BULLETIN

INTERNATIONAL CENTER FOR MATHEMATICS

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24

Contents

Coming Events

June 12-14, 2008: CIM/CRM Workshop on Financial Time Series

Organizers

Paulo Teles (Porto School of Economics) and Pilar Muñoz (Technical University of Catalonia, Barcelona, Spain).

Aims

This workshop brings together leading experts and several other researchers on financial time series, enabling them to present and discuss the latest research and case studies on this important issue and consequently providing knowledge exchange.

Financial time series is an increasingly important topic nowadays. In fact, modelling financial data recorded over time has achieved such high standards that it is now possible to deal with most data encountered in practice. Nevertheless, several important issues remain to be improved requiring new research and different approaches. Presenting and discussing them is extremely important and useful. To this purpose, some of the best researchers will present their latest developments and will be open for discussion with other participants. Exchanging own experiences, problems, views and approaches will undoubtedly bring an important contribution and will stimulate new research in the field.

This Workshop takes place at the premises of CIM – Centro Internacional de Matemática.

Invited speakers

Daniel Peña (Univ. Carlos III de Madrid, Spain)
Independent Component Analysis for Financial Time Series

Esther Ruiz (Univ. Carlos III de Madrid, Spain)
Bootstrap forecast in state space models

Feridun Turkman (Univ. Lisbon)
Extremes of Continuous-discrete Time Series

Frank Diebold (Univ. of Pennsylvania, USA)
Yield Curve Modelling and Forecasting

João Nicolau (ISEG, Lisbon Technical Univ.)
Modelling Financial Time Series through Second Order Stochastic Differential Equations

Nuno Crato (ISEG, Lisbon Technical Univ.)
Parametric and Nonparametric Methods for Comparing Financial Time Series

Philip Hans Franses (Erasmus Univ. Rotterdam, The Netherlands)
A simple test for GARCH against a Stochastic Volatility Model

For more information about the event, see

www.cim.pt/wfts2008

June 16-21, 2008: GAP VI - Workshop on Geometry and Physics

Organizers

Carlos Currás-Bosch (Univ. de Barcelona, Spain)

Rui Loja Fernandes (IST, Lisbon)

David Iglesias (CSIC, Madrid, Spain)

Eva Miranda (Univ. Autònoma de Barcelona, Spain)

San Vu Ngoc (Univ. de Rennes, France)

Ping Xu (Penn State Univ., USA)

Aims

GAP VI is the sixth edition of a series of "Séminaires Itinerants" that have taken place in different locations. This year the topic is Integrable Systems. There will be four mini-courses and several talks by participants.

The aim is to bring together young researchers from the two areas of Geometry and Physics and to fill in the GAP dividing this two deeply interconnected research fields.

The event is a joint CRM-CIM Workshop and will take place at CRM - Centre de Recerca Matemàtica, Barcelona (Bellaterra), Spain.

Mini-courses

Yves Colin de Verdière (Inst. Fourier, France)
Semi-classical Analysis of Integrable systems

Johannes Duistermaat (Univ. Utrecht, The Netherlands)
QRT and elliptic surfaces

Hakan Eliasson (Inst. Math. de Jussieu, France)
KAM for the non-linear Schrödinger equation

San Vu Ngoc (Univ. de Rennes, France)
Sympletic invariants of integrable Hamiltonian systems

For more information about the event, see

www.crm.cat/GAPVI

June 26-28, 2008: Workshop on Nonparametric Inference - WNI2008

Organizers

Carla Henriques (CMUC & Escola Superior de Tecnologia de Viseu)

Carlos Tenreiro (CMUC & Univ. of Coimbra)

Paulo Eduardo Oliveira (CMUC & Univ. of Coimbra)

Scientific committee

Antonio Cuevas (Univ. Autónoma de Madrid, Spain)

Emmanuel Candès (California Inst. of Techn., USA)

Enno Mammen (Univ. of Mannheim, Germany)

Irène Gijbels (Katholieke Univ. Leuven, Belgium)

Lászlo Györfi (Budapest Univ. of Technology and Economics, Hungary)

Paulo Eduardo Oliveira (CMUC & Univ. of Coimbra)

Phillippe Vieu (Univ. Paul Sabatier, Toulouse, France)

Aims

The goals of this workshop are:

- to illustrate active trends in a number of subjects in nonparametric statistics, including curve estimation, model checking, functional data, survival analysis, adaptive bandwidth choice and bootstrap;

- to give an opportunity for research students to develop their competence in nonparametric methods;
- to provide a meeting point for researchers in nonparametric inference, intending to contribute for the establishment of new links;
- additionally, it also hopes to contribute to incentive national research in non-parametric statistical topics.

The event will take place at the Department of Mathematics of the University of Coimbra.

INVITED SPEAKERS

Antonio Cuevas (Univ. Autónoma de Madrid, Spain)
On nonparametric estimation of boundary measures

Emmanuel Candès (California Inst. of Techn., USA)
Computationally tractable statistical estimation when there are more variables than observations

Irène Gijbels (Katholieke Univ. Leuven, Belgium)
to be announced

Lászlo Györfi (Budapest Univ. of Technology and Economics, Hungary)
Nonparametric prediction of time series

Phillippe Vieu (Univ. de Paul Sabatier, Toulouse, France)
On nonparametric functional data analysis

For more information about the event, see

www.mat.uc.pt/~wni2008

July 7-9, 2008: WEAA - Workshop on Estimating Animal Abundance

ORGANIZERS

Russell Alpizar-Jara (Dep. of Mathematics and CIMA, Univ. of Évora)

Anabela Afonso (Dep. of Mathematics and CIMA, Univ. of Évora)

João Filipe Monteiro (Dep. of Mathematics and CIMA, Univ. of Évora)

AIMS

This is an interdisciplinary workshop that intends to narrow the gap between statistical estimation theory for animal populations, and wildlife and fisheries applications of this methodology. The workshop will be an introductory overview of capture-recapture and distance sampling models and will include estimation of population size, survival rates and birth numbers. An emphasis will be placed on real examples and the importance of validation of model assumptions. Recent developments of capture-recapture applications to Epidemiological estimation problems could also be addressed.

Three days in a computer lab so participants will try out the programs: MARK, M-SURGE and U-CARE, and DISTANCE. Participants are encouraged to bring their own laptop and data sets for analyses. Additional selected data sets from fieldwork will also be available.

Topics will include:

- Closed and open capture-recapture models,
- The robust design,
- Designing capture-recapture studies,
- Multi-state capture-recapture models,
- Distance sampling methods.

The workshop will be held at the University of Évora.

INSTRUCTORS

Kenneth Pollock (North Carolina State Univ., USA)

Jean-Dominique Lebreton (CNRS, CEFE, France)

Theodore R. Simons (Cooperative Fish and Wildlife Research Unit, North Carolina State Univ., USA)

For more information about the event, see

www.eventos.uevora.pt/~weaa

July 21-25, 2008: CIM/UC Summer School in Dynamical Systems

SCIENTIFIC COORDINATOR

José Ferreira Alves (Univ. Porto)

AIMS

This Summer School is a joint venture of Centro Internacional de Matemática (CIM) and the University of Coimbra (UC), and is sponsored by the Gulbenkian Foundation and the Foundation for Science and Technology (FCT). It will gather a group of specialists working on Dynamical Systems and Ergodic Theory, and will highlight emerging trends and issues of main research topics.

The event will take place at the Department of Mathematics of the University of Coimbra.

Conference Program

Lorenzo J. Díaz (PUC-Rio, Brazil)
Partially hyperbolic dynamics

Carlangelo Liverani (Univ. Roma Tor Vergata, Italy)
Probability and uniformly hyperbolic systems

Nuno Luzia (New Univ. Lisbon)
Fractional dimensions

Marcelo Viana (IMPA, Brazil)
Geodesic flows on flat surfaces

For more information about the event, see

www.cim.pt/sds08

September 8-10, 2008: Calculus of Variations and its Applications: from Engineering to Economy

Local organizers

Ana Luísa Custódio, Marta Faias, Luísa Mascarenhas, Ana Margarida Ribeiro, Luís Trabucho (all New Univ. Lisbon)

Scientific committee

Luísa Mascarenhas (New Univ. Lisbon)

Luís Trabucho (New Univ. Lisbon)

Luís Nunes Vicente (Univ. Coimbra)

Aims

The aim of this event is to promote the scientific exchange of ideas and methods in such a broad and useful area as the Calculus of Variations. With the goal of applications to different areas such as Mathematics, Mechanics, Engineering, Economy, Finances, Chemistry, Biology, just to name a few, models and methods have been developed, with apparent different languages but that are susceptible of an unified analytical and numerical treatment.

Taking into account the most recent developments in this area of Mathematics, we wish to address problems associated with Partial Differential Equations, Optimal Control, Finite or Infinite Dimension Optimization, Shape Optimization in Structural Engineering, together with the associated computational aspects.

The event will take place at the Department of Mathematics of the New University of Lisbon.

Main lecturers

Guy Bouchitte (Univ. Toulon et du Var, France)
to be announced

Giuseppe Buttazzo (Univ. of Pisa, Italy)
Optimal Dirichlet regions for some elliptic problems

Bernard Dacorogna (École Polyt. Fédérale Lausanne, Switzerland)
On the pullback equation

Irene Fonseca (Carnegie Mellon Univ., Pittsburgh, USA)
Variational methods in materials and imaging

Joaquim J. Júdice (Univ. of Coimbra)
to be announced

Boris Mordukhovich (Wayne State Univ., USA)
Variational problems for evolution and control systems

Mário Páscoa (New Univ. of Lisbon)
to be announced

Invited speakers

Paula Amaral (New Univ. Lisbon), Nadir Arada (New Univ. Lisbon), Margarida Baía (Tech. Univ. Lisbon), Cristian Barbarosie (Univ. Lisbon), Ana Cristina Barroso (Univ. Lisbon), Fabio Chalub (New Univ. Lisbon), Fernanda Cipriano (Univ. Lisbon), João Correia da Silva (Univ. Porto), Isabel Narra Figueiredo (Univ. Coimbra), Raquel Gaspar (Tech. Univ. Lisbon), Diogo Gomes (Tech. Univ. Lisbon), Maria do Rosário Grossinho (Tech. Univ. Lisbon), João Pedro Nunes (ISCTE), António Ornelas (Univ. Évora), Helder Rodrigues (Tech. Univ. Lisbon), Delfim Torres (Univ. Aveiro)

For more information about the event, see

ferrari.dmat.fct.unl.pt/cva2008

For updated information on these events, see

www.cim.pt/?q=events

CIM News

New Administration of CIM

During the meeting of the General Assembly of CIM held on April 5, 2008 the Administration team for 2008-2012 was elected:

Executive Board:

José Francisco Rodrigues, Univ. of Lisbon (President)
Cristina Sernadas, Tech. Univ. of Lisbon (Vice-Pres.)
José Ferreira Alves, Univ. of Porto (Vice-Pres.)
Assis Azevedo, Univ. of Minho (Secretary)
Isabel Figueiredo, Univ. of Coimbra (Treasurer)

General Assembly:

Nuno Crato, ISEG and SPM (President)
Rafael Santos, Univ. of Algarve (Secretary)
Eugénio Rocha, Univ. of Aveiro (Secretary)

Statutory Audit Committee:

Alfredo Egídio dos Reis, Tech. Univ. of Lisbon (Pres.)
Carlos Braumann, Univ. of Évora (Secretary)
Rui Cardoso, New Univ. of Lisbon (Secretary)

New Associate of CIM

During the same Assembly of April 5, 2008 a new associate was welcomed: the *Centro de Matemática e Aplicações Fundamentais* of the University of Lisbon (cmaf.fc.ul.pt)

CIM collaborates in ICMI/ICIAM study

The International Commission on Mathematical Instruction (ICMI), established by the IMU (International Mathematical Union) has recently approved a new Study, the 20th, jointly with the International Council for Industrial and Applied Mathematics (ICIAM) on "Educational Interfaces between Mathematics and Industry (EIMI)". This Study will be coordinated by Alain Damlamian (Paris), Rudolf Strässer (Giessen) and José Francisco Rodrigues (Lisbon). This ICMI/ICIAM study, that was proposed by the Portuguese National Committee of Mathematics, will be launched in 2008 in Portugal with the collaboration of CIM and aims, in particular, the publication of a book to be presented to the next ICIAM Congress of Vancouver in July 2011. The following are some highlights of key ideas included in the proposal:

– Scientific and technological research is the basis for industrial innovation and mathematics plays an essential and driving role.

– A recent report prepared for the OECD Global Science Forum on "Mathematics in Industry", not only has recognized the intimate connections between innovation, science and mathematics, but also demands new strategy for education of students, including more interdisciplinary training.

