Daniel Augusto da Silva, Poet of Mathematics

by **Carlos Florentino***

Dedicated to the memory of Jaime de Lima Mascarenhas

Daniel da Silva was a remarkable Scientist and Mathematician of the mid 19th century. Working in Portugal, isolated from the main scientific centers of the time, his investigations in pure mathematics had almost no impact. Apart from giving a short biography of his life and work, this article makes the case for considering him an unavoidable character in the History of Science and one of the founders of Discrete Mathematics, through his introduction of a key method in Enumerative Combinatorics: the *Principle of Inclusion-Exclusion*.

> *Ninguem está authorisado a capitular quaesquer theorias mathematicas como destituidas de applicação vantajosa, como um mero recreio de elevadas intelligencias, e como inuteis trabalhos em relação à verdadeira sciencia.*
>
> —Daniel da Silva

1 Introduction

A Daniel Augusto da Silva (1814–1878) is one of the greatest Portuguese scientists of the 19th Century. His contributions are remarkable, for their quality and originality, not only in the context of Portuguese Science, but also internationally. Even though his life and mathematical work is documented by historians and some researchers, his important contributions have not yet received the deserved recognition from the mathematical community.

Daniel da Silva produced only a few manuscripts of scientific nature, most of them published by the Lisbon Academy of Sciences between the years 1851 and 1876. These articles belong to the fields of Statics (more precisely, what we call today Geometric or Rational Mechanics), Physics/Chemistry, Statistics and Actuarial Sciences, and Number Theory.

The book by Francisco Gomes Teixeira (1851–1933) [T1], published in 1934, and considered to be the most important reference about History of Mathematics in Portugal up to the end of the 19th century, singles out the four most important mathematical characters, according to the author: Pedro Nunes (1502–1578), Anastácio da Cunha (1744–1787), Monteiro da Rocha (1734–1819), and Daniel da Silva. Gomes Teixeira refers to Daniel in this way:

> Daniel da Silva, poet of mathematics, searched in these sciences what they have of beautiful; [. . .] he gave to the world of numbers his Statics, without worrying with the applications of this chapter of rational mechanics, that others later did, and gave it also his beautiful investigations about binomial congruences.

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Figure 1.—Daniel Augusto da Silva

Figure 2.— Dom Pedro II, Brasil's emperor, docks at Lisbon's Navy Arsenal in 1876

As happened with other great names of Science, the work of Daniel da Silva did not take place without unfortunate moments and drama. Indeed, two facts played a major role: his enormous creative capabilities ended up being constrained by a major illness he suffered for several years; and the fact that he always used the Portuguese language made it very difficult for the recognition of his contributions in other European countries with much more solid and developed scientific communities.

In this short article, we are going to summarise his biography and his work, concentrating on his most important innovations in Pure Mathematics, area which represented, in his own words, his great passion (sections 2 and 3). For their relevance in today's mathematics, we finish by analyzing with further detail (section 4) da Silva's contributions to Discrete Mathematics and Number Theory, which deserve to be widely known.

Acknowledgements

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2 Education and Professional Life

Daniel Augusto da Silva was born in Lisbon on the 16th of May of 1814, being the second son of Roberto José da Silva and Maria do Patrocínio. Roberto Silva was a merchant although his specific business does not seem to be documented.

Daniel's formative years took place against a difficult background of tensions and wars in Portugal. In fact, historians agree that the whole first half of the 19th century was not auspicious for the development of scientific culture in the country: this period included the Napoleon invasions, between 1807 and 1811, the violent campaigns opposing liberals and absolutists, and a civil war between 1828 and 1834.

At the age of 15, Daniel enrolled in the Royal Navy Academy (Academia Real da Marinha, ARM), and took mathematics courses ranging from Arithmetics to Calculus, as well as some courses on Mechanics, Physics and Navigation. He also attended courses at the Lisbon Royal Naval Observatory. He immediately showed special talent for mathematics and was awarded a distinction in each of the three years there. In 1832, he entered, by merit, the Royal Academy of Marine Guards (Academia Real dos Guardas-Marinhas, ARGM), an academy typically reserved for sons of officials, and was appointed Navy Officer in 1833. As he became interested in mathematics, after finishing the ARGM degree in 1835, he asked for permission and for a small fellowship, to enrol in the Mathematics Faculty of the Coimbra University (the only Portuguese University at the time). Being approved by the Navy, he moved to Coimbra, and his performance in the University was no less brilliant than in both Academies: many years later some of his old professors could still remember the brightness of da Silva as a student.

