FEATURE ARTICLE

Convolution operators on intervals and their use in diffraction theory

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1 Introduction

Classically speaking, convolution operators $\mathscr{W}$ on intervals Ω are one-dimensional linear integral operators where the integration kernels depend on the difference of the arguments and the domain of integration as well as the range of the independent variable are given by the same interval:

$$\mathscr{W}\varphi(x) = c\varphi(x) + \int_\Omega k(x-y)\varphi(y)\,dy = f(x), \quad x \in \Omega$$

Ω may be bounded or semi-infinite, or even consist of a union of intervals. Here and in various other papers $\mathscr{W}$ is briefly called "convolution type operator", although this name stands sometimes for the wider class of convolution type operators with variable coefficients, considered in the Lebesgue spaces $L^p(\Omega)$, for instance.

In applications, the composition with differential operators is very important, then naturally considered in Sobolev spaces, Bessel potential spaces, etc. This leads us rapidly to distribution theory and the world of pseudo-differential operators, to admit an adequate generality of settings. However we shall consider in this article only convolution operators on intervals in spaces of Bessel potentials in order to demonstrate some new important aspects of their theory and use for applications in mathematical physics, explained in the context of certain diffraction problems.

The crucial key of our recent approach is the study of operator relations (presented in the form of operator matrix identities) between the operators associated with linear boundary value problems and their boundary integral (or pseudo-differential) equations. Particular interest is devoted to the construction of relations of "very good quality" that allow explicit analytical solution and, e.g., an exact description of their singularities.

2 Convolution type operators

2.1 The general setting

We start giving the formal definition of the class of operators that we shall consider. These are the so-called *convolution type operators*

$$\mathscr{W}_{\Phi_{\mathscr{A}},\Omega} = r_\Omega \mathscr{A}_{|\widetilde{H}^{r,p}(\Omega)} : \widetilde{H}^{r,p}(\Omega) \to H^{s,p}(\Omega)\,, \tag{2.1}$$

acting between Bessel potential spaces, where $r, s \in \mathbb{R}$, $p \in]1,\infty[$, Ω is a finite interval or a half-line, $\mathscr{A}$ denotes a bounded, translation invariant operator from $H^{r,p}(\mathbb{R})$ into $H^{s,p}(\mathbb{R})$ (which can therefore be represented as a distributional convolution due to *Hörmander's theorem*) and r_Ω stands for the restriction of distributions from $\mathscr{S}'(\mathbb{R})$ to Ω. More precisely we have

$$\Omega =]0,a[\qquad \text{or} \qquad \Omega =]0,+\infty[\qquad (0<a<+\infty) \tag{2.2}$$

$$\mathscr{A}\varphi = \mathscr{K} * \varphi = \mathscr{F}^{-1}\Phi_{\mathscr{A}} \cdot \mathscr{F}\varphi, \qquad \varphi \in \mathscr{S} \tag{2.3}$$

$$H^{r,p} = H^{r,p}(\mathbb{R}) = \Lambda^{-r}L^p = \mathscr{F}^{-1}\lambda^{-r}\cdot\mathscr{F}L^p(\mathbb{R}) \tag{2.4}$$

$$\widetilde{H}^{r,p}(\Omega) = \left\{\varphi \in H^{r,p} : \operatorname{supp}\varphi \subset \overline{\Omega}\right\},\ H^{s,p}(\Omega) = r_\Omega H^{s,p}\,.$$

As usual, for $1 \le p < +\infty$, $L^p(\mathbb{R})$ denotes the Lebesgue space of all measurable functions ϕ on $\mathbb{R}$ with finite norm

$$\|\phi\|_{L^p(\mathbb{R})} = \left(\int_\mathbb{R} |\phi(y)|^p\,dy\right)^{1/p}.$$

$\mathscr{S} = \mathscr{S}(\mathbb{R})$ denotes the Schwartz space of rapidly decreasing smooth functions, $\mathscr{S}'$ the dual space of generalized functions of slow growth, and $\mathscr{K}*$ indicates a convolution (operator) with $\mathscr{K} \in \mathscr{S}'$. $\mathscr{F}$ is the Fourier transformation first of all defined by

$$\mathscr{F}\phi(\xi) = \int_{\mathbb{R}} \phi(x) \exp(ix\xi)\, dx, \qquad \xi \in \mathbb{R},$$

as a bijection from $\mathscr{S}$ onto $\mathscr{S}$, and secondly extended to the much larger set of distributions in $\mathscr{S}'$ due to the rule

$$\langle \mathscr{F}u, \phi\rangle = \langle u, \mathscr{F}\phi\rangle, \qquad u \in \mathscr{S}', \quad \phi \in \mathscr{S}$$

(where $\langle v, \varphi\rangle$ is the value of the functional $v \in \mathscr{S}'$ on $\varphi \in \mathscr{S}$).

We would like to point out that this last distributional definition of $\mathscr{F}$ is very useful for making a great range of calculations legal and, additionally, for providing a uniform definition of the Bessel potential spaces $H^{r,p}$ in (2.4), for all real values of smoothness orders r and $1 < p < \infty$, due to the use of the Bessel potential operator $\Lambda^{-r} = \mathscr{F}^{-1}\lambda^{-r}\cdot\mathscr{F}$, where $\lambda(\xi) = (\xi^2+1)^{1/2}$ for $\xi \in \mathbb{R}$.

$H^{r,p}$ is equipped with the norm of the corresponding functions in L^p, according to (2.4), $\widetilde{H}^{r,p}(\Omega)$ with the subspace topology and $H^{s,p}(\Omega)$ with the quotient space topology, respectively. Multi-indexed spaces ($r, s \in \mathbb{R}^n$) can be considered by analogy, as well as scales of Sobolev-Slobodečkiĭ spaces. For the above indicated range of indices, all those spaces are Banach spaces.

In formula (2.3) (sometimes called *convolution theorem*), $\Phi_{\mathscr{A}} \in L^\infty_{loc}(\mathbb{R})$ is known as the *Fourier symbol* of the convolution operator $\mathscr{A} = \mathscr{K}*$.