– Classically students on all levels have been taught the tools of mathematics which have been considered important by the teacher and high-school students are often taught as if mathematics is a dead science. When applications have been done, these have been often mostly artificial. Nowadays one needs the solution of much more complex problems and hence some training to solve such problems, in particular real life problems, has to be given.

– An international study on Education and Training on Applied and Industrial Mathematics on the secondary and tertiary level is therefore necessary and timely.

CIM on the Web

For updated information about CIM and its activities, see

www.cim.pt

On elliptic equations with superlinear nonlinearities

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Abstract

We survey some results on the solution set of equations of the form $-\Delta u = |u|^{p-2}u + f(x)$ $(p > 2)$ with Dirichlet boundary conditions on a smooth bounded domain of $\mathbb{R}^N$, from the point of view of the calculus of variations and critical point theory. We focus on the so called *perturbation from symmetry* problem.

1. Introduction

In the following Ω is a smooth bounded domain of $\mathbb{R}^N$. We shall be concerned with equations of the form

$$-\Delta u = g(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $g \in C^1(\mathbb{R};\mathbb{R})$ is *superlinear* in the sense that $g(s)/s \to +\infty$ as $|s| \to \infty$. The model nonlinearity is the homogeneous function

$$g(s) = |s|^{p-2}s \qquad \text{with } p > 2.$$

Classical methods based upon fixed point theorems do not apply easily to this problem because there are no a priori bounds for the solutions.

In the one-dimensional case $N = 1$, one can apply the shooting method to the ODE, and the existence of an unbounded sequence $(u_k)_k$ of solutions can be proved; moreover, the number of their nodal domains increases with k.

However, numerous open problems subsist in the case $N > 1$; among others, they concern the existence of solutions, the uniqueness in a prescribed class of functions (positive solutions, ground-state solutions, radially symmetric solutions, etc), their possible symmetry, the sign of the solutions as well as the number of their nodal domains.

In Section 2 we list a number of known results for the above problem. This list is not intended to be exhaustive but rather to provide the reader a flavor of the state of art. Since we aim mostly at the discussion in Section 4, we do not include complete references in Section 2; these can be found e.g. in [10, 31]. In Section 3 we comment briefly on the most fruitful framework that has been used so far to prove a number of such results. Section 4 is devoted to a special case where, although some basic questions still remain unsolved, new results have been obtained recently. Hereafter we will restrict our attention to the case $N \geqslant 3$; the critical Sobolev exponent $2^* := 2N/(N-2)$ will play an important role. Also, in order to keep the paper simple, we do not attempt in presenting the results in its most general form.

2. Some known results

Given $p \in \mathbb{R}$, $p > 2$, consider the problem

$$-\Delta u = |u|^{p-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{2.1}$$

In case $p < 2^*$, there is a compact embedding of the Sobolev space $H_0^1(\Omega)$ into $L^p(\Omega)$. Using various methods one can prove that (2.1) admits a solution different from the trivial one $u \equiv 0$; in fact, a positive solution $u > 0$.

The situation is different if $p \geqslant 2^*$. For example, if Ω is star-shaped and $p \geqslant 2^*$ then (2.1) has no solutions.

While if Ω has a nontrivial homology ($H_k(\Omega;\mathbb{Z}_2) \neq 0$ for some $k \geqslant 1$) and $p = 2^*$, then a positive solution does exist (Bahri and Coron, 1988). On the other hand, in any domain Ω the problem

$$-\Delta u = \lambda u + |u|^{p-2}u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (2.2)$$

with $p = 2^*$ admits a positive solution provided $N \geqslant 4$ and $0 < \lambda < \lambda_1(\Omega)$, the first eigenvalue of $(-\Delta, H_0^1(\Omega))$ (Brezis and Nirenberg, 1983).

The problem may admit several positive solutions. For example, if $p < 2^*$ is sufficiently close to 2^* then (2.1) has at least $\mathrm{cat}(\Omega)+1$ positive solutions, provided Ω has a nontrivial Ljusternik-Schnirelmann category $\mathrm{cat}(\Omega) > 1$ (Benci, Cerami and Passaseo, 1991). While, for example, if we remove k small balls from a given ball (so that the resulting set Ω has a fixed category $\mathrm{cat}(\Omega)$ independent of k) then, for p close to 2^*, the number of positive solutions of (2.1) increases up to $2k + 1$, provided we count them with their multiplicity in the sense of the critical groups (Benci and Cerami, 1994).

On the other hand, the uniqueness of positive solutions of (2.1) does hold if Ω is a ball and $p < 2^*$ (Gidas, Ni and Nirenberg, 1979). Similarly to the problem (2.2), if $\lambda < 0$ and $p < 2^*$ (Kwong, 1989), or if $\lambda > 0$ and $p \leqslant 2^*$ (Srikanth, 1993). In contrast, if Ω is an annulus and $p < 2^*$ is sufficiently close to 2^* then (2.1) admits one positive radial solution and a further positive nonradial solution (Brezis and Nirenberg, 1983).

As for sign-changing solutions, the number $N(u)$ of nodal domains of a solution u of (2.1) can be estimated by its Morse index $m(u)$ (see Section 3). It is an elementary fact that the inequality $N(u) \leqslant m(u)$ always holds, while if Ω is a ball or an annulus and u is radially symmetric then $N(u) \leqslant 1 + \frac{m(u)}{N+1}$ (Aftalion and Pacella, 2004).

Other, more specialized questions were studied in the past decades. It is also of great interest to consider more general nonlinearities, and in fact some of the results above hold for more general equations than (2.1). In the sequel, by a *superlinear and subcritical* nonlinearity g we will mean a function $g \in C^1(\mathbb{R};\mathbb{R})$ such that:

(i) $g'(s)s^2 \geqslant g(s)s > 0 \ \forall s \neq 0$;

(ii) $g(s)s \geqslant \mu G(s)$ for large $|s|$, where $\mu > 2$;

(iii) $|g(s)| \leqslant C(1 + |s|^{p-1}) \ \forall s$, with $2 < p < 2^*$.

We have used the notation $G(s) := \int_0^s g(\xi)\, d\xi$ (so $G(s) = |s|^p/p$ if $g(s) = |s|^{p-2}s$). We stress that, in strong contrast with the one-dimensional problem, the mere existence of solutions for a more general equation

$$-\Delta u = g(u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (2.3)$$

with g superlinear and subcritical, is not settled. In spite of the numerical evidence suggesting the existence of many solutions for (2.3) (Ding, Costa and Chen, 1999), only a three-solutions theorem is established so far (Wang, 1991): (2.3) admits a positive solution, a negative solution, and a further sign-changing solution.

However, in case g is superlinear, subcritical and *odd symmetric* ($g(-s) = -g(s) \ \forall s$) then the existence of an infinite number of solutions can be proved. In particular,

Theorem 1. [3] *For any $p < 2^*$ and any domain Ω, problem (2.1) admits an unbounded sequence of solutions.*

It is not known whether the number of nodal domains of these solutions is arbitrary large, neither whether a solution having at least three nodal domains does exist. In Section 3 we develop the content of Theorem 1 from a general point of view, while in Section 4 we present some new results in this direction.

3. Minimax theorems

In the rest of the paper g is a superlinear and subcritical nonlinearity. Solutions of (2.3) can be seen as *critical points* of the energy functional

$$I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(u), \qquad u \in H_0^1(\Omega), \quad (3.1)$$

where $G(s) := \int_0^s g(\xi)\, d\xi$, that is, points u such that $I'(u) = 0$. A number $c \in \mathbb{R}$ is a *critical value* of I if $I(u) = c$ for some critical point u. Since the functional is unbounded both from below and from above, global minimization/maximization is excluded. One can find critical points of I by either using constrained minimization or a minimax theorem.

The most celebrated minimax theorems in critical point theory are the Mountain Pass Theorem and the Saddle Point Theorem, due to Ambrosetti and Rabinowitz [3] and Rabinowitz [22] respectively, in the 70s. This followed earlier work which can be traced back to Birkhoff (1917), Ljusternik and Schnirelmann (1934) and M.A. Krasnosel'skiĭ (1964), among others.

We specialize the underlying idea to our problem. For every $k \in \mathbb{N}$, let us denote by E_k the finite dimensional space spanned by the first k eigenfunctions of

$(-\Delta, H_0^1(\Omega))$. For a given number $R_k > 0$, let $Q_k := B_{R_k}(0) \cap E_k$ be the ball of radius R_k in E_k, centered at the origin, and consider the class of (continuous) maps,

$$\Gamma_k := \{\gamma : Q_k \to H_0^1(\Omega) : \ \gamma \text{ odd}, \ \gamma|_{\partial Q_k} = Id\}.$$

Then we define the number

$$b_k := \inf_{\gamma \in \Gamma_k} \sup_{u \in Q_k} I(\gamma(u)). \tag{3.2}$$

It can be proved that $b_k > 0$ if R_k is sufficiently large, in particular $b_k \in \mathbb{R}$. Moreover,

$$b_k \leqslant b_{k+1} \ \forall k \quad \text{and} \quad \lim_{k \to \infty} b_k = +\infty.$$

The numbers b_k are natural candidates for being critical values of I. However, this will not be the case unless I is an *even* functional (i.e. $I(-u) = I(u) \ \forall u$). This amounts to ask that $g(s)$ is an odd nonlinearity.

So, in this case I admits indeed an infinite number of critical values; the corresponding critical points $(u_k)_k$ constitute a sequence of solutions to problem (2.3) whose $H_0^1(\Omega)$-norms tend to infinity as $k \to \infty$, and this settles Theorem 1 above.

Moreover, regardless of its symmetry, if g is asymptotically dominated by a pure-power nonlinearity $|s|^{p-2}s$, then one has the following estimates on the growth of b_k.

Proposition 2. [6, 30] *If* $g(s)s - |s|^p = \mathrm{o}(|s|^p)$ *as* $|s| \to \infty$ *then*

$$c_1 k^{2p/N(p-2)} \leqslant b_k \leqslant c_2 k^{2p/N(p-2)}$$

for some $c_1, c_2 > 0$ *independent of* k.

The second inequality is related to the asymptotic behavior of the eigenvalues of $(-\Delta, H_0^1(\Omega))$ and follows from the very definition of b_k. As for the first inequality, it arises from a semiclassical inequality of Cwikel, Lieb and Rosenbljum [16, 21, 28], which is used here in the context of *Morse index estimates.*

If u is a solution of (2.3), its Morse index $m(u)$ is defined as the number of negative eigenvalues of the linearized problem

$$-\Delta v = g'(u)v + \lambda v, \qquad v \in H_0^1(\Omega). \tag{3.3}$$

In an equivalent way, $m(u)$ is the supremum of the dimensions of the subspaces Z of $H_0^1(\Omega)$ where the quadratic form $I''(u)$ is definite negative (i.e. $I''(u)(\varphi, \varphi) < 0 \ \forall \varphi \in Z$, $\varphi \neq 0$).

Loosely speaking, for a general functional I, Morse theory is concerned with relating the structure of the critical point set of I in $\{a \leqslant I \leqslant b\}$ $(a, b \in \mathbb{R})$ with the homology, homotopy, homeomorphism, and diffeomorphism type of the pair $(\{I \leqslant b\}, \{a \leqslant I \leqslant b\})$.

The pioneering work of M. Morse on compact manifolds goes back to the 30s, followed by later developments and extensions to the infinite dimensional case by Palais, Rothe, Sard and Smale among others, in the 60s. As mentioned before, a decade later critical point theory in the framework of PDEs was giving its first steps. In the 80s these two methods were put aside; we quote the following paragraph from [6]:

"The two main methods in critical point theory are probably Morse theory (including Morse inequalities) and minimax variational approaches (as initiated by Ljusternik and Schnirelman). Morse theory usually provides (in some cases) critical points with a local information (i.e., the Morse index) but requires nondegenerate functionals and does not give precise indications on the energy levels. On the other hand, minimax critical point theory usually yields critical values by explicit formulas but lacks real local understanding of the structure of associated critical points. [...] More recently, attempts to understand the local nature of minimax critical points have been made."

This led to a huge literature on the subject, mostly in the 90s (see e.g. [15, 19, 24]). This is the context which relates the definition of the minimax levels b_k with the first estimate in Proposition 2. In our final section we explain the relevance of such type of estimates for problem (2.3).

4. Perturbation from symmetry

As mentioned above, (2.3) admits an unbounded sequence of solutions in case g is (superlinear, subcritical and) odd symmetric. In the one-dimensional case ($N = 1$, $\Omega = (-1, 1)$), under these assumptions the picture is rather clear. It is known ([12, 30]) that the solution set of the corresponding ODE consists precisely of a sequence $u_0, \pm u_1, \pm u_2, \ldots$ where $u_0 = 0$ and, up to the sign of $u_k'(0)$, u_k is completely determined by the condition: u_k possesses exactly $k - 1$ zeros in $(-1, 1)$. Moreover, going back to the numbers b_k defined in (3.2), we have that $b_k = I(u_k) < I(u_{k+1}) = b_{k+1}$ for every k. Finally, each solution u_k has Morse index k and is non-degenerate, in the sense that 0 is not an eigenvalue of the linearized (ordinary differential) equation in (3.3).

