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Contents

Editorial

In this issue of the bulletin, we present an article about the topological complexity of surfaces, which is a notion introduced by M. Farber in order to classify the complexity of the motion planning problem in robotics. We also include a paper regarding the representation of commutative rings and respective modules by means of the prime spectrum, which is the set of prime ideals of a ring. We present an article about small tubular neighbourhoods of the n-twisted Möbius strip described as affine varieties associated to explicit polynomials in three real variables, exhibiting illustrations produced with the software *surfer*.

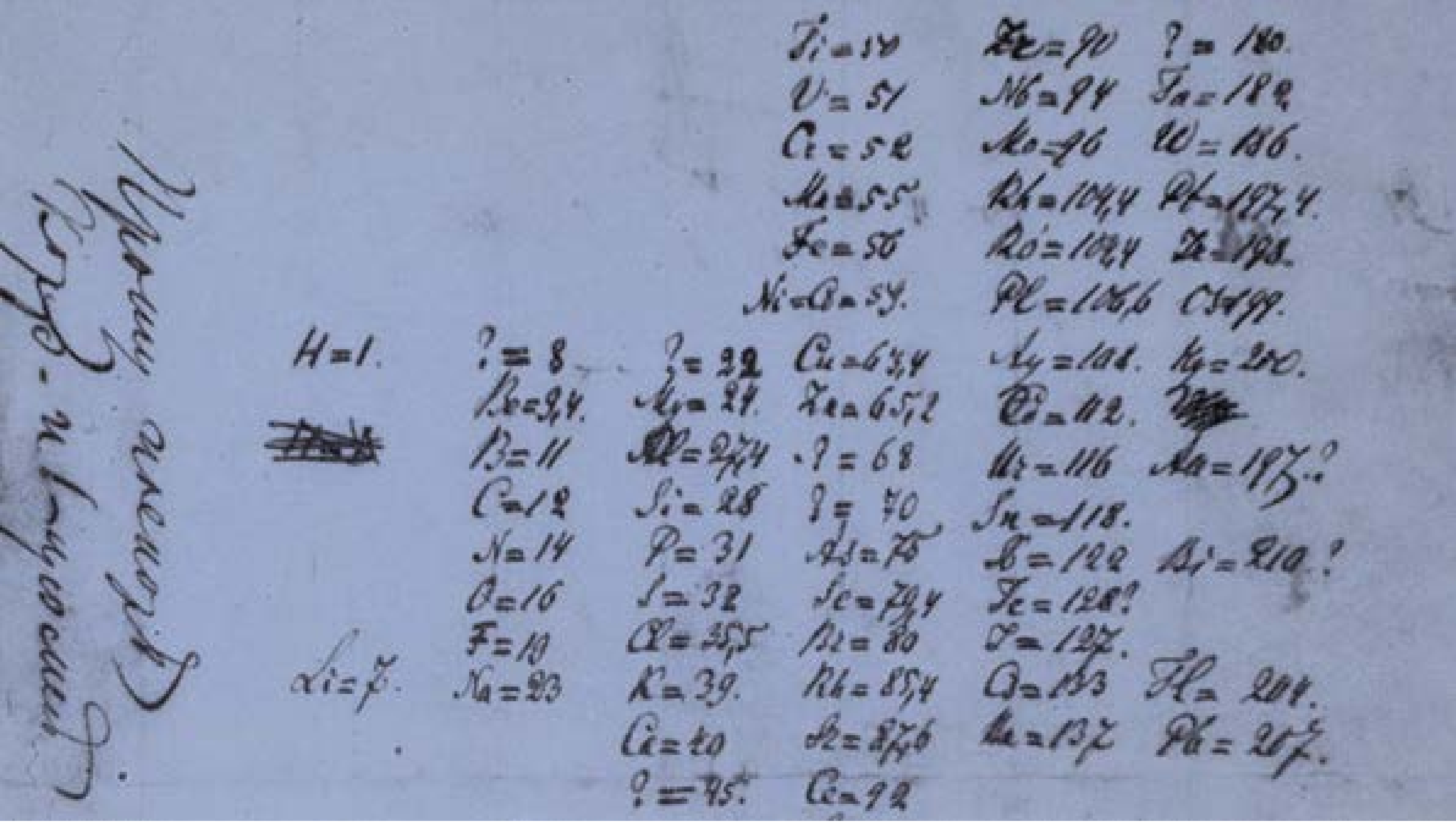
Inserted in the cycle of historical articles we feature a short biography of Luis Woodhouse, which highlights, in particular, his efforts to pursue recognition for the work and research of Portuguese mathematicians such as José Maria Dantas Pereira and João Evangelista Torriani.

