Curvature and the growth of cells.

A mathematics article was published, April 26, 2007, in the general science journal *Nature*. This unusual occurrence is due to the prominence and wide applicability of the result. Robert MacPherson and David Srolovitz solved the 50-year old problem of generalizing to three dimensions John von Neumann's work on the growth of cells in planar tesselations. The hypotheses in both cases are that *cell walls move with a velocity proportional to their mean curvature*, and that domain walls meet at 120°, hypotheses which are realized in many physical and biological contexts.

Von Neumann showed that the rate of change dA/dt of the area A of such a cell can be expressed in terms of γ the surface tension of a domain wall, M a kinetic coefficient describing the walls' mobility and n the number of vertices where distinct walls intersect, by

$$dA/dt = -2\pi M\gamma(1-n/6).$$

So for example in the tessellation portion shown in Fig. 1, the 8-vertex regions A and B will grow at the expense of the 2-vertex region C.

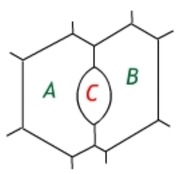


Fig. 1. With the common factor $2\pi M\gamma$ set to 1, von Neumann's formula tells us that $dA/dt = dB/dt = 1/3$, while $dC/dt = -2/3$.

MacPherson and Srolovitz's formula for the rate of change of the volume of a domain D in a 3-dimensional tessellation is formally analogous but requires the new and ingeniously defined *mean width* $\mathscr{L}(D)$, which they describe as "a natural measure of the linear size" of D. In terms of $\mathscr{L}(D)$, their formula reads

$$\frac{dV}{dt} = -2\pi M\gamma\Big(\mathscr{L}(D) - \frac{1}{6}\sum_i e_i\Big),$$

where e_i is the length of the i-th 1-dimensional edge of D, and the sum is taken over all the edges. Note that following our initial requirement, faces meet 3 by 3 along an edge with dihedral angles 120°.

The mean width $\mathscr{L}(D)$ is computed in two steps. First, for each line ℓ through the origin, the *Euler width* $\omega(D,\ell)$ of D along ℓ is the integral along ℓ of the Euler characteristic $\chi(\ell_p^\perp \cap D)$ of the intersection of D with the plane perpendicular to ℓ (see Fig. 2):

$$\omega(D,\ell) = \int_\ell \chi(\ell_p^\perp \cap D)\,dp.$$

So if D is convex (χ always $= 1$), $\omega(D,\ell)$ is exactly the length of the projection of D on ℓ.

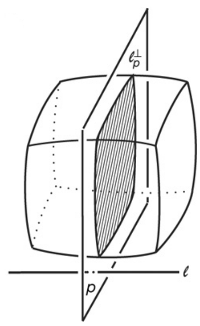


Fig. 2. For D a 3-dimensional domain, and ℓ a line through the origin, the Euler width $\omega(D,\ell)$ of D along ℓ is calculated by measuring, for each point p on ℓ, the Euler characteristic $\chi(\ell_p^\perp \cap D)$ of the intersection of D with the plane through p perpendicular to ℓ, and integrating along ℓ. Image reprinted by permission from Macmillan Publishers Ltd: *Nature* (Vol. **446**, 26 April 2007, p. 1054), copyright (2007).

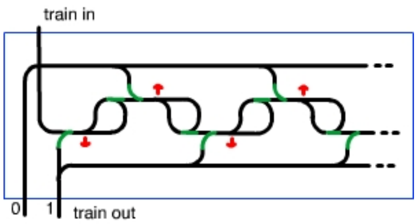
Then $\mathscr{L}(D)$ is computed as twice $\omega(D,\ell)$, averaged over the space RP^2 of lines through the origin:

$$\mathscr{L}(D) = 2\int_{RP^2} \omega(D,\ell)\,d\ell,$$

where $d\ell$ is normalized to have total integral 1.