When the nonlinear term of the equation is no longer odd symmetric but rather behaves asymptotically like one, there seems to be no reason why a great number of solutions should cease to exist. Consider for example the following basic perturbed problem

$$-\Delta u = |u|^{p-2}u + f(x), \qquad u \in H_0^1(\Omega), \tag{4.1}$$

where $2 < p < 2^*$ and, say, $f \in L^2(\Omega)$. This problem was first studied in [5, 6, 23, 29, 30]. In particular, the following holds.

Theorem 3. [6, 30] *If*

$$p < \frac{2N-2}{N-2} \tag{4.2}$$

then (4.1) *admits an unbounded sequence of solutions.*

We mention that in the case where non-homogeneous Dirichlet boundary conditions are considered, a similar conclusion holds provided $p < 2N/(N-1)$, cf. [13].

It remains an open and challenging problem to know if the full range $p < 2^* = 2N/(N-2)$ can be allowed in Theorem 3. The following two results somehow suggest that this is the case.

Theorem 4. [2] *Given $f \in L^2(\Omega)$ and $k \in \mathbb{N}$ there exists $\varepsilon_0 = \varepsilon_0(k)$ such that for $|\varepsilon| < \varepsilon_0$ the problem*

$$-\Delta u = |u|^{p-2}u + \varepsilon f(x), \qquad u \in H_0^1(\Omega),$$

with $p < 2^*$ admits at least k solutions.

Theorem 5. [4] *If $p < 2^*$ then the set of $f \in H^{-1}(\Omega)$ such that the problem* (4.1) *has infinitely many weak solutions is a dense residual set in $H^{-1}(\Omega)$.*

Let us describe roughly the underlying idea in the proof of Theorem 3. As in (3.1), let

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p - \int_\Omega f(x)u, \qquad u \in H_0^1(\Omega).$$

We denote by S the unit sphere in $H_0^1(\Omega)$ and by J the functional

$$J(u) := \max_{t>0} I(tu), \qquad u \in S.$$

It can be proved that there is a one-to-one correspondence between critical points of I and critical points of J. Moreover, the numbers b_k constructed above can also be defined as

$$b_k = \inf_{\gamma \in \mathscr{A}_k} \sup_{u \in S_k} J(\gamma(u)),$$

where $S_k = S \cap E_k$ and

$$\mathscr{A}_k := \{\gamma : S_k \to S,\ \gamma \text{ continuous and odd}\}.$$

Since J is not an even functional, b_k is not expected to be a critical value of J. However, let

$$I^*(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p, \qquad u \in H_0^1(\Omega),$$

and consider the corresponding functional J^* and minimax levels b_k^*. Then b_k^* is a critical value for J^*. Moreover it can be proved that given $a > 0$ then J has a critical value $c > a$ provided

$$\{J^* \leqslant b_k^* + \varepsilon\} \subset \{J \leqslant a\} \subset \{J \leqslant a + \varepsilon\} \subset \{J^* \leqslant b_{k+1}^* - \varepsilon\}$$

for some $\varepsilon > 0$.

So we see that the existence of infinitely many critical values for J will be a consequence of proving that the intervals (b_k^*, b_{k+1}^*) are large enough with respect to the difference $|J - J^*|$. Here is where Proposition 2 and the condition (4.2) come into play.

Going back to the symmetric problem (2.1), a further natural question concerns the sign of these solutions. The following theorem complements Theorem 1.

Theorem 6. [8, 11, 20] *If $p < 2^*$ then* (2.1) *admits a sequence of unbounded sign-changing solutions.*

The proof of Theorem 6 uses an homological description of the minimax levels. In order to deal with the perturbed symmetric problem, in [27] an elementary approach, based upon the ideas described in the previous section, was proposed. Now we deal with perturbations such as

$$-\Delta u = |u|^{p-2}u + f(x,u), \qquad u \in H_0^1(\Omega). \tag{4.3}$$

Theorem 7. [27] *If $p < (2N-2q)/(N-2)$ and $f(x,s)$ is a continuous function such that $f(x,s)/ \to s \to 0$ as $s \to 0$ uniformly in x and $0 \leqslant f(x,s)s \leqslant C(1+|s|^q)$, $0 < q < p$, then* (4.3) *admits a sequence of unbounded sign-changing solutions.*

A related problem concerns the case where

$$-\Delta u = V(x)|u|^{p-2}u + f(x), \qquad u \in H_0^1(\Omega). \tag{4.4}$$

and $V \in C^1(\overline{\Omega})$ changes sign in Ω. This new feature of the nonlinearity causes a lack of compactness (in fact if we deal with the homogeneous function $g(s) = |s|^{p-1}s$ this is not a serious problem, but it becomes a serious one whenever, say, $g(s) = |s|^{p-1}s + |s|^{r-1}r$ with $p \neq r$). The following was proved by means of a variational technique which relates a priori bounds of the solutions with a priori bounds of their Morse indices, an idea which goes back to [7].

Theorem 8. *Assume that $V(x)$ has only nondegenerate zero points in Ω. Then problem (4.4) has an unbounded sequence of solutions in the following two situations:*

(a) [26] $f \equiv 0$ *and* $p < 2^*$*;*

(b) [25] $f \in C(\overline{\Omega})$ *and* $p < (2N-2)/(N-2)$.

Concerning the symmetry of the solutions, we state the following deep result. A corresponding one for the perturbed problem (with, say, $f(x)$ radially symmetric) is not known.

Theorem 9. [17] *If Ω is a ball or an annulus and $p < 2^*$ then problem (2.1) admits an unbounded sequence of radially symmetric solutions and a further unbounded sequence of nonradially symmetric solutions.*

We also mention that the above problems have a natural extension to systems of the form

$$-\Delta u = |v|^{q-2}v + f_1(x), \quad -\Delta v = |u|^{p-2}u + f_2(x), \quad (4.5)$$

with $u, v \in H_0^1(\Omega)$, $p, q > 2$ and, say, $p \leqslant q$ (this reduces to (4.1) if $p = q$ and $f_1 = f_2$).

Theorem 10. *Problem (4.5) has an unbounded sequence of solutions $u, v \in H_0^1(\Omega)$ in the following two situations:*

(a) [1] $f_2 \equiv f_2 \equiv 0$ *and* $\frac{1}{p} + \frac{1}{q} > \frac{N-2}{N}$*;*

(b) [14] $f_1, f_2 \in L^2(\Omega)$ *and* $\frac{N}{2}(1 - \frac{1}{p} - \frac{1}{q}) < \frac{p-1}{p}$.

We observe that if $p = q$ then case (a) reduces to the assumption that $p < 2^*$, while (b) is precisely (4.2). No "generic" results in the spirit of Theorems 4 and 5 are known for (4.5).

Finally, we mention two directions of research on this type of problems: the case where Ω is the entire space $\mathbb{R}^N$ (see [9] for sign-changing solutions in the symmetric case; the non-symmetric case seems to be open); the case where the problem is sublinear rather than superlinear, i.e. $g(s) = |s|^{p-2}s$ with $p < 2$ in the model equation (see [18] for the non-symmetric case).

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Math in the Media

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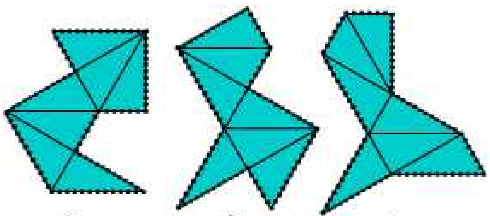
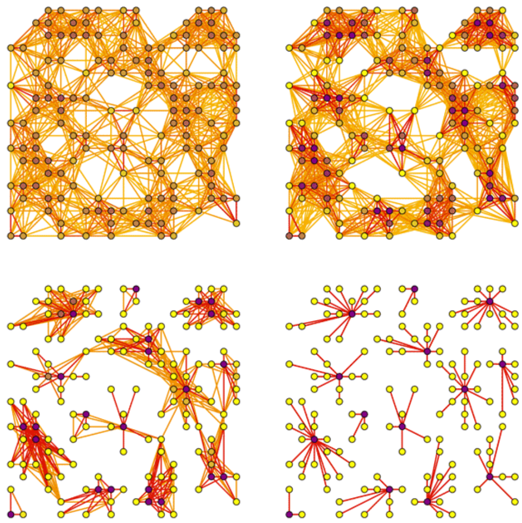
Hearing molecular drums. *Science* for February 8, 2008 ran a Report by a 6-member Stanford team entitled "Quantum Phase Extraction in Isospectral Electronic Nanostructures." The team, led by Hari Manoharan, took advantage of the discovery (Carolyn Gordon, David Webb, Scott Wolpert, 1992) of pairs of distinct polygonal shapes *isospectral* in the sense that they had exactly the same vibrational profile: identical responses at every frequency. This discovery was the long-awaited answer to Mark Kac's 1966 question "Can you hear the shape of a drum?" The *Science* authors use carbon monoxide molecules to draw a pair of different but geometrically isospectral shapes on the surface of a copper crystal. Each has area about 57 square nanometers, and encloses about 30 of "the 2D Fermi sea of electrons" that inhabit the surface; this pond of electrons will function as a "vibrating medium."



Three quantum nano-resonators assembled from carbon monoxide molecules. **a** "Bilby," **b** "Hawk," **c** "Broken Hawk." Each is assembled on a copper crystal by placing 90 CO molecules (black dots) around a polygonal contour. The polygon is geometrically the union of seven identical 30-60-90-degree triangles. In polygons **a** and **b**, any two adjacent triangles are related by reflection across their common border; this does not hold for **c**. The polygonal shapes **a** and **b** are known to be mathematically isospectral (but different in this respect from **c** even though **c** matches, for example, their area and perimeter); the authors exploit this feature to directly access the phase of the quantum-mechanical system formed by the surface electrons trapped inside the CO walls. Image after Manoharan *et al.*

The authors remark that "the time-independent Schrödinger equation is also a wave equation defined by the Laplacian and boundary conditions," i.e. the same equation that governs the sound of a drum, and that therefore the electronic resonances of the set of captured electrons will be the same for the two structures. Their main result is showing that "the complete phase information of wave functions in both structures can be experimentally determined" by "harnessing the topological property of isospectrality as the additional degree of freedom." This is physically significant because the spatial variation of the phase of the wave functions is measured without the usual reliance on interference phenomena. The supplementary information for this report includes a movie with soundtrack where the Schrödinger vibrations of the Bilby, Hawk and Broken Hawk nano-structures can be "heard" (at the rate of 100 THz $\sim$ 1 KHz).

Markov Clusters in the tree of life. "Long-held ideas regarding the evolutionary relationships among animals have recently been upended by sometimes controversial hypotheses based largely on insights from molecular data." So begins the abstract of a paper in the April 10 2008 *Nature*. The authors, an 18-member international team led by Casey Dunn (Brown), present in "Broad phylogenomic sampling improves resolution of the animal tree of life" a new method for selecting the genes to analyze in order to more accurately understand the relative position of species on the evolutionary tree.



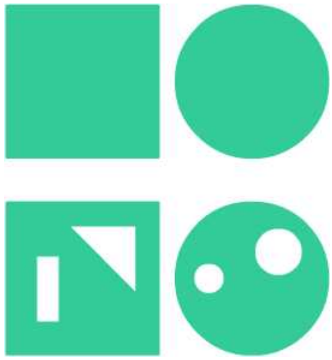
The Markov Cluster Algorithm at work. Red represents intensity. In Nieland's words: "... flow between different dense regions of nodes which are sparsely connected eventually evaporates, showing cluster structure present in the original input graph." Image courtesy of Stijn van Dongen.

"We present a new approach to identification of orthologous genes in animal phylogenomic studies that relies on a Markov cluster algorithm to analyse the structure of BLAST hits to a subset of the NCBI HomoloGene Database." BLAST (basic local alignment search tool) is a powerful algorithm, invented in 1990, for locating occurrences of a piece of genetic code in the NCBI (National Center for Biotechnology Information) database. The Markov Cluster Algorithm was devised in 2000 by Stijn van Dongen. It uses a stochastic, dynamic procedure to pinpoint the most significant part of a graph. In Henk Nieland's words: "Simulate many random walks (or flow) within the whole graph, and strengthen flow where it is already strong, and weaken it where it is weak. By repeating the process an underlying cluster structure will gradually become visible." (animated MCL algorithm simulation available in the main MCL website at `http://micans.org/mcl`).

