Solid n-twisted Möbius strips as real algebraic surfaces

by **Stephan Klaus***

We construct explicit polynomials p_n in three real variables x, y and z such that the associated affine variety $p_n^{-1}(0)$ gives a small tubular neighborhood of the n-twisted Möbius strips. The degree of p_n is given by $4+2n$. We give visualizations up to twisting number $n=6$ using the free software *surfer* of the open source platform Imaginary.

1 Introduction

It is a well-known fact from differential topology (e.g., see [1], chapter 1, §4) that for a smooth function $f : \mathbb{R}^m \to \mathbb{R}$ and a regular value $y \in \mathbb{R}$, the level set $M := f^{-1}(y)$ is a smooth $(m-1)$-dimensional hypersurface. Moreover, M has no boundary and is orientable. In case that M is compact and connected, it separates $\mathbb{R}^m$ in two regions, the *inside* and the *outside*, by the generalized Jordan-Brouwer separation theorem ([1], chapter 2, §5).

Hence, it is not possible to construct the Möbius strip as a smooth level set in this way as it has a 1-dimensional boundary and as it is non-orientable.

However, in [2] we have given polynomials of degree 6, 8 and 10 such that the visualization of the level sets $f^{-1}(0)$ (using the surfer-software [5] of the open source platform Imaginary) give Möbius strips with 1, 2 and 3 twists, respectively.

This apparent contradiction can be easily explained: These surfaces are not Möbius strips on the nose but thickened versions, i.e. boundaries of small tubular neighborhoods. We call them *solid* Möbius strips. Our method of construction (by *rotation with twisting*) will be explained in the next section. We remark that a similar method was used in [3] to construct the solid trefoil knot with a polynomial of order 14. An overview over these and other constructions of interesting surfaces can be found in [4].

The reason that we come back to the construction of Möbius strips is that we will present here a simplified construction which works for any number of twists and gives an explicit polynomial, whereas the method in [2] was ad-hoc in the degrees considered.

As a last remark we mention a theorem of Whitney [6]: For any closed subset $A \subset \mathbb{R}^m$ there exists a smooth function $f : \mathbb{R}^m \to \mathbb{R}$ such that $A := f^{-1}(0)$. Of course, in general 0 is then not a regular value of f.

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2 Rotation with n-twisting

We start with an affine real algebraic curve given as the level set of a polynomial $f(t,z) \in \mathbb{R}[t,z]$ for the value 0. In our case,

$$f(t,z) = \left(\frac{t}{a}\right)^2 + \left(\frac{z}{b}\right)^2 - 1$$

is an ellipse with semiaxes $0 < a < 1$ and $0 < b < 1$. We denote the eccentricity (with $a \leq b$) as the geometrically relevant parameter by

$$e := \frac{1}{b}\sqrt{b^2 - a^2}$$

and we are mainly interested in big eccentricity, e.g. $e \approx 1$.

Now, rotation with twisting denotes a mixture of two rotation motions. The first movement concerns the t-axis which we let rotate around the z-axis such that it

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spans the (x, y)-plane:

$$x = Ct, \qquad C := \cos\phi$$
$$y = St, \qquad S := \sin\phi$$

Here, ϕ denotes the angle between the t- and the x-axis. Note that $C^2+S^2 = 1$. At the same time we impose a second rotation ('twisting') within the (t, z)-plane around a center at $(1, 0)$ with angle ψ:

$$t' - 1 = c(t-1) + sz$$
$$z' = -s(t-1) + cz$$

Here, $c := \cos(\psi)$, $s := \sin(\psi)$ and $c^2 + s^2 = 1$. This variable transformation leads to the *master equation*

$$f(c(t-1) + sz, -s(t-1) + cz) = 0$$

where both rotation motions are coupled as an *n-twisted rotation*, i.e.

$$\psi = \frac{n}{2}\phi$$

because a twist is a *half* rotation.

