The Willmore Conjecture: a Celebration of Mathematics

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Cum enim Mundi universi fabrica sit perfectissima, atque a Creatore sapientissimo absoluta, nihil omnino in mundo contingit, in quo non maximi minimive ratio quaepiam eluceat[1]
— Leonhard Euler

1 Introduction

Established in 1961, the Oswald Veblen Prize in Geometry is an award granted by the American Mathematical Society in recognition of a notable research memoir in geometry and topology. Presented every three years, this year's edition of the prize distinguished the joint work of the Brazilian mathematician Fernando Codá Marques (Princeton University) and the Portuguese mathematician André Neves (Imperial College London), for their landmark achievement and major contribution to the use of variational methods in differential geometry, with a special highlight for the proof of the long-standing Willmore Conjecture.

Proposed in 1965 by the English geometer Thomas J. Willmore, the Willmore Conjecture concerned the quest for the torus with the lowest bending energy of all and predicted the equilibrium state of such curved surfaces. The problem has resisted proof for many years and inspired many mathematicians over time, borrowing ideas from several distinct areas from partial differential equations to algebraic geometry, conformal geometry, geometric measure theory and minimal surfaces. Willmore died on February 20, 2005, seven years before Marques and Neves posted a preprint of their 96-page proof on the arXiv, on February 27, 2012.

This article is dedicated to an overview of the history of the conjecture and its proof, in celebration of pursuit and achievement, through the works of Willmore, of Marques and Neves and of all those involved in this half century quest, as well as those of their precursors.

2 Willmore energy and the Willmore Conjecture

A central theme in Mathematics is the search for the optimal representative within a certain class of objects, often driven by the minimization of some energy, reflecting what occurs in many physical processes. From the early 1960s, Thomas Willmore devoted particular attention to the quest for the optimal immersion of a compact surface in Euclidean 3-space regarding the minimization of some natural energy motivated by questions on the elasticity of membranes and the energetic cost associated with membrane bending deformations.

We can characterize how much a membrane is bent at a particular point on the membrane by means of the curvature of the osculating circles of the planar curves obtained as perpendicular cross sections through the point (see Figure 2). The curvature of these circles consists of the inverse of their radii, with a positive or negative sign depending on whether the membrane curves upwards or downwards, respectively. The minimal and maximal values of the radii of the osculating circles associated with a particular point on the membrane define the principal curvatures, k_1 and k_2, and, from these, the mean curvature, $H = (k_1 + k_2)/2$, and the Gaussian curvature, $K = k_1 k_2$, at the point.

In modern literature on the elasticity of membranes (see, for example, [11] and [31]), a weighed sum $a\int H^2 + b\int K$, of the total squared mean curvature and the total Gaussian curvature, is considered as the elastic bending energy of a membrane. Having in consideration the Gauss-

[1] Leonhard Euler, *Methodus inveniendi lineas curvas Maximi Minimive proprietate gaudentes, sive solutio problematis isoperimetrici latissimo sensu accepti*, Lausannae & Genevae: Apud Marcum-Michaelem Bousquet & Socios (1744), p. 245.

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Figure 1.—Thomas J. Willmore. Portrait by Christine Choa (1999)

Bonnet theorem, according to which the total Gaussian curvature is a topological invariant and, therefore, negligible in deformations conserving the topological type, Willmore defined

$$\mathscr{W} = \int_\Sigma H^2 d\Sigma$$

as the Willmore [bending] energy of a compact, oriented [Riemannian] surface Σ [isometrically] immersed in $\mathbb{R}^3$.

The Willmore energy had already made its appearance early in the nineteenth century, through the works of Marie-Sophie Germain [6] and Siméon Denis Poisson [22] and their pioneering studies on elasticity and vibrating properties of thin plates. This energy had also appeared in the 1920s, in the works of Wilhelm Blaschke [4] and Gerhard Thomsen [27], but their findings were forgotten and only brought to light after the increased interest on the subject motivated by the work of Thomas Willmore.