The path to algebra: fractions. "News of the Week" in *Science* (March 21, 2008) was a story by Jeffrey Mervis about the National Mathematics Advisory Panel's release the week before of "a 120-page report on the importance of preparing students for algebra ... and its role as a gateway course for later success in high school, college, and the workplace." The report is available online (www.ed.gov/about/bdscomm/list/mathpanel). Mervis spoke with Larry Faulkner, the chair of the panel, and reports that the panel "avoided taking sides in a debilitating 2-decade-long debate on the appropriate balance between drilling students on the material and making sure they understand what they are doing." The recommendations are that "students should memorize basic arithmetic facts and spend more time on fractions and their meaning." But, as Mervis explains, "how teachers achieve those goals is up to them."

- Why do so many students have trouble with fractions? Faulkner: "Fractions have been downplayed." He mentions the perception that decimals and spreadsheets have eliminated the need for fractions. "But it's important to have an instinctual sense of what a third of a pie is, or what 20% of something is, to understand the ratio of numbers involved and what happens as you manipulate it."
- Q: Was the panel disappointed by the overall quality of existing education research? A: " ... We found a serious lack of studies with adequate scale and design for us to reach conclusions about their applicability for implementation."
- Q: Should the government be spending more money on this research? A:" ... If you want to get the value, you probably need to pay for it."
- Professional development programs? A: "There's tremendous variation in in-service programs. And the evidence is that many are not very effective."
- Calculators? "We feel strongly that they should not get in the way of acquiring automaticity [memorization of basic facts]. But the larger issue is the effectiveness of pedagogical software. At this stage, there's no evidence of substantial benefit or damage, but we wouldn't rule out products that could show a benefit."

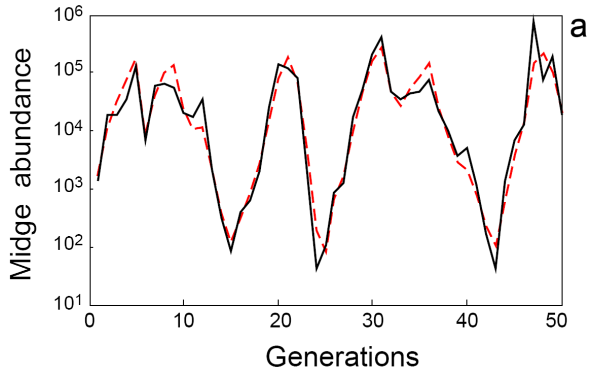
The hole truth?



The Schwarz-Christoffel formula (top) gives a conformal map from an arbitrary polygon to the unit disc. The generalization published in March 2007 by Darren Crowdy (below) applies to polygonally bounded domains of arbitrary topology. Image after Crowdy.

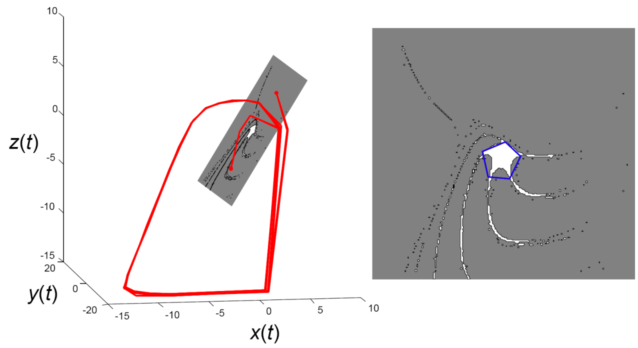
The Riemann mapping theorem guarantees a conformal map between any proper simply-connected planar domain and the open unit disc. In general, the map is constructed as the limit of an infinite process; but in case the domain is a polygon, an explicit, finite formula was found in the 1860s by Schwarz and Christoffel. An item by Adrian Cho, published March 6, 2008 on *Science*'s "Science Now" website, covers the publication a year ago (*Math. Proc. Camb. Phil. Soc.* 142 (2007) 319) of a generalization of the Schwarz-Christoffel formula to multiply-connected polygonal domains. The author was Darren Crowdy (Imperial College London); Cho quotes him: "If you give me any polygon with any number of polygonal holes, I can map it to a circle with the same number of circular holes." Crowdy's discovery "has been creating a buzz this week with coverage in several newspapers in the United Kingdom." For example, "140 Year-Old Schwarz-Christoffel Math Problem Solved" on `scientificblogging.com`. The point of Cho's piece, however, is not the mathematics but a priority controversy. Thomas DeLillo and Alan Elcrat (Wichita State), together with John Pfaltzgraff (Chapel Hill) published "Schwarz-Christoffel Mapping of Multiply Connected Domains" in the *Journal d'Analyse* (94 (2004) 17-47), and claim their share of the glory. According to Cho, "The Americans' formula ... involves the multiplication of an infinite number of terms, which goes haywire if the holes are too close together." Crowdy asserts that his method, which "replaces that product with an obscure beast known as Schottky-Klein prime function" (in Cho's words) is more reliable. Pfalzgraff is "very skeptical." Cho ends on a conciliatory note by quoting Michael Siegel (NJIT Newark) "It's a breakthrough, and all these people contributed." Cho's title: "Mathematicians Debate the Hole Truth."

Midge dynamics in Lake Myvatn.



50 generations of midge population in Lake Myvatn. The solid line represents observations, the dashed line output from the mathematical model with nine tuned parameters. Image courtesy of Anthony Ives.

"Mathematics Explains Mysterious Midge Behavior" is the title of an article by Kenneth Chang in the March 7 2008 *New York Times*. At Myrvatn ("Midge Lake") in northern Iceland, during mating season, the air can be thick with male midges (*Tanytarsus gracilentus*), billions of them. Chang quotes Anthony Ives (Wisconsin) "It's like a fog, a brown dense fog that just rises around the lake." And yet in other years, at the same time, there are almost none. Ives was the lead author on a report in *Nature* (March 6 2008) that gave an explanation for this boom-and-bust behavior in which, as Chang describes it, "the density of midges can rise or fall by a factor of a million within a few years." In the *Nature* report ("High-amplitude fluctuations and alternative dynamical states of midges in Lake Myvatn"), Ives and his co-authors characterize the midge ecology as one "driven by consumer-resource interactions, with midges being the consumers and algae/detritus the resources" and they set up a system of three coupled nonlinear difference equations, one each for midges, algae and detritus, to model it. The dynamics of this system include a stable state as well as a stable high-amplitude cycle; small variations in parameters can drive the system from one of those attractors to the other.



Alternative stable states of the midge-algae-detritus model. In the panel on the left, the plane is tangent to the manifold containing the cyclic component of the dynamics around the stationary point. The white region in the plane shows the domain of attraction to the invariant closed set, whereas the region in grey gives the domain of attraction to the outer stable cycle. The red lines give two examples of trajectories that converge to the outer stable cycle. The panel on the right shows the plane in more detail to illustrate the fine structure of the domain of attraction to the invariant closed set. The blue pentagon shows the unstable period 5 cycle that makes up part of the boundary between domains of attraction to the inner invariant closed set and the outer stable cycle. Image courtesy of Anthony Ives.

Parallel transport for qubits. *Science* for December 21, 2007 ran "Observation of Berry's Phase in a Solid-State Qubit." The authors are a team of ten from ETH, Sherbrooke and Yale, headed by Peter Leek and Andreas Wallraff. Their work falls under the rubric of what Seth Loyd called "holonomic quantum computation" (*Science* 292 5222): they use Berry phase changes produced by motion along paths (here the paths are in parameter space) to systematically manipulate the state of a qubit.

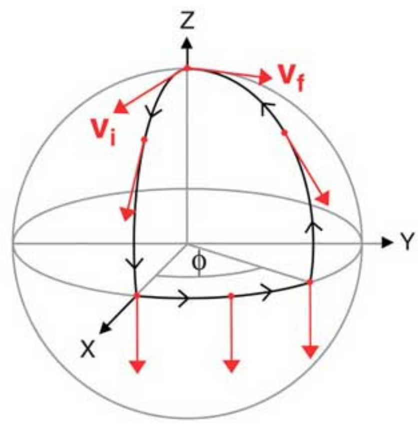


Fig. 1. Parallel transport of a vector around a geodesic triangle on the unit sphere, beginning (initial state v_i) and ending (final state v_f) at the North Pole. Image courtesy of Peter Leek and Andreas Wallraff.

Fig. 1 could come from a differential geometry text: it shows the parallel translation of a tangent vector around a $(\phi, \pi/2, \pi/2)$ geodesic triangle on the unit sphere. The final state $\mathsf{v_f}$ is rotated with respect to the initial state $\mathsf{v_i}$ by an angle which, when measured in radians, is exactly equal to the area enclosed by the path of the transport: in this case, ϕ. "The analogy of the quantum geometric phase with the above classical picture is particularly clear in the case of a two-level system (a qubit) in the presence of a bias field that changes in time," the authors write. In fact the set of states of a qubit may be represented as a sphere: an arbitrary superposition $z_0 < 0| + z_1 < 1|$ of its two base states corresponds to the point $[z_0 : z_1]$ in complex projective 1-space, which can be identified with the Riemann sphere by $[z_0 : z_1] \to z_0/z_1$ and stereographic projection.

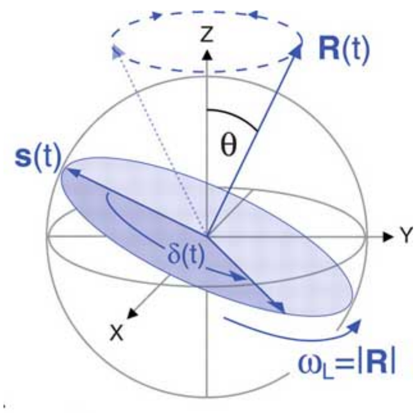
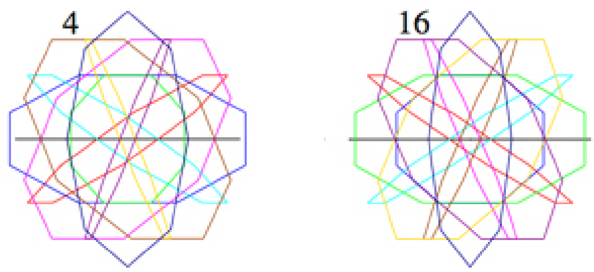


Fig. 2. The state space of a qubit can be represented as a sphere, with pure state $< 1|$ at the North Pole and $< 0|$ at the South Pole. Here the state s precesses at fixed speed about a vector R, and R itself is moving, at much slower speed, along a path of its own. Image courtesy of Peter Leek and Andreas Wallraff.

Suppose that as in Fig. 2, "the qubit state s continually precesses about the vector R, acquiring dynamic phase $\delta(t)$ at a rate $\mathsf{R} = |\mathsf{R}|$." "When the direction of R is now changed adiabatically in time (i.e., at a rate slower than R), the qubit additionally acquires Berry's phase while remaining in the same superposition of eigenstates with respect to the quantization axis R." When the axis R has been brought back to its original position after traversing a path C in its parameter space (via a two-stage maneuver that results in zero dynamic phase accumulation), "the geometric phase acquired by an eigenstate is $\pm\theta_{\mathsf{C}}/2$, where θ_{C} is the solid angle of the cone subtended by C at the origin." In the example illustrated, that cone is geometrically the same cone traced out by R in Fig. 2. As the authors remind us, its solid angle (i.e. the enclosed area intercepted on the unit sphere) "is given by $\theta_{\mathsf{C}} = 2n(1 - \cos\theta)$, depending only on the cone angle θ." [Note that the phase change is only 1/2 of the solid angle, in contrast with the purely geometric example. So a 360° planar rotation – enclosed area $2n$ – reverses the sign of the qubit.]

The article goes on to describe the experimental setup for implementing this phenomenon in real life. The qubit is a Cooper-pair box, the R-motions are driven by pulse-modulated microwave frequency signals, and the result is measured using quantum-state tomography.

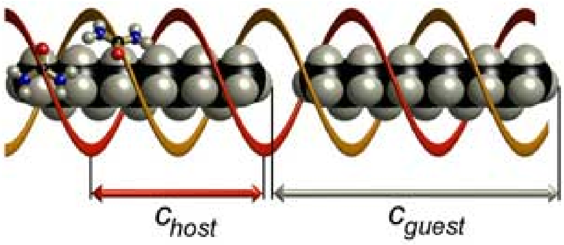
An 80-vertex polytope in *Physical Review*. Eric Altschuler and Antonio Pérez-Garrido published an article in *Physical Review* last year (E 76 016705 (2007)) in which they described "a four-dimensional polytope, new to our knowledge, with a high degree of symmetry in terms of the lengths of the sides." They found the configuration "by looking at the ... problem of finding the minimum energy configuration of 80 charges on the surface of the hypersphere S^3 in four dimensions" with the energy function $\Sigma(1/r_{ij})$ where r_{ij} is the distance between the i-th and j-th points, and the sum is taken over all pairs of distinct points. (They remark that they cannot prove this is actually a global minimum, but add that "even good local minima can be interesting or important configurations.") The other N for which they found symmetric configurations are 5, 8, 24 and 120; corresponding to the 4-simplex, the dual of the 4-cube, the 24-cell and the 600-cell. The authors give a method for visualizing their 80-vertex polytope in terms of the Hopf map $S^3 \to S^2$. They triangulate S^2 with 16 equal equilateral triangles: 4 abutting the North Pole, 4 the South, and a band of 8 around the Equator. This polyhedron has 10 vertices. Each of these vertices corresponds to a circle of the Hopf fibration, along which they describe explicitly how to place 8 of the polytope's vertices.



