Topological Methods in Dynamics with Applications in Ergodic Theory

Old and New

Paulo Varandas*

1 A brief introduction

After the notion of uniform hyperbolicity was coined in the seventies by Smale [26], it became the paradigm of chaotic dynamics. If, on the one hand, the local dynamics of a uniformly hyperbolic diffeomorphism is simple and conjugated to the one exhibited by linear saddles, on the other hand the global dynamics presents an unpredictable character due to sensibility to initial conditions, denseness of periodic trajectories and orbits with transitive and (ir)regular behavior. The geometric theory developed for uniformly hyperbolic dynamics guarantees the existence of local immersed submanifolds (called stable and unstable manifolds) that are invariant by the dynamics and that constitute a crucial ingredient in the construction of finite Markov partitions. In consequence, subshifts of finite type can be used as combinatorial models for the dynamics and the powerfull techniques and ideas from statistical mechanics extend to this context. The geometric and functional analytic approaches (via construction of Markov partitions and the description of the spectrum of transfer operators, respectively) play a key role in the construction of physical, Sinai-Ruelle-Bowen and equilibrium measures and the study of their statistical properties. We refer the interested reader to e.g. [5, 24] for a more complete account.

The aim of this text is to survey and to be an invitation to the use of topological methods in dynamics, as a valid and handy alternative to the aforementioned geometric and functional analytic approaches. The starting point is the notion of specification proposed by Rufus Bowen which consists of the ability of the dynamics to recreate with sharp proximity true orbits from any given number of finite pieces of orbits (see e.g. [11]). The relation between shadowing and specification, the description of the latter concept and its extensions, and its importance as a tool in ergodic theory will guide the exposition in the remaining sections. There is evidence that these topological methods may be used to describe partial hyperbolic dynamics and dynamics of group actions, contexts in which topological and functional analytic methods are still unavailable, contributing to one of the most important leading research directions and challenges in dynamical systems.

2 Basics on topological dynamics

Throughout this article we assume that M is a compact Riemannian manifold. Let f be a continuous map on M. Some of the main concerns of the characterization of the dynamics from a topological viewpoint involve the description of periodic and transitive behavior, and chaoticity. Let $\Omega(f) \subset M$ be the set of non-wandering points of f, that is, the points $x \in M$ so that every open neighborhood of x intersects a positive iterate of itself by the dynamics. A point $x \in M$ is *periodic* (of period n) if there exists $n \in \mathbb{N}$ so that $f^n(x) = x$. Let $\mathrm{Per}(f)$ denote the set of all periodic points of f. Recall that f is called *transitive* if there exists a point $x \in M$ so that its orbit $\{f^n(x) : n \in \mathbb{N}\}$ is dense in M, and it is called *topologically mixing* if for any pair of open sets U, V there exists $N \in \mathbb{N}$ so that $f^n(U) \cap V \neq \emptyset$ for every $n \geq N$. An intricate challenge that goes back to the sixties was to propose suitable mathematical notions of *chaos*. Historically, one refers to chaotic dynamics the ones that exhibit at least one of the following properties: sensitive dependence to initial conditions, expansiveness, strong recurrence and mixing conditions, shadowing, specification, exponential growth of periodic points or positive topological entropy (see e.g. [15]).

Our main interest here concerns chaotic dynamics in the sense that pieces of true orbits or pseudo-orbits can be well approximated by true orbits of the dynamical system. The first notion that we shall consider is that of shadowing, which we now describe. Given a metric space M, a continuous map $f : M \to M$ and $\delta > 0$, a sequence of points $(x_n)_{n\geq 0}$

* Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil,
E-mail: paulo.varandas@ufba.br
URL: www.pgmat.ufba.br/varandas

is a δ-*pseudo-orbit* if $d(f(x_n), x_{n+1}) < \delta$ for every $n \geq 0$. We say that $[x_i, t_i]_{i\in\mathbb{Z}}$ is a (δ, T)-*pseudo-orbit* for a flow $(X^t)_t$ if $d(X^{t_i}(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. A continuous map has the *shadowing property* if for any $\varepsilon > 0$ there exists $\delta > 0$ so that for any δ-pseudo-orbit $(x_n)_{n\geq 0}$ there exists $x \in M$ satisfying $d(f^n(x), x_n) < \varepsilon$ for every $n \geq 0$. In the case of homeomorphisms, pseudo-orbits and shadowing points are defined for both negative and positive iterates of the dynamics. A continuous flow $(X^t)_t$ satisfies the *shadowing property* if, for any $\varepsilon > 0$ and $T > 1$ there exists $\delta = \delta(\varepsilon, T) > 0$ such that for any (δ, T)-pseudo-orbit $[x_i, t_i]_{i\in\mathbb{Z}}$ there is $\tilde{x} \in \Lambda$ and a reparametrization $\tau \in \mathrm{R}(\varepsilon)$ (cf. 1) such that

$$d(X^{\tau(t)}(\tilde{x}), x_0 \star t) < \varepsilon, \text{ for every } t \in \mathbb{R}.$$

where for $t \in \mathbb{R}$, $x_0 \star t = X^{t-\sigma(i)}(x_i)$ if $\sigma(i) \leq t < \sigma(i+1)$.