A very interesting fact about the Willmore energy is that it is scale-invariant: if one dilates the surface by any factor, the Willmore energy remains the same. Think of a round sphere in $\mathbb{R}^3$ as an example: if one increases the radius, the surface becomes flatter and its squared mean curvature H^2 decreases, but, at the same time, its area gets larger, which increases the value of the integral in $\mathscr{W}$. One can show that these two phenomena counterbalance each other on any surface. In fact, the Willmore energy has the remarkable property of being invariant under any conformal transformation of $\mathbb{R}^3$, as established in the paper of White [32] and, actually, already known to Blaschke and Thomsen.

In view of the scale-invariance of the Willmore energy, the energy of round spheres coincides with the surface area of the round sphere of radius 1: 4π. Note, on the other hand, that $H^2 - K = \frac{1}{4}(k_1 - k_2)^2$, so that $H^2 \geq K$, with equality at umbilical points ($k_1 = k_2$). By the Gauss-Bonnet theorem, it follows that

$$\int_\Sigma H^2 d\Sigma \geq \int_\Sigma K d\Sigma = 4\pi(1-g),$$

where g denotes the genus of the surface. In particular, for surfaces of genus zero, we get $\int_\Sigma H^2 d\Sigma \geq 4\pi$, with equality only for the totally umbilical surfaces of $\mathbb{R}^3$. We conclude that round spheres are the minimizers of the Willmore energy among all topological spheres. Willmore showed, furthermore, that 4π is the absolute minimum of energy among all compact surfaces in $\mathbb{R}^3$:

Theorem 1 (Willmore [33, 35]).— *Let Σ be a compact surface in $\mathbb{R}^3$. Then*

$$\mathscr{W}(\Sigma) \geq 4\pi,$$

with equality if and only if Σ is a round sphere.

Having found the compact surfaces with least possible energy, and, with these, the energy minimizers within the class of surfaces of genus zero, Willmore embarked on the quest for the energy-minimizing shape among all topological tori. It seems reasonable that no obvious candidate stands out a priori. In order to develop some intuition on the problem, Willmore considered a particular type of torus: he fixed a circle of radius R on a plane and considered tubes Σ_r of constant radius $r < R$ around that circle. When r is very small, Σ_r is a very thin tube and so $\mathscr{W}(\Sigma_r)$ is very large. As we keep increasing the value of r, the hole of the torus decreases and eventually disappears, for $r = R$. Thus the function $r \mapsto \mathscr{W}(\Sigma_r)$ must reach an absolute minimum for some $r \in]0, R[$. Willmore [33] computed this minimum to be $2\pi^2$ and showed that, up to scaling, the optimal torus in this class has $r = 1$ and $R = \sqrt{2}$. Willmore conjectured