In our case of an ellipse we get (after multiplication with a^2b^2 in order to get rid of denominators)

$$\begin{aligned} a^2b^2 &= b^2(c(t-1) + sz)^2 + a^2(-s(t-1) + cz)^2 \\ &= c^2(b^2(t-1)^2 + a^2z^2) + 2cs(b^2 - a^2)(t-1)z + \\ &\quad + s^2(a^2(t-1)^2 + b^2z^2). \end{aligned}$$

3 Elimination of the rotation and twisting variables

Now we need to eliminate the variables C, S, c, s and t from the master equation in the section above in order to get a single equation $p(x, y, z) = 0$. At a first glance one could think that this is not possible with a polynomial p because of the transcendental functions cos and sin. However, with the de Moivre formula

$$\exp(\phi i)^n = \exp(n\phi i) = \exp(2\psi i) = \exp(\psi i)^2$$

we get for the real and the imaginary parts:

$$\begin{aligned} p_n(C, S) &:= \Re((C + iS)^n) = C^n - \binom{n}{2}C^{n-2}S^2 + \\ &\quad + \binom{n}{4}C^{n-4}S^4 \mp \cdots = c^2 - s^2 \\ q_n(C, S) &:= \Im((C + iS)^n) = \binom{n}{1}C^{n-1}S - \\ &\quad - \binom{n}{3}C^{n-3}S^3 \pm \cdots = 2cs \end{aligned}$$

with homogeneous polynomials p_n and q_n of order n. Because of $c^2 + s^2 = 1$ we obtain

$$c^2 = \frac{1}{2}(1 + p_n(C, S)), \qquad s^2 = \frac{1}{2}(1 - p_n(C, S)).$$

Now we use $C = x/t$ and $S = y/t$ and we insert the expressions for c^2, $2cs$ and s^2 into our master equation. Using the homogeneity of p_n and q_n we multiply the equation with t^n. This yields

$$(*) \qquad \begin{aligned} &\frac{1}{2}(t^n + p_n(x, y))(b^2(t-1)^2 + a^2z^2) + \\ &\quad + q_n(x, y)(b^2 - a^2)(t-1)z + \\ &+ \frac{1}{2}(t^n - p_n(x, y))(a^2(t-1)^2 + b^2z^2) - a^2b^2t^n = 0. \end{aligned}$$

Hence we have eliminated the rotation and twisting variables C, S, c and s. The last step is the algebraic elimination of the variable t. (Of course, we could just replace t by $\sqrt{x^2 + y^2}$ but this would not give a polynomial equation.)

4 Elimination of the variable t

This last step can be achieved in a more general context. Suppose we have given a polynomial $g(x, y, z, t) \in \mathbb{R}[x, y, z, t]$ and a polynomial $h(x, y, z) \in \mathbb{R}[x, y, z]$ and we want to eliminate t from the system

$$g(x, y, z, t) = 0$$
$$h(x, y, z) = t^2$$

We are in particular interested in the case of $h(x, y, z) = x^2 + y^2$. The algebraic elimination can be achieved by splitting g in even and odd powers of t:

$$g(x, y, z, t) = g_0(x, y, z, t^2) + tg_1(x, y, z, t^2)$$

Then from $g = 0$ we get $g_0 = -tg_1$ and squaring this equations yields

$$g_0(x, y, z, h(x, y, z))^2 = h(x, y, z)g_1(x, y, z, h(x, y, z))^2$$

which is the final solution of our elimination problem above.

In order to apply this procedure to equation $(*)$, we sort the terms according to powers of t. From the structure of the equation with its 4 terms, there appear only the powers t^{n+2}, t^{n+1}, t^n, t^2, $t^1 = t$ and $t^0 = 1$, and $(*)$ is equivalent to the equation in figure 1.

The matrix-like shape with 4 rows reflects the origin of each entry in the bracket from one the 4 terms of $(*)$. Of course, we use here the abbreviations $p_n = p_n(x, y)$ and $q_n = q_n(x, y)$. A further simplification with

$$A := a^2 + b^2 = b^2(1 + \epsilon), \qquad B := b^2 - a^2 = b^2(1 - \epsilon),$$
$$D := a^2b^2 = b^4\epsilon,$$

where e denotes the eccentricity and $\epsilon := 1-e^2$ (i.e., $\epsilon \approx 0$

$$
\begin{array}{llccccl}
t^{n+2} & (& \frac{1}{2}b^2 & & +\frac{1}{2}a^2 & &) \\
+t^{n+1} & (& -b^2 & & -a^2 & &) \\
+t^{n} & (& \frac{1}{2}(b^2+a^2z^2) & & +\frac{1}{2}(a^2+b^2z^2) & -a^2b^2 &) \\
+t^{2} & (& \frac{1}{2}p_nb^2 & & -\frac{1}{2}p_na^2 & &) \\
+t & (& -p_nb^2 & q_n(b^2-a^2)z & +p_na^2 & &) \\
+1 & (& \frac{1}{2}p_n(b^2+a^2z^2) & -q_n(b^2-a^2)z & -\frac{1}{2}p_n(a^2+b^2z^2) & &) \\
=0 & & & & & &
\end{array}
$$

Figure 1

is a thin ellipse and $\epsilon = 1$ gives a round torus), yields:

$$
(**) \qquad \frac{A}{2}t^{n+2} - At^{n+1} + (\frac{A}{2}(1+z^2) - D)t^n + \frac{B}{2}p_nt^2 - B(p_n - q_nz)t + \frac{B}{2}p_n(1-z^2) - Bq_nz = 0
$$

Now, in order to apply the above method of t-elimination, we have to distinguish the two cases of even and odd twisting numbers n.