3 Basics on ergodic theory

3.1 Invariant measures

The purpose of ergodic theory is to describe the asymptotic behavior of the orbits of 'almost every point' with respect to relevant measures for the dynamics. Given a σ-algebra $\mathscr{B}$ and a measurable map f on M, we say that a probability measure μ is f-*invariant* if $\mu(f^{-1}(A)) = \mu(A)$ for every $A \in \mathscr{B}$. We denote by $\mathscr{M}_1(f)$ the space of f-invariant probability measures. A set $A \in \mathscr{B}$ is f-*invariant* if $\mu(f^{-1}(A)\triangle A) = 0$. An invariant probability measure μ is *ergodic* if $\mu(A) \in \{0, 1\}$ for every f-invariant set A. By ergodic decomposition, the space $\mathscr{M}_1(f)$ is the convex hull of the space $\mathscr{M}_e(f)$ of f-invariant and ergodic probability measures (see e.g.[33]).

In what follows let μ be an f-invariant probability measure. Two pillars in ergodic theory are due to Poincaré (1890), and to von Neumann and Birkhoff (1931-1932). First, if $\mu(A) > 0$ then Poincaré recurrence theorem asserts that μ-almost every $x \in A$ is recurrent: there exists $n \geq 1$ so that $f^n(x) \in A$. Later, the ergodic theorems of von Neumann and Birkhoff brought the ideas present in Boltzman ergodic hypothesis in thermodynamics into the realm of dynamical systems. Birkhoff's ergodic theorem guarantees that if $\mu \in \mathscr{M}_1(f)$ is ergodic and $\phi \in L^1(\mu)$ then

$$\frac{1}{n}\sum_{j=0}^{n-1} \phi(f^j(x)) \longrightarrow \int \phi\, d\mu \quad \text{as } n \to \infty$$

for μ-almost every $x \in M$ and, thus, time averages of an observable for typical orbits coincide with the space average with respect to the underlying measure. von Neuman ergodic theorem guarantees the convergence of the previous time averages for observables ϕ on the Hilbert space $L^2(\mu)$.

3.2 Thermodynamic formalism

Some of the ideas of thermodynamic formalism, which aims the selection of invariant measures and the study of their statistical properties, where introduced from statistical mechanics into the realm of dynamical systems by pioneering contributions of Sinai, Bowen and Ruelle in the late seventies (see [11, 22] and references therein). Two particularly important classes of invariant measures are the so called equilibrium states and physical measures.

Given a potential $\phi \in C(M, \mathbb{R})$ the *topological pressure* $P_{\text{top}}(f, \phi)$ for f and ϕ can be defined by the variational principle

$$P_{\text{top}}(f, \phi) = \sup\Big\{h_\mu(f) + \int \phi\, d\mu : \mu \in \mathscr{M}_1(f)\Big\},$$

where $h_\mu(f)$ stands for the entropy of μ (see e.g. [33]). An invariant probability measure μ is called an *equilibrium state* for f with respect to ϕ if it attains the previous supremum. If $\phi \equiv 0$ the previous notion coincides with the topological entropy $h_{\text{top}}(f)$ of f. If there exists a unique equilibrium state then it is necessarily ergodic and we shall denote it by $\mu_{f,\phi}$. An f-invariant probability measure μ is a *physical measure* if its basin of attraction

$$B(\mu) = \Big\{ x \in M : \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)} \to \mu \text{ as } n \to +\infty \Big\}$$

has positive Lebesgue measure. There are many examples where equilibrium and physical measures coincide and are absolutely continuous with respect to some reference measures with some weak Gibbs property. We say that a probability measure ν is a *weak Gibbs measure* for f and ϕ if for ν-almost every x there are constants $(K_n)_{n\geq 1}$ so that $\limsup_n \frac{1}{n} \log K_n(x) = 0$ and

$$K_n^{-1}(x) \leq \frac{\nu(B(x, n, \epsilon))}{e^{-nP(\phi)+S_n\phi(x)}} \leq K_n(x)$$

for every $n \geq 1$, where $S_n\phi = \sum_{j=0}^{n-1} \phi \circ f^j$ and the dynamical ball $B(x, n, \epsilon)$ is the set of points $y \in M$ such that $d(f^j(y), f^j(x)) < \varepsilon$ for all $0 \leq j \leq n$. We say that ν is a *Gibbs measure with respect to* ϕ if there exists $K > 0$ such that the previous property holds with $K_n = K$ (independent of n and x).