We feature an interview honouring Graciano de Oliveira for his dedication to promoting mathematical research and dynamising the Portuguese scientific community.

We publish several summaries and reports of some of the activities partially supported by CIM, of which we highlight the celebration of the 150th anniversary of the Periodic Table and the second edition of the conference Global Portuguese Mathematicians that brought together Portuguese mathematicians working worldwide.

We feature an interview with Martin Hairer, who delivered a Pedro Nunes' Lecture, which is one of the most emblematic joint initiatives of CIM, in collaboration with SPM.

We recall that the bulletin welcomes the submission of review, feature, outreach and research articles in Mathematics and its applications.



A Mathematical Celebration of the 150th Anniversary of the Periodic Table

by **José Francisco Rodrigues***

In a joint initiative of the *Centro Internacional de Matemática* (CIM) and the *Instituto de Ciencias Matemáticas* (ICMAT) a simbolic mathematical celebration of the Periodic Table took place at the Academy of Sciences of Lisbon, the 21st November 2019. It consisted of four talks, by two mathematicians and two chemists: *Some mathematical aspects of the periodic table*, by José Francisco Rodrigues (CIM and FCiências/ULisboa), *The power of systematisation. The importance of precision*, by Manuel Yáñez and Otilia Mo (Universidad Autónoma de Madrid), *The periodic table: Are atoms the bricks of molecules?* by Adelino Galvão (ISTécnico/ULisboa) and *Counting lattice points and atomic energies oscillations*, by Antonio Córdoba (ICMAT and UAMadrid). They were streamed online and their record can be found at http://www.cim.pt/agenda/event/208.

The UNESCO decided to celebrate the year 2019 as the "*International Year of the Periodic Table of Chemical Elements*" — https://iypt2019.org/ — commemorating all around the world the 150th anniversary since Dmitry Mendeleev discovered the Periodic System only with 61 elements, but containing a strong prediction potential to accommodate new elements reaching nowadays 118. The Periodic Table is of central importance to chemistry and after one and half century of life it became one of the indispensable tools for science and an icon for scientific inquiry. Although until now mathematicians have little to say on the periodic table, the mathematical chemistry is an expanding interdisciplinary area. Recent works have shown the importance of the underlying mathematics of the periodic table in diverse areas such as group theory, topology, information theory and, of course, quantum mechanics. The introductory talk of the Lisbon meeting in-

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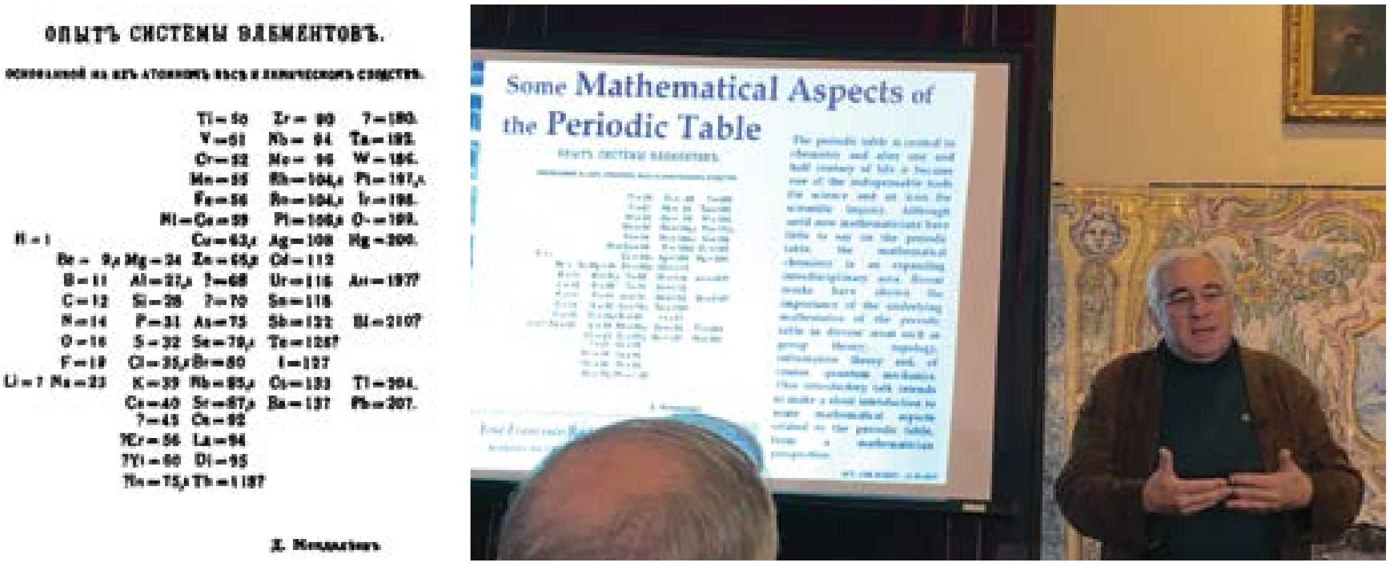