4.1 Even twisting number $n = 2m$

By sorting the odd t-powers to the right side we get from $(**)$ that

$$
\left(\frac{A}{2}\left(1+t^2+z^2\right) - D\right)t^n + \frac{B}{2}p_n\left(1+t^2-z^2\right) - Bq_nz = = t[At^n + B(p_n - q_nz)].
$$

Thus we have proved the following result by applying t-elimination:

Theorem 1.— For an even twisting number $n = 2m$, the n-twisted solid Möbius strip is given as an affine real algebraic surface for the following polynomial equation in x, y and z of degree $4 + 2n$:

$$
\left[\left(\frac{A}{2}\left(1+x^2+y^2+z^2\right) - D\right)\left(x^2+y^2\right)^m + \frac{B}{2}p_n\left(1+x^2+y^2-z^2\right) - Bq_nz\right]^2 = = (x^2+y^2)[A(x^2+y^2)^m + B(p_n - q_nz)]^2.
$$

4.2 Odd twisting number $n = 2m+1$

By sorting the odd t-powers to the right side we get from $(**)$ that

$$
-At^{n+1} + \frac{B}{2}p_n\left(1+t^2-z^2\right) - Bq_nz = = t\left[-\left(\frac{A}{2}\left(1+t^2+z^2\right) - D\right)t^{n-1} + B(p_n - q_nz)\right].
$$

Thus we have proved the following result by applying t-elimination:

Theorem 2.— For an odd twisting number $n = 2m+1$, the n-twisted solid Möbius strip is given as an affine real algebraic surface for the following polynomial equation in x, y and z of degree $4 + 2n$:

$$
\left[-A\left(x^2+y^2\right)^{m+1} + \frac{B}{2}p_n\left(1+x^2+y^2-z^2\right) - Bq_nz\right]^2 = = (x^2+y^2)\left[-\left(\frac{A}{2}\left(1+x^2+y^2+z^2\right) - D\right)(x^2+y^2)^m + +B(p_n - q_nz)\right]^2.
$$

5 Visualization for small values of n

The first three cases $n = 1, 2$ or 3 were already considered in our paper [2] with more clumsy computations. Our new general formula recovers our former results.

5.1 Twisting number $n = 0$

Our formula also works in the untwisted case $n = 0$. Then we have $p_0 = 1$ and $q_0 = 0$ and we get the following polynomial equation of order 4:

$$
\left[\frac{A}{2}\left(1+x^2+y^2+z^2\right) - D + \frac{B}{2}\left(1+x^2+y^2-z^2\right)\right]^2 = = (x^2+y^2)[A+B]^2.
$$

This gives not only the usual torus, but also for small a a surface of shape of a solid (finite) cylinder barrel and for small b a surface of shape of a solid annulus. See figure 2 (*surfer* code included).

5.2 Twisting number $n = 1$

This is the classical Möbius strip. We have $m = 0$, $p_1 = x$, $q_1 = y$ and we get the following polynomial equation of order 6:

$$
\left[-A\left(x^2+y^2\right) + \frac{B}{2}x\left(1+x^2+y^2-z^2\right) - Byz\right]^2 = = \left(x^2+y^2\right)\left[-\frac{A}{2}\left(1+x^2+y^2+z^2\right) + D + B(x - yz)\right]^2.
$$

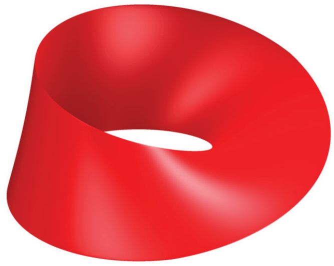
See figure 3 (*surfer* code included).

Note the small term $c^5 * 0.0001$. The reason for this modification is that the polynomial $(*)$ was constructed by multiplication of the preceding equation with t^n.