4 Uniform hyperbolicity

A compact f-invariant set $\Lambda \subset M$ is called *uniformly hyperbolic* for f if there exists a Df-invariant splitting $T_\Lambda M = E^s \oplus E^u$ and constants $C > 0$ and $\lambda \in (0, 1)$ so that

$$\|Df^n(x)\mid_{E^s_x}\| \leq C\lambda^n \;\&\; \|Df^{-n}(x)\mid_{E^u_x}\| \leq C\lambda^n$$

for every $x \in \Lambda$ and $n \geq 1$. We say that f is an *Anosov* diffeomorphism if M is a uniformly hyperbolic set. Given a hyperbolic set Λ, a point $x \in \Lambda$ and $\varepsilon > 0$, the *ε-stable set* of x is defined by $W^s_\varepsilon(x) = \{y \in M : d(f^n(y), f^n(x)) \leq \varepsilon \text{ for all } n \geq 0\}$. Similarly, the set $W^u_\varepsilon(x)$ of points $y \in M$ so that $d(f^{-n}(y), f^{-n}(x)) \leq \varepsilon$ for all $n \geq 0$ is the *ε-unstable set* of x. Given a hyperbolic set Λ for f there exists a uniform $\varepsilon > 0$ so that the stable and unstable sets $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ are C^r submanifolds tangent to E^s_x and E^u_x, respectively, for every $x \in \Lambda$. These are referred, respectively, as the *local stable* and *local unstable manifolds* at x of size ε. Uniform hyperbolicity is a C^1-open condition in the space $\mathrm{Diff}^1(M)$ of C^1-diffeomorphisms. We refer the reader to [24] for proofs.

Uniform hyperbolicity for flows is defined similarly. Given a C^1-flow $(X^t)_t$ on M and a compact $(X^t)_t$-invariant set $\Lambda \subseteq M$, we say that Λ is a *hyperbolic set* if there exists a DX^t-invariant and continuous splitting $T_\Lambda M = E^- \oplus E^0 \oplus E^+$ (E^0 subspace generated by the vector field $X(\cdot) = \frac{dX^t(\cdot)}{dt}\mid_{t=0}$) and constants $C > 0$ and $0 < \theta < 1$ such that

(i) $\|DX^t(x)\mid_{E^-_x}\| \leq C\theta^t$, and

(ii) $\|(DX^t(x)\mid_{E^+_x})^{-1}\| \leq C\theta^t$

for every $x \in M$ and $t \geq 0$. The flow $(X^t)_t$ is *Anosov* if the whole manifold M is a hyperbolic set. We refer the reader to [24] for more details on uniform hyperbolicity.

5 The notions: specification and gluing orbit properties

5.1 Discrete-time dynamics

A continuous map f on M satisfies the *specification property* if for any $\varepsilon > 0$ there exists an integer $N = N(\varepsilon) \geq 1$ such that: for every $k \geq 1$, any points $x_1, \dots, x_k$, and any sequence of positive integers $n_1, \dots, n_k$ and $p_1, \dots, p_k$ with $p_i \geq N(\varepsilon)$ there exists a point x in M such that $d(f^j(x), f^j(x_1)) \leq \varepsilon$ for every $0 \leq j \leq n_1$ and

$$d\big(f^{j+n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(x)\ ,\ f^j(x_i)\big) \leq \varepsilon$$

for every $2 \leq i \leq k$ and $0 \leq j \leq n_i$.

Among the maps that satisfy specification property one should refer topologically mixing subshifts of finite type, topologically mixing Anosov diffeomorphisms and topologically mixing continuous interval maps (see [10] and references therein). More flexible concepts include some measure theoretical non-uniform versions of the specification property that proved to hold for invariant measures with no zero Lyapunov exponents (cf. [18, 32]).

In the sequel we introduce two extensions of the notion of specification. Let μ be an f-invariant probability measure. We say that (f, μ) satisfies the *non-uniform specification property* if there exists $\delta > 0$ so that for μ-a.e. x and every $0 < \varepsilon < \delta$ there exists $p(x, n, \varepsilon) \in \mathbb{N}$ satisfying:

(i) $\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} p(x, n, \varepsilon) = 0$

(ii) given $x_1, \dots, x_k$ in a full μ-measure set and positive integers $n_1, \dots, n_k$, if $p_i \geq p(x_i, n_i, \varepsilon)$ then there exists z that ε-shadows the orbits of each x_i during n_i iterates with a time lag of $p(x_i, n_i, \varepsilon)$ in between $f^{n_i}(x_i)$ and x_{i+1}; that is, $z \in B(x_1, n_1, \varepsilon)$ and

$$f^{n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(z) \in B(x_i, n_i, \varepsilon)$$

for every $2 \leq i \leq k$.

