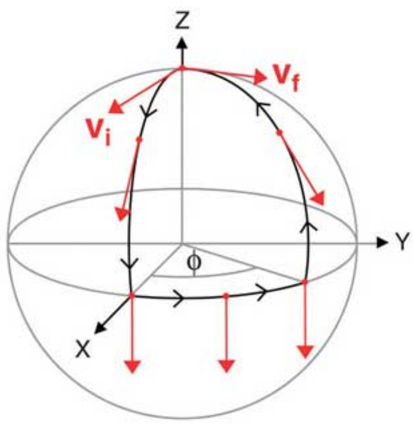


Fig. 1. Parallel transport of a vector around a geodesic triangle on the unit sphere, beginning (initial state v_i) and ending (final state v_f) at the North Pole. Image courtesy of Peter Leek and Andreas Wallraff.

Fig. 1 could come from a differential geometry text: it shows the parallel translation of a tangent vector around a $(\phi, \pi/2, \pi/2)$ geodesic triangle on the unit sphere. The final state v_f is rotated with respect to the initial state v_i by an angle which, when measured in radians, is exactly equal to the area enclosed by the path of the transport: in this case, ϕ. "The analogy of the quantum geometric phase with the above classical picture is particularly clear in the case of a two-level system (a qubit) in the presence of a bias field that changes in time," the authors write. In fact the set of states of a qubit may be represented as a sphere: an arbitrary superposition $z_0 < 0| + z_1 < 1|$ of its two base states corresponds to the point $[z_0 : z_1]$ in complex projective 1-space, which can be identified with the Riemann sphere by $[z_0 : z_1] \to z_0/z_1$ and stereographic projection.

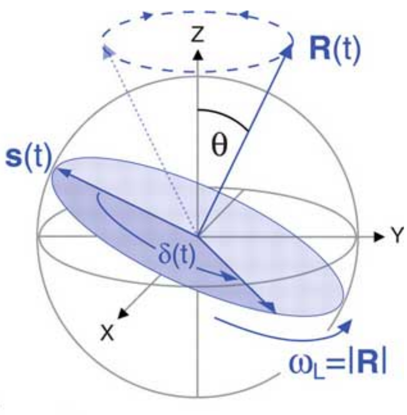
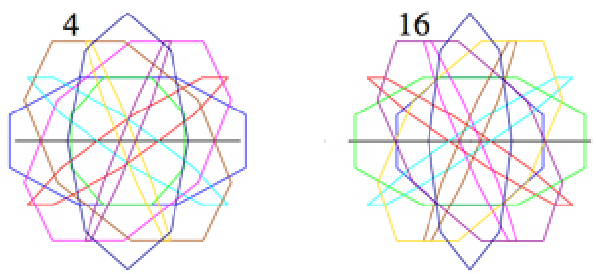


Fig. 2. The state space of a qubit can be represented as a sphere, with pure state $< 1|$ at the North Pole and $< 0|$ at the South Pole. Here the state s precesses at fixed speed about a vector R, and R itself is moving, at much slower speed, along a path of its own. Image courtesy of Peter Leek and Andreas Wallraff.

Suppose that as in Fig. 2, "the qubit state s continually precesses about the vector R, acquiring dynamic phase $\delta(t)$ at a rate $\mathsf{R} = |\mathsf{R}|$." "When the direction of R is now changed adiabatically in time (i.e., at a rate slower than R), the qubit additionally acquires Berry's phase while remaining in the same superposition of eigenstates with respect to the quantization axis R." When the axis R has been brought back to its original position after traversing a path C in its parameter space (via a two-stage maneuver that results in zero dynamic phase accumulation), "the geometric phase acquired by an eigenstate is $\pm\theta_\mathsf{C}/2$, where θ_C is the solid angle of the cone subtended by C at the origin." In the example illustrated, that cone is geometrically the same cone traced out by R in Fig. 2. As the authors remind us, its solid angle (i.e. the enclosed area intercepted on the unit sphere) "is given by $\theta_\mathsf{C} = 2n(1 - \cos\theta)$, depending only on the cone angle θ." [Note that the phase change is only 1/2 of the solid angle, in contrast with the purely geometric example. So a 360° planar rotation – enclosed area $2n$ – reverses the sign of the qubit.]