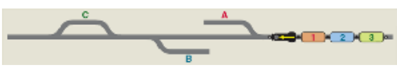
The authors state that their formula and von Neumann's are both special cases of a general n-dimensional formula, which they give. The Supplementary Information (see www.nature.com/nature/journal/v446/n7139/supp info/nature05745.html) for their article (entitled "The von Neumann relation generalized to coarsening of three-dimensional microstructures") gives the proof of their 3-dimensional formula and rules for computing $\mathscr{L}(D)$; for example the cube of side length a has mean width $3a$.

Computing with locomotives. "Trains of thought" is a piece by Brian Hayes in the March-April 2007 *American Scientist* (available online — www.americanscientist.org/AssetDetail/assetid/54774). He takes us to a "hump yard," where boxcars are sorted into trains by rolling through a series of switches: "... I can't shake the impression that the hump yard itself is a kind of computer–that the railroad cars are executing some secret algorithm." In fact *any* algorithm can be so executed. In 1994 Adam Chalcraft and Michael Greene, then Cambridge undergraduates, showed how to use a track layout to implement a given Turing machine (paper available online — www.monochrom.at/turingtrainterminal/Chalcraft.pdf). As Hayes explains it: "The machine is programmed by setting switch points in a specific initial pattern; then a locomotive running over the tracks resets some of the switches as it passes; the result of the computation is read from the final configuration of the switches." One of the trickier parts is what they call a *distributor*: it routes trains alternatively onto track 0 and onto track 1. They prove that this cannot be accomplished with a finite configuration, and exhibit the following open-ended layout to do the distribution.



Chalcraft and Greene's *Distributor* has two kinds of switches: *spring switches* which always direct an incoming car to the green track, and *lazy switches* which are reset by the last train through. The red arrow shows the current setting of each lazy switch. The first train through resets the leftmost lazy switch to "up" on its second pass and exits on track 1.

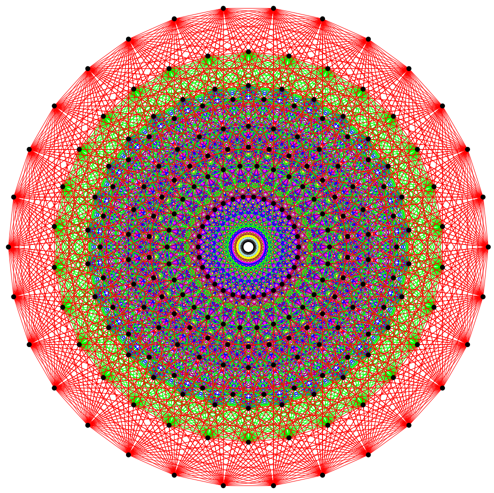
Train-track layouts turn out to have fascinated puzzle makers and computer scientists for quite some time. Hayes' illustrations include one of Sam Loyd's puzzles and Donald Knuth's "railroading interpretations of three important data structures: the stack, the queue and the double-ended queue, or deque." Hayes even gives us a puzzle of his own:



"The task is simply to deliver cars 1, 2 and 3 to destinations A, B and C. The cars are already in delivery order." The solution is given at www.ams.org/mathmedia/archive/05-2007-media.html#four.

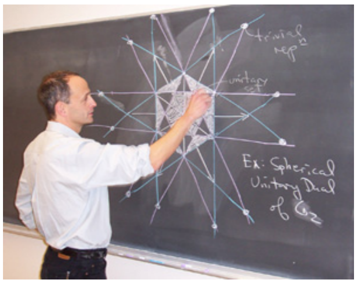
Chalcraft and Greene's work was picked up by Ian Stewart for his Mathematical Recreations column in the September 1994 *Scientific American*. That column ("A Subway Named Turing") is available online (www.fortunecity.com/emachines/e11/86/subway.html).

E_8 in the *New York Times*.