Starting with a distribution $\phi \in \mathscr{S}'$, we are able to write a convolution operator characterized by ϕ as $\mathscr{F}^{-1}\phi\cdot\mathscr{F} : \mathscr{S} \to \mathscr{S}'$. The set of Fourier symbols ϕ for which $\mathscr{F}^{-1}\phi\cdot\mathscr{F}$ has a bounded extension $\mathscr{F}^{-1}\phi\cdot\mathscr{F} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is usually denoted by $\mathscr{M}^p$, and their elements are called L^p*–Fourier multipliers*. The set $\mathscr{M}^p$ endowed with the norm $\|\phi\|_{\mathscr{M}^p} = \|\mathscr{F}^{-1}\phi\cdot\mathscr{F}\|_{\mathscr{L}(L^p(\mathbb{R}))}$, and point-wise multiplication, forms a Banach algebra. Knowing these facts, and considering the influence of the smoothness orders in the spaces in (2.1), it is possible to conclude that if $\lambda^{s-r}\phi \in \mathscr{M}^p$ then $\mathscr{W}_{\Phi_{\mathscr{A}},\Omega}$ is a well defined and bounded (linear) operator.

2.2 About the case $\Omega = \mathbb{R}_+ =]0, +\infty[$

The probably best known convolution type operator $\mathscr{W}_{\Phi_{\mathscr{A}},\Omega}$ is that one for which $r = s = 0$ and $\Omega = \mathbb{R}_+$. In this case we are working with the famous *Wiener-Hopf operators* acting between Lebesgue spaces. These operators received their name due to the pioneering work of Norbert Wiener and Eberhard Hopf [36] about the study of integral equations of the form

$$\mathscr{W}_{c+\mathscr{F}k,\mathbb{R}_+}\varphi(x) := c\varphi(x) + \int_0^\infty k(x-y)\varphi(y)\, dy = f(x)\,, \quad x \in \mathbb{R}_+\,, \tag{2.5}$$

for an unknown φ from $L^p(\mathbb{R}_+)$ where $f \in L^p(\mathbb{R}_+)$ is arbitrarily given and $c \in \mathbb{C}$ and $k \in L^1(\mathbb{R})$ are fixed and known. The *Wiener algebra* is defined by

$$\mathbb{W} = \left\{\phi \,:\, \phi = c + \mathscr{F}k\,,\ c \in \mathbb{C}\,,\ k \in L^1(\mathbb{R})\right\}$$

which is a Banach algebra when endowed with the natural norm $\|c + \mathscr{F}k\|_{\mathbb{W}} = |c| + \int_{\mathbb{R}} |k(y)|\, dy$ and the usual multiplication operation. The Wiener algebra is a subalgebra of $\mathscr{M}^p$.

For those who had the opportunity to read last year's *Feature Article* "Mathematical Finance - a glimpse from the past challenging the future" in CIM Bulletin 17, we would like to note that the Norbert Wiener mentioned there is the same person to whom we are referring here. In fact, Norbert Wiener appeared already in 1921 with a work about Brownian motion. Later on, he moved by invitation of his engineering colleagues to the *MIT* where he generalized his work on Browian motion to more general stochastic processes. This in turn led him to study harmonic analysis around 1930. In this way, his work on generalized harmonic analysis led him to study *Tauberian theorems* in 1932, and his contributions on this topic won the Böcher Memorial Prize in 1933 (a prize awarded in memory of Professor Maxime Böcher) [15].

Wiener-Hopf operators have a similar structure as the so-called *Toeplitz operators*. Therefore they are often studied together, see the famous work of Mark Kreĭn [19]. The basic result for L^p spaces reads as follows.

Theorem 2.1 (Kreĭn [19]). *The Wiener-Hopf operator $\mathscr{W}_{c+\mathscr{F}k,\mathbb{R}_+}$ in (2.5), acting between L^p spaces, is one-sided invertible if and only if*

$$c + (\mathscr{F}k)(\xi) \neq 0\,, \quad \text{for all } \xi \in \mathbb{R}\cup\{\infty\}\,. \tag{2.6}$$

Moreover, if (2.6) holds true, then $\mathscr{W}_{c+\mathscr{F}k,\mathbb{R}_+}$ is invertible, only left-sided invertible or only right-sided invertible in case of the integer $\varkappa = \operatorname{wind}(c + \mathscr{F}k)$ being zero, positive or negative, respectively.

Additionally, under the assumption (2.6), it follows that

$$\dim\ker \mathscr{W}_{c+\mathscr{F}k,\mathbb{R}_+} = \max\{-\varkappa, 0\}$$
$$\dim\operatorname{coker} \mathscr{W}_{c+\mathscr{F}k,\mathbb{R}_+} = \max\{\varkappa, 0\}\,.$$

In the last theorem the notation $\operatorname{wind}\phi$ refers to the winding number of the graph of ϕ. As the name suggests, that is the number of windings around the origin carried out by the point $\phi(\xi)$ when ξ runs from $-\infty$ to $+\infty$.

Knowledge about the kernel and co-kernel of an operator belongs to the so-called *Fredholm theory* of this operator. By definition, an operator with closed image and with a finite dimensional kernel and co-kernel is called a *Fredholm operator*. For Fredholm operators, the notion of Fredholm index is important. The *Fredholm index* of such an operator is the difference between the dimension of the kernel and the dimension of the co-kernel.

A matrix version of the last theorem was obtained by Gohberg and Kreĭn [14]. Instead of (2.6) a corresponding condition for the determinant of the matrix Fourier symbol characterizes the Fredholm property of the Wiener-Hopf operator $\mathscr{W}$, which is then not automatically one-sided invertible, but an index formula holds still true:

$$\dim \ker \mathscr{W} - \dim \operatorname{coker} \mathscr{W} = \operatorname{wind} \det(cI + \mathscr{F}k) .$$

In the last decades great advances were made in the Fredholm study of Wiener-Hopf operators, for much more complicated classes of Fourier symbols than the Wiener algebra. An example is the class of piecewise continuous functions for which, in the Fredholm case, an auxiliary new function can be constructed by filling up the gaps in the graph of the symbol that correspond with its discontinuities. Such extensions of the initially non-closed graphs are found with the help of some well defined arcs so that the winding number of the resulting continuous curve provides also the value for the Fredholm index of the corresponding initial Wiener-Hopf operator (following therefore the spirit of the above theorem of Kreĭn). As expected, the shape of the arcs depend on the integrability index p. For details see the work of Roland Duduchava [11].