Figure 1.—The Periodic Table of Mendeleev shown in the talk by J. F. Rodrigues.

tended to make a short introduction to some mathematical aspects related to the periodic table, from the perspective of a mathematician.

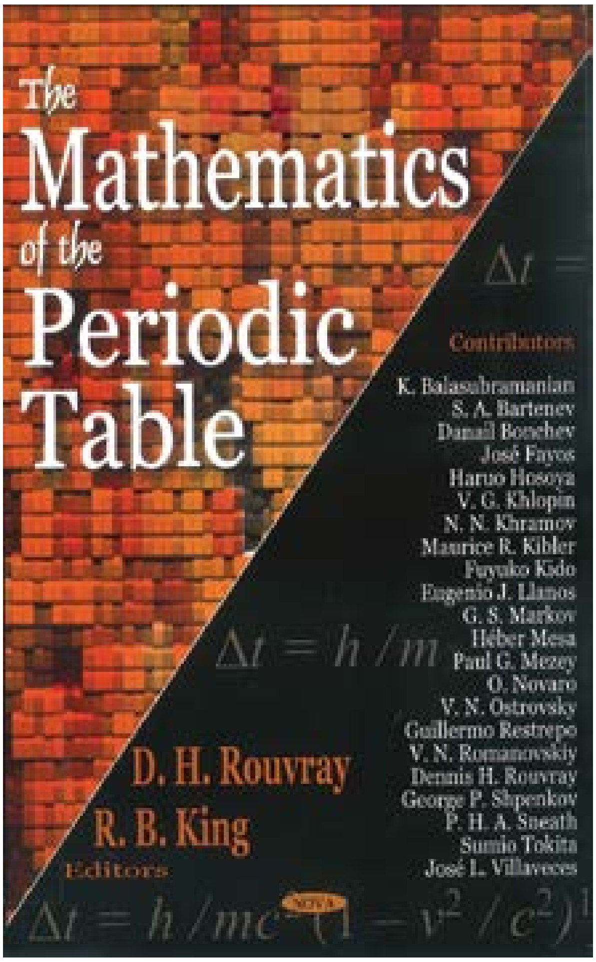
After referring that the initial idea of this mathematical meeting was originated by the reading of the article on *Matemáticas alrededor de la tabla periódica*, by Antonio Córdoba, published the 4th January 2019 in the section *Café y Teoremas* in the Spanish newspaper *El País*, and decided together at the ERCOM meeting held in March in Cambridge by the directors of CIM and ICMAT, Rodrigues has started his presentation by evoking the role of Mathematics in Quantum Mechanics, and vice versa, by displaying the equation

$$H\Psi = E\Psi$$

relating the Hamiltonian H applied to the wave function Ψ and the eigenvalue E associated with the quantic number, and referring the classical work of John von Neumann (1903–1957) on the *Foundations of Quantum Mechanics* (1932). He quoted a significant statement of 1951 by Tosio Kato: *The fundamental quality required of operators representing physical quantities in quantum mechanics is that hey be hypermaximal or self-adjoint in the strict sense employed in the theory of Hilbert space, which is equivalent to saying that the eigenvalue problem is completely solvable for them, that is, there exists a complete set (discrete or continuous) of eigenfunctions. (…) The main purpose of the present paper is to show that the Schrödinger Hamiltonian operator of every atom, molecule, or ion, in short, of every system composed of a finite number of particles interacting with each other through a potential energy, for instance of Coulomb type, is essentially self-adjoint. Thus, our result serves as a mathematical basis for all theoretical works concerning nonrelativitic quantum mechanics* [K].