The *Times* printed a black and white version of this image, giving a glimpse of the size and complexity of the Lie group E_8. The configuration (projected here into 2 dimensions) shows part of the arrangement of closest packed balls in 8-dimensional space; the vertices represent a ball's 240 nearest neighbors in 8-space, with bonds drawn between nearest neighbors among the neighbors. E_8 contains a discrete subgroup mapping 256-to-one *onto* the 696,729,000-element symmetry group of this configuration. The image given here was made by John Stembridge, who explains it in www.math.lsa.umich.edu/~jrs/coxplane.html.

The most straightforward Lie groups are groups of n by n matrices characterized by some linear algebraic condition preserved in products, e.g. determinant nonzero, determinant $= 1$. The product of two matrices is a matrix whose entries are analytic functions (actually sums of products) of the entries in the factors. That's all it takes to make a Lie group. The building blocks of Lie theory, the simple Lie groups, fall into four infinite families of larger and larger matrices, plus five *exceptional* groups F_4, G_2, E_6, E_7, E_8. The last, largest (248-dimensional) and gnarliest of the exceptionals, E_8, has been in the news recently. Kenneth Chang reported, in the March 20 2007 *New York Times*, the culmination of a four-year effort by a team of 18 mathematicians, led by Jeffrey Adams (Maryland), to work out the details of its algebraic structure. His description of exactly what they were calculating is very vague, perhaps inevitably, but he clearly conveys the message that the task was enormous. "To understand using E_8 in all its possibilities requires calculation of 200 billion numbers," Chang tells us. "Possibilities" presumably refers to the set of unitary representations of E_8: the main way a group can be analyzed is through *representations* (projections which preserve multiplication) onto finite or infinite-dimensional matrix groups. The many episodes of the huge computation are laid out in David Vogan's narrative (atlas.math.umd.edu/kle8.narrative.html), a good story well told.



"Jeffrey D. Adams and a Lie group," as seen in the *Times*. Photo by Mark Tilmes, used with permission.

Intel silver and bronze for math projects. Second and third place in this year's Intel Science Talent Search went to mathematics projects, as reported by Aimee Cunningham in *Science Online* (www.sciencenews.org/articles/20070317/fob7.asp) for March 17, 2007. "Second place and a $75,000 scholarship went to John Vincent Pardon, a 17-year-old from Durham Academy in Chapel Hill, N.C. In his mathematical project, Pardon proved that a closed curve can be made convex without permitting any two points on the curve to get closer to one another.

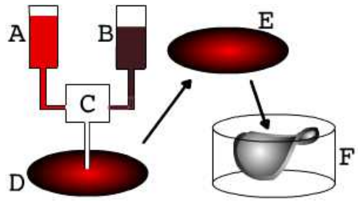
Mathematics research also won the third-place prize, which comes with a $50,000 scholarship. Eighteen-year-old Dmitry Vaintrob of South Eugene High School in Eugene, Ore., found a connection between different descriptions of certain mathematical shapes."

Vaintrob's project was reported on the Intel site in more detail: the award was "for his sophisticated investigation of ways to associate algebraic structures to topological spaces. Dmitry proved that loop homology and Hochschild cohomology coincide for an important class of spaces." Pardon's Intel citation also mentioned that his project had "solved a classical open problem in differential geometry."]

Pardon and Vaintrob's scholarship awards were also reported in the March 14 2007 *New York Times*.

"Journeys to the Distant Fields of Prime". Kenneth Chang's article took up the top of the first page in the *New York Times* Science section for March 13, 2007. It is a "Scientist at Work" profile of Terence Tao (UCLA), one of this year's Fields Medal winners. Don't be put off by the absurd title; Chang gives us a balanced and sympathetic look at this mathematical star. He takes us to Tao's public lecture on prime numbers (slides available in www.math.ucla.edu/~tao/preprints/Slides/primes.pdf, video in http://164.67.141.39:8080/ramgen/specialevents/math/tao/tao-20070117.smil), but then focuses on a "real-world" area of Tao's research, his work on compressed sensing. In a digital camera millions of sensors record an image which then gets compressed. Tao: "Compressed sensing is a different strategy. You also compress the data, but you try to do it in a very dumb way, one that doesn't require much computer power at the sensor end." In fact, Chang tells us, Tao and Caltech professor Emmanuel Candès have shown that "even if most of the information were immediately discarded, the use of powerful algorithms could still reconstruct the original image."