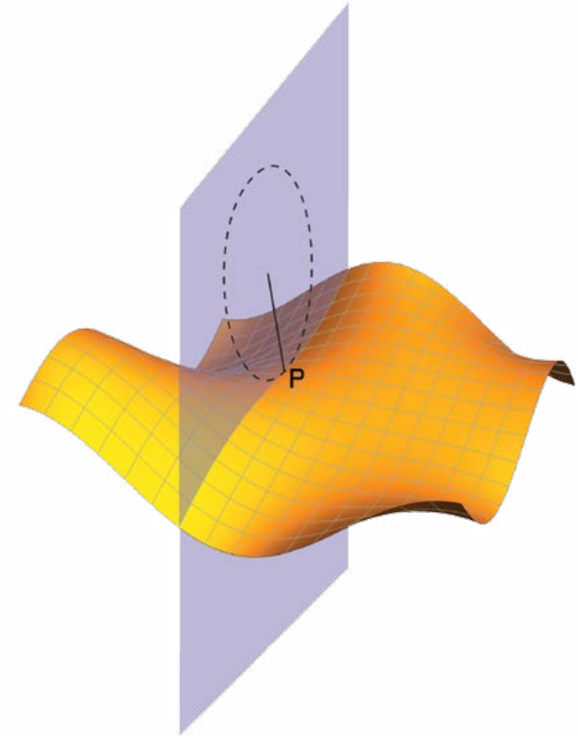

Figure 2.—Osculating circle to a surface at a point *P* on the surface

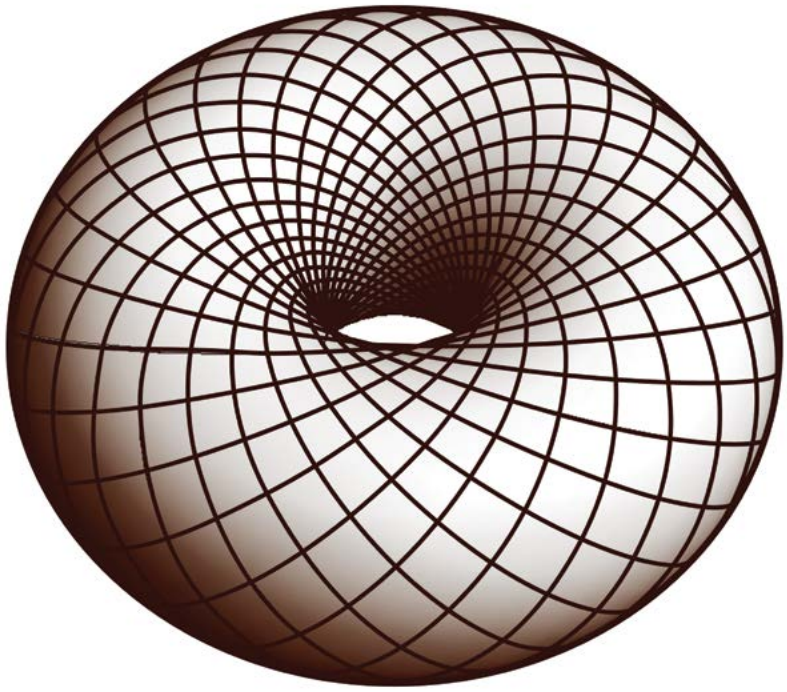

Figure 3 .—Torus of revolution with $W = 2\pi^2$

that this torus of revolution should minimize the Willmore energy among all tori:

Willmore Conjecture

(Willmore [33]) *Let Σ be a compact surface of genus one in $\mathbb{R}^3$. Then*

$$\mathscr{W}(\Sigma) \geq 2\pi^2.$$

3 On the quest for the optimal torus

The Willmore Conjecture has been verified in many special cases. Willmore himself [34] and, independently, Katsuhiro Shiohama and Ryoichi Takagi [25] proved it when the torus is a tube of constant radius around an arbitrary space curve in $\mathbb{R}^3$. Over the decades, more and more classes of tori were proven to have bending energy greater than or equal to $2\pi^2$, through the works of Rémi Langevin and Harold Rosenberg [9], Bang-Yen Chen [5], Joel Langer and David Singer [8], Peter Li and Shing-Tung Yau [10], Sebastián Montiel and Antonio Ros [18, 23, 24] and Peter Topping [28, 29]. In 1991, the biophysicists David Bensimon and Michael Mutz [3] have experimentally verified the conjecture in membranes of toroidal vesicles produced in laboratory. In 1993, Leon Simon [26] established the existence of a torus that minimizes the Willmore energy. An overview of partial results can be found in [14]. We select the following, which, in particular, reduced the verification of the Willmore Conjecture to embedded tori:

Theorem 2 (Li-Yau [10]).— Compact surfaces with self-intersections have Willmore energy greater than or equal to 8π.

A key to the proof of the Willmore Conjecture was moving the problem from $\mathbb{R}^3$ to the unit 3-sphere $S^3 \subset \mathbb{R}^4$, having in mind that the two are conformally related by stereographic projection. The torus found by Willmore is mapped onto the Clifford torus $S^1\left(\frac{1}{\sqrt{2}}\right) \times S^1\left(\frac{1}{\sqrt{2}}\right)$, which is a classical example of a minimal surface in S^3.

Minimal surfaces are defined variationally as the stationary configurations for the area functional, surfaces that locally minimize the area. In general, these surfaces admit ambient deformations that can decrease their area and are, therefore, not (globally) area-minimizing.

Minimal surfaces were first considered by Joseph-Louis Lagrange [7], in 1762, who raised the question of existence of surfaces of least area among all those spanning a given closed curve in Euclidean 3-space as the boundary. Earlier, in a work published in 1744,[2] Leonhard Euler had already discussed minimizing properties of the surface now known as the catenoid, although he only considered variations within a certain class of surfaces. The problem raised

[2] Ibidem.

by Lagrange became known as the Plateau's Problem, referring to Joseph Antoine Ferdinand Plateau, who first experimented with soap films [21].