2.3 About the case $\Omega =]0, a[$

In view of the above background, the idea to treat the more difficult case of $\Omega =]0, a[$ consists of reducing it somehow to the situation where $\Omega = \mathbb{R}_+$. That is why one of the objectives in the theory of convolution type operators is to construct operator relations between convolution type operators with $\Omega \neq \mathbb{R}_+$ and others with $\Omega = \mathbb{R}_+$, although both structures are of great difference in general.

Due to the interest originating from mathematical physics applications, several papers were directly devoted to the study of the Fredholm property, index formulas, and invertibility of $\mathscr{W}_{\Phi_{\mathscr{A}},\Omega}$, for the case where Ω is a finite interval; see for instance the work of M. P. Ganin [13], B. V. Pal'cev [22, 23], V. Yu. Novokshenov [29], Yu. I. Karlovich, I. M. Spitkovsky [17, 18], M. A. Bastos, A. F. dos Santos [3], M. A. Bastos, A. F. dos Santos, R. Duduchava [4] and L. P. Castro, F.-O. Speck [8]. While the early work of M. P. Ganin is written in a classical form focusing explicit solution in certain special cases and reduction to Riemann-Hilbert boundary value problems on $\mathbb{R}$, most of the recent work is based on the construction of an *algebraic equivalence after extension relation*, see [20, 21] and [4] as well,

$$\begin{bmatrix} \mathscr{W}_{\Phi_{\mathscr{A}},\Omega} & 0 \\ 0 & I_Y \end{bmatrix} = E \begin{bmatrix} \mathscr{W}_{\Phi,\mathbb{R}_+} & 0 \\ 0 & I_Z \end{bmatrix} F \qquad (2.7)$$

with a matrix Wiener-Hopf operator $\mathscr{W}_{\Phi,\mathbb{R}_+}$ according to the previous case where $\Omega = \mathbb{R}_+$, i.e. to find, beside of $\mathscr{W}_{\Phi,\mathbb{R}_+}$, additional Banach spaces Y, Z and invertible linear operators E, F acting between dense subspaces of the corresponding direct topological sums such that (2.7) holds. If E and F are homeomorphisms, (2.7) is said to be a *(topological) equivalence after extension relation* and we write in this case

$$\mathscr{W}_{\Phi_{\mathscr{A}},\Omega} \quad \overset{*}{\sim} \quad \mathscr{W}_{\Phi,\mathbb{R}_+} . \qquad (2.8)$$

This is equivalent to the fact that $\mathscr{W}_{\Phi_{\mathscr{A}},\Omega}$ and $\mathscr{W}_{\Phi,\mathbb{R}_+}$ are *matricially coupled* and $\mathscr{W}_{\Phi,\mathbb{R}_+}$ is an *indicator* for $\mathscr{W}_{\Phi_{\mathscr{A}},\Omega}$, see [1, 2]. In fact, it is known from [2, Theorem 1] that two (general) bounded linear operators acting between Banach spaces, T and S, are (topologically) equivalent after extension if and only if they are matricially coupled, that is, if and only if there are additional operators T_j and S_j (with $j = 0, 1, 2$) so that

$$\begin{bmatrix} T & T_2 \\ T_1 & T_0 \end{bmatrix} \; : \; X_1 \oplus Y_2 \to X_2 \oplus Y_1$$

$$\begin{bmatrix} S_0 & S_1 \\ S_2 & S \end{bmatrix} \; : \; X_2 \oplus Y_1 \to X_1 \oplus Y_2$$

are bounded invertible linear operators satisfying

$$\begin{bmatrix} T & T_2 \\ T_1 & T_0 \end{bmatrix}^{-1} = \begin{bmatrix} S_0 & S_1 \\ S_2 & S \end{bmatrix} . \qquad (2.9)$$

The notion of matricial coupling was introduced and used in [1], already within the spirit of finding solutions for integral equations, and we can also find some of the roots of this notion in the early work of Allen Devinatz and Marvin Shinbrot [10].

Since 1984 it is known that matricial coupling implies (topological) equivalence after extension, but only in 1992 it was proved by Bart and Tsekanovskiĭ [2] that the converse holds also true.

Thinking of the related work of several Portuguese researchers on *matrix completion problems* [5, 9, 30, 32, 33], we would like to point out that matrices T and S of complex numbers (of size $m_T \times n_T$ and $m_S \times n_S$, respectively), are matricially coupled if and only if

$$\operatorname{rank} T - \operatorname{rank} S = m_T - m_S = n_T - n_S . \qquad (2.10)$$

This means that once given matrices T and S of arbitrary size such that (2.10) holds true, we can solve the *completion problem* of constructing additional matrices

T_0, T_1, T_2, S_0, S_1 and S_2 such that (2.9) occurs (and vice versa).

Both relations, the algebraic equivalence after extension relation (2.7) and the topological one (2.8), are reflexive, symmetric and transitive. But evidently (2.8) has much stronger *transfer properties*:

- $\mathscr{W}_{\Phi_{\mathscr{A}},\Omega}$ belongs to the same regularity class (invertibility, Fredholm property, generalized invertibility, normal solvability etc.) as $\mathscr{W}_{\Phi,\mathbb{R}_+}$ does, since the operators have isomorphic kernels and co-kernels; indices and defect numbers are the same;
- explicit formulas for generalized inverses or regularizers of $\mathscr{W}_{\Phi,\mathbb{R}_+}$ imply corresponding formulas for those of $\mathscr{W}_{\Phi_{\mathscr{A}},\Omega}$, and vice versa;
- qualitative properties of solutions can be concluded, dependent on the particular form of E and F: singular behavior, asymptotic expansion etc. [7, 31];
- operator theoretical conclusions are possible: description of the spectrum, numerical range, reduction of order, perturbation, positivity, application of the fixed point principle, normalization [27] etc.

For the finite interval case, Kuijper [20] proposed an extension method to construct algebraic equivalence after extension relations (2.7) based on certain injective and surjective operators which are determined by the geometry of Ω. Kuijper's method guarantees the existence of invertible linear operators E and F that are constructed just by an algebraic decomposition of the domain and image spaces into the corresponding defect spaces of the two operators and their algebraic complements.