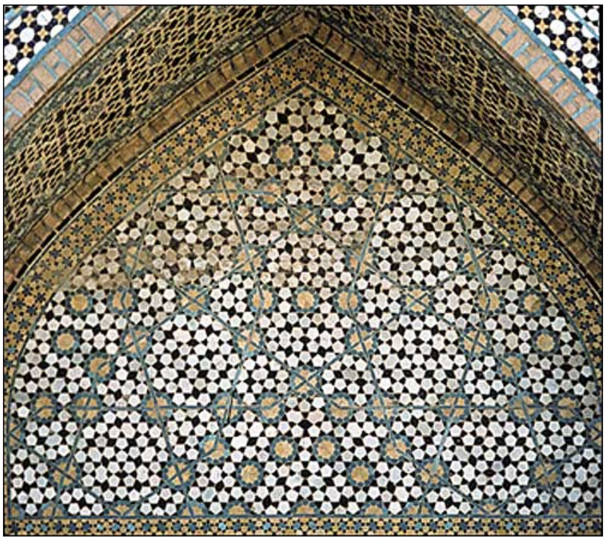
Cooking Gaussian curvature.



Gaussian cuisine. Low-concentration solution (A) and high-concentration solution (B) of N-isopropylacrylamide (NIPA) are mixed (C) in continuously varying proportion and extruded centrally between parallel plates (D) to form a gelatinous disc (E) with radially varying NIPA concentration, which is placed (F) in a hot bath; the heat makes the low-concentration areas shrink faster than the high, resulting in a non-Euclidean metric. Adapted from *Science* **315** 1117.

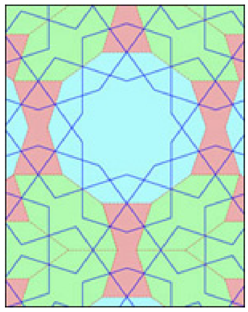
Anyone who has considered a potato chip mathematically has seen how Gaussian curvature can be produced by cooking. A team at the Hebrew University have found a way to control this process so as to produce (within a certain range) discs whose Gaussian curvature is a prescribed function of the radial coordinate. Their report, in the February 23 2007 *Science*, is entitled: "Shaping of Elastic Sheets by Prescription of Non-Euclidean Metrics." The authors (Yael Klein, Efi Efrati and Eran Sharon) present their project as a "novel shaping mechanism" for 2-dimensional objects. "Rather than aiming at a specific embedding, one prescribes on the sheet only a 2D metric, the 'target metric' g_{tar} ... . The free sheet will settle to a 3D configuration that minimizes its elastic energy. In this mechanism, the selected configuration is set by the competition between bending and stretching energies, and its metric will be close to (but different from) g_{tar}." Bending energy comes into the picture because the gel is not a 2-dimensional object: it has a finite thickness and resists bending. Nevertheless, "We show that the construction of elastic sheets with various target metrics is possible and results in spontaneous formation of 3D structures." The authors spend some time discussing the difference between the positive curvature case ("The surfaces of $K_{tar} > 0$ preserve the radial symmetry of g_{tar}, generating surfaces of revolution") and the negative ("The surfaces of $K_{tar} < 0$ break this symmetry, forming wavy structures"). They report: "A more surprising observation is the asymmetric distribution of the Gaussian curvature. Instead of the negative, rotationally symmetric K_{tar}, $K(\rho, \theta)$ varies periodically in θ, attaining positive and negative values." [It looks to me like they are measuring normal curvature here. -TP]

Medieval Islamic quasi-periodic tilings.



A quasi-periodic tiling from the Darb-i Imam shrine in Isfahan. Image courtesy K. Dudley and M. Elliff.

