Chaotic Dynamics: physical measures and statistical features

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Abstract

We present some results on the statistical features of certain chaotic dynamical systems. We shall focus on the existence of physical measures, decay of correlations and statistical stability.

1. Introduction

Take a mathematical space M and think of its points as representing physical, biological or some other variables. Endow this space with a function (rule) $f\colon M \circlearrowleft$ which, given any point in M, comes up with another point in M. The combination is a *discrete-time dynamical system* for which M is the *phase space*, and the function gives the *evolution law*. The *orbit* (or *trajectory*) of a given point $x \in M$ is the sequence of successive iterates $(f^n(x))_n$, where $f^n = f \circ \cdots \circ f$ (n times).

In broad terms, one may refer two main goals of Dynamical Systems theory: *i)* to describe the typical behavior of trajectories, specially as time goes to infinity, and *ii)* to understand how this behavior changes when the law that governs the system is sightly modified. Even in cases of simple evolution laws, orbits may have a rather complicated behavior, which makes its description a very difficult task, specially when the system has *sensitivity to initial conditions*: a small change in the initial state produces large variations in the long term behavior of the trajectory. A well succeeded strategy for studying this kind of systems is through a probabilistic viewpoint: if one is not able to predict the future configuration of the system, let us try at least to find out the probability of certain configurations. In this approach we are particularly interested in *physical measures*, which characterize asymptotically, in time average, a large set of orbits.

Starting with classical results, in this work we present recent developments on the probabilistic theory of chaotic dynamical systems, specially about the existence of physical measures and some of their statistical features.

2. Physical measures

Let $(M, \mathscr{A}, \mu)$ be a probability space and $f : M \to M$ be such that $f^{-1}(A) \in \mathscr{A}$ for each $A \in \mathscr{A}$. We say that f *preserves the measure* μ, or μ is an f*-invariant measure*, if $\mu(f^{-1}(A)) = \mu(A)$ for all $A \in \mathscr{A}$. A direct consequence of this definition is that $\{x \in M\colon x \in A\}$ and $\{x \in M\colon f^n(x) \in A\}$ have the same μ measure for every $n \in \mathbb{N}$. This means that the probability of finding a point in a measurable set does not depend on the moment we are considering. One of the first results on the probabilistic features of dynamical systems was obtained by Poincaré for conservative systems, and can be translated to our context in the following way:

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Poincaré Recurrence Theorem. *Assume that f preserves a probability measure μ. If A is a measurable set, then for almost every $x \in A$, there are infinitely many $n \in \mathbb{N}$ for which $f^n(x) \in A$.*

The previous result says nothing about the *frequency* on which typical orbits visit A, i.e. it gives no information on

$$\lim_{n\to\infty} \frac{\#\{0 \le j < n\colon f^j(x) \in A\}}{n}. \tag{2.1}$$

Does this limit exist? Where does it converge to? Birkhoff Ergodic Theorem gives answers to these questions and, in fact, much more general conclusions. Before we state it, let us introduce some important concept on this subject. Assume that f preserves a measure μ. We say that f (or μ) is *ergodic* if $\mu(A) = 0$ or $\mu(M \setminus A) = 0$ for any $A \in \mathscr{A}$ with $f^{-1}(A) = A$. Observing that $f^{-1}(A) = A$ implies that $f(A) \subset A$ and $f(M \setminus A) \subset M \setminus A$, this means that the space cannot be decomposed into two significant parts that do not interact.

Birkhoff Ergodic Theorem. *Assume that f preserves a probability measure μ. If φ is integrable, then there is an integrable function φ^* such that for μ almost every $x \in M$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \varphi^*(x).$$

Moreover, $\varphi^*(x) = \int \varphi d\mu$ for μ almost every $x \in M$, provided μ is ergodic.

Taking φ as the characteristic function of a measurable set A, we easily deduce that the limit in (2.1) exists for μ almost every $x \in M$. Furthermore, if μ is ergodic, then it is precisely $\mu(A)$. This means that the frequency of visits to A coincides with the proportion that A occupies in the phase space.

The results we have presented so far concern dynamics over a probability measure space with no additional structure on the underlying phase space M. Frequently M has a Riemannian manifold structure and a volume form on it which gives rise to a Lebesgue measure m on the Borel sets of M. Birkhoff Ergodic Theorem gives that asymptotic time averages exist for almost every point, with respect to an invariant measure μ, and they coincide with the spatial average, provided μ is ergodic. However, an invariant measure can lack of physical meaning, in the sense that sets with full μ measure may have zero Lebesgue measure. This problem can be overcome by the notion that we present below, which has been introduced by Sinai, Ruelle and Bowen in the context of hyperbolic dynamical systems.