The article goes on to describe the experimental setup for implementing this phenomenon in real life. The qubit is a Cooper-pair box, the R-motions are driven by pulse-modulated microwave frequency signals, and the result is measured using quantum-state tomography.

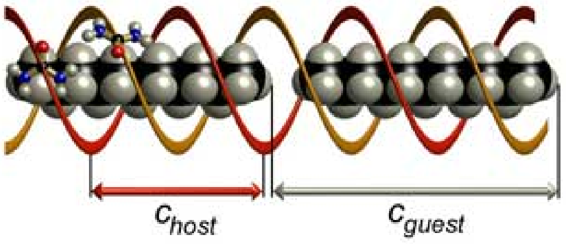
An 80-vertex polytope in *Physical Review*. Eric Altschuler and Antonio Pérez-Garrido published an article in *Physical Review* last year (E 76 016705 (2007)) in which they described "a four-dimensional polytope, new to our knowledge, with a high degree of symmetry in terms of the lengths of the sides." They found the configuration "by looking at the ... problem of finding the minimum energy configuration of 80 charges on the surface of the hypersphere S^3 in four dimensions" with the energy function $\Sigma(1/r_{ij})$ where r_{ij} is the distance between the i-th and j-th points, and the sum is taken over all pairs of distinct points. (They remark that they cannot prove this is actually a global minimum, but add that "even good local minima can be interesting or important configurations.") The other N for which they found symmetric configurations are 5, 8, 24 and 120; corresponding to the 4-simplex, the dual of the 4-cube, the 24-cell and the 600-cell. The authors give a method for visualizing their 80-vertex polytope in terms of the Hopf map $S^3 \to S^2$. They triangulate S^2 with 16 equal equilateral triangles: 4 abutting the North Pole, 4 the South, and a band of 8 around the Equator. This polyhedron has 10 vertices. Each of these vertices corresponds to a circle of the Hopf fibration, along which they describe explicitly how to place 8 of the polytope's vertices.



Two from a sequence of 20 projections of the 80-vertex polytope from 4-space into the plane. Each of the 10 Hopf-fibration circles has a different color, and appears as an octagon linking its 8 polytope vertices. Entire sequence, each projection composed with an additional rotation by 30° about a fixed plane in 4-space, available in `www.ams.org/mathmedia/images/altschuler-complete.jpg`. Images courtesy of Eric Altschuler.

Another description of the 80-vertex polytope was published by Johannes Roth later in the same journal (E 76 047702 (2007)).

Physical Chemistry in 4D.

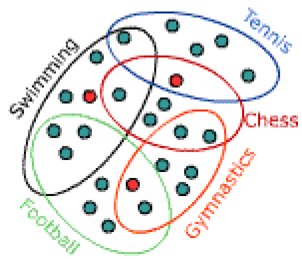


The structure of an alkane-urea channel-inclusion compound. C_{host} = 1.102 nm at room temperature. The "host" urea subsystem (spirals) and the "guest" alkane have irrationally related periodicity, which leads to the material presenting phase transitions that can only be explained in a 4-dimensional "superspace." Image courtesy of Bertrand Toudic.

"Hidden Degrees of Freedom in Aperiodic Materials" is a report in the January 4 2008 *Science*. The first author is Bertrand Toudic (Rennes); of the other nine, seven are based in France, one at Kansas State and one in Bilbao. The geometric structure of a crystal is described by listing its planes of reflection symmetry, labelled by triples (h, k, l) of integers (Miller indices) that describe their slopes in coordinates (a, b, c) adapted to the crystal. These planes give rise to characteristic patterns of peaks of intensity in photographic (or other) records of how the crystal scatters radiation. Toudic and his team investigate "aperiodic" materials like the alkane-urea compound illustrated above. This compound consists of a framework of nanotubes ("urea molecules are connected by hydrogen bonds to form helical ribbons, which are woven together to form a honeycomb array of linear, nonintersecting, hexagonal tunnels") inside which "Guests such as nonadecane pack end to end within van der Waals contact of each other." In case the repeat length C_{host} of the helical structure of the tunnels and the packing distance C_{guest} of the alkane guests are not rationally related [presumably, on an appropriate scale], we need an additional Miller index to explain the diffraction patterns. The sets of Miller indices are now vectors (h, k, l, m) in a "superspace" of which the first three dimensions are the familiar ones.