"... by the 15th century, the tessellation approach was combined with self-similar transformations to construct nearly perfect quasi-crystalline Penrose patterns, five centuries before their discovery in the West." This text appears in the abstract for "Decagonal and Quasi-Crystalline Tilings in Medieval Islamic Architecture," by Peter J. Lu and Paul J. Steinhardt, in the February 23 2007 *Science.* It is known that 5-fold rotational symmetry is incompatible with translational periodicity, but Peter Lu seems to have been the first one to notice that medieval Islamic artists went ahead, used motifs with 5-fold symmetry, and produced "quasi-periodic" patterns long before that concept was born. As he told NPR (*All Things Considered*, February 22 2007; transcript and images available online — www.npr.org/templates/story/story.php?storyId=7544360), he made this observation during a trip to Uzbekistan. When he got back to Harvard, where he is a graduate student in Physics, he did some investigation and discovered that Islamic geometers had devised a set of five polygonal building-blocks, each one decorated with polygonal lines; when the blocks were used to tile an area the lines fit together to give the intricate knot-like patterns called *girih.* One of these "girih blocks" is in fact identical to the "fat rhombus" we use in Penrose tilings.



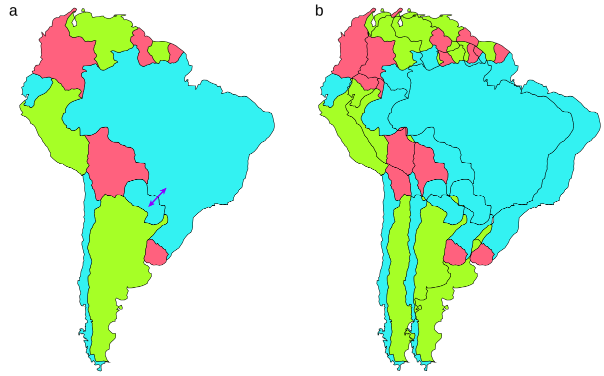
Part of a *girih*-pattern tiling from a Turkish mosque, with its analysis in terms of three of the decorated blocks (bowtie, decagon and flat hexagon) used by Islamic geometers. The other two are a pentagon and our "fat rhombus." Photographic image courtesy W. B. Denny, geometric analysis image courtesy Peter J. Lu.

Self-similarity is the one of the hallmarks of Penrose-type quasi-periodic tilings; this fact also seems not to have escaped the Islamic geometers: "Perhaps the most striking innovation arising from the application of girih tiles was the use of self-similarity transformation (the subdivision of large girih tiles into smaller ones) to create overlapping patterns at two different length scales, in which each pattern is generated by the same girih tile shapes." An example: the tiling from the Darb-i Imam shrine shown above.

Mathematical tools needed. "Bringing cartoons to life" is an essay by John J. Tyson under the "Connections" rubric in the February 22 2007 *Nature.* Abstract: "To understand cells as dynamic systems, mathematical tools are needed to fill the gap between molecular interactions and physiological consequences." Tyson, university distinguished professor of biological sciences at Virginia Tech, makes the point that "a network of interacting genes and proteins is a dynamic system evolving in space and time according to fundamental laws of reaction, diffusion and transport." He focuses on *programmed cell death* as an example of a a nonlinear system: "its molecular regulatory network is bistable (either off or on) at zero signal strength and monostable (on) for signals above the threshold." He posits a chemical feedback loop which "might generate" this kind of dynamic responses, and asks "But can we be sure our intuition is correct? ... How might the regulatory system fail? What are the most effective ways to intervene pharmaceutically to repair the cell-death pathway?" The answers, he proposes, will come from computer modeling. "The network of reactions ... can be cast into a set of kinetic [differential] equations. ... By following the arrows, a computer can simulate the temporal evolution of the control system under any specified experimental conditions." Tyson points to the existence of "a well-developed mathematical theory" with qualitative concepts such as bifurcation points, which "accord well with our intuitive notions," a theory which "forges a rigorous chain of deductions from molecular interactions to kinetic equations to vector fields to physiological consequences." He ends by predicting that in particular the uncertainties about "the molecular correlates of programmed cell death" will "be resolved largely by experiments driven by theoretical issues such as the importance of bistability, the roles of feedback and feed forward, and robustness in the face of noise."