Theorem 2.2 (Kuijper). *For $\Omega =]0,a[$, the convolution type operator $\mathscr{W}_{\Phi_{\mathscr{A}},\Omega}$ introduced in (2.1) is algebraically equivalent after extension to a new Wiener-Hopf matrix operator*

$$\mathscr{W}_{\Phi_{\mathscr{C}},\mathbb{R}_+} = r_{\mathbb{R}_+}\mathscr{C} : [L^p_+(\mathbb{R})]^2 \to [L^p(\mathbb{R}_+)]^2 \tag{2.11}$$

where $\mathscr{C} = \mathscr{F}^{-1}\Phi_{\mathscr{C}} \cdot \mathscr{F}$, $\Phi_{\mathscr{C}} \in [L^\infty(\mathbb{R})]^{2\times 2}$, and

$$\Phi_{\mathscr{C}} = \begin{bmatrix} \tau_{-a}\zeta^r & 0 \\ \lambda_-^s \Phi_{\mathscr{A}} \lambda_+^{-r} & \tau_a \zeta^s \end{bmatrix},$$

with $\tau_a(\xi) = \exp(ia\xi)$, $\lambda_\pm(\xi) = \xi \pm i$, for $\xi \in \mathbb{R}$, and $\zeta = \lambda_-/\lambda_+$.

Here we used the abbreviations $L^p_+(\mathbb{R}) := \widetilde{H}^{0,p}(\mathbb{R}_+)$ and $L^p(\mathbb{R}_+) := H^{0,p}(\mathbb{R}_+)$.

For most cases of the smoothness orders r and s, the *Kuijper theorem* was already improved, in the sense that it is possible to present a *topological* equivalence after extension relation in explicit form; namely for non-critical orders, i.e., if $s - 1/p \in \mathbb{R}\backslash\mathbb{Z}$. In the critical cases only existence of a stronger relation could be proved. We will not go into details here but the interested reader can proceed into this direction by consulting [1, 8]. However, until now there is no general unifying method to obtain a topological equivalence after extension relation for all orders $r, s \in \mathbb{R}$ (and $1 < p < +\infty$) in the finite interval variant.

It is clear that explicit relations (in the form of operator matrix identities) have their direct profits. One of the consequences of such explicit formulas can be seen, e.g., in the fact that if we know a (generalized) inverse of $W_{\Phi_{\mathscr{C}},\mathbb{R}_+}$, say $W^-_{\Phi_{\mathscr{C}},\mathbb{R}_+}$, then the explicit equivalence after extension relation

$$\begin{bmatrix} \mathscr{W}_{\Phi_{\mathscr{A}},\Omega} & 0 \\ 0 & I_Y \end{bmatrix} = E \begin{bmatrix} W_{\Phi_{\mathscr{C}},\mathbb{R}_+} & 0 \\ 0 & I_Z \end{bmatrix} F \tag{2.12}$$

allows a quick way to find an explicit (generalized) inverse of $\mathscr{W}_{\Phi_{\mathscr{A}},\Omega}$ since

$$\begin{bmatrix} \mathscr{W}^-_{\Phi_{\mathscr{A}},\Omega} & 0 \\ 0 & I_Y \end{bmatrix} = F^{-1} \begin{bmatrix} W^-_{\Phi_{\mathscr{C}},\mathbb{R}_+} & 0 \\ 0 & I_Z \end{bmatrix} E^{-1}$$

is a (generalized) inverse of the matricial operator in the right-hand side of (2.12), and therefore $\mathscr{W}^-_{\Phi_{\mathscr{A}},\Omega}$ is a generalized inverse of $\mathscr{W}_{\Phi_{\mathscr{A}},\Omega}$. Naturally, a similar reasoning is obtained for one-sided inverses if they exist.

One can say that Theorem 2.2 provides a reduction of complexity: We start with a convolution type operator acting between Bessel potential spaces on the interval Ω and arrive at a Wiener-Hopf operator acting between Lebesgue spaces on the interval $\mathbb{R}_+$. The price consists of a larger size of the operator and a more complicated Fourier symbol, which contains terms oscillating at infinity.

2.4 On Fourier symbols $\Phi_{\mathscr{A}}$ from the Wiener algebra

Let us now consider operator (2.1) when $\Phi_{\mathscr{A}}$ is an invertible element in the Wiener algebra on the real line

$$\Phi_{\mathscr{A}} \in \mathscr{G}\,\mathbb{W} \tag{2.13}$$

and

$$\Omega =]0,a[, \quad p = 2, \quad r = s = k \in \mathbb{Z}. \tag{2.14}$$

In this case, the Fourier symbol $\Phi_{\mathscr{C}}$ of the last theorem takes the particular form

$$\Phi_{\mathscr{P}} = \begin{bmatrix} \tau_{-a}\zeta^k & 0 \\ \Phi_{\mathscr{A}}\zeta^k & \tau_a\zeta^k \end{bmatrix}.$$

A so-called *Wiener-Hopf factorization* [34] of $\Phi_{\mathscr{P}}$ is well-known in the case of $\Phi_{\mathscr{A}} \equiv 1$ and $k = 0$, because

$$\begin{bmatrix} \tau_{-a} & 0 \\ 1 & \tau_a \end{bmatrix} = \begin{bmatrix} \tau_{-a} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \tau_a \\ 0 & -1 \end{bmatrix}, \tag{2.15}$$

which implies a formula for the inverse of $\mathscr{W}_{\Phi_{\mathscr{A}},\Omega}$ in the case of a *sectorial symbol*

$$\Phi_{\mathscr{A}} = c\,(1+\epsilon), \qquad \|\epsilon\|_{L^\infty(\mathbb{R})} < 1 \tag{2.16}$$

in terms of a Neumann series (provided $c \in \mathbb{C}\setminus\{0\}$, $k=0$ and $p=2$ are satisfied).

We will now describe a procedure to obtain a generalized inverse of (2.1) under the assumptions (2.13)–(2.14) for general $k \in \mathbb{Z}$ and some restrictions; more precisely the formulas are:

(a) explicit in closed analytical form, if $\Phi_{\mathscr{A}}$ is rational;

(b) explicit in analytical form plus Neumann series, if $\Phi_{\mathscr{A}}$ is not rational.