The authors bring this extra dimension into salience by exhibiting a phase transition that cannot exist without it. As Philip Coppens explains it, in a "Perspectives" piece in the same issue of Science, when the c axis is pointing along the tube, " the average structure of the urea ... is described by the $hkl0$ reflections, and the average structure of the alkane ... by the $hk0m$ reflections, whereas the remaining $hklm$ reflections are due exclusively to the mutual interaction between the two lattices. This implies that [the urea lattice] imposes a distortion on [the alkane lattice], and vice versa." "At temperatures above 149 K, all nonadecane columns in the crystal distort in an identical way. However, below this temperature, the extra $hklm$ reflections that appear in the diffraction pattern show that the relative modulation of the host and guest lattices alternates from channel to channel in the a-axis direction ... even though the periodicity of the average structures of the host and the guest in this direction does not change, as indicated by the absence of additional $hkl0$ and $hk0m$ reflections." He concludes "Such a transition, which only affects the mutual interaction, can only be described properly in super-space, even though the physical reality is obviously three-dimensional."

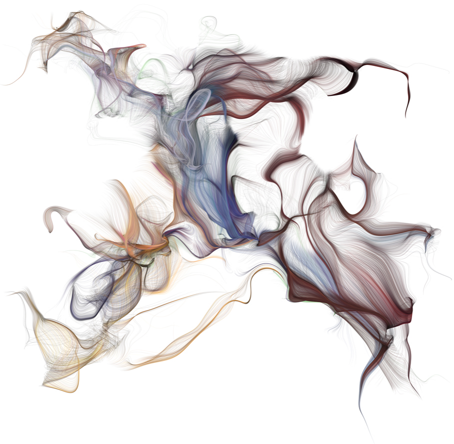
Physical insight into a hard combinatorial problem.



The hitting-set problem. Here the job is to find a minimal-size set of students (discs) representing all five sports. The red discs are a "hitting set." Image after Selman.

Since work of Mitchell, Selman and Levesque in 1992 it has been understood that some hard computational problems can undergo phase transitions at critical values of their parameters. Recently this statistical mechanical behavior has been harnessed to yield information about the solutions of some hitherto intractable problems. The work, by Marc Mézard and Marco Tarzia, appeared (*Phys Rev* E76 041124) last year, and was picked up by Bart Selman in a "News and Views" piece for *Nature*, February 7, 2008. Selman tells us that the authors "demonstrate an innovative approach to solving one well-known NP-complete problem, known as the hitting-set problem." A hitting set picks out from the union of a collection of sets a subset that contains at least one element of each; the problem is to find a hitting set with the smallest number of elements. Mézard and Tarzia, realizing "that tools from statistical physics developed to study physical phase transitions might help in developing more efficient algorithms for solving combinatorial problems," adapted the calculation of ground-state properties of certain condensed-matter systems to give the *survey-propagation method*. "Mézard and Tarzia use the survey-propagation method to compute statistical properties of the solutions of instances of the hitting-set problem." This is considered even harder than finding a single solution, but survey propagation, working near a phase boundary, can get the information "... by iteratively solving a large set of coupled equations, modelling the local interactions between variables probabilistically. This solution process can be performed in a parallel, distributed fashion using many different processors, and generally converges to an answer extremely quickly –in seconds for equations with thousands of variables."

"The computational realization of gesture".