An invariant probability measure μ is said to be a *physical measure* for $f\colon M \to M$ if for a positive Lebesgue measure set of points $x \in M$

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi\, d\mu, \tag{2.2}$$

for all continuous $\varphi : M \to \mathbb{R}$. This means that the averages of Dirac measures over the orbit of x converge in the weak* topology to the measure μ. We define the *basin* of μ as the set of points $x \in M$ for which (2.2) holds for all continuous φ.

It easily follows from Birkhoff Ergodic Theorem that if μ is an ergodic probability measure which is *absolutely continuous with respect to the Lebesgue measure*, i.e. it does not give positive weight to sets with zero Lebesgue measure, then μ is a physical measure. Indeed, if μ is ergodic, then by Birkhoff Ergodic Theorem its basin has full μ measure. By absolute continuity, the basin of μ cannot have zero Lebesgue measure.

3. Low dimensional dynamics

There is no need of great complexity in evolution laws for which intricate dynamical behavior occurs. To illustrate this, the basic model is the family of quadratic maps $q_a(x) = 1 - ax^2$, where $x \in [-1, 1]$ and $a \in [0, 2]$ is a parameter[2]. In spite of its simple appearance, the dynamics of these maps presents many remarkable phenomena. From the topological point of view, the situation is quite well understood in most situations.

Theorem 3.1 ([Ly1], [GS])**.** *There is an open and dense set of parameters $a \in [0, 2]$ for which q_a has a periodic orbit[3] attracting Lebesgue almost every point.*

In spite of its simple formulation, this remained as a long term conjecture in one dimensional dynamics. From a probabilistic point of view, the situation is completely different. Its richness first became apparent with the work of Jakobson, where it was shown that a positive measure set of parameters corresponds to quadratic maps with chaotic behavior.

Theorem 3.2 ([Ja])**.** *There is a positive Lebesgue measure set of parameters $a \in [0, 2]$ for which q_a has an absolutely continuous ergodic measure μ_a.*

By the considerations at the end of Section 2 we have that μ_a is a physical measure. Some extra knowledge on the properties of μ_a allows us to show that $\log |q_a'|$ is μ_a

[2] Here is where the rich part of the dynamics lies. For parameters out of this range or points out of this domain the dynamics is well understood.

[3] The orbit of a given point x is called *periodic* if some positive iterate of it coincides with x.

integrable and $\int \log |q_a'| d\mu_a > 0$. By Birkhoff Ergodic Theorem

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} \log |q_a'((q_a^j(x))| = \int \log |q_a'| d\mu,$$

and so, using the chain rule, we have a positive *Lyapunov exponent* at almost every x:

$$\lim_{n\to\infty} \log |(q_a^n)'(x)| > 0.$$

The existence of this positive Lyapunov exponent gives one pervasive feature of chaos: *sensitivity to the initial conditions.*

As we have seen, at least two types of distinct behavior are present on the quadratic family, and they alternate in a complicate way. Besides these two types, different behaviors were shown to exist, including examples with bad statistics, like absence of a physical measure or a physical measure concentrated on a hyperbolic repeller. Finally Lyubich depicted a nice picture of the global situation.

Theorem 3.3 ([Ly2]). *For Lebesgue almost every $a \in [0,2]$ the map q_a has either a periodic attracting orbit or an absolutely continuous ergodic measure.*

Though we have used the absolutely continuous ergodic measure to obtain a positive Lyapunov exponent, the existence of this exponent can be deduced directly for a positive Lebesgue measure subset of parameters. The big difficulty in carrying this out is that quadratic maps combine regions of the phase space where the dynamics expands, together with a critical region where the derivative becomes arbitrarily small. In [BC1], Benedicks and Carleson implemented a strategy which enabled them to prove the existence of a positive Lyapunov exponent not only for quadratic maps, but also for the Hénon maps

$$\begin{array}{rccc} f_{a,b}: & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x,y) & \longmapsto & (1-ax^2+y, bx). \end{array}$$

In [He], Hénon proposed this two parameter family as a model for non-linear two dimensional dynamics. This can be thought as a simplified discrete-time version of the Lorenz flow and interpreted as an unfolding of the quadratic family[4]. Based on numerical experiments for $a = 1.4$ and $b = 0.3$, Hénon conjectured that this system should have a *strange attractor*. It was not at all *a priori* clear that the attractor detected experimentally by Hénon was not a long stable periodic orbit. Benedicks and Carleson managed to prove that Hénon's conjecture was true for small $b > 0$[5].