Having finished his studies in Coimbra in 1839, he immediately returned to Lisbon, and followed a career in the Navy. He was promoted to Brigadier on 1840, and, later that year, to Second Lieutenant of the Navy. Following the French trend of *Grands Écoles*, in 1845, the ARGM was transformed into the Navy School (Escola Naval), and Daniel was appointed as a professor there. He taught Mechanics; Astronomy and Optics; Artillery and Fortification, and Geography and Hydrography. Initially, he was hired as *Lente Substituto* (Assistant Professor), and became *Lente Proprietario* (Full Professor) in 1848.

There were two unfortunate moments when da Silva lost the opportunity of becoming a Professor at the blossoming Polytechnic School of Lisbon. This School had just been created in 1837, in the context of a higher education reform, by a Royal Decree, to replace the ARM. In 1839 he applied to a teaching position through a competitive process but unfortunately, for health reasons, Daniel da Silva could not be present in a kind of interview/examination. Even after justifying his absence, the panel decided to cancel the placement, afraid of a possible impugnation. The second occasion was in 1848 when the Directing Body of the Polytechnic School of Lisbon, acknowledging Daniel da Silva's value, directly asked the Government to authorise his appointment, something rejected on the grounds that, by law, all places should be filled by public competition.

Nonetheless, while in the Navy School, it is between 1848 and 1852 that Daniel da Silva experiences his first main creative period with the completion of his 3 first manuscripts, sent for publication by the Royal Lisbon Academy of Sciences (Academia Real das Ciências, ARC[1]). In 1851, he becomes a *free member* of this Society, and is elected as full member the following year.

In late 1852 his health problems became so severe, and magnified by his overwork and his great dedication to research, that he applied for a leave and went to Madeira, hoping to recover there. However, his poor health persisted and he was unable to carry out his duties until in 1859 the Naval Health Board classified him "unfit for active duty."

This same year, he was ellected *honorary member* of the ARC, and married Zefferina d'Aguiar (1825–1913) from the town of Funchal. Daniel and Zefferina had a single child, Júlio Daniel da Silva who was born in 1866. Sadly, Júlio would die at the age of 25 without descendants.

Even without teaching duties, he continued to hold his Navy position until retiring in 1868. In his latest years, worried that his passion by Pure Mathematics would worsen his health condition, Daniel continued to do research, but dedicated himself to more applied Sciences, publishing works in actuarial sciences and on the theory of the flame. In his words:

> The passion for the study of mathematics, that was in me greatly disordered by excess, many years now has been reduced to the modest proportions of a platonic love.

In 1871, the young Francisco Gomes Teixeira, a third year student in Coimbra heard his professor José Queirós mention da Silva's theory of couples in Statics with high praises, recalling his brightness as a student, more than 30 years before. An excellent mathematician himself, Teixeira became acquainted with Daniel's work, and decided to write an essay on continued fractions, a subject of Daniel's incomplete chapter 10 of [dS]. He then wrote a letter to Daniel including this essay, and this started an excellent and joyful relationship between the two. Da Silva soon invited Gomes Teixeira to become a member of the ARC and tried to get him a position in the Lisbon Astronomic Observatory. After Daniel died, Teixeira presented his eulogy to the ARC and he would become da Silva's most complete biographer [T2]. Further accounts on Daniel da Silva life and work can be found in [Di, Du, Ma, O, Sa1, Va].

3 Scientific Work

In the period 1849–51, Daniel da Silva wrote 3 manuscripts concerning investigations on geometric methods in statics of rigid bodies, and on number theory. These show that he was an avid reader of the classics, being inspired by names such as Euler (1707–83), Lagrange (1736–1813), Legendre (1752–1833), Gauss (1777–1855) and Poinsot (1777–1859). He would obtain mathematical articles published in European Academies of Sciences, especially from the one in Paris.

Daniel's first paper *On the transformation and reduction of binaries of forces* was written before 1850, but only published in 1856 by the ARC. Closely following an article of Louis Poinsot on the same theme, this article contains no original results, but presents a new treatment and some simplified proofs.