A physical model of a minimal surface can be obtained by dipping a wire frame into a soap solution. The resulting soap film is minimal in the sense that it always tries to organize itself so that its surface area is as small as possible whilst spanning the wire contour. This minimal surface area is, naturally, reached for the flat position,[3] which happens to be a position of vanishing mean curvature. This does not come as a particular feature of this rather simple example of minimal surface. In fact, the Euler-Lagrange equation of the variational problem underlying minimal surfaces turns out to be precisely the zero mean curvature equation, as discovered by Jean Baptiste Meusnier [17].

With the characterization of minimal surfaces by identically vanishing mean curvature, the theory of minimal submanifolds has been developed and extended to other ambient geometries and ended up playing a crucial role in the understanding of the Willmore energy.

On the sphere, the Willmore energy becomes area plus the total squared mean curvature: if $\pi : S^3 \setminus \{(0,0,0,1)\} \to \mathbb{R}^3$ denotes the stereographic projection, then

$$\int_\Sigma H^2 d\Sigma = \int_{\tilde\Sigma} (1 + \tilde H^2) d\tilde\Sigma,$$

for $\tilde H$ the mean curvature of $\tilde\Sigma := \pi^{-1}(\Sigma) \subset S^3$ (with respect to the standard metric on S^3). In particular, the Willmore energy of a minimal surface in S^3 coincides with its area.

Crucially, Marques and Neves reduced the quest for an optimal embedding in S^3 to the class of minimal embeddings in S^3:

Theorem 3 (Marques-Neves [14]).— Let $\Sigma \subset S^3$ be an embedded closed surface with positive genus. Then there exists an embedded closed minimal surface $\tilde\Sigma \subset S^3$ such that $\mathscr{W}(\Sigma) \geq \text{area}(\tilde\Sigma)$.

Next they established the Clifford torus as a surface of least area among all minimal embeddings of closed surfaces in S^3 with genus (at least) one:

Theorem 4 (Marques-Neves [14]).— Let $\Sigma \subset S^3$ be an embedded closed minimal surface with positive genus. Then $\text{area}(\Sigma) \geq 2\pi^2$, and equality holds if and only if Σ is the Clifford torus, up to isometries of S^3.

With Theorems 3 and 4, Marques and Neves established, in particular, the following:

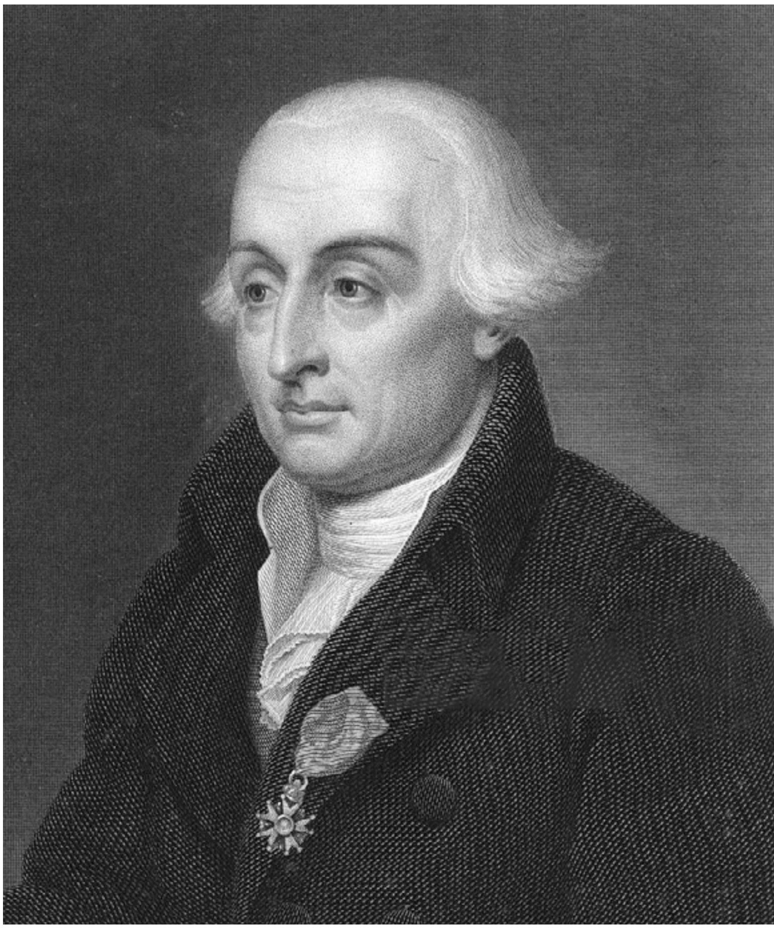

Figure 4.—Joseph Louis Lagrange. Engraving by Robert Hart (ca. 1834-1837), from a bust in the Library of the Institute of France

Theorem 5 (Marques-Neves [14]).— Let $\Sigma \subset S^3$ be an embedded compact surface with positive genus. Then $\mathscr{W}(\Sigma) \geq 2\pi^2$, and the equality holds if and only if Σ is the Clifford torus, up to conformal transformations of S^3.

With this, and in the light of Theorem 2, Fernando Codá Marques and André Neves have proved the Willmore Conjecture:

Corollary 6.— The Willmore Conjecture holds.

The milestone step achieved in Theorem 3 comes as an aplication of the Min-max Theory developed by Frederick Almgren [1] and Jon Pitts [20]. Driven by the problem of existence of minimal submanifolds of dimension higher than 2, Almgren introduced the notion of *varifold* and developed a general scheme to produce minimal manifolds in Riemannian manifolds. The question of regularity of these objects was later treated by Pitts, in the case of codimension one. Their combined works established, remarkably, the existence of an embedded, closed minimal hypersurface for any

[3] This is also the position in which the membrane is the most relaxed. In fact, minimal surfaces are examples of *Willmore surfaces*, surfaces that satisfy the equation $\Delta H + 2(H^2 - K)H = 0$ which, in the particular case of compact surfaces, characterizes the stationary configurations for the Willmore functional (see, for example, [35]).
Unlike flat soap films, soap bubbles exist under a certain surface tension, in an equilibrium where slightly greater pressure

Figure 5.—Fernando Codá Marques

Figure 6.—André Neves

given n-dimensional compact Riemannian manifold, with $3 \leq n \leq 6$, cf. [20].

As with many groundbreaking results in Mathematics, the work of Marques and Neves has provided new insights and suggested new approaches to other significant questions. Their contribution includes, in particular, two sequels of a similar spirit, namely, the proof of the Freedman-He-Wang conjecture for links [2], jointly with Ian Agol, and the proof of Yau's conjecture on the existence of infinitely many minimal hypersurfaces in manifolds of positive Ricci curvature [16] (see also [12, 13, 15, 19]).

4 The recipients of the 2016 Oswald Veblen Prize in Geometry

Fernando Codá Marques was born in São Carlos, Brazil, in 1979. He received a BS from the Federal University of Alagoas and an MS from IMPA, both in 1999, and his PhD from Cornell University in 2003. He became a Professor at IMPA in 2010 and, four years later, a Professor at Princeton University. In 2012, he was distinguished with the TWAS (The World Academy of Sciences for the advancement of science in developing countries) Prize in Mathematics, the Ramanujan Prize and the UMALCA (Unión Matemática de América Latina y el Caribe) Prize.

André Neves was born in Lisbon, Portugal, in 1975. He received his first degree from Instituto Superior Técnico in 1999 and his PhD from Stanford University in 2005. He has held positions at Princeton University from 2005 to 2009, the year he moved to Imperial College London, where he became a Professor in 2013. He received the Philip Leverhulme Prize in 2012, the LMS Whitehead Prize in 2013, the Royal Society Wolfson Merit Award in 2015 and the New Horizons Prize in Mathematics, also in 2015.

inside the bubble is balanced by the area-minimizing forces of the bubble itself. With their spherical shape, soap bubbles are area-minimizing surfaces under the constraint of volume enclosed. These are surfaces of (non-zero) constant mean curvature and, therefore, examples of *constrained Willmore surfaces*, the generalization of Willmore surfaces that arises when we restrict to infinitesimally conformal variations (for more details, see, for example, [30]).