Figure 2

```
(0.5*(a^2+b^2)*(1+x^2+y^2+z^2)-a^2*b^2+0.5*(b^2-a^2)
*(1+x^2+y^2-z^2))^2-4*b^4*(x^2+y^2)
```

Figure 3



```
(-(a^2+b^2)*(x^2+y^2)+0.5
*(b^2-a^2)*x*(1+x^2+y^2-z^2)
-(b^2-a^2)*y*z)^2-(x^2+y^2)
*(-0.5*(a^2+b^2)
*(1+x^2+y^2+z^2)+a^2*b^2
+(b^2-a^2)*(x-y*z))^2
+c^5*0.0001
```

Figure 4

```
((0.5*(a^2+b^2)
*(1+x^2+y^2+z^2)-a^2*b^2)
*(x^2+y^2)+0.5*(b^2-a^2)
*(x^2-y^2)*(1+x^2+y^2-z^2)
-2*(b^2-a^2)*x*y*z)^2
-(x^2+y^2)*((a^2+b^2)
*(x^2+y^2)+(b^2-a^2)
*(x^2-y^2-2*x*y*z))^2
+c^7*0.0001
```

This adds the z-axis as a singular 1-dimensional set to the smooth 2-dimensional level let. Because this introduces a numerically critical behavior in a small neighborhood of the z-axis, the surfer software produces a *ghost image* there. Note that this effect becomes more dominant with larger twisting numbers n. Now, the small extra term allows a smoothing of the level set. With the right sign of c, the smoothing eliminates the z-axis as a singular set.

5.3 Twisting number $n = 2$

Thus we have $m = 1$, $p_2 = x^2 - y^2$, $q_2 = 2xy$ and we get the following polynomial equation of order 8:

$$\left[\left(\frac{A}{2}\left(1+x^2+y^2+z^2\right)-D\right)\left(x^2+y^2\right)+\frac{B}{2}\left(x^2-y^2\right)\left(1+x^2+y^2-z^2\right)-2Bxyz\right]^2 =$$

$$= \left(x^2+y^2\right)\left[A\left(x^2+y^2\right)+B\left(x^2-y^2-2xyz\right)\right]^2.$$

See figure 4 (*surfer* code included).

5.4 Twisting number $n = 3$

Thus we have $m = 1$, $p_3 = x^3 - 3xy^2$, $q_3 = 3x^2y - y^3$ and we get the following polynomial equation of order 10:

$$\left[-A\left(x^2+y^2\right)^2+\frac{B}{2}\left(x^3-3xy^2\right)\left(1+x^2+y^2-z^2\right)-B\left(3x^2y-y^3\right)z\right]^2 =$$

$$= \left(x^2+y^2\right)\left[-\left(\frac{A}{2}\left(1+x^2+y^2+z^2\right)-D\right)\left(x^2+y^2\right)+B\left(x^3-3xy^2-\left(3x^2y-y^3\right)z\right)\right]^2.$$

See figure 5 (*surfer* code included).

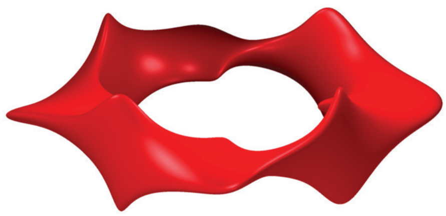
5.5 Twisting number $n = 4$

Thus we have $m = 2$, $p_4 = x^4 - 6x^2y^2 + y^4$, $q_4 = 3x^3y - 3xy^3$ and we get the following polynomial

Figure 5

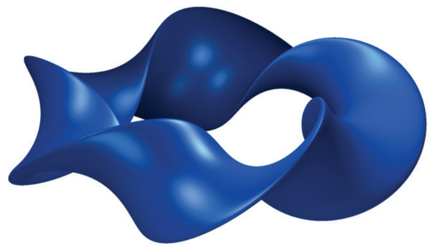
```
(-(a^2+b^2)*(x^2+y^2)^2+0.5
*(b^2-a^2)*(x^3-3xy^2)
*(1+x^2+y^2-z^2)
-(b^2-a^2)*(3x^2y-y^3)*z)^2
-(x^2+y^2)*(-(0.5*(a^2+b^2)
*(1+x^2+y^2+z^2)-a^2*b^2)
*(x^2+y^2)+(b^2-a^2)
*(x^3-3x*y^2
-(3*x^2*y-y^3)*z))^2
+c^9*0.0001
```