His second paper, *Memoir on the rotation of forces about their points of application*, was read to the Academy in 1850 and published the following year. Here, da Silva considers a system of forces turning around their points of application, but maintaining their relative angles during the rotation. The article was written without knowledge of the results of A. F. Möbius (1790–1868) on this subject. Möbius had incorrectly stated that if a system is in equilibrium in four different orientations, then it is in equilibrium in all possible positions. In this memoir of da Silva describes correctly the equilibrium properties of a system of forces, and

[1] Nowadays called *Academia das Ciências de Lisboa*.

also proves that, in general, there are only four equilibrium positions. The third of da Silva's memoirs, on number theory, was read to the ARC on March 1852, but his illness prevented the completion of the published version [dS] (see below). We dedicate section 4 to an exposition (of the initial part) of this article, and its important achievements. About it, Teixeira writes:

> The main subject he considered was the resolution of binomial congruences, a theory which belongs simultaneously to the domain of higher arithmetic and higher algebra, and he enriched it with such important and general results that his name deserves to be included in the list of those who founded it. It was indeed Daniel da Silva who first gave a method to solve systems of linear congruences, an honour which has been unduly attributed to the distinguished English arithmetician Henry Smith,[2] who only in 1861 dealt with this subject, and was also the one who first undertook the general study of binomial congruences.[T1]

As mentioned, da Silva's health prevented him from doing any serious mathematical research for quite some time. Only many years later, his health did improve a little and he again undertook research. This new period started in 1866, with a very short article on Statics and, shortly after, two articles on Statistics and Actuarial Sciences. Daniel proposed mathematical models of demography and applied them to the financial structuring of pension funds, and in particular to one of the oldest portuguese Welfare Institutions called Montepio Geral (founded in 1840). These two articles, *Average annual amortization of pensions in the main Portuguese Welfare Institutes*, and *Contribution to the comparative study of population dynamics in Portugal* are described in detail in a recent PhD thesis [Ma], which is also a very important source of information on Daniel da Silva, and on Actuarial Calculus and the Navy Schools, at the epoch.

In 1872, he published *On several new formulae of Analytic Geometry relative to oblique coordinate axes*, which generalizes certain well known formulae in Analytic Geometry to a setup based on non-orthogonal frames.

He also carried out studies in the area of Physics and Chemistry, that could have been motivated by his previous lectures in the Navy School. He performed several experiments, with the collaboration of António Augusto Aguiar (1838–1887), professor at the Polithecnic School of Lisbon, and studied the speed of transmission of a gas flame in its blueish and brightest part in the 1873 article *Considerations and experiments about the flame*.

In 1877, the last year of his life, Daniel received some unpleasant news from France. As he had worked in scientific isolation, and used the portuguese language, J. G. Darboux (1842–1917) had just published results very similar to Daniel's own investigations on Statics (including the same correction of Möbius' mistake) without acknowledging da Silva's work. His letter to Teixeira denotes his disappointment:

Figure 3.—Logo of *Academia das Ciências de Lisboa*

> Almost all propositions of Darboux are published twenty six years ago in the Memoirs of the Lisbon Academy of Sciences, in my work on the rotation of forces about their points of application! [. . .] My memoir, which contains many other things, besides those considered by Möbius, including a correction of one mistake he did, the same one which Darboux proudly claims correcting, lies ignored, for nearly twenty six years, in the Libraries of almost all Academies of the world! What worth it is writing in portuguese!

Daniel da Silva knew of Darboux's article [Da] in the French journal "Les Mondes", and sent to it a "Réclamation de Priorité"; this reclamation was published in this periodical on March that same year, but, as L. Saraiva writes in [Sa1]:

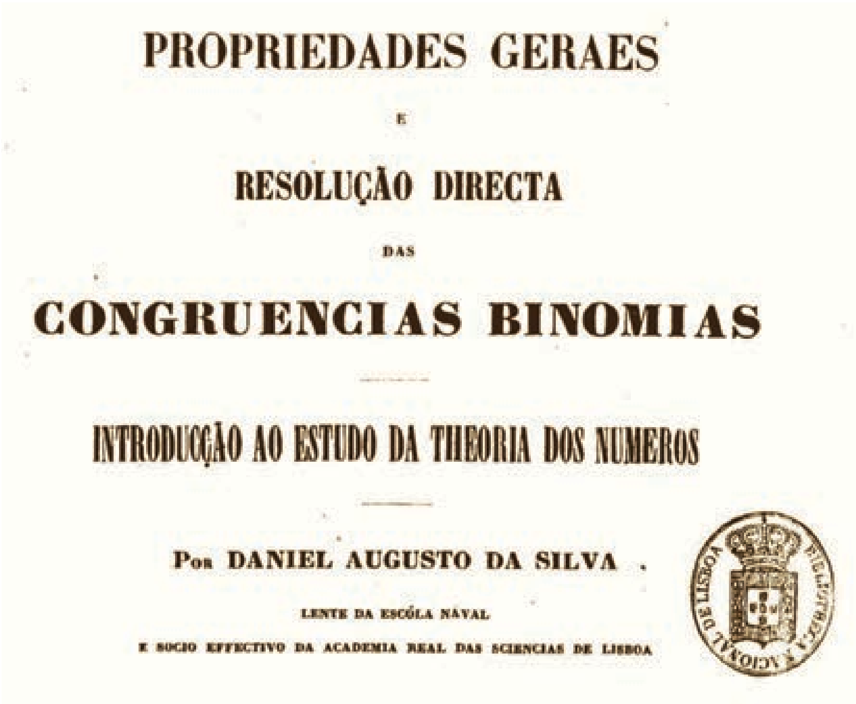
> It certainly would have been better if he had written directly to the French Academy of Sciences, where his work could have been more widely discussed.

After Daniel's death, this whole story had such a profound influence on Gomes Teixeira, that he became one of the first portuguese scientists and mathematicians to continuously promote the interaction of portuguese academics with foreign researchers. He founded the first mathematical journal, independent of any academic institution, printed in the Iberian Peninsula (Jornal de Sciencias Mathematicas e Astronomicas, see [R]) which substantially contributed to the dissemination of portuguese research. And he also stimulated the analysis of Daniel's work in the international scene, inviting some of his students and collaborators to review and to continue da Silva's work.

For a comparative analysis of the results of da Silva on Statics and those of A. F. Möbius and F. Minding (1806–1885), see F. A. Vasconcelos [Va], who was encouraged by Gomes Teixeira to perform this detailed study.

Daniel's research on the propagation of the flame was rediscovered by the German chemist and professor in Zurich, Karl Heumann (1850–1894) who, recongnizing the priority of Daniel da Silva in some of these investigations, wrote him a letter in 1878. Unfortunately, Daniel would never read it, as the letter arrived right after he passed away. For a complete list of da Silva's publications, see [DdS].

[2] *Henry J. S. Smith (1826–1883), see [Sm].*



PROPRIEDADES GERAES

E

RESOLUÇÃO DIRECTA

DAS

CONGRUENCIAS BINOMIAS

INTRODUCÇÃO AO ESTUDO DA THEORIA DOS NUMEROS

Por DANIEL AUGUSTO DA SILVA

LENTE DA ESCÓLA NAVAL

E SOCIO EFFECTIVO DA ACADEMIA REAL DAS SCIENCIAS DE LISBOA

Figure 4.—Cover page of Daniel da Silva's memoir on Congruences

4 Discrete Mathematics and Number Theory

The memoir entitled *General properties and direct resolution of binomial congruences* [dS] was presented to the Lisbon Academy of Sciences in 1852 and published two years later. There are many reasons to consider this as Daniel da Silva's masterpiece. Right in the first pages, da Silva develops a case for the importance of Pure Mathematics and its relationship with Applications, for the relevance and elegance of Number Theory, citing and praising several famous mathematicians such as Fermat (1601–65), Euler, Lagrange, Legendre, Poinsot and Gauss, instead of going directly to the results as in his other articles. Even though it includes many original results, this memoir appears also to have a pedagogical goal, as indicated by the subtitle *Introduction to the study of number theory*. This (as well as the expression *General Properties*) hints that Daniel da Silva had in mind the foundations of a whole new theory of mathematics, naturally abstract, but that could provide, in his opinion, numerous applications in many contexts.