The strategy is as follows

(i) Letting $w = \operatorname{wind}\Phi_{\mathscr{A}}$, we consider, instead of

$$\mathscr{W}_{\Phi_{\mathscr{A}},\Omega} = r_\Omega \mathscr{F}^{-1}\Phi_{\mathscr{A}}\cdot\mathscr{F} : \widetilde{H}^{k,p}(\Omega) \to H^{k,p}(\Omega),$$

the restricted or continuously extended operator

$$\mathscr{W}^{(s)}_{\Phi_{\mathscr{A}},\Omega} : \widetilde{H}^{s,p}(\Omega) \to H^{s,p}(\Omega),$$

$$\mathscr{W}^{(s)}_{\Phi_{\mathscr{A}},\Omega} = \begin{cases} \operatorname{Rst}\, r_\Omega \mathscr{F}^{-1}\Phi_{\mathscr{A}}\cdot\mathscr{F} \;, & s > k \\ \operatorname{Ext}\, r_\Omega \mathscr{F}^{-1}\Phi_{\mathscr{A}}\cdot\mathscr{F} \;, & s < k \\ \mathscr{W}_{\Phi_{\mathscr{A}},\Omega} \;, & s = k \end{cases} \tag{2.17}$$

respectively, where $s = -w$.

(ii) We relate the operator (2.17) (in the sense of Theorem 2.2) with a Wiener-Hopf operator

$$\begin{aligned} \mathscr{W}^{(s)}_{\Phi_{\mathscr{A}},\Omega} \;&\overset{*}{\sim}\; \mathscr{W}_{\Phi_{\mathscr{P}}{}^{(s)},\mathbb{R}_+} = r_{\mathbb{R}_+}\mathscr{F}^{-1}\Phi_{\mathscr{P}}{}^{(s)}\cdot\mathscr{F} \\ &\qquad : \left[L^2_+(\mathbb{R})\right]^2 \to \left[L^2(\mathbb{R}_+)\right]^2 \\ \Phi_{\mathscr{P}}{}^{(s)} \;&=\; \begin{bmatrix} \tau_{-a}\zeta^s & 0 \\ \Phi_{\mathscr{A}}\zeta^s & \tau_a\zeta^s \end{bmatrix} \end{aligned}$$

where $\operatorname{wind}(\Phi_{\mathscr{A}}\zeta^s) = 0$.

(iii) Now we consider the particular cases $\Phi_{\mathscr{A}}\zeta^s = \Phi_0\, r_+$ if $s \geq 0$ or $\Phi_{\mathscr{A}}\zeta^s = \Phi_0\, r_-$ if $s \leq 0$ where Φ_0 satisfies (2.16) and $r_\pm \in \mathscr{G}\mathscr{R}_\pm(\dot{\mathbb{R}})$, i.e., $r_\pm$ are invertible rational functions which are restrictions to $\dot{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$ of holomorphic functions in the upper/lower half-plane and continuous in the union of the upper/lower half-plane with the real line. It is known [13] that the previous factorization exists (due to the sign of s) and that it can be constructed by means of Weierstrass approximation.

(iv) So we have to factor (after elementary factorization)

$$\begin{aligned} G &= \begin{bmatrix} \tau_{-a}\zeta^s r_+^{-1} & 0 \\ \Phi_0 & \tau_a\zeta^s r_-^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \tau_{-a}\rho_1 & 0 \\ \Phi_0 & \tau_a\rho_2 \end{bmatrix} \end{aligned}$$

where either $r_- = 1$ (for $s \geq 0$) or $r_+ = 1$ (for $s \leq 0$), and all non-oscillating symbols are 1 at ∞

$$\Phi_0(\infty) = r_+(\infty) = r_-(\infty) = \rho_1(\infty) = \rho_2(\infty) = 1.$$

(v) We reduce G to a non-oscillating symbol G_0 by using (2.15)

$$G = \begin{bmatrix} \tau_{-a} & 1 \\ 1 & 0 \end{bmatrix} G_0 \begin{bmatrix} 1 & \tau_a \\ 0 & -1 \end{bmatrix}$$

with

$$G_0 = \begin{bmatrix} \Phi_0 & \tau_a(\Phi_0 - \rho_2) \\ \tau_{-a}(\rho_1 - \Phi_0) & \rho_1 - \Phi_0 + \rho_2 \end{bmatrix}.$$

(vi) Consider the principal part of G_0

$$G_1 = \begin{bmatrix} 1 & \tau_a(1-\rho_2) \\ \tau_{-a}(\rho_1 - 1) & \rho_1 - 1 + \rho_2 \end{bmatrix}$$

separately in the two cases $s \geq 0$ or $s \leq 0$, respectively:

(vi$_+$) $s \geq 0$, $\rho_1 = \zeta^s r_+^{-1}, \rho_2 = \zeta^s$

$$\begin{aligned} G_1 &= \begin{bmatrix} 1 & \tau_a(1-\zeta^s) \\ \tau_{-a}(\zeta^s r_+^{-1} - 1) & \zeta^s r_+^{-1} - 1 + \zeta^s \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ \tau_{-a}(\zeta^s r_+^{-1} - 1) & \zeta^{2s} r_+^{-1} \end{bmatrix} \\ &\qquad\qquad \times \begin{bmatrix} 1 & \tau_a(1-\zeta^s) \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ b_- & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \zeta^{2s} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b_+ & r_+^{-1} \end{bmatrix} \\ &\qquad\qquad \times \begin{bmatrix} 1 & \tau_a(1-\zeta^s) \\ 0 & 1 \end{bmatrix} \end{aligned}$$

where $b_- + b_+\zeta^{2s} = \tau_{-a}(\zeta^s r_+^{-1} - 1)$, i.e., $b_- = P_-\,\tau_{-a}(\zeta^s r_+^{-1} - 1)$ and $b_+ = P_+\,\tau_{-a}(\zeta^{-s} r_+^{-1} - \zeta^{-2s})$, with $P_\pm = \mathscr{F}\chi_\pm\mathscr{F}^{-1}$ being the Cauchy projection operators due to the characteristic functions $\chi_\pm$ of $\mathbb{R}_\pm$;

(vi$_-$) $s \leq 0$, $\rho_1 = \zeta^s, \rho_2 = \zeta^s r_-^{-1}$

$$G_1 = \begin{bmatrix} 1 & \tau_a(1-\zeta^s r_-^{-1}) \\ \tau_{-a}(\zeta^s - 1) & \zeta^s - 1 + \zeta^s r_-^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ \tau_{-a}(\zeta^s - 1) & 1 \end{bmatrix} \begin{bmatrix} 1 & \tau_a(1-\zeta^s r_-^{-1}) \\ 0 & \zeta^{2s} r_-^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ \tau_{-a}(\zeta^s - 1) & 1 \end{bmatrix} \begin{bmatrix} 1 & c_- \\ 0 & r_-^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \zeta^{2s} \end{bmatrix} \times \begin{bmatrix} 1 & c_+ \\ 0 & 1 \end{bmatrix}$$

where $c_+ + c_- \zeta^{2s} = \tau_a(1-\zeta^s r_-^{-1})$, i.e., $c_+ = P_+ \tau_a(1-\zeta^s r_-^{-1})$ and $c_- = P_- \tau_a(\zeta^{-2s} - \zeta^{-s} r_-^{-1})$.

