Math in the Media

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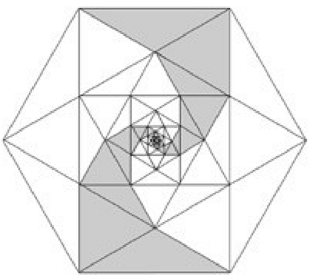
Spidrons.

This spidroball, a rhombic triacontahedron assembled from 30 spidron pairs, appeared on the cover of the October 21 *Science News*. Image courtesy Marc Pelletier, Walt van Ballegooijen, Dániel Erdély and Amina Buhler Allen.

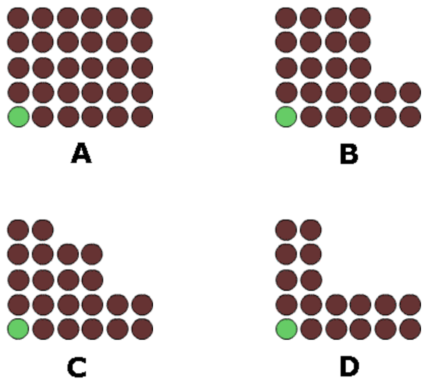
The easiest way to draw spidrons is to start with a hexagon, inscribe a six-pointed star, and repeat with the star's interior hexagon ad infinitum. Then attaching to each flat isosceles triangle the equilateral triangle to its right, and to each equilateral triangle the isosceles triangle below it, you get six spiralling polygonal chains: each one is half a spidron. (Alternatively, always attach to the left; this construction makes it clear that the area of a spidron is one third of the area of the hexagon you started with).



Two semi-spidrons assemble into one full spidron.

Spidrons were recently featured in *Science News Online* (October 21, 2006), where Ivars Peterson tells us that they were invented and named in the early 1970s by Dániel Erdély, a Hungarian industrial designer and a student of Rubik, and that Erdély soon discovered that when creased properly a spidron array takes on interesting 3-dimensional behavior, with potential practical applications. Erdély has recently been collaborating with other graphic designers and with sculptors; some of their work was presented at last summer's Bridges conference (www.lkl.ac.uk/bridges). One sample: the spidroball shown above. Additional images are available on Erdély's Spidron website (www.spidron.hu).

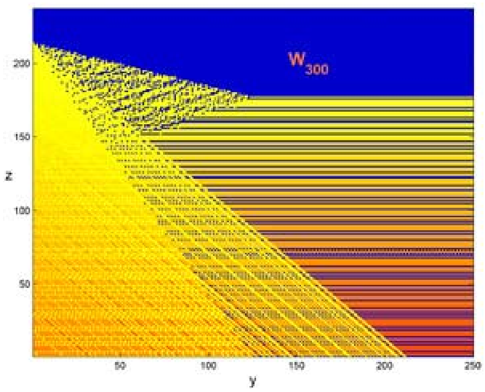
Next year in Marienbad: chaos. Chomp is a 2-dimensional version of Nim (www.csm.astate.edu/Nim.html), the game popularized in *L'année dernière à Marienbad*.



The first 3 moves in a 5 x 6 game of Chomp. A: the initial configuration; the object is not to be forced to select the green cookie. B: after Player 1's first bite. C: after Player 2's first bite. D: after Player 1's second bite. Each bite takes a cookie and all the cookies north and east of it.

But while a simple strategy exists for Nim, Chomp is much harder. It is known that there is always a winning strategy for Player 1 but the strategy itself is unknown in general, except for a few special cases like $n \times n$, $2 \times n$, and $n \times 2$. In "Chaotic Chomp" (*Science News Online*, July 22, 2006, www.sciencenews.org/articles/20060722/bob10.asp) Ivars Peterson reports on developments in the analysis of the $3 \times n$ case. Chomp dates back to 1974 (in fact, it is equivalent to a game discovered in 1950) but was taken up a few years ago by Doron Zeilberger, a mathematician at Rutgers, who decided it would be "an ideal problem for illustrating the role that computers can play in mathematical research."