A second reason is that a big portion of the basics of what we call today Discrete Mathematics are literally present in this work, constituting a great advance for the epoch. A typical syllabus for a freshman Discrete Mathematics course includes:

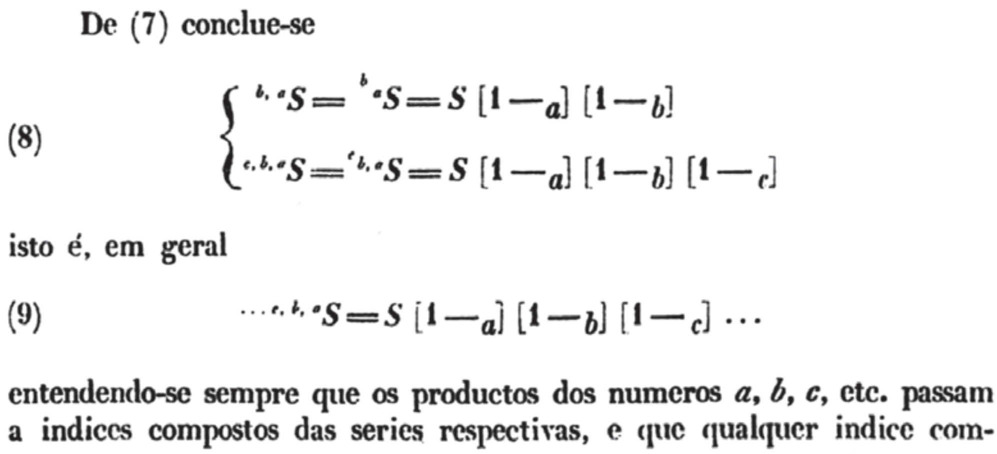
- Some Logic and Set Theory (including operations with sets (intersections, unions, etc), cardinality, examples such as $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}/n\mathbb{Z}$ etc);
- Basics of Number Theory (including modular arithmetic, divisibility, the Euler φ function, the theorems of Fermat/Euler, etc);
- Some Enumerative Combinatorics (including the binomial formula, the principle of inclusion-exclusion, generating functions, etc).

It is remarkable that da Silva's memoir from 1854 treats, in a clear, elegant and modern way, most of the above list of topics and subtopics. Moreover, some of the methods used bear a striking coincidence with those of textbooks for first year courses of Discrete/Finite Mathematics adopted nowadays in Colleges and Universities around the world. This happens, e.g, in Daniel's proof of the Euler's formula for $\varphi(n)$, the function that counts the number of positive integers less than $n \in \mathbb{N}$ and prime with it (see below).

A final argument for considering [dS] as Daniel's masterpiece are the original results and their relevance today. Indeed, two very important results introduced here are unanimously attributed to da Silva: the famous Principle of Inclusion-Exclusion, and a formula for congruences that generalises the well-known formula of Euler involving his φ function. Let us recall this material, from a modern perspective, and compare with the way Daniel introduces it.

4.1 The Principle of Inclusion-Exclusion

The Principle of Inclusion-Exclusion (abbreviated PIE) is one of the essential counting methods in Combinatorics, allowing a multitude of applications. It is ubiqui-


De (7) conclue-se

$$(8)\quad \begin{cases} {}^{b,a}S = {}^{b}{}^{a}S = S\,[1-_a]\,[1-_b] \\ {}^{c,b,a}S = {}^{c}{}^{b,a}S = S\,[1-_a]\,[1-_b]\,[1-_c] \end{cases}$$

isto é, em geral

$$(9)\quad {}^{\cdots c,b,a}S = S\,[1-_a]\,[1-_b]\,[1-_c]\cdots$$

entendendo-se sempre que os productos dos numeros ***a, b, c***, **etc. passam a indices compostos das series respectivas, e que qualquer indice com-**

Figure 5.—Part of page 10 of [dS] where Daniel's symbolic formula is written

tous in most books dedicated to this area.[3] The PIE generalises the formula for the cardinality of the union of two finite sets A and B:

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

where $|A|$ denotes the cardinality of A. This is also widely used in Probability or Measure Theory since, with appropriate interpretation, we can replace cardinality by probability or by measure. To state the PIE in modern terms, for $i = 1, \cdots, N$ ($N \in \mathbb{N}$) consider finite subsets $A_1, \cdots, A_N$ of a given finite set X, and let $A = \cup_{i=1}^{N} A_i$ be their union inside X, then the PIE is the formula:

$$|A| = \sum_i |A_j| - \sum_{i<j} |A_{ij}| + \sum_{i<j<k} |A_{ijk}| - \cdots + (-1)^{N-1}|A_{12\cdots N}| \tag{1}$$

where $A_{ij} := A_i \cap A_j$, $A_{ijk} := A_i \cap A_j \cap A_k$, etc. Equivalently, the PIE determines the cardinality of the complement $A^c := X \setminus A$ of A, as:

$$\begin{aligned} |A^c| &= |X| - |A| = \\ &= |X| - \textstyle\sum_i |A_i| + \sum_{i<j} |A_{ij}| + \cdots + (-1)^N |A_{12\cdots N}| \end{aligned} \tag{2}$$

For us, the most interesting aspect of da Silva's proof of (1) is that it requires the introduction of the *notion of set*, at least of a finite one, a couple of decades before the foundations of set theory laid by George Cantor (1845–1918)! Even though Daniel believes the concept of set is of major importance, he refers to this just as a convenient "notation". In da Silva's own words:

To prove this formula we will employ a notation, that may advantageously serve in other cases. Suppose that in a sequence S of numbers (that we consider *united* and not *summed*, since if even some of them could be negative, there wouldn't result any *subtraction*) one asks which are the ones that satisfy some property a; we will denote by S_a the *reunion* of those numbers; Similarly S_b, $S_{b,c}$, $S_{a,b,c}$ etc, the reunion of those terms of S verifying property b, or simultaneously the properties b, c, etc.[4]

using also the notation ${}^{\cdots,c,b,a}S$ for the elements of S that do not possess any of the properties $a, b, c, \ldots$, da Silva's presents his *symbolic formula* as follows:

$${}^{\cdots,c,b,a}S = S[1-_a][1-_b][1-_c]\cdots \tag{3}$$

and he clarifies that, on the rigth hand side, "in such a product the letters a, b etc., become indices, and any composed index such as a_{b_c} becomes a simple index a, b, c" *since* "it is easy to see that $S_{a_{b_c}} = S_{a,b,c}$" and similarly for upper left indices.

Daniel continues, using ψ for the cardinality of a set:

The same formula also immediately gives us the number of numbers contained in ${}^{\cdots,c,b,a}S$; denoting this number by $\psi^{\cdots,c,b,a}S$, and letting

[3]One standard textbook [St] explicitly mentions the PIE more that 50 times.

[4]We reproduced here Daniel's own emphasis in the 4 words: *united*, *summed*, *subtraction* and *reunion*.

the sign ψ have an analogous meaning applied to the additive and subtractive series in 3 it is clear that we will have:

$$\psi^{\cdots,c,b,a}S = \psi S[1-_a][1-_b][1-_c]\cdots. \tag{4}$$

One of the most common modern proofs of PIE uses the characteristic function of a subset $A \subset X$, defined by:

$$\chi_A(x) := \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

It is easy to see that characteristic functions are multiplicative for intersections, and additive for disjoint unions:

$$\chi_{A\cap B} = \chi_A \chi_B, \quad \chi_{A\sqcup B} = \chi_A + \chi_B.$$

This implies, upon observing that $A^c = \cap_{i=1}^N A_i^c$, when $A = \cup_{i=1}^N A_i$:

$$\begin{aligned} \chi_{A^c} &= \prod_{i=1}^N \chi_{A_i^c} = \prod_{i=1}^N (\mathbb{1} - \chi_{A_i}) = \\ &= \mathbb{1} - \sum_i \chi_{A_i} + \sum_{i<j} \chi_{A_{ij}} - \cdots + (-1)^N A_{12\cdots N}, \end{aligned} \tag{5}$$

where $\mathbb{1} = \chi_X$ (the constant function 1 defined on X). Noting that cardinality is just given by summing over X (recall this is a finite set):

$$|A| = \sum_{x\in X} \chi_A(x),$$

equation (5) is transformed into the PIE, in the form (2). By replacing a with χ_{A_1}, b with χ_{A_2} etc., the similarity of this proof with Daniel's formulae (3) and (4) is manifest!