(vii) Thus, since one-sided inverses are stable against small perturbations, we obtain a one-sided inverse of $\mathscr{W}_{\Phi_{\mathscr{P}^{(s)}},\mathbb{R}_+}$ in both cases and $\operatorname{ind} \mathscr{W}_{\Phi_{\mathscr{P}^{(s)}},\mathbb{R}_+} = -2s = 2w$. From (ii) we have a one-sided inverse of $\mathscr{W}^{(s)}_{\Phi_{\mathscr{A}},\Omega}$. Consequently, due to the so-called *Shift Theorem* [6, 12] and (i), we are able to obtain a generalized inverse of $\mathscr{W}_{\Phi_{\mathscr{A}},\Omega}$, for the present case.

3 Applications in diffraction theory

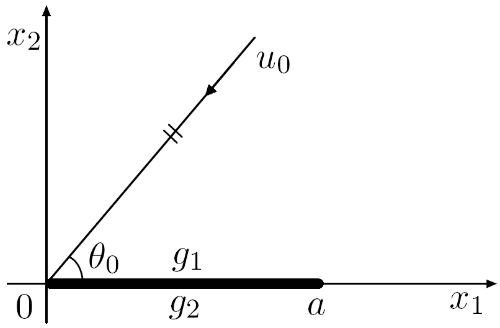
In this section we would like to exemplify the use of convolution type operators in some boundary value and/or transmission problems in weak formulation which originate from diffraction of time-harmonic waves by an infinite strip, see [16, 25, 26, 28, 35] for a detailed background. The proofs of the results presented below can be found in [8].

A. Sommerfeld was the first to formulate and solve a canonical boundary value problem for the Helmholtz equation which governs time-harmonic scalar waves. In his famous Habilitation Thesis of 1896 he was dealing with geometries formed by half-planes and wedges. He used series expansions and Riemann surface concepts to arrive at the solutions of corresponding Dirichlet boundary value problems. The so-called Sommerfeld integrals were afterwards systematically used by authors from Soviet Union and culminated in what is now known as the *Maliuzhinets method* [24]. Western authors preferred using the so-called *Wiener-Hopf method*, based on the Fourier transformation and factorization of the Fourier symbol of the corresponding convolution type operators – in the spirit of the last section.

Here we will consider the diffraction by a strip of an incoming plane wave u_0 of the form

$$u_0 = \exp\left[-ik(x_1 \cos\theta_0 + x_2 \sin\theta_0)\right],$$

where θ_0 is the angle of incidence (see the Figure), and we have omitted the time harmonic factor $\exp(-i\omega_0 t)$.



A plane wave incident upon a strip located on the x_1 axis, between 0 and a, and having boundary data g_1 and g_2 on its banks.

The wave number $k = \omega_0\sqrt{\varepsilon\mu}$ is assumed to be complex satisfying $\Im m\,(k) > 0$, i.e., ε and μ are parameters of a lossy medium. The electromagnetic theory yields, for a large spectrum of materials, a quasi-homogeneous refracted wave, which propagates perpendicularly to the boundary regardless of the incident angle. The x_3–dependence is therefore cancelled due to the perpendicular wave propagation, leading us to the consideration of a $\mathbb{R}^2$ situation with the strip Σ and its x_1–complement Σ' here represented by

$$\Sigma = \left\{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \in [0,a], x_2 = 0\right\} = [0,a] \times \{0\} \tag{3.18}$$
$$\Sigma' = (\mathbb{R} \backslash]0,a[) \times \{0\}\,. \tag{3.19}$$

The diffracted or scattered field then satisfies the Helmholtz equation as well as the total field does, and several possible boundary conditions can be valid on the banks of Σ corresponding to different material behavior.

These considerations lead us to the problem of finding an element $u \in L^p(\mathbb{R}^2)$ such that

$$u^\pm = u_{|\mathbb{R}\times\mathbb{R}_\pm} \in H^{l,p}(\mathbb{R}\times\mathbb{R}_\pm) \tag{3.20}$$
$$Lu^\pm = (\Delta + k^2)u^\pm = 0 \qquad \text{in} \quad \mathbb{R}\times\mathbb{R}_\pm \tag{3.21}$$
$$[u]_{\Sigma'} = (u^+(x) - u^-(x))_{|x\in\Sigma'} = 0 \tag{3.22}$$
$$\left[\frac{\partial u}{\partial x_2}\right]_{\Sigma'} = \left(\frac{\partial u^+}{\partial x_2}(x) - \frac{\partial u^-}{\partial x_2}(x)\right)_{|x\in\Sigma'} = 0 \tag{3.23}$$

where $p \in]1,\infty[$, $l > 1/p$, $l - 1/p \notin \mathbb{N}$, $k \in \mathbb{C}$ with $\Im m\,(k) > 0$ are given and

$$B_j u(x_1) = \Sigma_{\sigma_1+\sigma_2\leq m_j} b^+_{\sigma,j} D^\sigma u^+(x_1,0) + b^-_{\sigma,j} D^\sigma u^-(x_1,0) = g_j(x_1), \qquad x_1 \in [0,a], \quad j = 1,2 \tag{3.24}$$

where $\sigma = (\sigma_1,\sigma_2), m = (m_1,m_2) \in \mathbb{N}_0^2$, $b^\pm_{\sigma,j} \in \mathbb{C}$ and $g_j \in H^{l-\frac{1}{p}-m_j,p}(]0,a[)$ are assumed to be known.

From the physical point of view, one is mainly interested in solutions in the energy space, $u \in H^{1,2}(\mathbb{R}^2 \setminus \Sigma)$ [26]. But we know already from the study of half-plane problems that many of these are ill-posed in that space setting [27]. They need a normalization which is often implemented by change of the space parameters l, p of $H^{l,p}$. Another reason to consider the problem in a scale of spaces is to look for regularity results and asymptotic expansion [31].