Thomas Briggs Veils # 73. A larger image is available in www.salientimages.com/Veils73.htm but, as Briggs explains: "In order to represent these images on a web site they must be reduced in resolution by 99%. The works are a minimum of 3 feet square. The actual line weight is equivalent to that of a 0.2 - 0.3 millimeter pen nib, yet the large scale structure holds up when seen from a distance. This disparity of scale is an essential element of the experience of the works." Image used with permission.

On Thomas Briggs' website (www.salientimages.com) the artist details his methods, and the way mathematics enters into them: "The computational realization of gesture in my practice entails the construction of a spatial field of action. In this space various mathematical functions which represent small aspects of movement are distributed. The sum of the various functions is recorded for millions of points in space. These data are collated and translated into thousands of drawing primitives which are written into an image file for printing and archiving."

Numeral cognition and language. What is the relation between our concepts of number and the words we have in our language to express them? The old question was recently thrown into relief by Peter Gordon's report in *Science* (October 15, 2004) on the Pirahã, an extremely inscrutable Amazonian tribe whose language seems almost completely devoid of number-words. Gordon (Biobehavioral Sciences, Columbia) was categorical: "... the Pirahã's impoverished counting system limits their ability to enumerate exact quantities when set sizes exceed two or three items." Gordon was taken to task by Daniel Casasanto (Brain and Cognitive Sciences, MIT) who argues (Letters, *Science*, March 18, 2005) that "[the] results are no less consistent with the opposite claim [i.e., that they lack number words because they lack number concepts], which is arguably more plausible." The Pirahã controversy is the background for "The Limits of Counting: Numerical Cognition Between Evolution and Culture" (*Science*, January 11, 2008). The authors, Sieghard Beller and Andrea Bender (Psychology, Freiburg), focus on the evolution of numbering systems, for which they distinguish two properties: extent and degree of abstractness. They take their examples from Austronesian languages; Adzera is one of them. "Its number words for 1 to 5 are composed of numerals for 1 and 2 only: bits, iru$^?$, iru$^?$ da bits (= 2 + 1), iru$^?$ da iru$^?$ (= 2 + 2), and iru$^?$ da iru$^?$ da bits (= 2 + 2 + 1)." This is a system with small *extent*. The authors contrast Adzera with Mangarevan, where besides a general counting sequence there is another one used for tools, sugar cane, pandanus (a fruit) and breadfruit, while ripe breadfruit and octopus are counted with a different sequence, and the first breadfruit and octopus of a season are counted with yet another. This system lacks *abstractness*. The point the authors emphasize is that both these languages "belong to the same linguistic cluster ... and inherited a regular and abstract decimal numeration system with (at least) two powers of base 10 from their common ancestor, Proto-Oceanic." As they state in their conclusion, "Numeration systems do not always evolve from simple to more complex and from specific to abstract systems."

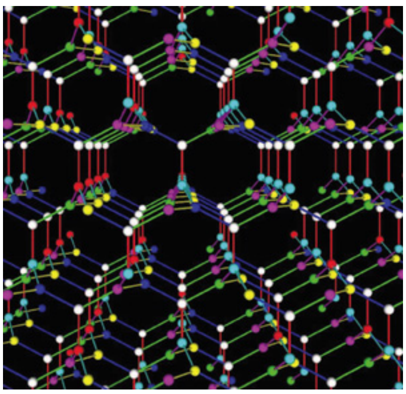
The mathematics of choosiness. "The coevolution of choosiness and cooperation," a Letter in the January 10 2008 *Nature*, describes a mechanism for the evolution of cooperative behavior. The Bristol-Debrecen team of John McNamara, Zoltan Barta, Lutz Fromhage and Alasdair Houston ran simulations of the "continuous snowdrift game," where in each round an individual, playing against one other, incurs a cost $C(x)$ depending on its own cooperativeness x, and receives a benefit $B(x + x')$ depending on the summed cooperativeness of both players. [The "snowdrift game" gets its name from an example where two drivers are stuck on opposite sides of a snowdrift, and have to choose between waiting in the car and shovelling]. Here are some details of the simulations: Along with cooperativeness (x), each player has a trait y called choosiness. Choosiness specifies the minimum degree of cooperativeness that the player will accept from its co-player.