We say a continuous map f on M satisfies the *gluing orbit property* if for any $\varepsilon > 0$ there exists an integer $N = N(\varepsilon) \geq 1$ so that for any points $x_1, x_2, \dots, x_k \in M$ and positive integers $n_1, \dots, n_k$, there are $p_1, \dots, p_k \leq N(\varepsilon)$ and $x \in M$ so that $d(f^j(x), f^j(x_1)) \leq \varepsilon$ for every $0 \leq j \leq n_1$ and

$$d\big(f^{j+n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(x)\ ,\ f^j(x_i)\big) \leq \varepsilon$$

for every $2 \leq i \leq k$ and $0 \leq j \leq n_i$. The latter property is satisfied e.g. by irrational rotations, which are far from having any mixing property and it is sometimes referred also as a transitive specification property [8, 34]. Similar flavored notions of linkability and closeability were introduced by Gelfert and Kwietniak [14]. We refer the reader to [17] and references therein for a more exhaustive description of the state of the art.

5.2 Continuous-time dynamics

In opposition to the discrete-time setting, the mixing properties of continuous-time dynamical systems are harder to analyze. For instance, while for uniformly hyperbolic diffeomorphisms every Hölder continuous potential admits a unique equilibrium state, which is a Gibbs measure and mixes exponentially fast, not all hyperbolic flows have exponential mixing (see e.g. [5]). Moreover, not all hyperbolic flows have the specification property, which is an indicator that a suitable notion should be more flexible to hold for a larger class of dynamics. Recall a continuous flow $(X^t)_{t\in\mathbb{R}}$ has the *specification property* on $\Lambda \subset M$ if for any $\epsilon > 0$ there exists a $T = T(\epsilon) > 0$ such that: given any finite colection τ of intervals $I_i = [a_i, b_i]$ $(i = 1 \dots m)$ of the real line satisfying $a_{i+1} - b_i \geq T(\epsilon)$ for every i and every map $P : \bigcup_{I_i\in\tau} I_i \to \Lambda$ such that $X^{t_2}(P(t_1)) = X^{t_1}(P(t_2))$ for any $t_1, t_2 \in I_i$ there exists $x \in \Lambda$ so that $d(X^t(x), P(t)) < \epsilon$ for all $t \in \bigcup_i I_i$.

In continuous-time setting the shadowing property of the finite pieces of orbits should reflect the speed at which different points travel in their trajectories. For that reason let $\mathscr{R}$ be the set of all increasing homeomorphisms $\tau : \mathbb{R} \to \mathbb{R}$ so that $\tau(0) = 0$ and, given $\varepsilon > 0$, set

$$\mathscr{R}(\varepsilon) = \Big\{\tau \in \mathscr{R} : \Big|\frac{\tau(t) - \tau(s)}{t - s} - 1\Big| < \varepsilon, s \neq t \in \mathbb{R}\Big\}. \qquad (1)$$

We say that a continuous flow $(X^t)_t$ has the *reparametrized gluing property* if for any $\varepsilon > 0$ there exists $K = K(\varepsilon) \in$

$\mathbb{R}^+$ such that for any points $x_0, x_1, \dots, x_k \in M$ and times $t_0, t_1, \dots, t_k \geq 0$ there are $p_0, p_1, \dots, p_{k-1} \leq K(\varepsilon)$, a reparametrization $\tau \in \mathscr{R}(\varepsilon)$ and a point $y \in M$ so that

$$d(X^{\tau(t)}(y)), X^t(x_0)) < \varepsilon \quad \forall t \in [0, t_0]$$

and

$$d(X^{\tau(t+\sum_{j=0}^{i-1} p_j + t_j)}(y), X^t(x_i)) < \varepsilon \quad \forall t \in [0, t_i]$$

for every $1 \leq i \leq k$. If, in addition, the point y can be chosen periodic we say that $(X^t)_t$ satisfies the periodic reparametrized gluing orbit property. Criteria for (semi)flows to satisfy gluing orbit properties can be found in [8, 10].

6 Specification and gluing orbit properties: some consequences

In this section we shall focus on the analysis of continuous-time dynamics (since proofs are technically more demanding and results are in many cases harder to find in the literature) and on the comparison between continuous and discrete time dynamics.