Theorem 3.4 ([BC2]). *There is a positive Lebesgue measure set $\mathscr{BC}$ of parameters such that for each $(a,b) \in \mathscr{BC}$ the map $f = f_{a,b}$ has the following properties:*

(1) *there is an open set $U \subset \mathbb{R}^2$ such that $\overline{f(U)} \subset U$ and $\Lambda = \bigcap_{n=0}^{\infty} f^n(U)$ attracts the orbit of every $x \in U$;*

(2) *there is $z_0 \in \Lambda$ whose orbit is dense on Λ, and there is $c > 0$ such that $\|Df^n(z_0)(0,1)\| \geq e^{cn}$ for all $n \geq 1$;*

(3) *f has a unique physical measure supported on Λ.*

The physical measure was obtained by Benedicks and Young in [BY1]. The second item of the theorem gives the existence of a positive Lyapunov exponent in a dense orbit, thus showing that the attractor displays sensitive dependence to the initial conditions for the parameters in $\mathscr{BC}$.

4. Non-uniformly expanding maps

As seen in the previous section, for one-dimensional maps the existence of absolutely continuous invariant measures is intimately connected with the existence of a positive Lyapunov exponent. Inspired by the remarkable progress for the one dimensional case, one presently aims at obtaining similar conclusions in higher dimensions. The first result we present in this direction is for uniformly expanding maps. A map $f\colon M \to M$ is called *uniformly expanding* if there is $\sigma < 1$ such that $\|Df(x)^{-1}\| < \sigma$ for every $x \in M$.

Theorem 4.1 ([KS]). *Let $f : M \to M$ be a C^2 uniformly expanding map. Then f has a unique ergodic absolutely continuous invariant probability measure whose basin has full Lebesgue measure.*

We are also interested in maps admitting (critical) sets where the derivative is not an isomorphism or simply does not exist. We say that $\mathscr{C} \subset M$ is a *non-degenerate critical set* if the derivative of f behaves as a power of the distance close to $\mathscr{C}$. Staying away from technicalities, we refer [ABV] for a precise definition of this concept. Let us just mention that it captures the flavor of non-flat critical points in dimension one.

Let $f\colon M \to M$ be a local diffeomorphism in $M \setminus \mathscr{C}$, where $\mathscr{C}$ is a non-degenerate critical set with zero Lebesgue measure. We say that f is *non-uniformly expanding* if the following conditions hold:

[4] For $b = 0$ orbits eventually lie on $\{y = 0\}$ and dynamics can be thought as that of quadratic maps.

[5] It remains an interesting open question to know if the chaotic attractor exists for Hénon's choice of parameters $a = 1.4$ and $b = 0.3$.

(c_1) there is $\lambda > 0$ such that for Lebesgue almost every $x \in M$

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| < -\lambda;$$

(c_2) for all $\epsilon > 0$ there is $\delta > 0$ such that for Lebesgue almost every $x \in M$

$$\limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(f^j(x), \mathscr{C}) < \epsilon.$$

Condition (c_1) allows points where the derivative does not expand. Expansion is only attained asymptotically in average for most orbits. We shall refer to (c_2) as *slow recurrence* to $\mathscr{C}$. It essentially says that generic orbits do not hit small neighborhoods of the critical set too frequently.

Theorem 4.2 ([ABV]). *Let $f\colon M \to M$ be a C^2 non-uniformly expanding map. There are absolutely continuous ergodic probability measures μ_1, ..., μ_p whose basins cover a full Lebesgue measure subset of M.*

Uniqueness can be obtained if f is *transitive*, i.e. with a dense orbit in M. Uniformly expanding maps are always transitive.

Condition (c_1) assures that the *expansion time* function $\mathscr{E}(x)$, defined as the minimum $N \geq 1$ such that for all $n \geq N$

$$\frac{1}{n}\sum_{i=0}^{n-1} \log \|Df(f^i(x))^{-1}\| < -\lambda,$$

is well defined and finite Lebesgue almost everywhere.