- After each round, a player earns the payoff $B(x, x') - C(x) + A - S$, where B and C are as above, A is a fixed component and S is the startup cost for an individual in a newly formed pair.

- At the same time, the players learn their partner's cooperativeness, and choose whether to look for new partners (if $x < y'$ or $y < x'$) or to play again (otherwise).
- After each round an individual also produces offspring (clones except for "occasional small changes caused by mutation") proportionally to the size of the payoff;
- Between rounds the players "incur some risk of mortality." "Individuals that die are replaced by individuals selected at random from all offspring produced in the previous round."

Among the main conclusions of the experiments: "in a situation where individuals have the opportunity to engage in repeated pairwise interactions, the equilibrium degree of cooperativeness depends critically on the amount of behavioural variation that is being maintained in the population by processes such as mutation." Additionally, "The results suggest an important role of lifespan in the evolution of cooperation." The authors give heuristic arguments to interpret these results: in a uniform population nothing can be gained by being choosy, and therefore there is no incentive for individuals to be cooperative. "This situation changes profoundly if significant variation is maintained in the population by processes such as mutation." Moreover, high mortality counteracts the evolution of cooperation: "If the cooperative associations ... are soon disrupted by mortality, then establishing them is not worth the associated costs."

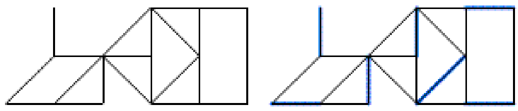
"A Mathematical Gem". is how Constance Holden (Random Samples, *Science*, January 18, 2008) describes this image, gleaned from the February 2008 issue of the *AMS Notices*.



The K-4 crystal is the maximal abelian covering of the tetrahedron, with the inherited geometry. For a larger and higher-resolution image, visit the February 2008 Notices. Image credit Hisashi Naito.

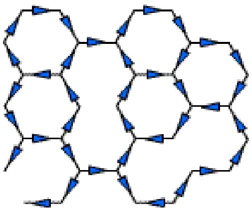
There it illustrates an article by Toshikazu Sunada, who shows that this crystalline structure shares with the diamond the "strong isotropy property," and that these are the only two such structures in three dimensions. (The strong isotropy property states that for any two vertices V and W of the crystal, any ordering of the edges adjacent to V and any ordering of the edges adjacent to W, there is a lattice-preserving congruence taking V to W and each V-edge to the similarly ordered W-edge). Sunada states that the K-4 crystal, beautiful as it is, is purely a mathematical object. Holden begs to differ: "In fact, it shows up in inorganic compounds, lipid networks, and liquid crystals and has been known for decades by other names."

Holographic algorithms. The January-February 2008 *American Scientist* features a report by Brian Hayes on holographic (or "accidental") algorithms, a recent phenomenon in computational mathematics. "Their computational power comes from the mutual cancellation of many contributions to a sum, as in the optical interference pattern that creates a hologram," according to their inventor, Leslie Valiant (Harvard); hence the name. The primordial "holographic" algorithm is the determinant of an n by n matrix: in principle it is a sum of $n!$ terms, but in practice, using row-reduction, it can be computed with only about n^3 operations. In fact, determinants turn out to be at the heart of all the examples Hayes presents. For example, the "perfect matching" problem: on a given graph, is there a set of edges linking each vertex to exactly one other vertex? And the associated counting problem: if so, how many such matchings are there?



A graph with one of its perfect matchings. After Hayes, *American Scientist* 96, No. 1.

This question seems to require looking at all possible choices of edges (a number growing factorially with the number of vertices) to see which ones work, but for a planar graph the Fisher-Kasteleyn-Temperley (FKT) algorithm, dating back to the early 1960s, equates the calculation of the number of perfect matchings on a planar graph with n vertices to the calculation of the determinant of a certain n by n matrix. Valiant's new holographic algorithms go one step further, and relate calculations in one context to the perfect matching problem in an associated graph. One such context, the "Three-ice problem," is illustrated below. Hayes explains: "The strategy is to build a new planar graph called a matchgrid, which encodes both the structure of the ice graph and the not-all-equal constraints that have to be satisfied at each vertex. Then we calculate a weighted sum of the perfect matchings in the matchgrid, using the efficient FKT algorithm. Although there may be no one-to-one mapping between individual matchings in the matchgrid and valid assignments of bond directions in the ice graph, the weighted sum of the perfect matchings is equal to the number of valid assignments."