6.1 Topological aspects

The space of homeomorphisms are often described in terms of topological classes, where we say that the homeomorphisms f and g are topologically conjugate if there exists a homeomorphism h so that $f \circ h = h \circ g$. Hence, the dynamics of homeomorphisms in the same topological class is the same up to a continuous change of coordinates. Similarly, flows are usually classified up to topological equivalence, that is, homeomorphisms that preserve orbits and their orientation but not necessarily the speed of the trajectories. If, on the one hand, it is not hard to check that the specification and the gluing orbit property are topological invariants, on the other hand topological equivalence may fail to preserve the gluing orbit properties for flows since these may affect the kind of reparametrizations that are considered at the shadowing process.

The strong contrast between discrete and continuous time dynamics is also present in the relation between shadowing and specification. While topologically mixing expansive continuous maps on compact metric spaces with shadowing property satisfy the specification property, this may not hold even for very simple Anosov flows. Moreover, minimal flows on $\mathbb{T}^2$ satisfy gluing orbit properties but fail to be topologically mixing. See [1, 8] for more details. Nevertheless, flows with the reparametrized gluing orbit property satisfy some 'weak mixing' conditions [8]. More precisely:

Theorem 1.— If $(X^t)_t$ satisfies the reparametrized gluing orbit property then $(X^t)_t$ has positive lower frequency $\tau(B_1, B_2)$ of visits to balls B_1, B_2 of radius ε given by

$$\liminf_{t\to+\infty} \frac{1}{t} \text{Leb}(\{s \in [0,t] : B_1 \cap X_{-s}(B_2) \neq \emptyset\})$$

is strictly positive. Moreover, for all balls B_1, B_2 of radius ε centered at points with closed orbits there exists $C > 0$ such that $\tau(B_1, B_2) \geq C\varepsilon$.

We also note that if the flow $(X^t)_t$ is expansive then the topological entropy is bounded by the exponential growth rate of periodic orbits, a result which also holds in the context of semigroups of expanding maps (cf. [8, 12]).

6.2 Space of invariant measures

The push-forward $f_\sharp$ acting on the space of probability measures in M is defined by $(f_\sharp \mu)(A) = \mu(f^{-1}(A))$ for every $A \in \mathscr{B}$. This map inherits some of the the topological characteristics of the original dynamics. First, if f has a specification property then so does $f_\sharp$ and these are equivalent in the context of continuous interval maps (see e.g. [21]). Second, the simplex of invariant measures for maps with specification is the Poulsen simplex (see [25, 17]). Given a continuous flow $(X^t)_t$ we denote by $\mathscr{M}_1((X^t)_t)$ the space of $(X^t)_t$-invariant probabilities. In [8] one could recover part of the "richness" for the simplex of invariant probability measures for dynamics with gluing orbit properties. More precisely,

Theorem 2.— If $(X^t)_t$ satisfies the periodic reparametrized gluing orbit property then periodic measures are dense in $\mathscr{M}_1((X^t)_t)$, and the set of ergodic measures forms a residual subset of $\mathscr{M}_1((X^t)_t)$.

As continuous flows with shadowing and a dense set of periodic orbits satisfy the reparametrized gluing orbit property (cf. [7]) we obtain the following consequence:

Corollary 3.— Assume $(X^t)_t$ is a continuous and volume preserving flow. If $(X^t)_t$ satisfies the periodic shadowing property and the periodic points are dense in M then periodic measures are dense in $\mathscr{M}_1((X^t)_t)$.

6.3 Large deviations.

In the early nineties, L.-S. Young [35] addressed the question of the velocity of convergence of ergodic averages on Birkhoff's ergodic theorem in the case of Gibbs measures. Here, given a potential $\phi : M \to \mathbb{R}$ and a probability μ, we say that μ is *weak Gibbs* with respect to ϕ, with constant $P_\mu \in \mathbb{R}$, if for any $\varepsilon > 0$ there exists $K_t(\varepsilon)$ (depending only on ε and on the time t) so that $\lim_{t\to\infty} \frac{1}{t} \log K_t(\varepsilon) = 0$ and

$$\frac{1}{K_t(\varepsilon)} \leq \frac{\mu(B(x,t,\varepsilon))}{\exp\left[\int_0^t \phi(X^s(x))ds - tP_\mu\right]} \leq K_t(\varepsilon)$$

for μ-almost every $x \in M$ and every $t \geq 0$. A continuous observable $\psi : M \to \mathbb{R}$ is called of *tempered variation* if there is $\delta > 0$ such that $\lim_{t\to\infty} \frac{1}{t}\gamma(\psi, t, \delta) = 0$, where

$$\gamma(\psi, t, \delta) = \sup_{y\in B(x,t,\delta)} \Big| \int_0^t \psi(X^s(x)) - \psi(X^s(y))ds \Big|.$$