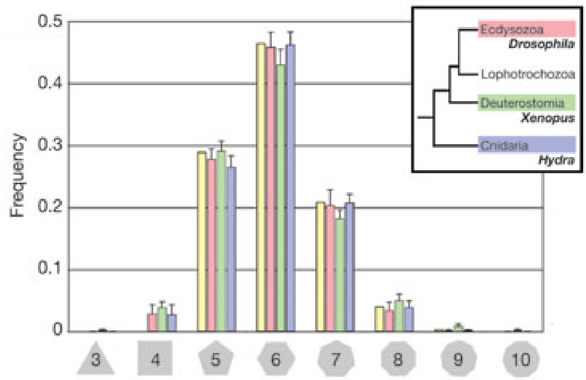
Zeilberger introduced the notation (x, y, z) to describe the position in $3 \times n$ Chomp which has x columns of 3 cookies, y columns of 2, and z columns of 1, and published in 2000 an algorithm generating for each x an algorithm for playing the game with an arbitrary y and z. He returned to the problem in 2003 with faster algorithms and on the basis of the results speculated "It seems that we have 'chaotic' behavior, but in a vague, yet-to-be-made-precise sense." Peterson focuses on the recent work of Eric Friedman (Computer Science, Cornell) and Adam Landsberg (Physics, Claremont colleges), who have fleshed out this intuition: "By using mathematical tools originally developed for calculating properties of physical systems, Friedman and Landsberg show that the exact location of winning and losing cookies in Chomp varies unpredictably with small changes in the size of the initial array." The figure below uses Zeilberger's notation and shows in yellow/red, for $x = 300$, the "instant winner" positions (positions from which you can leave your opponent in a losing position with smaller x). The chaotic region is clearly visible.



Winning positions (yellow/red) for a 3-row Chomp game with 300 height-3 columns. The y and z coordinates refer to the number of height-2 and height-1 columns, respectively. Image courtesy Adam Landsberg.

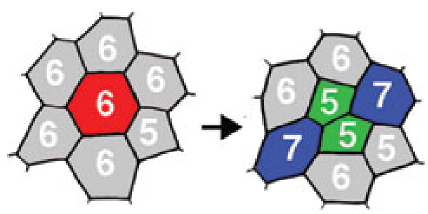
Furthermore, they made the remarkable discovery that Chomp is renormalizable. As Peterson explains it, "the geometry of winning positions for small values of x and winning positions for large values of x is roughly the same, after a suitable change in scale." Specifically, the W600 figure, scaled down by a factor of 2 in each direction, is essentially indistinguishable from the W300 shown here. Zeilberger's papers (excellent reading) are available at www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/chomp.html and www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/byrnes.html. Friedman and Landsberg's paper is also available online, as a PDF file (people.cornell.edu/pages/ejf27/pfiles/chomptr.pdf). For a history of the problem, see Andries Brouwer's page (www.win.tue.nl/~aeb/games/chomp.html) on the game.

Epithelial topology. Epithelial tissue is typically a 2-dimensional array of cells. Topologically the average cell shape must be a hexagon. Remarkably, an identical, asymmetric distribution of polygonal shapes shows up over an enormous range of organisms. Drosophila is a fly, Xenopus is a frog and Hydra is a tiny fresh-water relative of jellyfish.



"Drosophila wing disc (pink), Xenopus tail epidermis (green) and Hydra epidermis (blue) all exhibit a similar non-gaussian distribution of epithelial polygons with less than 50% hexagonal cells and high (and asymmetric) percentages of pentagonal and heptagonal cells. The inset indicates relative phylogenetic positions for Drosophila, Xenopus and Hydra." Yellow bars represent the theoretical distribution derived in this article (see below). Image from *Nature* 442 1038-1041, used with permission.

A theoretical explanation for this phenomenon is given in "The emergence of geometric order in proliferating metazoan epithelia," by Matthew Gibson (Harvard) and collaborators, in *Nature* for August 31, 2006. It relies on the observation that when a cell in a 2-dimensional array divides, each of its daughters typically has one fewer neighbor, while two of its neighbors pick up an extra side.

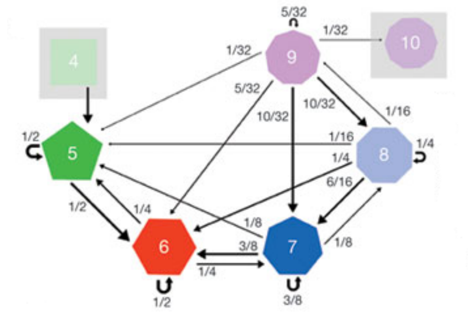


Typically the daughters of a hexagonal cell are pentagons, while two hexagonal neighbors become heptagons. Image from *Nature* 442 1038-1041, used with permission.