The boundary values of u are taken in the sense of the *trace theorem*. The choice of the other data spaces results from the *representation formula* [26] (cf. Prop. 3.1 later on) as a consequence of (3.24). The orders of the boundary operators B_j are arbitrary (from the mathematical point of view).

We associate an operator with the problem, say

$$\mathscr{P}^{(l,p)} : u \mapsto g = (g_1, g_2) \tag{3.25}$$

where the domain $\mathscr{D}(\mathscr{P}^{(l,p)})$ of $\mathscr{P}^{(l,p)}$ is characterized by (3.20)–(3.23), the action and the image space of $\mathscr{P}^{(l,p)}$ are described by (3.24) with the corresponding norms. Evidently the problem (3.18)–(3.24) is well-posed in this space setting if and only if the operator

$$\begin{aligned} &\mathscr{P}^{(l,p)} : \mathscr{X} \to \mathscr{Y}, \\ &\mathscr{X} = H^{l,p}(\mathbb{R} \times \mathbb{R}_+) \times H^{l,p}(\mathbb{R} \times \mathbb{R}_-), \\ &\mathscr{Y} = \times_{j=1}^{2} H^{l-\frac{1}{p}-m_j,p}(]0,a[) \end{aligned} \tag{3.26}$$

is boundedly invertible.

The main objectives are:

(i) to find the spaces in which the operator $\mathscr{P}^{(l,p)}$ is boundedly invertible and those where it is normally solvable (which implies the Fredholm property in the elliptic case and the existence of a generalized inverse in terms of factorization);

(ii) to determine the defect numbers (not only the index) of $\mathscr{P}^{(l,p)}$ by computing the partial indices of a matrix symbol, which appears in a topological (not only algebraic) equivalence after extension relation in the above-mentioned sense;

(iii) to get a generalized inverse of $\mathscr{P}^{(l,p)}$, if possible

  (a) in closed analytical form, or

  (b) in terms of a uniformly convergent series under physically reasonable assumptions on the parameters. As a matter of fact it is not possible to deduce these results only from an algebraic equivalence after extension relation between $\mathscr{P}^{(l,p)}$ and a Wiener-Hopf operator, cf. [4].

The convolution type operators enter here in the scene because we are able to recognize a relation between $\mathscr{P}^{(l,p)}$ and such an operator in form of an operator matrix identity.

Proposition 3.1. *The operator $\mathscr{P}^{(l,p)}$ is (algebraically and topologically) equivalent to a convolution type operator on the interval $[0,a]$ acting in the corresponding boundary data spaces of Bessel potentials. More precisely $\mathscr{P}^{(l,p)} = \mathscr{W}_{\Phi_{\mathscr{A}},\Omega}\, B_-\, \mathscr{T}_0$ where the trace operator*

$$\begin{aligned} &\mathscr{T}_0 : \mathscr{D}(\mathscr{P}^{(l,p)}) \to \left[H^{l-\frac{1}{p},p}(\mathbb{R})\right]^2, \\ &\mathscr{T}_0 u = u_0 = (u_0^+, u_0^-)^T = (u^+_{|x_2=0}, u^-_{|x_2=0})^T \end{aligned}$$

is bounded invertible by the representation formula

$$\begin{aligned} u &= \mathscr{K}\, u_0 \\ u(x_1,x_2) &= \mathscr{F}^{-1}_{\xi\mapsto x_1} \left\{ \exp[-t(\xi)x_2]\, \widehat{u}_0^+(\xi)\, \chi_+(x_2) \right. \\ &\qquad \left. + \exp[t(\xi)x_2]\, \widehat{u}_0^-(\xi)\, \chi_-(x_2) \right\}. \end{aligned}$$

Here $\mathscr{K}$ is called a Poisson operator, $\widehat{\varphi}$ denotes the Fourier transform of φ, and $t(\xi) = (\xi^2 - k^2)^{1/2}$. Further

$$\begin{aligned} B_- &= \mathscr{F}^{-1} \begin{bmatrix} 1 & -1 \\ -t & -t \end{bmatrix} \cdot \mathscr{F} \\ &: \left[H^{l-\frac{1}{p},p}(\mathbb{R})\right]^2 \to \mathscr{X}_0 = H^{l-\frac{1}{p},p}(\mathbb{R}) \times H^{l-\frac{1}{p}-1,p}(\mathbb{R}) \end{aligned}$$

maps the Dirichlet trace vector $u_0 = (u_0^+, u_0^-)^T = \mathscr{T}_0 u$ (for $u \in \mathscr{D}(\mathscr{P}^{(l,p)})$) into the jump vector of Dirichlet and Neumann data

$$f = (u_0^+ - u_0^-, u_1^+ - u_1^-)^T = \left([u]_{\Sigma\cup\Sigma'}, \left[\frac{\partial u}{\partial x_2}\right]_{\Sigma\cup\Sigma'} \right)^T.$$

The operator $\mathscr{W}_{\Phi_{\mathscr{A}},\Omega} : \mathscr{X}_0 \to \mathscr{Y}$ is an operator of the form (2.1) where $r = (l - 1/p, l - 1 - 1/p)$, $s = (l - m_1 - 1/p, l - m_2 - 1/p)$ and

$$\Phi_{\mathscr{A}}(\xi) = \begin{bmatrix} \sum_{|\sigma|\le m_1} \frac{1}{2} \left(b^+_{\sigma,1} - b^-_{\sigma,1}\right) (-i\xi)^{\sigma_1} (-t(\xi))^{\sigma_2} & \sum_{|\sigma|\le m_1} \frac{-1}{2t(\xi)} \left(b^+_{\sigma,1} + b^-_{\sigma,1}\right) (-i\xi)^{\sigma_1} (t(\xi))^{\sigma_2} \\ \sum_{|\sigma|\le m_2} \frac{1}{2} \left(b^+_{\sigma,2} - b^-_{\sigma,2}\right) (-i\xi)^{\sigma_1} (-t(\xi))^{\sigma_2} & \sum_{|\sigma|\le m_2} \frac{-1}{2t(\xi)} \left(b^+_{\sigma,2} + b^-_{\sigma,2}\right) (-i\xi)^{\sigma_1} (t(\xi))^{\sigma_2} \end{bmatrix} \tag{3.27}$$

Corollary 3.2. *The system of equations $\mathscr{W}_{\Phi_{\mathscr{A}},\Omega} f = g$ decouples, i.e. $\Phi_{\mathscr{A}}$ is triangular after multiplication with a constant matrix, if and only if some linear combination of the two boundary conditions (3.24) contains only a linear combination of either "difference" or "sum data", i.e. it can be written as*

$$Bu(x_1) = \sum_{|\sigma|\le m_0} b_\sigma D^\sigma (u^+ \pm u^-)(x_1, 0), \quad x_1 \in [0,a].$$

Using the theory of the last section we can now proceed with finding the concrete form of the related Wiener-Hopf operator.