Two from a sequence of 20 projections of the 80-vertex polytope from 4-space into the plane. Each of the 10 Hopf-fibration circles has a different color, and appears as an octagon linking its 8 polytope vertices. Entire sequence, each projection composed with an additional rotation by 30° about a fixed plane in 4-space, available in `www.ams.org/mathmedia/images/altschuler-complete.jpg`. Images courtesy of Eric Altschuler.

Another description of the 80-vertex polytope was published by Johannes Roth later in the same journal (E 76 047702 (2007)).

Physical Chemistry in 4D.

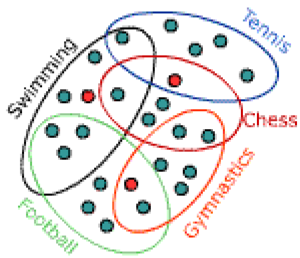


The structure of an alkane-urea channel-inclusion compound. $C_{host} = 1.102$ nm at room temperature. The "host" urea subsystem (spirals) and the "guest" alkane have irrationally related periodicity, which leads to the material presenting phase transitions that can only be explained in a 4-dimensional "superspace." Image courtesy of Bertrand Toudic.

"Hidden Degrees of Freedom in Aperiodic Materials" is a report in the January 4 2008 *Science*. The first author is Bertrand Toudic (Rennes); of the other nine, seven are based in France, one at Kansas State and one in Bilbao. The geometric structure of a crystal is described by listing its planes of reflection symmetry, labelled by triples (h, k, l) of integers (Miller indices) that describe their slopes in coordinates (a, b, c) adapted to the crystal. These planes give rise to characteristic patterns of peaks of intensity in photographic (or other) records of how the crystal scatters radiation. Toudic and his team investigate "aperiodic" materials like the alkane-urea compound illustrated above. This compound consists of a framework of nanotubes ("urea molecules are connected by hydrogen bonds to form helical ribbons, which are woven together to form a honeycomb array of linear, nonintersecting, hexagonal tunnels") inside which "Guests such as nonadecane pack end to end within van der Waals contact of each other." In case the repeat length C_{host} of the helical structure of the tunnels and the packing distance C_{guest} of the alkane guests are not rationally related [presumably, on an appropriate scale], we need an additional Miller index to explain the diffraction patterns. The sets of Miller indices are now vectors (h, k, l, m) in a "superspace" of which the first three dimensions are the familiar ones.

The authors bring this extra dimension into salience by exhibiting a phase transition that cannot exist without it. As Philip Coppens explains it, in a "Perspectives" piece in the same issue of Science, when the c axis is pointing along the tube, " the average structure of the urea ... is described by the $hkl0$ reflections, and the average structure of the alkane ... by the $hk0m$ reflections, whereas the remaining $hklm$ reflections are due exclusively to the mutual interaction between the two lattices. This implies that [the urea lattice] imposes a distortion on [the alkane lattice], and vice versa." "At temperatures above 149 K, all nonadecane columns in the crystal distort in an identical way. However, below this temperature, the extra $hklm$ reflections that appear in the diffraction pattern show that the relative modulation of the host and guest lattices alternates from channel to channel in the a-axis direction ... even though the periodicity of the average structures of the host and the guest in this direction does not change, as indicated by the absence of additional $hkl0$ and $hk0m$ reflections." He concludes "Such a transition, which only affects the mutual interaction, can only be described properly in super-space, even though the physical reality is obviously three-dimensional."

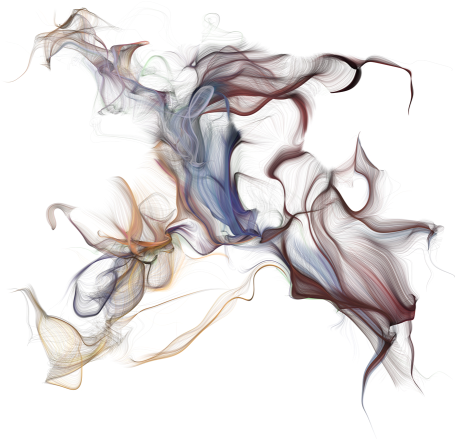
Physical insight into a hard combinatorial problem.



The hitting-set problem. Here the job is to find a minimal-size set of students (discs) representing all five sports. The red discs are a "hitting set." Image after Selman.

Since work of Mitchell, Selman and Levesque in 1992 it has been understood that some hard computational problems can undergo phase transitions at critical values of their parameters. Recently this statistical mechanical behavior has been harnessed to yield information about the solutions of some hitherto intractable problems. The work, by Marc Mézard and Marco Tarzia, appeared (*Phys Rev* E76 041124) last year, and was picked up by Bart Selman in a "News and Views" piece for *Nature*, February 7, 2008. Selman tells us that the authors "demonstrate an innovative approach to solving one well-known NP-complete problem, known as the hitting-set problem." A hitting set picks out from the union of a collection of sets a subset that contains at least one element of each; the problem is to find a hitting set with the smallest number of elements. Mézard and Tarzia, realizing "that tools from statistical physics developed to study physical phase transitions might help in developing more efficient algorithms for solving combinatorial problems," adapted the calculation of ground-state properties of certain condensed-matter systems to give the *survey-propagation method.* "Mézard and Tarzia use the survey-propagation method to compute statistical properties of the solutions of instances of the hitting-set problem." This is considered even harder than finding a single solution, but survey propagation, working near a phase boundary, can get the information "... by iteratively solving a large set of coupled equations, modelling the local interactions between variables probabilistically. This solution process can be performed in a parallel, distributed fashion using many different processors, and generally converges to an answer extremely quickly –in seconds for equations with thousands of variables."

"The computational realization of gesture".



Thomas Briggs Veils # 73. A larger image is available in www.salientimages.com/Veils73.htm but, as Briggs explains: "In order to represent these images on a web site they must be reduced in resolution by 99%. The works are a minimum of 3 feet square. The actual line weight is equivalent to that of a 0.2 - 0.3 millimeter pen nib, yet the large scale structure holds up when seen from a distance. This disparity of scale is an essential element of the experience of the works." Image used with permission.

On Thomas Briggs' website (www.salientimages.com) the artist details his methods, and the way mathematics enters into them: "The computational realization of gesture in my practice entails the construction of a spatial field of action. In this space various mathematical functions which represent small aspects of movement are distributed. The sum of the various functions is recorded for millions of points in space. These data are collated and translated into thousands of drawing primitives which are written into an image file for printing and archiving."

Numeral cognition and language. What is the relation between our concepts of number and the words we have in our language to express them? The old question was recently thrown into relief by Peter Gordon's report in *Science* (October 15, 2004) on the Pirahã, an extremely inscrutable Amazonian tribe whose language seems almost completely devoid of number-words. Gordon (Biobehavioral Sciences, Columbia) was categorical: "... the Pirahã's impoverished counting system limits their ability to enumerate exact quantities when set sizes exceed two or three items." Gordon was taken to task by Daniel Casasanto (Brain and Cognitive Sciences, MIT) who argues (Letters, *Science*, March 18, 2005) that "[the] results are no less consistent with the opposite claim [i.e., that they lack number words because they lack number concepts], which is arguably more plausible." The Pirahã controversy is the background for "The Limits of Counting: Numerical Cognition Between Evolution and Culture" (*Science*, January 11, 2008). The authors, Sieghard Beller and Andrea Bender (Psychology, Freiburg), focus on the evolution of numbering systems, for which they distinguish two properties: extent and degree of abstractness. They take their examples from Austronesian languages; Adzera is one of them. "Its number words for 1 to 5 are composed of numerals for 1 and 2 only: bits, iru$^?$, iru$^?$ da bits ($= 2 + 1$), iru$^?$ da iru$^?$ ($= 2 + 2$), and iru$^?$ da iru$^?$ da bits ($= 2 + 2 + 1$)." This is a system with small *extent*. The authors contrast Adzera with Mangarevan, where besides a general counting sequence there is another one used for tools, sugar cane, pandanus (a fruit) and breadfruit, while ripe breadfruit and octopus are counted with a different sequence, and the first breadfruit and octopus of a season are counted with yet another. This system lacks *abstractness*. The point the authors emphasize is that both these languages "belong to the same linguistic cluster ... and inherited a regular and abstract decimal numeration system with (at least) two powers of base 10 from their common ancestor, Proto-Oceanic." As they state in their conclusion, "Numeration systems do not always evolve from simple to more complex and from specific to abstract systems."

The mathematics of choosiness. "The coevolution of choosiness and cooperation," a Letter in the January 10 2008 *Nature*, describes a mechanism for the evolution of cooperative behavior. The Bristol-Debrecen team of John McNamara, Zoltan Barta, Lutz Fromhage and Alasdair Houston ran simulations of the "continuous snowdrift game," where in each round an individual, playing against one other, incurs a cost $C(x)$ depending on its own cooperativeness x, and receives a benefit $B(x + x')$ depending on the summed cooperativeness of both players. [The "snowdrift game" gets its name from an example where two drivers are stuck on opposite sides of a snowdrift, and have to choose between waiting in the car and shovelling]. Here are some details of the simulations: Along with cooperativeness (x), each player has a trait y called choosiness. Choosiness specifies the minimum degree of cooperativeness that the player will accept from its co-player.

- After each round, a player earns the payoff $B(x, x') - C(x) + A - S$, where B and C are as above, A is a fixed component and S is the startup cost for an individual in a newly formed pair.

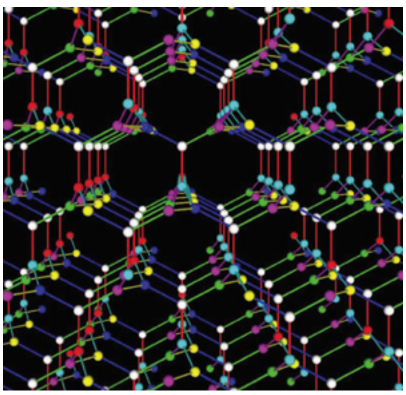
- At the same time, the players learn their partner's cooperativeness, and choose whether to look for new partners (if $x < y'$ or $y < x'$) or to play again (otherwise).

- After each round an individual also produces offspring (clones except for "occasional small changes caused by mutation") proportionally to the size of the payoff;

- Between rounds the players "incur some risk of mortality." "Individuals that die are replaced by individuals selected at random from all offspring produced in the previous round."

Among the main conclusions of the experiments: "in a situation where individuals have the opportunity to engage in repeated pairwise interactions, the equilibrium degree of cooperativeness depends critically on the amount of behavioural variation that is being maintained in the population by processes such as mutation." Additionally, "The results suggest an important role of lifespan in the evolution of cooperation." The authors give heuristic arguments to interpret these results: in a uniform population nothing can be gained by being choosy, and therefore there is no incentive for individuals to be cooperative. "This situation changes profoundly if significant variation is maintained in the population by processes such as mutation." Moreover, high mortality counteracts the evolution of cooperation: "If the cooperative associations ... are soon disrupted by mortality, then establishing them is not worth the associated costs."

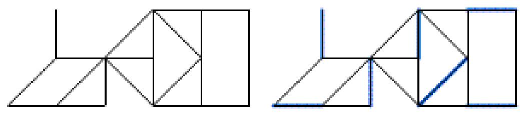
"A Mathematical Gem". is how Constance Holden (Random Samples, *Science*, January 18, 2008) describes this image, gleaned from the February 2008 issue of the *AMS Notices*.



The K-4 crystal is the maximal abelian covering of the tetrahedron, with the inherited geometry. For a larger and higher-resolution image, visit the February 2008 Notices. Image credit Hisashi Naito.

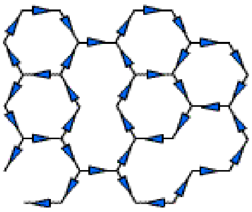
There it illustrates an article by Toshikazu Sunada, who shows that this crystalline structure shares with the diamond the "strong isotropy property," and that these are the only two such structures in three dimensions. (The strong isotropy property states that for any two vertices V and W of the crystal, any ordering of the edges adjacent to V and any ordering of the edges adjacent to W, there is a lattice-preserving congruence taking V to W and each V-edge to the similarly ordered W-edge). Sunada states that the K-4 crystal, beautiful as it is, is purely a mathematical object. Holden begs to differ: "In fact, it shows up in inorganic compounds, lipid networks, and liquid crystals and has been known for decades by other names."