Gluing orbit properties were first introduced in [10] with the motivation of obtaining large deviations principles for *all* hyperbolic flows:

THEOREM 4.— Assume the semiflow $(X^t)_{t\geq 0}$ satisfies the gluing orbit property, ϕ is a bounded potential with tempered variation and μ is a weak Gibbs probability with respect to ϕ. If $a < b$ and $\psi : M \to \mathbb{R}$ is a bounded observable with tempered variation then

$$\liminf_{t\to\infty} \frac{1}{t} \log \mu\Big(\frac{1}{t} \int_0^t \psi \circ X^s(\cdot) ds \in (a, b) \Big) \\ \geq - \inf \Big\{ P_\mu - h_\nu(X^1) - \int \phi \, d\nu \Big\},$$

where the infimum is taken over all $(X^t)_t$-invariant probability measures ν so that $\int \psi \, d\nu \in (a, b)$. If, in addition, M is compact and $\psi : M \to \mathbb{R}$ is continuous then

$$\limsup_{t\to\infty} \frac{1}{t} \log \mu\Big(\frac{1}{t} \int_0^t \psi \circ X^s(\cdot) ds \in [a, b] \Big) \\ \leq - \inf \Big\{ P_\mu - h_\nu(X^1) - \int \phi \, d\nu \Big\},$$

where the infimum is taken over all $(X^t)_t$-invariant probability measures ν so that $\int \psi \, d\nu \in [a, b]$.

A surprising connection between large derivations and multifractal analysis (cf. Subsection 6.4 below) allows to use the large deviations estimates to study the size of the level sets and irregular set in the multifractal decomposition [9].

6.4 Some other aspects

For shortness, in what follows we give a more direct and informal presentation of other important characterizations of dynamics with some gluing orbit property and their use as an important tool.

A characterization for uniform hyperbolicity

The relation between specification, gluing and uniform hyperbolicity among smooth dynamics is well understood. If the specification property holds in a C^1-open neighborhood of diffeomorphisms or vector fields then these are Anosov [23, 4]. Similarly any C^1-open subset of diffeomorphisms (resp. vector fields) with the gluing orbit property is formed by transitive Anosov diffeomorphisms (resp. Anosov flows) [34, 10]. So, from the C^1-robust viewpoint, uniform hyperbolicity, specification and the gluing orbit properties coincide. The picture is radically different beyond the scope of uniform hyperbolicity. Indeed, specification is rare even among partially hyperbolic diffeomorphisms [27, 28].

Thermodynamic formalism

Bowen [11] proved that expansive homeomorphisms with specification have a unique equilibrium state with respect to all continuous potentials with tempered variation. More recently, Climenhaga and Thompson extended Bowen's approach to deal with dynamical systems where the set of points with obstructions to either specification or expansiveness do not carry full topological pressure (we refer the reader to [13] for a precise formulation and applications). More recently, Pavlov [19] showed that expansive maps with non-uniform specification may have more than one equilibrium state.

Multifractal formalism

The general idea of multifractal analysis, that can be traced back to Besicovitch, is to decompose the phase space in subsets of points which have a similar dynamical behavior and to describe the size of each of such subsets from the dimensional or topological viewpoint. Given a continuous map f on M and $\phi : M \to \mathbb{R}$ continuous, decompose

$$M = \bigcup_{\alpha\in\mathbb{R}} M_\alpha \,\cup\, I_\phi(f)$$

where $M_\alpha = \{x \in M : \lim_n \frac{1}{n} S_n\phi(x) = \alpha\}$ are level sets of convergence for Birkhoff averages and the irregular set $I_\phi(f)$ is the set of points for which the Birkhoff averages for ϕ do not converge. The irregular set for continuous observables and maps with specification is either empty or carries full topological entropy. Moreover, the topological pressure of level sets can be characterized by the supremum for invariant measures supported on them (see [31] and references therein). A much harder situation is to describe the topological entropy of saturated sets. Given a subset $K \subset \mathcal{M}_1(f)$ of f-invariant probability measures, a saturated set in M is the subset $G_K \subset M$ of points $x \in M$ whose accumulation points $V_T(x)$, in the weak* topology, of the empirical measures

$$\mathcal{E}_n(x) := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(x)}$$

coincides with the prescribed subset K of invariant probability measures. Saturated sets can be used to describe convergence properties of Birkhoff averages with respect to every continuous observable. Clearly $V_T(x)$ is a singleton if and only if the Birkhoff averages of every continuous observable are convergent at x. Some extensions of the original notion of specification can be used to estimate the topological pressure of saturated sets for some non-uniformly hyperbolic maps [20, 29, 30].