We observe that slow recurrence condition is not needed in all its strength. Actually, it is enough that it holds for some sufficiently small $\epsilon > 0$ and $\delta > 0$ conveniently chosen; see [Al, Remark 3.8]. We fix once and for all $\epsilon > 0$ and $\delta > 0$ in those conditions. This allows us to define the *recurrence time* $\mathscr{R}(x)$, as the minimum $N \geq 1$ such that for all $n \geq N$

$$\frac{1}{n}\sum_{i=0}^{n-1} -\log \operatorname{dist}_\delta(f^j(x), \mathscr{C}) < \epsilon,$$

which is finite Lebesgue almost everywhere. We define the *tail set* (at time n) as

$$\Gamma_n = \{x \in M\colon \mathscr{E}(x) > n \text{ or } \mathscr{R}(x) > n\}.$$

This is the set of points that at time n have not reached the exponential growth or slow recurrence assured by (c_1) and (c_2). Non-uniform expansion guarantees that the Lebesgue measure of this set converges to zero when $n \to \infty$. The speed of this convergence plays an important role in the statistical features of non-uniformly expanding dynamical system, as we shall see later on.

Next we present a family of maps, introduced by Viana in [Vi], that has served as a model for many general results on non-uniformly expanding maps.

Example 4.3 (Viana maps). Let a_0 be a parameter conveniently chosen and take $b : S^1 \to \mathbb{R}$ a Morse function. Consider the cylinder transformation $\hat{f} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ given by

$$\hat{f}(s, x) = \big(\hat{g}(s), \hat{q}(s, x)\big),$$

where $\hat{g}$ is an expanding map of the circle $\hat{g}(s) = ds$ (mod $\mathbb{Z}$), for some $d \geq 2$, and $\hat{q}(s, x) = a(s) - x^2$ with $a(s) = a_0 + \alpha b(s)$, for small $\alpha > 0$.

Theorem 4.4 ([Vi]). *If f is close to $\hat{f}$ in the C^3 topology, then f is non-uniformly expanding. Moreover, there is $c > 0$ such that $m(\Gamma_n) \lesssim e^{-c\sqrt{n}}$.*

Viana maps reveal some new phenomenon if comparing to the family one dimensional quadratic maps: the non-uniformly expanding behavior holds for an open set of transformations. Recall that by Theorem 3.1 we have density of parameters for which the corresponding quadratic map has a periodic attractor.

5. Mixing rates

There are several possible ways of measuring the chaoticity of a given dynamical system. One of them is analyzing its mixing rates. An invariant probability measure μ is said to be *mixing* if

$$\mu(f^{-n}(A) \cap B) \to \mu(A)\mu(B), \tag{5.1}$$

when $n \to \infty$, for any measurable sets A, B. We leave it as an easy exercise to the reader to show that mixing implies ergodicity[6].

Roughly speaking, mixing indicates that, as long as sufficiently large iterates are taken, the proportion of points in B arising from A tends to the proportion that A occupies in the whole space. In general there is no specific rate at which the convergence in (5.1) occurs. However, defining the *correlation function* of *observables* $\varphi, \psi\colon M \to \mathbb{R}$,

$$C_n(\varphi, \psi) = \left| \int (\varphi \circ f^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu \right|,$$

it is sometimes possible to obtain specific rates at which $C_n(\varphi, \psi)$ decays to zero, provided φ and ψ have sufficient regularity. Observe that choosing the observables

[6] The converse is not true: irrational rotations of the circle are ergodic and not mixing with respect to the length measure, which is obviously invariant.

as characteristic functions we get the definition of mixing.

Given $\varphi\colon M \to \mathbb{R}$, consider the random variables φ, $\varphi \circ f$, $\varphi \circ f^2$,... . The exponential decay of correlations tells in particular that $\varphi \circ f^n$ and φ become uncorrelated exponentially fast as n tends to infinity.

Theorem 5.1 ([BY2]). *Hénon maps have exponential decay of correlations (with respect to the unique physical measure) for parameters in $\mathscr{BC}$.*

A key ingredient in the proof of this result is the existence of a direction of non-uniform expansion. However, there is a well localized set of "critical" points where orbits suffer setbacks in expansion when they pass near this set. The decay of correlations takes into account the set of points approaching in a counterproductive way the source of non-expansion. The measure of this set decays exponentially fast to zero.

For non-uniformly expanding maps, *a priori* we have no knowledge on the source of "critical" behavior. The decay of correlations ultimately depends on the speed that the Lebesgue measure of the tail set converges to zero, at least for some specific rates.