The "Three-ice problem." For a planar graph where each vertex abuts 1, 2 or 3 edges, how many ways can the edges be oriented (blue arrows) with no in-in-in or out-out-out configurations? The image shows one admissible assignment of orientations. After Hayes.

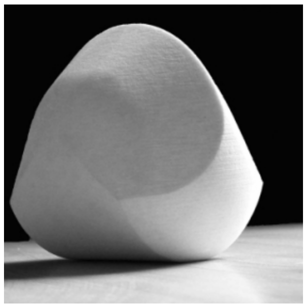
"Everything in our world is purely mathematical - including you". This startling quotation occurs about halfway through Dennis Overbye's "Laws of Nature, Source Unknown" in the December 18 2007 *New York Times.* Overbye attributes it to Max Tegmark, a cosmologist at MIT, whom he considers "The ultimate Platonist ... In talks and papers recently he has speculated that mathematics does not describe the universe - it is the universe. Dr. Tegmark maintains that we are part of a mathematical structure, albeit one gorgeously more complicated than a hexagon, a multiplication table or even the multidimensional symmetries that describe modern particle physics. Other mathematical structures, he predicts, exist as their own universes in a sort of cosmic Pythagorean democracy, although not all of them would necessarily prove to be as rich as our own."

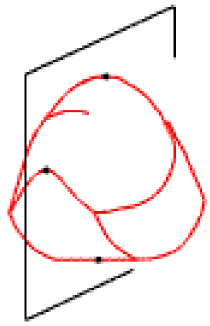
Tegmark's thesis is expounded in "Mathematical cosmos: why numbers rule" (*New Scientist*, September 15, 2007). The main argument is this: "If we assume that reality exists independently of humans, then for a description to be complete, it must also be well defined according to non-human entities – aliens or supercomputers, say – that lack any understanding of human concepts. ... This is where mathematics comes in. To a modern logician, a mathematical structure is precisely this: a set of abstract entities with relations between them. ... So ... If you believe in an external reality independent of humans, then you must also believe in what I call the mathematical universe hypothesis: that our physical reality is a mathematical structure."

But just as you were thinking that this would make life simpler, you read on. "The hypothesis also makes a much more dramatic prediction: the existence of parallel universes." The explanation: "Most physicists hope for a theory of everything that ... can be specified in few enough bits to fit in a book, if not on a T-shirt. The mathematical universe hypothesis implies that such a simple theory must predict a multiverse. Why? Because this theory is by definition a complete description of reality: if it lacks enough bits to completely specify our universe, then ... the extra bits that describe our universe simply encode which universe we are in, like a multiversal phone number."

If you are scratching your head in stunned disbelief, that's perfectly OK: "Evolution endowed us with intuition only for those aspects of physics that had survival value for our distant ancestors, such as the parabolic trajectories of flying rocks. Darwin's theory thus makes the testable prediction that whenever we look beyond the human scale, our evolved intuition should break down. ... To me, an electron colliding with a positron and turning into a Z-boson feels about as intuitive as two colliding cars turning into a cruise ship. The point is that if we dismiss seemingly weird theories out of hand, we risk dismissing the correct theory of everything, whatever it may be."

The Gömböc.