Figure 7.— Leonhard Euler. Portrait by Jakob Emanuel Handmann (1753)

Fernando Codá Marques and André Neves were awarded the 2016 Oswald Veblen Prize in Geometry at the 122nd Annual Meeting of the American Mathematical Society in Seattle, Washington, on January 7, 2016.

"It is an honor and an immense pleasure to be a recipient, together with my friend André, of the prestigious Oswald Veblen Prize in Geometry.

I am thankful to the committee for this recognition of our work. I am grateful to my family, especially my parents, Severino and Dilze, my wife Ana, and my siblings Gustavo and Clarissa. I am sure that without their love and support I would not be here today. I also look forward to meeting my baby son, Pedro, who is joining us.

I thank also my late advisor, José Fernando Escobar (Chepe), who was always kind and supportive of me, and Richard Schoen, whose influence has been fundamental in my career. The year I spent with Rick was decisive and helped shape my vision of what is important in mathematics. I thank all my teachers, especially Professor Manfredo do Carmo. His lessons inspired me to choose the beautiful field of geometry. I am also grateful to Harold Rosenberg for the many mathematical discussions and to my students, who provide further motivation in my life. The collaboration and friendship with André has been a constant source of joy to me over the last ten years.

The study of minimal varieties is an old subject that began with the work of Lagrange on the foundations of the calculus of variations. The solution of the Plateau problem for mappings of the disk (Douglas and Rado, 1930) and for rectifiable currents (Federer and Fleming, 1960) are milestones of the field. But the question of existence of closed minimal varieties in general compact Riemannian manifolds is not a problem of minimization. This inspired Almgren (1965) to develop a deep min-max theory for the area functional. His work was improved by his PhD student J. Pitts (1981), but remained largely untouched until the last few years.

André and I were extremely delighted when we discovered that this old theory would play a major role in the solution of the Willmore conjecture. This required a change of perspective: instead of trying to minimize the conformally invariant Willmore functional, as originally proposed, we used conformal transformations to convert the problem into a question of minimizing the *maximum of the area of certain five-parameter families of surfaces in the three-sphere. Our work was done mainly while we were both visiting Stanford University at the end of 2011, and the main breakthrough came when we realized how to prove such families are topologically nontrivial. We were very amazed. A few months later we wrote a paper with Ian Agol in which we used similar ideas to solve a conjecture of Freedman, He, and Wang on the Moebius energy of links. Then we turned our attention to the general min-max theory and used it to prove Yau's conjecture about the existence of infinitely many minimal hypersurfaces in the positive Ricci curvature setting. The ideas of Gromov and a paper of Guth on multiparameter sweepouts were very influential. There have been several articles on min-max theory recently, especially by young people, and this makes us very happy. Major questions remain open, such as understanding the index, topology, and multiplicity of these minimal varieties. We hope to contribute further to the field."*[4] (Fernando Codá Marques)

"It is a great honor to receive the Oswald Veblen Prize in Geometry along with my dear friend Fernando.

Working and developing min-max theory together with Fernando has been a tremendous experience: it started with an academic interest in conformal deformations of surfaces, but soon we realized that we were discovering some new rich topology in the space of all surfaces. Coupling that with principles of Morse theory and ideas from minimal surfaces theory, we were able to answer some long-standing open questions in geometry. Since its beginnings, variational methods have had great influence in geometry, and I am delighted that our work made some contributions on that front. This is a beautiful subject, and I hope that its contributions will keep increasing for many years to come.

I consider myself very fortunate to have had Richard Schoen — one of the pioneers of geometric analysis — as my PhD advisor. His mathematical work and sharp intuition have been a towering influence on my research. I would also like to thank my collaborators and friends, from whom I have undoubtedly learned a lot, and my colleague Sir Simon Donaldson for all his support and encouragement throughout my career.

Finally, none of this would have been possible without the constant love and unyielding support of my parents Nelsa and Custódio, my wife Filipa, and our two adorable children, Eva and Tomás. In one way or another, they have all made sacrifices for the pursuit of my career."[5] (André Neves)

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