Holographic algorithms. The January-February 2008 *American Scientist* features a report by Brian Hayes on holographic (or "accidental") algorithms, a recent phenomenon in computational mathematics. "Their computational power comes from the mutual cancellation of many contributions to a sum, as in the optical interference pattern that creates a hologram," according to their inventor, Leslie Valiant (Harvard); hence the name. The primordial "holographic" algorithm is the determinant of an n by n matrix: in principle it is a sum of $n!$ terms, but in practice, using row-reduction, it can be computed with only about n^3 operations. In fact, determinants turn out to be at the heart of all the examples Hayes presents. For example, the "perfect matching" problem: on a given graph, is there a set of edges linking each vertex to exactly one other vertex? And the associated counting problem: if so, how many such matchings are there?



A graph with one of its perfect matchings. After Hayes, *American Scientist* 96, No. 1.

This question seems to require looking at all possible choices of edges (a number growing factorially with the number of vertices) to see which ones work, but for a planar graph the Fisher-Kasteleyn-Temperley (FKT) algorithm, dating back to the early 1960s, equates the calculation of the number of perfect matchings on a planar graph with n vertices to the calculation of the determinant of a certain n by n matrix. Valiant's new holographic algorithms go one step further, and relate calculations in one context to the perfect matching problem in an associated graph. One such context, the "Three-ice problem," is illustrated below. Hayes explains: "The strategy is to build a new planar graph called a matchgrid, which encodes both the structure of the ice graph and the not-all-equal constraints that have to be satisfied at each vertex. Then we calculate a weighted sum of the perfect matchings in the matchgrid, using the efficient FKT algorithm. Although there may be no one-to-one mapping between individual matchings in the matchgrid and valid assignments of bond directions in the ice graph, the weighted sum of the perfect matchings is equal to the number of valid assignments."



The "Three-ice problem." For a planar graph where each vertex abuts 1, 2 or 3 edges, how many ways can the edges be oriented (blue arrows) with no in-in-in or out-out-out configurations? The image shows one admissible assignment of orientations. After Hayes.

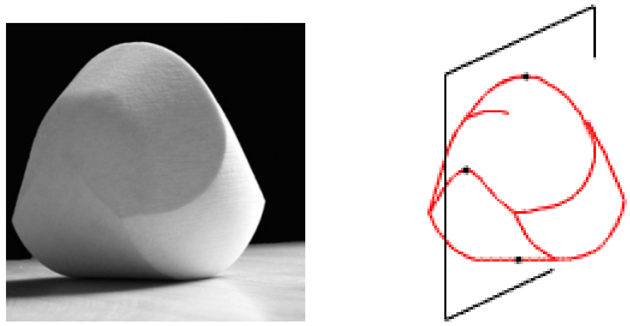
"Everything in our world is purely mathematical - including you". This startling quotation occurs about halfway through Dennis Overbye's "Laws of Nature, Source Unknown" in the December 18 2007 *New York Times*. Overbye attributes it to Max Tegmark, a cosmologist at MIT, whom he considers "The ultimate Platonist ... In talks and papers recently he has speculated that mathematics does not describe the universe - it is the universe. Dr. Tegmark maintains that we are part of a mathematical structure, albeit one gorgeously more complicated than a hexagon, a multiplication table or even the multidimensional symmetries that describe modern particle physics. Other mathematical structures, he predicts, exist as their own universes in a sort of cosmic Pythagorean democracy, although not all of them would necessarily prove to be as rich as our own."

Tegmark's thesis is expounded in "Mathematical cosmos: why numbers rule" (*New Scientist*, September 15, 2007). The main argument is this: "If we assume that reality exists independently of humans, then for a description to be complete, it must also be well defined according to non-human entities – aliens or supercomputers, say – that lack any understanding of human concepts. ... This is where mathematics comes in. To a modern logician, a mathematical structure is precisely this: a set of abstract entities with relations between them. ... So ... If you believe in an external reality independent of humans, then you must also believe in what I call the mathematical universe hypothesis: that our physical reality is a mathematical structure."

But just as you were thinking that this would make life simpler, you read on. "The hypothesis also makes a much more dramatic prediction: the existence of parallel universes." The explanation: "Most physicists hope for a theory of everything that ... can be specified in few enough bits to fit in a book, if not on a T-shirt. The mathematical universe hypothesis implies that such a simple theory must predict a multiverse. Why? Because this theory is by definition a complete description of reality: if it lacks enough bits to completely specify our universe, then ... the extra bits that describe our universe simply encode which universe we are in, like a multiversal phone number."

If you are scratching your head in stunned disbelief, that's perfectly OK: "Evolution endowed us with intuition only for those aspects of physics that had survival value for our distant ancestors, such as the parabolic trajectories of flying rocks. Darwin's theory thus makes the testable prediction that whenever we look beyond the human scale, our evolved intuition should break down. ... To me, an electron colliding with a positron and turning into a Z-boson feels about as intuitive as two colliding cars turning into a cruise ship. The point is that if we dismiss seemingly weird theories out of hand, we risk dismissing the correct theory of everything, whatever it may be."

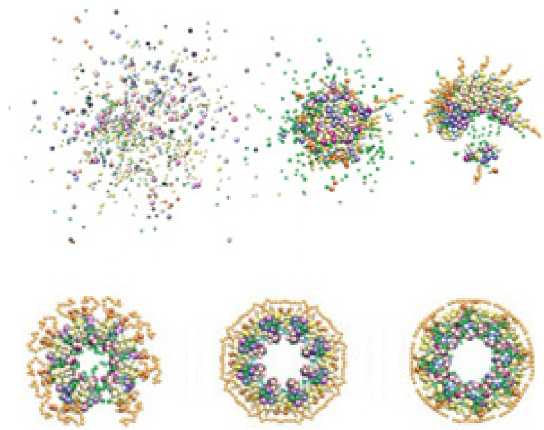
The Gömböc.



The Gömböc looks something like this. It has back-front symmetry as well as symmetry in the plane shown. The prototype given to Arnol'd was about 4 inches wide.

"The Self-Righting Object" was among the items chosen for the *New York Times Magazine*'s 7th Annual Year in Ideas (December 9, 2007). Named "the Gömböc" by its inventors – Gábor Domokos and Péter Várkonyi of Budapest – it is the result, according to the *Magazine*, of "a long mathematical quest," starting with a problem posed to Domokos in 1995 by the celebrated Russian mathematician V. I. Arnol'd: to construct a "mono-monostatic" object. This would be a convex, homogeneous object with such a geometry that it would have exactly one stable position when placed on a flat surface. "Homogeneous" rules out toys of the "Comeback Kid" class which rely on a weighted bottom to keep them coming back up. The Gömböc also got play in a piece by Julie Rehmeyer in *Science News Outline* for April 7, 2007. Neither of these accounts gives any hint of the mathematics involved in Domokos and Várkonyi's solution, although Rehmeyer reminds us that "flat toys cut from a piece of plywood" always have at least two stable positions (see the Gömböc website for details). And she's funnier. It turns out that the Gömböc looks a bit like a turtle, and the question arises whether turtles might have evolved mono-monostaticity to avoid getting stranded on their backs. "So far, they've tested 30 turtles and found quite a few that are nearly self-righting. Várkonyi admits that most biology experiments study many more animals than that but, he says, 'it's much work, measuring turtles.' "

Math and macromolecular architecture. "The Molecular Architecture of the Nuclear Pore Complex" was the cover story in the November 29 2007 *Nature* and highlighted there ("News and Views," "Making the Paper") as a substantial achievement by its authors, a Rockefeller-UCSF team led by Michael Rout, Brian Chait and Andrej Sali. A striking feature of the research was the essential involvement of mathematical methods developed by physicists for handling problems with a very high number of degrees of freedom. The Nuclear Pore Complex is a large (molecular mass around 50 million) assembly of 456 proteins (in yeast) that spans the nuclear envelope and controls movement of material into and out of the nucleus. It was known that about 30 different proteins are involved, and the general shape was understood: "a doughnut-shaped structure, consisting of eight spokes arranged radially around a central channel." But the exact way the pieces fit together was a mystery. The article spells it out completely. An accompanying article, "Determining the architectures of macromolecular assemblies," explains how the puzzle was solved. The phrase the authors use to describe their method is "integrating spatial restraints." The spatial restraints are all the available data about the shapes and affinities of the constituent proteins, encoded into a set of functions that give 0 when "the restraint is satisfied" and higher values if it is violated. "In essence, restraints can be thought of as generating a 'force' on each component in the assembly, to mould them into a configuration that satisfies the data used to define the restraints." This "force" is essentially (minus) the gradient of a scoring function cooked up from the restraints. The "integration" is an optimization process: "The optimization starts with a random configuration of the constituent proteins' beads, and then iteratively moves them so as to minimize violations of the restraints." (The beads are points representing the location of each protein). The configuration is periodically shaken up by "simulated annealing" to "minimize the likelihood of getting caught in local scoring function minima."



Representative configurations at various stages of the optimization process from top (very large scores) to lower right (with a score of 0). Adapted from *Nature* 450 690; used with permission.

Approximately 200,000 different initial configurations were tested, and used to yield "an ensemble of 1000 structures satisfying the input restraints." Then the structures from the ensemble were superposed, and used to generate a single structure for the entire pore. From the end of the abstract: "The present approach should be applicable to many other macromolecular assemblies."

An Interview with F. William Lawvere - Part Two

You studied in Columbia from February 1960 to June 1961, returning there for the Ph.D. defense in May 1963. In the interim you went to Berkeley and Los Angeles. Why?

Even though I had had an excellent course in mathematical logic from Elliott Mendelson at Columbia, I felt a strong need to learn more set theory and logic from experts in that field, still of course with the aim of clarifying the foundations of category theory and of physics. In order to support my family, and also because of my deep interest in mathematics teaching, I had taken up employment over the summers of 1960 and 1961 with TEMAC, a branch of the Encyclopedia Britannica, which was engaged in producing high school text books in modern mathematics in a new stepwise interactive format. In 1961, TEMAC built a new building near the Stanford University campus devoted to that project. Thus the further move was not due to having lost a grant, but rather for those two purposes: in the Bay area I could reside in Berkeley, follow courses by Tarski, Feferman, Scott, Vaught, and other leading set theorists, and also commute to Palo Alto to process the text book which I was writing mainly at home. Nor was my first destination in California the think tank referred to in Mac Lane's book. Rather, since my slow progress in writing my second programmed textbook was not up to the speed which I thought TEMAC expected, I resigned from that job. A friend from the Indiana days now worked for the think tank near Los Angeles, and was able to persuade them to give me a job. At the beginning I understood that the job would involve design of computer systems for verifying possible arms control agreements; but when I finally got the necessary secret clearance, I discovered that other matters were involved, related with the Vietnam war. Mac Lane's account is essentially correct concerning the way in which my friend and fellow mathematician Bishop Spangler in the think tank became my supervisor and then gave me the opportunity to finish my thesis on categorical universal algebra. In February 1963, wanting very much to get out of my Los Angeles job to take up a teaching position at Reed College, I asked Eilenberg for a letter of recommendation. His very brief reply was that the request from Reed would go into his waste basket unless my series of abstracts be terminated post haste and replaced by an actual thesis. This tough love had the desired effect within a few weeks.

Having defended the Ph.D. in May 1963, I was able to leave the think tank and re-enter normal life as an assistant professor at Reed College for the academic year 1963-64. En route to Portland I attended the 1963 Model Theory meeting in Berkeley, where besides presenting my functorial development of general algebra, I announced that quantifiers are characterized as adjoints to substitution.

So, you spent the academic year 1963-64 as an assistant professor at Reed College.

At Reed I was instructed that the first year of calculus should concentrate on foundations, formulas there being taught in the second year. Therefore, in spite of already having decided that the category of categories is the appropriate framework for mathematics in general, I spent several preparatory weeks trying to devise a calculus course based on Zermelo-Fraenkel set theory. However, a sober assessment showed that there are far too many layers of definitions, concealing differentiation and integration from the cumulative hierarchy, to be able to get through those layers in a year. The category structure of Cantor's structureless sets seemed both simpler and closer. Thus, the Elementary Theory of the Category of Sets arose from a purely practical educational need, in a sort of experience that Saunders also noted: the need to explain daily for students is often the source of new mathematical discoveries.

A theory of a category of Cantorian abstract sets has the same proof-theoretic strength as the theory of a Category of Categories that I had initiated in the Introduction to my thesis. More objectively, sets can be defined as discrete categories and conversely categories can be defined as suitable finite diagrams of discrete sets, and the relative strengths thus compared. The category of categories is to be preferred for the practical reason that all mathematical structures can be constructed as functors and in the proper setting there is no need to verify in every instance that one has a functor or natural transformation.