"Proof at a roll of the dice." That's the title of a News and Views piece contributed by Bernard Chazelle (Computer Science, Princeton) to the December 28 2006 *Nature*. His subject is probabilistically checkable proof, or PCP: "the curious phenomenon that the mere ability to toss coins makes it possible to check the most complex of mathematical proofs at no more than a passing glance." The underlying theorem is about ten years old, and has recently been given an "elementary" proof ("the latest chapter in one of the most engrossing chronicles of computer science") by Irit Dinur (Hebrew University).

Here is Chazelle's statement of the PCP Theorem: "any statement S whose validity can be ascertained by a proof P written over n bits also admits an alternative proof, Q. This proof Q has two appealing features: it can be derived from P in a number of steps proportional to n^c, where c is some constant; and P can be verified by examining only three bits of Q picked at random. If S is true, a correct P will satisfy the verifier with a probability of 99%. If it is not true, any alleged proof P will trigger a rejection from Q with a probability higher than 50%." To suggest how P and Q are related, Chazelle has us imagine figure **a** below as P: a proof of the (false) statement that a map of South America can be colored with 3 colors so that no adjacent countries are colored the same. To check the validity of this proof one has to check all the boundaries of all the countries; eventually one finds that Brazil and Paraguay are colored the same. Q corresponds to the coloring **b** of the "smeared out" map on the right. Dinur's construction guarantees that if the first map is not 3-colorable, then **b** "will leave at least a fixed fraction of its edges monochromatic." And so a random probe has a good chance of detecting an error. Chazelle reminds us towards the end to "Keep in mind that this is all about verifying proofs, not about understanding them — with only three bits! — let alone discovering them. That must still be done the hard way."

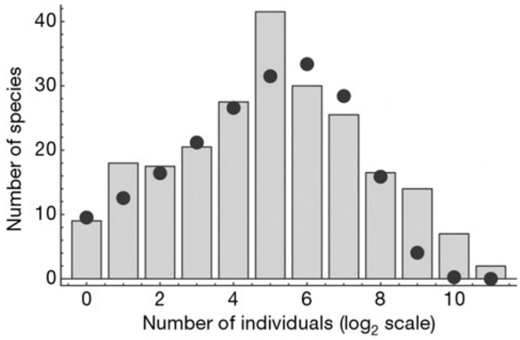


The putative proof **a** that South America is 3-colorable (false) has just one error. In the proof **b**, "smeared" in analogy to Dinur's PCP transformation, the error appears in many places. "Establishing the validity –or not– of the original map with high statistical certainty thus requires the checking of only a small, randomly chosen subregion of the smeared map." Image reprinted by permission from Macmillan Publishers Ltd: *Nature* (Vol. **444**, 21/28 December 2006, p. 1018), copyright (2007).

xe^{-x} in a tropical rainforest. "Dynamical evolution of ecosystems" ran in the December 14 2006 *Nature*. The authors, a team led by Jayanth Banavar (Penn State) and Amos Maritan (Padua), start their report with the sentence: "We present an analytical model that allows one to probe the characteristic timescales of evolving tropical forests and to evaluate the consequences of anthropogenic processes." In this and in a previous paper with a different team, Banavar and Maritan explore how *density dependence* (an effect that "disfavours the population growth of locally abundant species relative to uncommon species") impacts species diversity and relative species abundance (RSA). Here they show how a factor (b) representing density dependence fits into an analytic expression