To study the way the distribution of polygonal types changes under repeated subdivisions, Gibson and his colleagues axiomatize the situation (each of these statements is given an experimental justification):

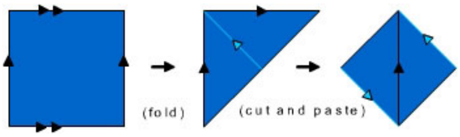
- cells are polygons with a minimum of four sides.
- cells do not re-sort.
- mitotic siblings retain a common junctional interface.
- cells have asynchronous but roughly uniform cell cycle times.
- cleavage planes always cut a side rather than a vertex of the mother polygon.
- mitotic cleavage orientation randomly distributes existing tricellular junctions to both daughter cells.

They use these axioms to construct a Markov-chain model for the distribution of polygonal types. This model predicts the yellow bars in our first image, and also predicts a rapid evolution to this distribution regardless of the initial set of polygonal types.



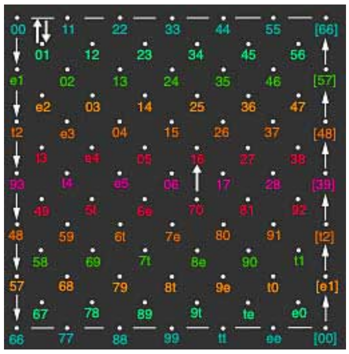
Markov-chain model for the change in polygonal type from one generation to the next. Image from *Nature* 442 1038-1041, used with permission.

The Geometry of Musical Chords. This is the title of a report (music.princeton.edu/~dmitri/voiceleading.pdf) in the July 7 2006 *Science*, written by Dmitri Tymoczko, Professor of Music at Princeton. The abstract begins: "A musical chord can be represented as a point in a geometrical space called an orbifold. Line segments represent mappings from the notes of one chord to those of another." The simplest example of this representation is for the case of intervals, or two-note chords. As Tymoczko says, "Human pitch perception is both logarithmic and periodic." We judge the distance between tones in terms of the ratio of their pitches, and identify tones when that ratio is 2. So the psychological space of tones is a circle, where we can mark off 12 equidistant points corresponding to the pitch classes $C, C\#, D, \ldots, A\#, B$. It is convenient to identify this circle with $T^1 = \mathbb{R}/12\mathbb{Z}$ and to place the equal-tempered pitches at the integral points $0(C), \ldots, 11(B)$. Then the space of pairs of tones is the torus $T^1 \times T^1$ and the space of intervals (unordered pairs) is the quotient of this torus by the relation $(x, y) \sim (y, x)$. The result is a Möbius strip, a manifold with boundary and thus an *orbifold*.



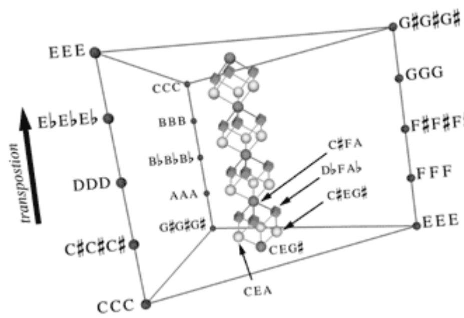
The identification $(x, y) \sim (y, x)$ makes the torus into a Möbius strip, a manifold with boundary.

Here is how the intervals appear on the Möbius strip:



The 2-note chords, or intervals, as they appear on the Möbius strip of unordered tone pairs. "t" is 10 and "e" is 11. $70 = 07$ corresponds to the fifth chord $C - G$. Transposition corresponds to sideways motion. "Voice leading," i.e. motion through chords, is represented by paths on the surface: e.g. $C - G \to D - F\#$ is represented by the arrow $70 \to 16$. Note that the voice leading $C - C\# \to C\# - C$ reflects off the upper boundary. Image courtesy Dmitri Tymoczko.

For three or four-note chords the topology becomes more complicated. For example three-note chords live on the 3-dimensional orbifold constructed by taking a 3-dimensional prism with base a triangle, twisting the base so as to cyclically permute the vertices, and identifying it with the opposite face. But it makes musical sense: "Chords that divide the octave evenly lie at the center of the orbifold and are surrounded by the familiar sonorities of Western tonality."



The 3-dimensional orbifold representing the space of 3-note chords. "Chords that divide the octave evenly lie at the center of the orbifold and are surrounded by the familiar sonorities of Western tonality." Image courtesy Dmitri Tymoczko.

It is clearly Tymoczko's intent for these representations to serve not only as a tool in musical analysis, but also as a stimulus for new directions in composition. More information, including free ChordGeometries software, on his website (music.princeton.edu/~dmitri).