After Reed I spent the summer of 1964 in Chicago, where I reasoned that Grothendieck's theory of Abelian categories should have a non-linear analogue whose examples would include categories of sheaves of sets; I wrote down some of the properties that such categories should have and noted that, on the basis of my work on the category of sets, such a theory would have a greater autonomy than the Abelian one could have (it was only in the summer of 1965 on the beach of La Jolla that I learned from Verdier that he, Grothendieck and Giraud had developed a full-blown theory of such "toposes", but without the autonomy). Later, at the ETH in Zurich ...

... where you stayed from September 1964 through December 1966 as visiting research scientist at Beno Eckmann's Forschungsinstitut für Mathematik *...*

... there I was able to further simplify the list of axioms for the category of sets in a paper that Mac Lane then communicated to the *Proceedings of the National Academy of Sciences USA*. There I also wrote up for publication the talk on "the category of categories as a foundation for mathematics" which I gave at the first international meeting on category theory at La Jolla, California, 1965.

A. Kock and F. W. Lawvere in Cafe Odeon, Zurich (Fall of 1966; photo courtesy of A. Kock).

Which were the purposes of your elementary theory of the category of sets ?

It was intended to accomplish two purposes. First, the theory characterizes the category of sets and mappings as an abstract category in the sense that any model for the axioms that satisfies the additional non-elementary axiom of completeness, in the usual sense of category theory, can be proved to be equivalent to the category of sets. Second, the theory provides a foundation for mathematics that is quite different from the usual set theories in the sense that much of number theory, elementary analysis, and algebra can apparently be developed within it even though no relation with the usual properties of $\in$ can be defined.

Philosophically, it may be said that these developments supported the thesis that even in set theory and elementary mathematics it was also true as has long been felt in advanced algebra and topology, namely that the substance of mathematics resides not in Substance, as it is made to seem when $\in$ is the irreducible predicate, but in Form, as is clear when the guiding notion is isomorphism-invariant structure, as defined, for example, by universal mapping properties. As in algebra and topology, here again the concrete technical machinery for the precise expression and efficient handling of these ideas is provided by the Eilenberg-Mac Lane theory of categories, functors and natural transformations.

Let us return to Zurich.

At Zurich I had many discussions with Jon Beck and we collaborated on doctrines. The word "doctrine" itself is entirely due to him and signifies something which is like a theory, except appropriate to be interpreted in the category of categories, rather than, for example, in the category of sets. The "algebras" for a doctrine deserve to be called "theories" because dualizing into a fixed algebra defines a semantics functor relating abstract generals and corresponding concrete generals. Jon was insistent on mathematical clarity and did much to encourage precision in discussions and in the formulation of mathematical results. He noted that my structure functor adjoint to semantics is analogous to Grothendieck's cocycle definition of descent in that both partially express the structure that inevitably arises when objects are constructed by a functorial process, and which if hypothesized helps to reverse the process and discern the origin. Implementing this general philosophical notion of descent requires the choice of an appropriate "doctrine" of theories in which the induced structure can be expressed.

Also from Zurich I attended a seminar in Oberwolfach where I met Peter Gabriel and learned from him many aspects not widely known even now of the Grothendieck approach to geometry. In general the working atmosphere at the *Forschungsinstitut* was so agreeable, that I later returned during the academic year 1968/69.

As an assistant professor in Chicago, in 1967, you taught with Mac Lane a course on Mechanics, where "you started to think about the justification of older intuitive methods in geometry"[7]*. You called it "synthetic differential geometry". How did you arrive at the program of Categorical Dynamics and Synthetic Differential Geometry ?*

From January 1967 to August 1967 I was Assistant Professor at the University of Chicago. Mac Lane and I soon organized to teach a joint course based on Mackey's book "Mathematical Foundations of Quantum Mechanics".

So, Mackey, a functional analyst from Harvard mainly concerned with the relationship between quantum me-

[7] Saunders Mac Lane, *A Mathematical Autobiography*, A K Peters, 2005.

chanics and representation theory, had some relation to category theory.

His relation to category theory goes back much further than that, as Saunders and Sammy had explained to me. Mackey's Ph.D. thesis displayed remarkable thinking of a categorical nature, even before categories had been defined. Specifically, the fact that the category of Banach spaces and continuous linear maps is fully embedded into a category of pairings of abstract vector spaces, together with the definition and use of "Mackey convergence" of a sequence in a "bornological" vector space were discovered there and have played a basic role in some form in nearly every book on functional analysis since. What is perhaps unfortunately not clarified in nearly every book on functional analysis, is that these concepts are intensively categorical in character and that further enlightenment would result if they were so clarified.

And the referee who, despite initial skepticism, permitted the first paper giving an exposition of the theory of categories to see the light of day in the TAMS in 1945, was none other than George Whitelaw Mackey.

Back to the origins of Synthetic Differential Geometry, where did the idea of organizing such a joint course on Mechanics originate ?

Apparently, Chandra had suggested that Saunders give some courses relevant to physics, and our joint course was the first of a sequence. Eventually Mac Lane gave a talk about the Hamilton-Jacobi equation at the Naval Academy in summer 1970 that was published in the *American Mathematical Monthly.*

In my separate advanced lecture series, which was attended by my then student Anders Kock, as well as by Mac Lane, Jean Bénabou, Eduardo Dubuc, Robert Knighten, and Ulrich Seip, I began to apply the Grothendieck topos theory that I had learned from Gabriel to the problem of simplified foundations of continuum mechanics as it had been inspired by Truesdell's teachings, Noll's axiomatizations, and by my 1958 efforts to render categorical the subject of topological dynamics.

Beyond what I had learned from Gabriel at Oberwolfach on algebraic geometry as a gros topos, my particular contribution was to elevate certain ingredients, such as the representing object for the tangent bundle functor, to the level of axioms so as to permit development unencumbered by particular construction. That particular ingredient had apparently never been previously noted in the C-infinity category. It was immediately clear that the program would require development, in a similar axiomatic spirit, of the topos theory of which I had heard in 1965 from Verdier on the beach at La Jolla. Indeed, my appointment at Chicago had been encouraged also by Marshall Stone who was enthusiastic about my 1966 observation that the topos theory would make mathematical both the Boolean-valued models in general and the independence of the continuum hypothesis in particular. That these apparently totally different toposes, involving infinitesimal motion and advanced logic, could be part of the same simple axiomatic theory, was a promise in my 1967 Chicago course. It only became reality after my second stay at the *Forschungsinstitut* in Zurich, Switzerland 1968-69 during which I discovered the nature of the power set functor in toposes as a result of investigating the problem of expressing in elementary terms the operation of forming the associated sheaf, and after 1969-1970 at Dalhousie University in Halifax, Nova Scotia, Canada, through my collaboration with Myles Tierney.

You went to Dalhousie in 1969 with one of the first Killam professorships.

Indeed, and was able to have a dozen collaborators at my discretion, also supported by Killam.

And then you arrived, together with the algebraic topologist Myles Tierney, to the concept of elementary topos. Could you describe us that collaboration with Myles Tierney ?

Myles presented a weekly seminar in which the current stage of the work was described and indeed some of the work was in the form of discussions in the seminar itself: remarks by students like Michel Thiebaud and Radu Diaconescu were sometimes key steps.

Myles Tierney and Dana Scott
(1971 Conference at Dalhousie, photo courtesy of Robert Paré).

Although I had been able to convince myself in Zurich, Rome, and Oberwolfach, that a finite axiomatization was possible, it required several steps of successive simplification to arrive at the few axioms known now. The criterion of sufficiency was that by extending any given category satisfying the axioms, it should be possible to build others by presheaf and sheaf methods. The "fundamental theorem" of slices, followed by our discovery that left exact comonads also yield toposes, more than covered the presheaf aspect. The concept of sheaves led to the conjecture that subtoposes would be precisely parametrized by certain endomaps of the subobject classifier, and this was verified; those endomaps are now known as Lawvere-Tierney modal operators, and correspond classically to Grothendieck topologies. That the corresponding subcategory of sheaves can be described in finite terms is a key technical feature, which was achieved by making explicit the partial-map classifier. That the theory is elementary means that it has countable models and other features making it applicable to independence results in set theory and to higher recursion, etc, but on the other hand Grothendieck's theory of U-toposes is precisely included through his own technique of relativization together with additional axioms, such as the splitting of epimorphisms and 2-valuedness, on U itself.

(By the way, those two additional axioms are positive – or geometrical– so that there is a classifying topos for models of them, a fact still awaiting exploitation by set theory.)

MATHEMATICIANS' CONFERENCE

A major conference on "Connections between category theory and algebraic geometry and intuitionistic logic" was held at Dalhousie last weekend. It attracted approximately 50 mathematicians from Canada, the United States and Europe.

Leading mathematicians from Europe and the North American continent who accepted invitations to lecture were I. Bucur of Academie de la Republic Socialiste de Roumanie; J. Giraud of Paris and Dalhousie; L. Illusie of Paris and M.I.T.; J. Lambek of McGill; H. Lauchli of ETH Zurich; F.W. Lawvere of Dalhousie; D.S. Scott of Princeton; M. Tierney of Rutgers; J. Verdier of Paris and Brandeis, and others.

Also participating were:- J. Beck of Cornell; N. Kelly of Chicago; F. Szabo of Sir George Williams; G. Takeuti of Illinois; and W. Wraith of Aarhus.

Formal lectures were given each morning, with the afternoons and evenings free for discussions and talks arranged by the participants. The conference was sponsored by the university's department of mathematics.

Fred Linton and F. William Lawvere
(photo courtesy of Robert Paré).

In 1971, official date of the birth of topos theory, unfortunately the dream team at Dalhousie was dispersed. What happened, that made you go to Denmark ?

Some members of the team, including myself, became active against the Vietnam war and later against the War Measures Act proclaimed by Trudeau. That Act, similar in many ways to the Patriot Act 35 years later in the US, suspended civil liberties under the pretext of a terrorist danger. (The alleged danger at the time was a Quebec group later revealed to be infiltrated by the RCMP, the Canadian secret police.) Twelve communist bookstores in Quebec (unrelated to the terrorists) were burned down by police; several political activists from various groups across Canada were incarcerated in mental hospitals, etc. etc. I publicly opposed the consolidation of this fascist law, both in the university senate and in public demonstrations. The administration of the university declared me guilty of "disruption of academic activities". Rumors began to be circulated, for example, that my categorical arrow diagrams were actually plans for attacking the administration building. My contract was not renewed.

And after a short period in Aarhus, you went to Italy. Why ?

Conditions in the *Matematisk Institut* were very agreeable, and the collaboration with Anders Kock was very fruitful and enjoyable. However when the long northern night set in, it turned out to be bad for my health, so I accepted an invitation from Perugia. I still enjoy visiting Denmark in the summer.

After a few years in Europe, you returned to the United States, for SUNY at Buffalo ...

John Isbell and Jack Duskin were able to persuade the dean that (contrary to the message sent out by one of the Dalhousie deans) I was not a danger and might even be an asset.

In spite of your return to the USA, you kept close ties with the Italian mathematical community. In November 2003 there was a conference in Firenze ("Ramifications of Category Theory") to celebrate the 40th anniversary of your Ph.D. thesis[8]*. Could you summarize the main ideas contained in it ?*

Details are given in my commentary to the TAC Reprint (these Reprints are an excellent source of other early material on categories). The main point was to present a categorical treatment of the relation between algebraic theories and classes of algebras, incorporating the previous "universal" algebra of Birkhoff and Tarski in a way applicable to specific cases of mathematical interest such as treated in books of Chevalley and of Cartan-Eilenberg. The presentation-free redefinition of both the theories and the classes required explicit attention to the category of categories.

In the Firenze conference there were talks both on mathematics and philosophy. You keep interested in the philosophy of mathematics ...

Yes. Since the most fundamental social purpose of philosophy is to guide education and since mathematics

[8] *Functorial Semantics of Algebraic Theories*, Reprinted in *Repr. Theory Appl. Categ.* 5 (2004) 1-121 (electronic).

is one of the pillars of education, accordingly philosophers often speculate about mathematics. But a less speculative philosophy based on the actual practice of mathematical theorizing should ultimately become one of the important guides to mathematics education.

Ramifications of Category Theory, 2003
(photo by Andrej Bauer, used with permission).

As Mac Lane wrote in his Autobiography, "The most radical aspect is Lawvere's notion of using axioms for the category of sets as a foundation of mathematics. This attractive and apposite idea has, as of yet, found little reflection in the community of specialists in mathematical logic, who generally tend to assume that everything started and still starts with sets". Do you have any explanation for that attitude ?