Theorem 5.2 ([ALP], [Go]). *Assume that $f : M \to M$ is a C^2 transitive non-uniformly expanding map. If $m(\Gamma_n)$ is summable, then some power of f is mixing with respect to the (unique) physical measure μ. Moreover, for Hölder continuous φ, ψ one has:*

(1) *if there is $\gamma > 1$ for which $m(\Gamma_n) \lesssim n^{-\gamma}$, then $C_n(\varphi,\psi) \lesssim n^{-\gamma+1}$;*

(2) *if there are $\gamma > 0$ and $0 < \eta \le 1$ for which $m(\Gamma_n) \lesssim e^{-\gamma n^\eta}$, then there is $\gamma' > 0$ such that $C_n(\varphi,\psi) \lesssim e^{-\gamma' n^\eta}$.*

Using Theorem 4.4 we easily deduce that the decay of correlations for Viana maps has order $e^{-c\sqrt{n}}$ at least[7].

Let us now give some consequence of the decay of correlations. Starting with the Lebesgue measure m, one may consider the sequence of push-forwards $f_*^n m$, for $n \ge 1$, where these measures are defined for each $n \ge 1$ as $f_*^n m(A) = m(f^{-n}(A))$. In many situations (e. g. uniformly expanding maps) the absolutely continuous invariant measure is actually equivalent to the Lebesgue measure m, in such a way that we may take $\psi = dm/d\mu$ in $C_n(\varphi,\psi)$ and, assuming m normalized, we obtain

$$C_n(\varphi,\psi) = \left| \int (\varphi \circ f^n) dm - \int \varphi d\mu \right|,$$

Supposing $C_n(\varphi,\psi) \to 0$ as $n \to \infty$, one has

$$\int (\varphi \circ f^n) dm \longrightarrow \int \varphi d\mu.$$

This means that $f_*^n m$ converges in the weak* topology to μ. Hence, the faster correlations decay, the better physical measure are approximated by the push-forwards of Lebesgue measure.

6. Statistical stability

One is interested in studying the variation of physical measures in certain classes of dynamical systems. Its continuous variation points in the direction of stability of the dynamical system, at least in terms of the statistical distribution of orbits for nearby dynamics.

Let $\mathscr{F}$ be a family of C^k maps, for some $k \ge 2$, from a manifold M into itself, and consider $\mathscr{F}$ endowed with the C^k topology. Assume that each $f \in \mathscr{F}$ admits a unique physical measure μ_f. We say that $\mathscr{F}$ is *statistically stable* if

$$\mathscr{F} \ni f \longmapsto \mu_f$$

is continuous with respect to the weak* topology on the space of probability measures.

As shown in Theorem 3.4, though highly unstable in terms of the evolution of its individual orbits, Hénon attractors at $\mathscr{BC}$ parameters are fairly regular in statistical terms. The next result shows that the statistics of the these maps does not change dramatically when one perturbs parameters in $\mathscr{BC}$.

Theorem 6.1 ([ACF]). *The family $\mathscr{BC}$ is statistically stable.*

The physical measures of Hénon maps at $\mathscr{BC}$ parameters are supported on attractors with zero bidimensional Lebesgue measure. Consequently, those physical measures are necessarily singular with respect to the Lebesgue measure. In cases where the physical measure is absolutely continuous with respect to the Lebesgue measure m on the phase space, we may even aim at *strong statistical stability*: the map

$$\mathscr{F} \ni f \longmapsto \frac{d\mu_f}{dm}$$

is continuous with respect to the $L^1(m)$ norm in the space of densities.

The following result holds for families $\mathscr{F}$ of non-uniformly expanding maps. We denote by Γ_n^f the tail set associated to $f \in \mathscr{F}$.

Theorem 6.2 ([AV],[Al]). *Assume that there are $C > 0$ and $\gamma > 1$ such that $m(\Gamma_n^f) \le Cn^{-\gamma}$ for all $f \in \mathscr{F}$ and $n \ge 1$. Then $\mathscr{F}$ is strongly statistically stable.*

Using Theorem 4.4 we easily deduce that the family of Viana maps is strongly statistically stable[8].

[7] It remains an interesting open question to know if the estimate for the measure of the tail set given by Theorem 4.4 is optimal.
[8] Though not explicitly stated in Theorem 4.4, the rate at which $m(\Gamma_n)$ decays to 0 is uniform on the set of Viana maps.

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