Mathematical error control. Barry Mazur has a "News and Views" piece in the September 7 2006 *Nature* about recent steps towards the proof of the Sato-Tate conjecture, which predicts the distribution of the error terms in good approximations for solutions of combinatorial number-theoretic problems. These are not the errors that plague natural scientists measuring things out in the field, but the report is a work of art. Mazur illustrates the conjecture with a nice, and elementary, example. The problem here is to count the number $N(p)$ of ways a prime number p can be written as a sum of 24 squares of integers. Note that zero and negative numbers are allowed to participate, and that all permutations of the terms in a sum must be counted as different "ways". So $N(2)$ is already 1,104. Now there exists a good approximation $A(p)$ for $N(p)$:

$$A(p) = \frac{16}{691}(p^{11} + 1),$$

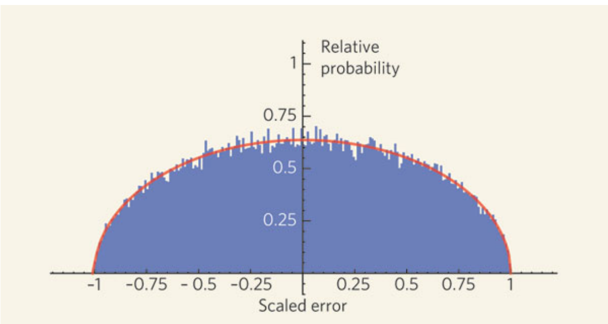
good in the sense that the error scales like the square root of N; in fact there is an explicit least upper bound for the error as a function of p:

$$\mid N(p) - A(p) \mid \leq \frac{66304}{691}\sqrt{p^{11}}.$$

For problems like this, the Sato-Tate conjecture predicts that the distribution of the scaled error

$$\frac{N(p) - A(p)}{\frac{66304}{691}\sqrt{p^{11}}}$$

should be governed by the distribution $\frac{2}{\pi}\sqrt{1-x^2}$, whose graph is a semi-circle normalized to have area 1.

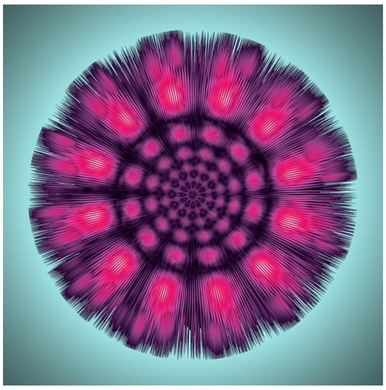


The scaled error distribution for $N(p)$ predicted by the Sato-Tate conjecture (red curve) and the actual distribution for primes less than one million. Image from *Nature*, 443 38-40, used with permission.

For this case of the conjecture the evidence is excellent but there is as yet no proof. Mazur mentions a class of problems, related to elliptic curves, where the conjecture has in fact been proved (through the efforts of his Harvard colleague Richard Taylor and Taylor's collaborators). As Mazur explains it, "The proof came by combining some wonderful pieces of mathematics, and the key to it all is the so-called representation theory. This branch of mathematics, in its various guises, studies abstract groups by representing them as groups of linear transformations of vector spaces. By understanding the profound number-theoretic structure behind enough of the symmetric tensor powers of a certain representation of a certain group, one can compute the probability distribution of the corresponding scaled error terms, and so confirm the Sato-Tate conjecture." Mazur concludes: "This is a magnificent achievement for at least two reasons. First, the method brings synthetic unity to deep results in quite distinct mathematical fields. ...Second, the work answers a question of delicate nature. Number theorists have long held the opinion that the 'error terms', despite the pejorative name, have a mesmerizingly rich structure ... and that the keys to some of the deepest issues in their subject lie hidden in that structure."

Math at the World Cup. According to a news report in the June 15, 2006 *Nature*, it has been established mathematically that soccer goals are contagious, statistically speaking: scoring one goal increases the probability that your team will score more. Michael Hopkin, who write the piece, calls this "one of soccer's classic clichés," and attributes the result to Martin Weigel (Herriot-Watt University, Edinburgh) and his colleagues Elmar Bittner, Andreas Nussbaumer and Wolfhard Janke, all at Leipzig University. The four have posted a preprint on arXiv.org (arxiv.org/abs/physics/0606016) with the title "Football fever: goal distributions and non-Gaussian statistics." As they put it: "modifying the Bernoulli random process underlying the Poissonian model to include a simple component of self-affirmation seems to describe the data surprisingly well and allows to understand the observed deviation from Gaussian statistics." They analyzed "historical football score data from many leagues in Europe as well as from international tournaments, including data from all past tournaments of the 'FIFA World Cup' series" and concluded: "The best fits are found for models where each extra goal encourages a team even more than the previous one: a true sign of *football fever*." The group paid special attention to three German soccer leagues: the East German *Oberliga*, the West German *Bundesliga* and the women's league, the *Frauen-Bundesliga*. They found that their self-affirmation factor κ was higher for the East German league and highest of all for the women.