$$P_{RSA}(x) = \frac{(D\tau)^{-b/D}}{\Gamma(b/D)} x^{b/D-1} e^{-x/D\tau}$$

for the probability distribution function giving relative species abundance, and match their calculation with a measured RSA distribution: that of trees in the stand of tropical rain-forest maintained by the Smithsonian Institution on Barro Colorado Island, Panama. The analytic expression is calculated using a symmetric model in which the species are interchangeable; they have birth and death rates $b(x) = b_1 x + p_0$, $d(x) = d_1 x - p_0$, where b_1 and d_1 are the per-capita rates and p_0 incorporates the density dependence. In the equation, $\tau = 1/(b_1 - d_1)$ is the characteristic timescale of the system (reflecting how fast the system returns to equilibrium after a perturbation); $D = (b_1+d_1)/2$ "accounts for demographic stochasticity" and $b = 2p_0$. The hairy coefficient is there to guarantee a total integral equal to 1.



Relative species abundance for trees in the Barro Colorado Island forest from the 1990 census, compared with predictions (dots) from the expression given above. Individuals of more than 1 cm in diameter were counted. Image reprinted by permission from Macmillan Publishers Ltd: *Nature* (Vol. **444**, 14 December 2006, p. 926), copyright (2007).

Mathematician becomes "Genome Sleuth". On December 12, 2006, the *New York Times* "Scientist at Work" series featured Nick Patterson, a mathematician. His PhD, from Cambridge, was in finite group theory. Patterson told the *Times*' Ingfei Chen: "I'm a data guy. What I know about is how to analyze big, complicated data sets." He honed this skill on code-breaking, first for the British, then for the U.S. Department of Defense. After some 20 years as a cryptographer, applying the Hidden Markov Model to "predict the next letter in a sequence of ... text" he turned this skill to predicting the next data point is a series of stock prices, working for the hedge fund managed by mathematician/financier Jim Simons. When he started, according to Chen, the fund was worth $200 million; seven years later, it was up to $4 billion. "Their methods apparently worked." But now the data guy is on to a third career: "Genome Sleuth Nick Patterson" was the caption for his photograph in the *Times*. And apparently the methods are still working. An article by him and four of his colleagues at the Broad Institute (Cambridge MA) ran in the June 29 2006 *Nature*. The title: "Genetic evidence for complex speciation of humans and chimpanzees." The team ran a comparison of the human, chimpanzee and related genomes on a much larger scale (by a factor of 800) than had ever been attempted. Chen: "Two strange patterns emerged. Some human DNA regions trace back to a much older common ancestor of humans and chimps than other regions do, with the ages varying by up to four million years. But on the X chromosome, people and chimps share a far younger common ancestor than on other chromosomes. ... the data appeared best explained if the human and chimp lineages split but later began mating again, producing a hybrid that could be a forebear of humans."

The math of swarms.

School of "silversides," Bonaire, N.A., March 2000. Image courtesy Kent Wenger.

"Math explains how group behavior is more than the sum of its parts" is the subtitle to Erica Klarreich's report "The Mind of the Swarm" (www.sciencenews.org/articles/20061125/bob10.asp) in the November 25 2006 *Science News*. Examples of the behavior in question: "a flock of birds swooping through the evening sky, ... a school of fish making a hairpin turn, an ant colony building giant highways, or locusts marching across the plains." One of Klarreich's sources is Iain Couzin (Oxford, Princeton) whose 2002 article (with several co-authors) "Collective Memory and Spatial Sorting in Animal Groups" (*J. theor. Biol.* **218**, 1-11) gave a simple mathematical Ising-type model (the "alignment zone" model) which duplicates some of the exotic behavior of schools of fish. Specifically, for a certain range of parameter values the simulated school would look like a torus, with all the fish swimming around a common axis. Klarreich quotes Couzin: "When we first saw [the doughnut] pattern in the simulations, I thought 'That's really weird!' But then we found in the literature that it really does appear in nature. ... There's nothing in the individual rules that says, 'Go in a circle,' but it happens spontaneously." The key to a general understanding of these collective phenomena, Klarreich tells us, seems to be "a trio of physics and engineering principles– nonlinearity, positive feedback, and phase transitions."