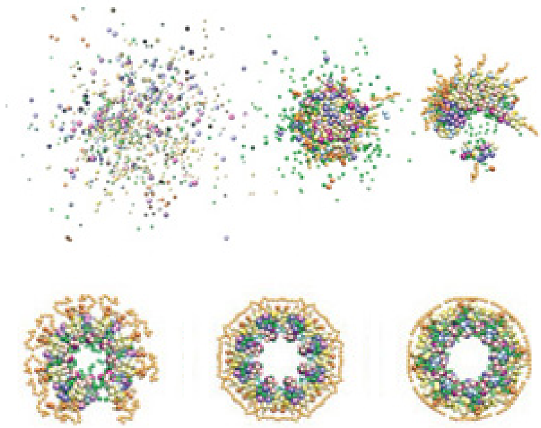




The Gömböc looks something like this. It has back-front symmetry as well as symmetry in the plane shown. The prototype given to Arnol'd was about 4 inches wide.

"The Self-Righting Object" was among the items chosen for the *New York Times Magazine*'s 7th Annual Year in Ideas (December 9, 2007). Named "the Gömböc" by its inventors – Gábor Domokos and Péter Várkonyi of Budapest – it is the result, according to the *Magazine*, of "a long mathematical quest," starting with a problem posed to Domokos in 1995 by the celebrated Russian mathematician V. I. Arnol'd: to construct a "mono-monostatic" object. This would be a convex, homogeneous object with such a geometry that it would have exactly one stable position when placed on a flat surface. "Homogeneous" rules out toys of the "Comeback Kid" class which rely on a weighted bottom to keep them coming back up. The Gömböc also got play in a piece by Julie Rehmeyer in *Science News Outline* for April 7, 2007. Neither of these accounts gives any hint of the mathematics involved in Domokos and Várkonyi's solution, although Rehmeyer reminds us that "flat toys cut from a piece of plywood" always have at least two stable positions (see the Gömböc website for details). And she's funnier. It turns out that the Gömböc looks a bit like a turtle, and the question arises whether turtles might have evolved mono-monostaticity to avoid getting stranded on their backs. "So far, they've tested 30 turtles and found quite a few that are nearly self-righting. Várkonyi admits that most biology experiments study many more animals than that but, he says, 'it's much work, measuring turtles.' "

Math and macromolecular architecture. "The Molecular Architecture of the Nuclear Pore Complex" was the cover story in the November 29 2007 *Nature* and highlighted there ("News and Views," "Making the Paper") as a substantial achievement by its authors, a Rockefeller-UCSF team led by Michael Rout, Brian Chait and Andrej Sali. A striking feature of the research was the essential involvement of mathematical methods developed by physicists for handling problems with a very high number of degrees of freedom. The Nuclear Pore Complex is a large (molecular mass around 50 million) assembly of 456 proteins (in yeast) that spans the nuclear envelope and controls movement of material into and out of the nucleus. It was known that about 30 different proteins are involved, and the general shape was understood: "a doughnut-shaped structure, consisting of eight spokes arranged radially around a central channel." But the exact way the pieces fit together was a mystery. The article spells it out completely. An accompanying article, "Determining the architectures of macromolecular assemblies," explains how the puzzle was solved. The phrase the authors use to describe their method is "integrating spatial restraints." The spatial restraints are all the available data about the shapes and affinities of the constituent proteins, encoded into a set of functions that give 0 when "the restraint is satisfied" and higher values if it is violated. "In essence, restraints can be thought of as generating a 'force' on each component in the assembly, to mould them into a configuration that satisfies the data used to define the restraints." This "force" is essentially (minus) the gradient of a scoring function cooked up from the restraints. The "integration" is an optimization process: "The optimization starts with a random configuration of the constituent proteins' beads, and then iteratively moves them so as to minimize violations of the restraints." (The beads are points representing the location of each protein). The configuration is periodically shaken up by "simulated annealing" to "minimize the likelihood of getting caught in local scoring function minima."



Representative configurations at various stages of the optimization process from top (very large scores) to lower right (with a score of 0). Adapted from *Nature* 450 690; used with permission.

Approximately 200,000 different initial configurations were tested, and used to yield "an ensemble of 1000 structures satisfying the input restraints." Then the structures from the ensemble were superposed, and used to generate a single structure for the entire pore. From the end of the abstract: "The present approach should be applicable to many other macromolecular assemblies."