Math: Whale Songs → Kaleidoscopic Images. "Subtle Math Turns Songs of Whales Into Kaleidoscopic Images" was the headline for a piece in the August 1 2006 *New York Times*, accompanied by four images like this one.



A periodic segment of the song of the Minke Whale *Balenoptera acutorostrata*; graphic generated using wavelet analysis; plotted in polar coordinates with time $=\theta$. Aguasonic image (www.neoimages.net/artistportfolio.aspx?pid=1421) by Mark Fischer, used with permission.

Gretchen Cuda tells how Mark Fischer, a California-based former engineer, has been using "wavelets — a technique for processing digital signals — to transform the haunting calls of ocean mammals into movies that visually represent the songs and still images that look like electronic mandalas." Cuda checked with Gil Strang of the MIT Math Department, and reports that wavelets, once relatively obscure, "are being used in applications as diverse as JPEG image compression, high definition television and earthquake research." The song and the video, where the pattern shown above can easily be recognized, are available at Minke-Boing on Google.uk (video.google.co.uk/videoplay?docid=-5022402114614151095&q=minke+boing). More from Mark Fischer on his website (www.aguasonic.com).

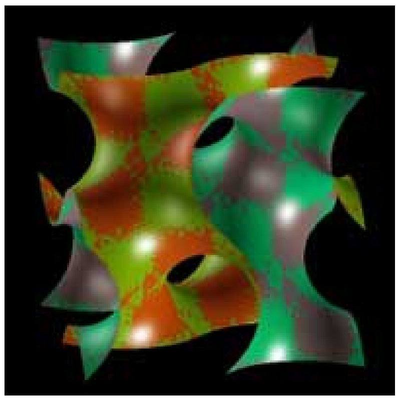
Nanoscale Minimal Surface? "Mesostructured germanium with cubic pore symmetry," by the MSU chemists Gerasimos Armatas and Mercouri Kanatzidis, appeared in the June 29, 2006 *Nature*. The article describes a preparation of germanium resulting in "two three-dimensional labyrinthine tunnels obeying $Ia\bar{3}d$ space group symmetry and separated by a continuous germanium minimal surface." The thickness of the walls of this germanium structure is given as one nanometer. The "minimal surface" separating the labyrinths is identified as the gyroid (www.msri.org/about/sgp/jim/geom/minimal/library/G), a triply periodic surface first described by Alan Schoen in an NASA Technical Note dated May 1970. Schoen gives the (x, y, z) coordinates of a point on the surface in terms of complex integrals:

$$x = \Re \int \frac{e^{i\theta_G}(1-\tau^2)}{\sqrt{1-14\tau^4+\tau^8}}\, d\tau$$

$$y = \Re \int i\, \frac{e^{i\theta_G}(1+\tau^2)}{\sqrt{1-14\tau^4+\tau^8}}\, d\tau$$

$$z = \Re \int 2\, \frac{e^{i\theta_G}\tau}{\sqrt{1-14\tau^4+\tau^8}}\, d\tau$$

where $\theta_G = 38.0147740^o$ approximately is calculated using elliptic integrals. (The 3-integral format goes back to Weierstrass; the specific $\sqrt{1-14\tau^4+\tau^8}$ was used, with 0^o and 90^o instead of θ_G, in H. A. Schwarz's 1865 construction of the first known triply periodic minimal surfaces). Armatas and Kanatzidis, on the other hand, use the much more simply defined level surface $\cos x \sin y + \cos y \sin z + \cos z \sin x = 0$. What is going on? As David Hoffman explained to me, these two surfaces, although extremely close, are not the same. The coincidence is mysterious. As I understand it, chemists start with the symmetry group, which they determine by Fourier analysis of transmission electron micrographs of their sample. From the symmetry group they calculate the equation of a periodic nodal surface as a Fourier series. Our level surface equation comes from setting the sum of the lowest order terms to zero.



The Gyroid (red) and the surface $\cos x \sin y + \cos y \sin z + \cos z \sin x = 0$ (green) plotted together. Image: James T. Hoffman and David Hoffman, Scientific Graphics Project (www.msri.org/about/sgp/jim/geom/level/minimal), used with permission.

Taking more terms gives better approximations to the gyroid, but why this procedure leads to a minimal surface is, as far as I can tell, unknown.