Figure 6



```
((0.5*(a^2+b^2)
*(1+x^2+y^2+z^2)-a^2*b^2)
*(x^2+y^2)^2+0.5
*(b^2-a^2)
*(x^4-6x^2y^2+y^4)
*(1+x^2+y^2-z^2)-3
*(b^2-a^2)
*(x^2-y^2)*x*y*z)^2
-(x^2+y^2)*((a^2+b^2)
*(x^2+y^2)^2+(b^2-a^2)
*(x^4-6*x^2*y^2+y^4-3
*(x^3*y-x*y^3)*z))^2
+c^9*0.0001
```

Figure 7



```
(-(a^2+b^2)
*(x^2+y^2)^3+0.5
*(b^2-a^2)
*(x^5-10x^3y^2+5xy^4)
*(1+x^2+y^2-z^2)-(b^2-a^2)
*(5x^4y-10x^2y^3+y^5)*z)^2
-(x^2+y^2)*(-(0.5*(a^2+b^2)
*(1+x^2+y^2+z^2)-a^2*b^2)
*(x^2+y^2)^2+(b^2-a^2)
*(x^5-10x^3y^2+5xy^4
-(5x^4y-10x^2y^3+y^5)*z))^2
+c^13*0.0001
```

equation of order 12:

$$\Big[\Big(\frac{A}{2}(1+x^2+y^2+z^2)-D\Big)(x^2+y^2)^2+ \frac{B}{2}(x^4-6x^2y^2+y^4)(1+x^2+y^2-z^2) -3B(x^3y-xy^3)z\Big]^2 = (x^2+y^2)\Big[A(x^2+y^2)^2+ B\big(x^4-6x^2y^2+y^4-3(x^3y-xy^3)z\big)\Big]^2.$$

See figure 6 (*surfer* code included).

5.6 Twisting number $n = 5$

Thus we have $m = 2$, $p_5 = x^5 - 10x^3y^2 + 5xy^4$, $q_5 = 5x^4y - 10x^2y^3 + y^5$ and we get the following polynomial equation of order 14:

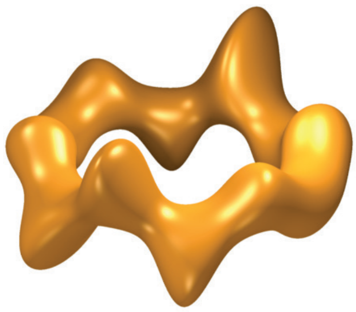
$$\Big[-A(x^2+y^2)^3+ +\frac{B}{2}(x^5-10x^3y^2+5xy^4)(1+x^2+y^2-z^2)- -Bq_nz\Big]^2 = (x^2+y^2)\Big[-\Big(\frac{A}{2}(1+x^2+y^2+z^2)-D\Big)(x^2+y^2)^2+ +B\big(x^5-10x^3y^2+5xy^4-(5x^4y-10x^2y^3+y^5)z\big)\Big]^2.$$

See figure 7 (*surfer* code included).

5.7 Twisting number $n = 6$

Thus we have $m = 3$, $p_6 = x^6 - 15x^4y^2 + 15x^2y^4 - y^6$, $q_6 = 6x^5y - 20x^3y^3 + 6xy^5$ and we get the following

Figure 8



```
((0.5*(a^2+b^2)*(1+x^2+y^2+z^2)-a^2*b^2)*(x^2+y^2)^3+0.5*(b^2-a^2)
*(x^6-15x^4y^2+15x^2y^4-y^6)*(1+x^2+y^2-z^2)-(b^2-a^2)*(6x^5y-20x^3y^3+6xy^5)
*x*y*z)^2-(x^2+y^2)*((a^2+b^2)*(x^2+y^2)^3+(b^2-a^2)*(x^6-15x^4y^2+15x^2y^4-y^6
-(6x^5y-20x^3y^3+6xy^5)*z))^2+c^15*0.0001
```

polynomial equation of order 16:

$$\Big[\Big(\frac{A}{2}(1+x^2+y^2+z^2)-D\Big)(x^2+y^2)^3+ \\ +\frac{B}{2}(x^6-15x^4y^2+15x^2y^4-y^6)(1+x^2+y^2-z^2)- \\ -B(6x^5y-20x^3y^3+6xy^5)z\Big]^2= \\ =(x^2+y^2)\Big[A(x^2+y^2)^3+B(x^6-15x^4y^2+15x^2y^4- \\ -y^6-(6x^5y-20x^3y^3+6xy^5)z)\Big]^2.$$

See figure 8 (*surfer* code included).

Note that for a twisting number larger than 6 the z-axis as a singular set is a numerically very unstable region such that a necessary correction strongly deforms the whole surface. Already for $n=6$ the deformation of the surface is quite strong.

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