7 A final invitation: some open questions

The use of topological methods in ergodic theory is nowadays a very active area of research. We will finish this short article with some open questions as an invitation for the reader to explore the underlying ideas presented here.

1. The relation between specification and the gluing orbit property is still not fully understood. Given the previous discussion it is natural to ask whether there exists a Baire residual subset $\mathscr{R}$ of the space of C^1-diffeomorphisms with the gluing orbit property so that every topologically mixing diffeomorphism $f \in \mathscr{R}$ satisfies the specification property.

2. Regarding the thermodynamic formalism of maps displaying some weak form of specification, it is natural to ask if an expansive map f with the gluing orbit property has a unique equilibrium state for every regular (e.g. Hölder continuous) potential. Are the related transfer operators quasi-compact on the Banach space of Hölder continuous observables? See [5] for definition of transfer operators. Similar question can be posed for flows with the reparametrized gluing orbit property.

3. The non-wandering set of a uniformly hyperbolic diffeomorphism can be decomposed in a finite number of pieces on which the dynamics acts as a subshift of finite type and each piece, up to an iterate of the dynamics, satisfies the specification property. On the converse direction, if f is a continuous expansive map with the gluing orbit property does there exist $N \geq 1$ and a disjoint union $M = \bigcup_{1\leq i\leq N} \Lambda_i$ of compact sets so that $f(\Lambda_i) = \Lambda_{i+1}$ for all $1 \leq i \leq N$ (with the convention that $\Lambda_{N+1} = \Lambda_1$) and the iterate $f^N : \Lambda_i \to \Lambda_i$ has the specification property? If so, which extra information can be given on each of the 'basic' pieces Λ_i?

4. The relation between specification and uniform hyperbolicity is well established (recall Subsection 6.4). However, much less is known on the relation between these topological concepts with the measure theoretical notions of non-uniform specification. Assume $\mathscr{U}$ is a C^1 open set of transitive diffeomorphisms on M so that all g-invariant measures satisfy the non-uniform specification property, for all $g \in \mathscr{U}$. Is $\mathscr{U}$ formed by maps with some gluing orbit property? We believe the C^1-robustness assumption should be crucial above.

5. The multifractal analysis of time averages for flows is much harder than for maps even when assuming the reparametrized gluing orbit property. In comparison with the discrete time setting, the difficulty relies on the fact that the reparametrization depends on the points that are being shadowed. Nevertheless we expect that if $(X^t)_t$ is a continuous flow with the reparametrized gluing orbit property and the Birkhoff irregular set of a continuous potential is non-empty then it should carry full topological entropy.

6. Geometric Lorenz attractors are among the simpler flows where regular orbits accumulate on singular orbits (see e.g. [2]). The coexistence of singular and regular orbits brings much complexity to the dynamics and imply, in particular, the absence of weak forms of shadowing for most geometric Lorenz attractors [16, 3]. In view of some criteria for non-uniform specification properties [10] it is natural to ask whether geometric Lorenz attractors enjoy a reparametrized gluing orbit property with respect to reparametrizations with a logarithmic singularity at the origin. This can be thought as a step in the direction of establishing a thermodynamic formalism for geometric Lorenz attractors.

7. Finally, the underlying ideas of the property of specification are expected to be applied in far more general situations. This property was proved to hold for C_0 semigroups on Banach spaces, including solutions of the hyperbolic heat equation and Black-Scholes equation (see e.g. [6] and references therein). Since most results addressed here require compactness as a crucial ingredient it is a challenge to understand up to which extent the ideas arising from multifractal formalism can extend to the context of partial differential equations and/or operators on infinite dimensional ambient spaces.

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References

[1] N. Aoki, Topological dynamics, in: Topics in General Topology, in: North-Holland Math. Library, vol. 41, Amsterdam, 1989.

[2] V. Araújo and M. Pacífico, Three-dimensional flows, Ergeb. Math. Grenz 3, Springer, Heidelberg, 2010.

[3] A. Arbieto, J.E. Reis, and R. Ribeiro *On various types of shadowing for geometric Lorenz flows*, Rocky Mountain J. Math. 45:4 (2015), 1067–1090.

[4] A. Arbieto, L. Senos and T. Sodero, *The specification property for flows from the robust and generic viewpoint*, J. Differential Equations 253 (2012), 1893–1909.

[5] V. Baladi. Positive transfer operators and decay of correlations. Advanced Series in Nonlinear Dynamics: Volume 16, 2000.