Next, Daniel applies his formula (4) to deduce the formula of Euler

$$\varphi(n) = n\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\cdots\left(1-\frac{1}{c}\right), \tag{6}$$

when $n = a^\alpha b^\beta \cdots c^\gamma$ is the prime factorization of $n \in \mathbb{N}$, and this is done in *the same way* as in many modern books: see, for example, the standard textbook [Bi], from 2005, that derives the formula (6) precisely as da Silva does, using PIE.

4.2 Euler's Theorem and Bézout identity

Euler used the function φ in his celebrated theorem:

$$a^{\varphi(n)} \equiv 1 \pmod{n}, \tag{7}$$

for $a \in \mathbb{Z}$ relatively prime to $n \in \mathbb{N}$. This formula generalizes Fermat's theorem:

$$a^{p-1} \equiv 1 \pmod{p},$$

for p a prime and a not multiple of p.

The following elegant and interesting generalization of (7) was proved by Daniel da Silva. Let $n = a_1 a_2 \cdots a_k$ where all factors ($k > 1$ in number) are pairwise relatively prime. Then:

$$a_1^{\varphi(n/a_1)} + a_2^{\varphi(n/a_2)} + \cdots + a_k^{\varphi(n/a_k)} \equiv k-1 \pmod{n}. \tag{8}$$

The proof can be found in the original memoir of Daniel da Silva [dS] or in [CRS, p. 211].

The particular case $k = 2$ of this formula connects beautifully with the Bézout identity. This identity (in a simplified form) states that, given two relatively prime natural numbers $a, b \in \mathbb{N}$, there are solutions $x, y \in \mathbb{Z}$ to:

$$ax + by = 1,$$

and it is well known that this can be solved by an ancestral method: the (extended) Euclidean algorithm. From (8), we see that Daniel's formula provides a direct solution:

$$x = a^{\varphi(b)-1}, \quad y = b^{\varphi(a)-1},$$

for the "congruence version" of Bézout's identity:

$$ax + by \equiv 1 \pmod{ab}$$

(under the same assumptions on a, b).

The monograph goes on with many interesting applications of these formulas and related questions. Among these, Daniel provides direct resolutions for linear congruences, for the chinese remainder theorem, and for many congruences of the form

$$ax^n \equiv b \pmod{N}.$$

Daniel's health problems intensified as he was approaching the end of his monograph: he wasn't able to revise neither the preface nor the final part (see [Sa2]). For the same reason, the last two chapters are incomplete: in the 9th some theorems he would like to add are missing; and sections 4 and 5 of the 10th chapter have only their title. The last section was supposed to be a study of continued fractions, the theme of Teixeira's first letter to Daniel, 20 years later.

Without any impact whatsoever at the time it was written, this memoir was discovered almost by accident, half a century later, by the italian mathematician Cristoforo Alasia (1864–1918) who, suprised by its depth, dedicated three articles to Daniel's work between 1903 and 1914 (all in italian, the first being [A]). However, as far as we know, only one conference proceedings (written in portuguese) has addressed the concept of set in da Silva's work [dC].[5]

We finish with a quote from [dS], a wonderful illustration of Daniel da Silva's passion for mathematics and his opinion on the importance of pure mathematics and its role in science:

[5] I thank J. F. Rodrigues for the indication of this reference.

In general, it can be said that nobody is authorized to capitulate any mathematical theory as deprived from advantageous applications, as a mere recreation of elevated minds, and as useless work towards true science. All acquired truths are that many elements of acumulated intelectual wealth. Soon or late, the day will come when concrete science will have to search within this vast arsenal the necessary tools for great discoveries, which in this way will pass from speculative theorems to the category of practical truths. Every day, one or another area of mathematical-physics or celestial or industrial mechanics, is observed to suddenly stop its development to implore assistance from the further improvements in pure analysis, without which those very important sciences cannot progress.

Daniel da Silva could have hardly guessed that this phrase would be, more than a century later, so appropriate to the subject of his own article. In fact, the algorithms that preserve the security of data across the internet, such as the famous RSA (Rivest–Shamir–Adleman) criptographic system that we use (even without noticing) on a daily basis, depend crucially on Euler's formula (7).

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