The past 100 years' tradition of "foundations as justification" has not helped mathematics very much. In my own education I was fortunate to have two teachers who used the term "foundations" in a common-sense way (rather than in the speculative way of the Bolzano-Frege-Peano-Russell tradition). This way is exemplified by their work in Foundations of Algebraic Topology, published in 1952 by Eilenberg (with Steenrod), and the Mechanical Foundations of Elasticity and Fluid Mechanics, published in the same year by Truesdell. Whenever I used the word "foundation" in my writings over the past forty years, I have explicitly rejected that reactionary use of the term and instead used the definition implicit in the work of Truesdell and Eilenberg. The orientation of these works seemed to be "concentrate the essence of practice and in turn use the result to guide practice". Namely, an important component of mathematical practice is the careful study of historical and contemporary analysis, geometry, etc. to extract the essential recurring concepts and constructions; making those concepts and constructions (such as homomorphism, functional, adjoint functor, etc.) explicit provides powerful guidance for further unified development of all mathematical subjects, old and new.

Could you expand a little bit on that ?

What is the primary tool for such summing up of the essence of ongoing mathematics? Algebra! Nodal points in the progress of this kind of research occur when, as in the case with the finite number of axioms for the metacategory of categories, all that we know so far can be expressed in a single sort of algebra. I am proud to have participated with Eilenberg, Mac Lane, Freyd, and many others, in bringing about the contemporary awareness of Algebra as Category Theory. Had it not been for the century of excessive attention given to alleged possibility that mathematics is inconsistent, with the accompanying degradation of the F-word, we would still be using it in the sense known to the general public: the search for what is "basic". We, who supposedly know the explicit algebra of homomorphisms, functionals, etc., are long remiss in our duty to find ways to teach those concepts also in high school calculus.

Having recognized already in the 1960s that there is no such thing as a heaven-given platonic "justification" for mathematics, I tried to give the word "Foundations" more progressive meanings in the spirit of Eilenberg and Truesdell. That is, I have tried to apply the living axiomatic method to making explicit the essential features of a science as it is developing in order to help provide a guide to the use, learning, and more conscious development of the science. A "pure" foundation which forgets this purpose and pursues a speculative "foundation" for its own sake is clearly a NON-foundation.

Foundations are derived from applications by unification and concentration, in other words, by the axiomatic method. Applications are guided by foundations which have been learned through education.

You are saying that there is a dialectical relation between foundations and applications.

Yes. Any set theory worthy of the name permits a definition of mapping, domain, codomain, and composition; it was in terms of those notions that Dedekind and later mathematicians expressed structures of interest. Thus, any model of such a theory gives rise to a category and whatever complicated additional features may have been contemplated by the theory, not only common mathematical properties, but also most interesting "set theoretical" properties, such as the generalized continuum hypothesis, Dedekind finiteness, the existence of inaccessible or Ulam cardinals, etc. depend only on this mere category.

During the past forty years we have become accustomed to the fact that foundations are relative, not absolute. I believe that even greater clarifications of foundations will be achieved by consciously applying a concentration of applications from geometry and analysis, that is, by pursuing the dialectical relation between foundations and applications.

More recently, you have given algebraic formulations of such distinctions as 'unity vs. identity' of opposites, 'extensive vs. intensive' variable quantities, 'spatial vs. quantitive' categories ...

Yes, showing that through the use of mathematical category theory, such questions lead not to fuzzy speculation, but to concrete mathematical conjectures and results.

It has been one of the characteristics of your work to dig down beneath the foundations of a concept in order to simplify its understanding. Here you are truly a descendant of Samuel Eilenberg, in his "insistence on getting to the bottom of things". We vividly remember a lecture you presented in Coimbra to our undergraduate students. You have recently published a couple of textbooks[9]*. Why do you find it important enough to dedicate a significant amount of your time and effort to it ?*

Many of my research publications are the result of long study of the two problems: (1) How to effectively teach calculus to freshmen. (2) How to learn, develop, and use physical assumptions in continuum thermomechanics in a way which is rigorous, yet simple.

F. William Lawvere and Stephen Schanuel
(Sydney, 1988; photo courtesy of R. Walters).

In other words, the results themselves can only be building blocks in an answer to the question: "How can we take concrete, pedagogical steps to narrow the enormous gap in 20th century society between the fact that: (a) everybody must use technology which rests on science, which in turn depends on mathematics; yet (b) only a few have a working acquaintance with basic concepts of modern mathematics such as retractions, fixed-point theorems, morphisms of directed graphs and of dynamical systems, Galilean products, functionals, etc."

Only armed with such concepts can one hope to respond with confidence to the myriad of methods, results, and claims which in the modern world are associated with mathematics. With Stephen Schanuel I have begun to take up the challenge of that question in our book Conceptual Mathematics which reflects the ongoing work of many mathematicians.

What is your opinion on the Wikipedia article about you ?

The disinformation in the original version has been largely removed, but much remains in other articles about category theory.

We have recently celebrated Kurt Gödel's 100th birthday. What do you think about the extra-mathematical publicity around his incompleteness theorem ?

In *Diagonal arguments and Cartesian closed categories*[10] we demystified the incompleteness theorem of Gödel and the truth-definition theory of Tarski by showing that both are consequences of some very simple algebra in the Cartesian-closed setting. It was always hard for many to comprehend how Cantor's mathematical theorem could be re-christened as a "paradox" by Russell and how Gödel's theorem could be so often declared to be the most significant result of the 20th century. There was always the suspicion among scientists that such extra-mathematical publicity movements concealed an agenda for re-establishing belief as a substitute for science. Now, one hundred years after Gödel's birth, the organized attempts to harness his great mathematical work to such an agenda have become explicit[11].

You have always been concerned in explaining how to describe relevant mathematical settings and facts in a categorical fashion. Is category theory only a language ?

No, it is more than a language. It concentrates the essential features of centuries of mathematical experience and thus acts as indispensible guide to further development.

[9]F. W. Lawvere and R. Rosebrugh, *Sets for Mathematics*, Cambridge University Press, Cambridge, 2003; F. W. Lawvere and S. Schanuel, *Conceptual Mathematics. A First Introduction to Categories*, Cambridge University Press, Cambridge, 1997.

[10]Reprinted in *Repr. Theory Appl. Categ.* 15 (2006) 1-13 (electronic).

[11]The controversial John Templeton Foundation, which attempts to inject religion and pseudo-science into scientific practice, was the sponsor of the international conference organized by the Kurt Gödel Society in honour of the celebration of Gödel's 100th birthday. This foundation is also sponsoring a research fellowship programme organized by the Kurt Gödel Society.

What have been for you the major contributions of category theory to mathematics?

First, the work of Grothendieck in his Tohoku's paper[12]. Nuclear spaces was one of the great inventions of Grothendieck. By the way, Silva worked a lot on these spaces and Grothendieck's 1953 paper on holomorphic functions[13] was inspired by a 1950 paper of Silva[14].

The concept of adjoint functors, discovered by Kan in the mid 1950's, was also a milestone, rapidly incorporated as a key element in Grothendieck's foundation of algebraic geometry and in the new categorical foundation of logic and set theory.

I may also mention Cartesian closedness, the axiomatization of the category of categories, topos theory ... Cartesian closed categories appeared the first time in my Ph.D. thesis, without using the name. The name appeared first in Kelly and Eilenberg's paper[15]. I don't exactly agree with the word "Cartesian". Galileo is the right source, not Descartes.

You are regarded by many people as one of the greatest visionaries of mathematics in the beginning of the twentieth first century. What are your thoughts on the future development of category theory inside mathematics?

I think that category theory has a role to play in the pursuit of mathematical knowledge. It is important to point out that category theorists are still finding striking new results in spite of all the pessimistic things we heard, even 40 years ago, that there was no future in abstract generalities. We continue to be surprised to find striking new and powerful general results as well as to find very interesting particular examples.

We have had to fight against the myth of the mainstream which says, for example, that there are cycles during which at one time everybody is working on general concepts, and at another time anybody of consequence is doing only particular examples, whereas in fact serious mathematicians have always been doing both.

F. William Lawvere and Maria Manuel Clementino (Braga, March 2007).

One should not get drunk on the idea that everything is general. Category theorists should get back to the original goal: applying general results to particularities and to making connections between different areas of mathematics.

Interview by Maria Manuel Clementino and Jorge Picado (University of Coimbra)

Francis William Lawvere (born February 9, 1937 in Muncie, Indiana) is a mathematician well-known for his work in category theory, topos theory, logic, physics and the philosophy of mathematics. He has written more than 60 papers in the subjects of algebraic theories and algebraic categories, topos theory, logic, physics, philosophy, computer science, didactics, history and anthropology, and has three books published (one of them with translations into Italian and Spanish), with three more in preparation at this moment. He also edited three volumes of the Springer series *Lecture Notes in Mathematics* and supervised twelve Ph.D. theses. The electronic series *Reprints in Theory and Applications of Categories* includes reprints of seven of his fundamental articles, with author commentaries, among them his Ph.D. dissertation and his full treatment of the category of sets.

At the 1970 International Congress of Mathematicians in Nice he introduced an algebraic version of topos theory which unified geometry and set theory. Worked out in collaboration with Myles Tierney, this theory has since been developed further by many people, with applications to several fields of mathematics. Two of those fields had previously been introduced by Lawvere: (1) His 1967 Chicago lectures (published 1978) on categorical dynamics had shown how toposes with specified infinitesimal objects can provide a flexible geometric background for models of

[12] A. Grothendieck, Sur quelques points d'algèbre homologique, *Tohoku Math. J.* 9 (1957) 119-121.
[13] A. Grothendieck, Sur certains espaces de fonctions holomorphes, I, *J. Reine Angew. Math.* 192 (1953) 35-64.
[14] J. Sebastião e Silva, Analytic functions and functional analysis, *Portugaliae Math.* 9 (1950) 1-130.
[15] S. Eilenberg and G. M. Kelly, Closed categories, in: *Proc. Conf. Categorical Algebra* (La Jolla, Calif., 1965), pp. 421-562, Springer, 1966.

continuum physics, which led to a new subject known as Synthetic Differential Geometry; (2) In his 1967 Los Angeles lecture, and his 1968 papers on hyperdoctrines and adjointness in foundations, Lawvere had launched and developed the field of categorical logic, which has since been widely applied to geometry and computer science. Those ideas were indispensable for his 1983 simplified proof of the existence of entropy in non-equilibrium thermomechanics.

Many of Lawvere's research publications result from efforts to improve the teaching of calculus and of engineering thermomechanics. In particular, it was his 1963 Reed College course in the foundations of calculus which led to his 1964 axiomatization of the category of sets and ultimately to the elementary theory of toposes.

Professor Lawvere studied with Clifford Truesdell and Max Zorn at Indiana University and completed his Ph.D. at Columbia in 1963 under the supervision of Samuel Eilenberg. Before completing his Ph.D., Lawvere spent a year in Berkeley as an informal student of model theory and set theory, following lectures by Alfred Tarski and Dana Scott. During 1964-1966 he was a visiting research professor at the *Forschungsinstitut für Mathematik* at the ETH in Zurich. He then taught at the University of Chicago, working with Mac Lane, and at the City University of New York Graduate Center (CUNY), working with Alex Heller. Back in Zurich for 1968-69 he proposed elementary (first-order) axioms for toposes generalizing the concept of the Grothendieck topos. Dalhousie University in 1969 set up a group of Killam-supported researchers with Lawvere at the head; but in 1971 it terminated the group because of Lawvere's political opinions (namely his opposition to the 1970 use of the War Measures Act).

Then Lawvere went to the *Institut for Matematiske* in Aarhus (1971-72) and ran a seminar in Perugia, Italy (1972-1974) where he especially worked on various kinds of enriched category. From 1974 until his retirement in 2000 he was professor of mathematics at the University at Buffalo, often collaborating with Stephen Schanuel. There he held a Martin professorship (1977-82). He was also a visiting research professor at the IHES Paris (1980-81). He is now Professor Emeritus of Mathematics and Adjunct Professor Emeritus of Philosophy at the State University of New York at Buffalo and continues to work on his 50-year quest for a rigorous and flexible framework for the physical ideas of Truesdell and Walter Noll, based on category theory.

His personal view of mathematics and physics, based on a broad and deep knowledge, keeps influencing mathematicians and attracting experts from other areas to Mathematics. This influence was very apparent in the honouring session that took place in the last International Category Theory Conference (Carvoeiro, Portugal, June 2007), on the occasion of his 70th Birthday, through spontaneous and intense testimonies of both senior mathematicians and young researchers. Indeed, besides his extraordinary qualities as a mathematician, we wish to stress the care and efforts he puts into the guidance of students and young researchers, which we could confirm in Coimbra when he gave a lecture on Category Theory to undergraduate students, and again in the dialog we were very honoured to be part of, during the preparation of this interview.

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