Theorem 3.3. *Let $\mathscr{P}^{(l,p)}$ be the operator defined by (3.18)–(3.26). Then we have the equivalence after extension relation*

$$\mathscr{P}^{(l,p)} \overset{*}{\sim} \mathscr{W}_{\Phi_{\mathscr{U}},\mathbb{R}_+} \in \mathscr{L}\left(\left[L^p_+(\mathbb{R})\right]^{2n}, [L^p(\mathbb{R}_+)]^{2n}\right) \quad (3.28)$$

where $n = 2$ or 1 (in certain decomposing cases described below) and

$$\Phi_{\mathscr{U}} = \begin{bmatrix} \tau_{-a}\zeta^r I_n & 0 \\ \lambda^s_- \Phi_{\mathscr{A}} \lambda^{-r}_+ & \tau_a \zeta^s I_n \end{bmatrix} \quad (3.29)$$

where $\Phi_{\mathscr{A}}$ is given by (3.27) or can be replaced by a scalar symbol according to Corollary 3.2 in the case $n = 1$, respectively. The orders are $r = (l-1/p, l-1-1/p)$, $s = (l - m_1 - 1/p, l - m_2 - 1/p)$ or, for $n = 1$, are components of these two vectors.

In the last result the equivalence after extension is an algebraic one but for certain cases we can perform a topological relation. This is the case of $p = 2$, by using Theorem 3 of [2].

We notice that in the decomposing case, $\Phi_{\mathscr{A}}$ is triangular (say upper). Therefore, one can identify and use (in the following way) an operator $\mathscr{W}_{\Phi_{\mathscr{U}_2},\mathbb{R}_+}$ that has the form (3.28)–(3.29) with $n = 1$. If it is invertible we have

$$\mathscr{P}^{(l,p)} \overset{*}{\sim} \begin{bmatrix} \mathscr{W}_{\Phi_{\mathscr{U}_1},\mathbb{R}_+} & * \\ 0 & \mathscr{W}_{\Phi_{\mathscr{U}_2},\mathbb{R}_+} \end{bmatrix} =$$
$$= \begin{bmatrix} I & * \\ 0 & \mathscr{W}_{\Phi_{\mathscr{U}_2},\mathbb{R}_+} \end{bmatrix} \begin{bmatrix} \mathscr{W}_{\Phi_{\mathscr{U}_1},\mathbb{R}_+} & 0 \\ 0 & I \end{bmatrix},$$

i.e., equivalence after extension to $\mathscr{W}_{\Phi_{\mathscr{U}_1},\mathbb{R}_+}$ and the remainder operator has the same form (3.28)–(3.29) with $n = 1$.

For technical reasons we extend in this final part the definition of $\lambda_\pm(\xi) = \xi \pm i$ and work now with

$$\lambda_\pm(\xi) = \xi \pm k\,, \qquad \Im m\,(k) > 0\,.$$

This makes the Fourier symbols simpler since we can combine factors $\lambda^s_\pm$ with t, but does not change the principal nature of factorizations or the topology of Bessel potential spaces $H^{s,p} = \lambda^{-s}_+ L^p = \lambda^{-s}_- L^p$.

If $\mathscr{W}_{\Phi_{\mathscr{U}_2},\mathbb{R}_+}$ is not invertible, but a shifted one $W^{(w)}_{\Phi_{\mathscr{U}_2},\mathbb{R}_+}$: $\widetilde{H}^{w,p}(\mathbb{R}_+) \to H^{w,p}(\mathbb{R}_+)$ (defined by restriction or continuous extension) is invertible for some $w = (w_1, w_1) \in \mathbb{R}^2$, one can try to consider $W^{(w,w)}_{\Phi_{\mathscr{U}},\mathbb{R}_+}$ first and then "shift back", i.e. express results for $\mathscr{W}_{\Phi_{\mathscr{U}},\mathbb{R}_+}$ in terms of results for $W^{(w,w)}_{\Phi_{\mathscr{U}},\mathbb{R}_+}$.

Corollary 3.4. *The symbol $\Phi_{\mathscr{U}}$ can be written in the form*

$$\Phi_{\mathscr{U}} = \zeta^r \begin{bmatrix} \tau_{-a} I_n & 0 \\ \lambda^{s-r}_- \Phi_{\mathscr{A}} & \tau_a \zeta^{s-r} I_n \end{bmatrix}$$

where $s - r \in \mathbb{Z}^n$, i.e. λ^{s-r}_- and ζ^{s-r} are rational, precisely $s - r = (-m_1, 1 - m_2)$ if $n = 2$, which admits integer components up to 1. Moreover, the elements of $\lambda^{s-r}_- \Phi_{\mathscr{A}}$ are:

- *Hölder continuous functions with a possible jump at infinity, and*
- *algebraic compositions of $\zeta^{1/2}$ and rational functions.*

Arriving at this point, a Wiener-Hopf factorization of $\Phi_{\mathscr{U}}$ and the operator relations of the former section would lead us to (generalized) inverses of $\mathscr{P}$, and therefore to solutions of the initial boundary value problem (3.24). However the factorization problem *in general* is not solved and remains an open problem for challenging future research.

4 Conclusion

The operator theoretic approach enables us to identify clearly the relations between the (operator associated with the) given problem and the (operator associated with the) boundary pseudodifferential equations. One can analyze simultaneously classes of problems with different boundary conditions and space settings with respect to the questions mentioned before: Solvability, explicit analytical presentation (in particular cases) and qualitative results like regularity, singular behavior and asymptotic expansion. The method was demonstrated for a prototype class of problems from diffraction theory.

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