[6] S. Bartoll, F. Martínez-Giménez, A. Peris and F. Rodenas, *The specification property for C_0-semigroups*, Preprint ArXiv 2016.

[7] M. Bessa, M. J. Torres and P. Varandas, *On periodic orbits, shadowing and strong transitivity of continuous flows*, Preprint 2016.

[8] T. Bomfim, M. J. Torres and P. Varandas, *Topological features of flows with the reparametrized gluing orbit property*, Preprint 2016.

[9] T. Bomfim and P. Varandas, *Multifractal analysis for weak Gibbs measures: from large deviations to irregular sets*, Ergod. Th. & Dynam. Sys. (to appear)

[10] T. Bomfim and P. Varandas, *The gluing orbit property, uniform hyperbolicity and large deviation principles for semiflows*, Preprint ArXiv:1507.03905

[11] R. Bowen, *Some systems with unique equilibrium states*, Math. Syst. Theory 8 (1974), 193–202.

[12] M. Carvalho, F. Rodrigues and P. Varandas. *Semigroups of expanding maps*, Preprint ArXiv:1601.04275.

[13] V. Climenhaga and D. Thompson *Equilibrium states beyond specification and the Bowen property*, J. London Math. Soc. 87 (2013), 401–427.

[14] K. Gelfert and D. Kwietniak, *On density of ergodic measures and generic points*, Ergod. Th. Dynam. Sys. (to appear).

[15] J. Li and X. D. Ye, *Recent development of chaos theory in topological dynamics*, Acta Math. Sinica, 32:1 (2016) 83–114.

[16] M. Komuro, Lorenz attractors do not have the pseudo-orbit tracing property, J. Math. Soc. Japan, 37 (1985) 489–514.

[17] D. Kwietniak, M. Lacka, P. Oprocha. *A panorama of specification-like properties and their consequences*, Dynamics and Numbers, Contemporary Mathematics, vol. 669 (2016) 155–186.

[18] K. Oliveira and X.Tian. *Non-uniform hyperbolicity and non-uniform specification*, Trans. Amer. Math. Soc., 365 (2013), 4371–4392.

[19] R. Pavlov *On intrinsic ergodicity and weakenings of the specification property.* Adv. Math., 295 (2016) 250–270.

[20] C. Pfister, W. Sullivan, *On the topological entropy of saturated sets*, Ergod. Th. Dynam. Sys. 27 (2007) 929–956.

[21] F. Rodrigues and P. Varandas, *Specification properties for group actions and thermodynamics of expanding semigroups*, J. Math. Phys. 57, (2016) 052704

[22] D. Ruelle, *Statistical mechanics on a compact set with $\mathbb{Z}^p$ action satisfying expansiveness and specification*, Trans. Amer. Math. Soc., 185 (1973) 237–251.

[23] K. Sakai, N. Sumi and K. Yamamoto. *Diffeomorphisms satisfying the specification property*, Proc. Amer. Math. Soc., 138 (2009), 315–321.

[24] M. Shub. *Global stability of dynamical systems*, Springer Verlag, (1987).

[25] K. Sigmund. *On dynamical systems with the specification property*, Trans. Amer. Math. Soc. 190 (1974), 285–299.

[26] S. Smale. *Differentiable dynamical systems*, Bull. Amer. Math. Soc., 73 (1967) 747–817.

[27] N. Sumi, P. Varandas and K.Yamamoto, *Partial hyperbolicity and specification*, Proc. Amer. Math. Soc. 144:3 (2016) 1161–1170.

[28] N. Sumi, P. Varandas and K.Yamamoto, *Partial hyperbolicity and specification for flows*, Dynam. Sys, 30:4, (2015) 501–524.

[29] F. Takens and E. Verbitski, *Multifractal analysis of local entropies for expansive homeomorphisms with specification*, Comm. Math. Phys., 203 (1999) 593–612.

[30] X. Tian and P. Varandas, *Topological entropy of level sets of empirical measures for non-uniformly expanding maps* Preprint 2016.

[31] D. Thompson, *The irregular set for maps with the specification property has full topological pressure*, Dynam. Sys. 25(1), 2010, 25–51.

[32] P. Varandas, *Non-uniform specification and large deviations for weak Gibbs measures*, J. Statist. Phys., 146 (2012), 330–358.

[33] P. Walters, An introduction to ergodic theory, *Springer-Verlag*, New York, 1982.

[34] W. Sun and X. Tian, *Diffeomorphisms with various C^1-stable properties.* Acta Mathematica Scientia 32B:2 (2012) 552–558.

[35] L.-S. Young, *Some large deviation results for dynamical systems*, Trans. Amer. Math. Soc., 318(2) (1990), 525–543.