Some chapters of the recent collective book *The Mathematics of the Periodic Table* [RK], which contains eclectic material on some mathematical methods and a range of ideas, concepts and different approaches applied to the study of the Periodic Table, were commented on the first talk. The first chapter, written by one of the editors, raises the interesting question of the ultimate size of the periodic table, after displaying a list of all the chemical elements discovered since the 17th century, when Phosphorus was first isolated, followed by the 19, the 51 and the 33 elements isolated, respectively, in the 18th, the 19th and the 20th century. The history of the prediction about the upper limit of the periodic table have had several variations with the discovery of the superheavy elements, and nowadays the highest is the Oganesson (Og), the 118th element. It remains a conceivable possibility that the development of new technologies will enlarge this number.

Among the other chapters of [RK], the one on a topological study of the periodic system, by defining topologies on the set of chemical elements based on similarity trees and introducing a space of physicochemical properties, has shown a robust way of classifying the elements into metals, metametals, semimetals and non-metals. Based on the Shannon's information theory applied to finite discrete sets, another chapter described a number of information indices for characterizing the electronic and nuclear structure of atoms of the chemical elements, which similarity in



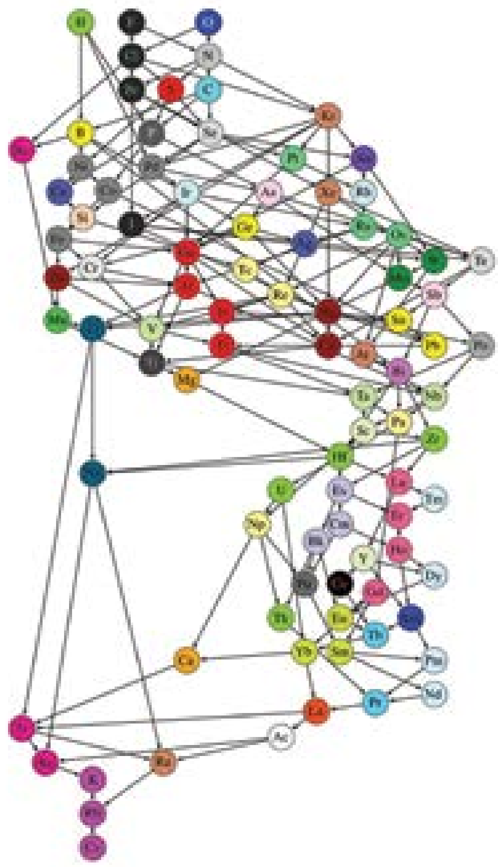


Figure 2.—The cover page of [RK] and the hypergraph of W. Leal and G. Restrepo from [LR].

their periodic trends were used by D. Bonchev to predict the binding energies of 45 unsynthesized isotopes of the elements 101 to 118. Most of them were confirmed with a high accuracy. Finally, three chapters of [RK], almost a third of the book, describe the applications of Group Theory to the Periodic Table. Its relevance is well known since the works of V.A. Fock and V. Bargman in the 1930's demonstrated that the O(4) symmetry of the hydrogen atom stems from the conservation of two constants of motion. Nowadays there are several group-theoretical approaches to the Periodic Table, but all of them are based on the fact that the problems of classification within Quantum Mechanics are closed related to symmetry questions that can be treated using Group Theory and one of them, the Elementary Particle Approach, according to the theoretical physicist V.N. Ostrovsky, "*claims to treat an element as a whole, as some non-split entity. In particular, a dynamic group of the Periodic Table implements a dream of the alchemists, namely transmutation of elements. A mere application of the dynamic group generators transforms one chemical element into another, thus implementing the ambitious goal. The remaining problem is that the chemical elements are defined as vectors in some abstract Hilbert space, and nobody knows how to connect this with physicochemical reality.*" [RK, page 305].

Recently, in another interesting mathematical description of the Periodic System based on the relations of order and similarity of chemical elements, in [LR] it is proposed an ordered hypergraph (see figure 2), where the hyperedges are similarity classes to describe the structure of the Periodic Table, which has been at the core of chemistry for more than 150 years.

The first talk ended with a completely different view on the Periodic Table by the mathematician and musician Tom Lehrer, who lectured mathematics and music theater at the University of Califormia, Santa Cruz, and sang "*The Elements*" [L], in which he set names of the chemical elements to the tune of a music by Gilbert and Sullivan.

Manuel Yáñez started his presentation on the power of systematisation and the importance of precision, stating clearly that the revolution of Mendeleev Table was, one

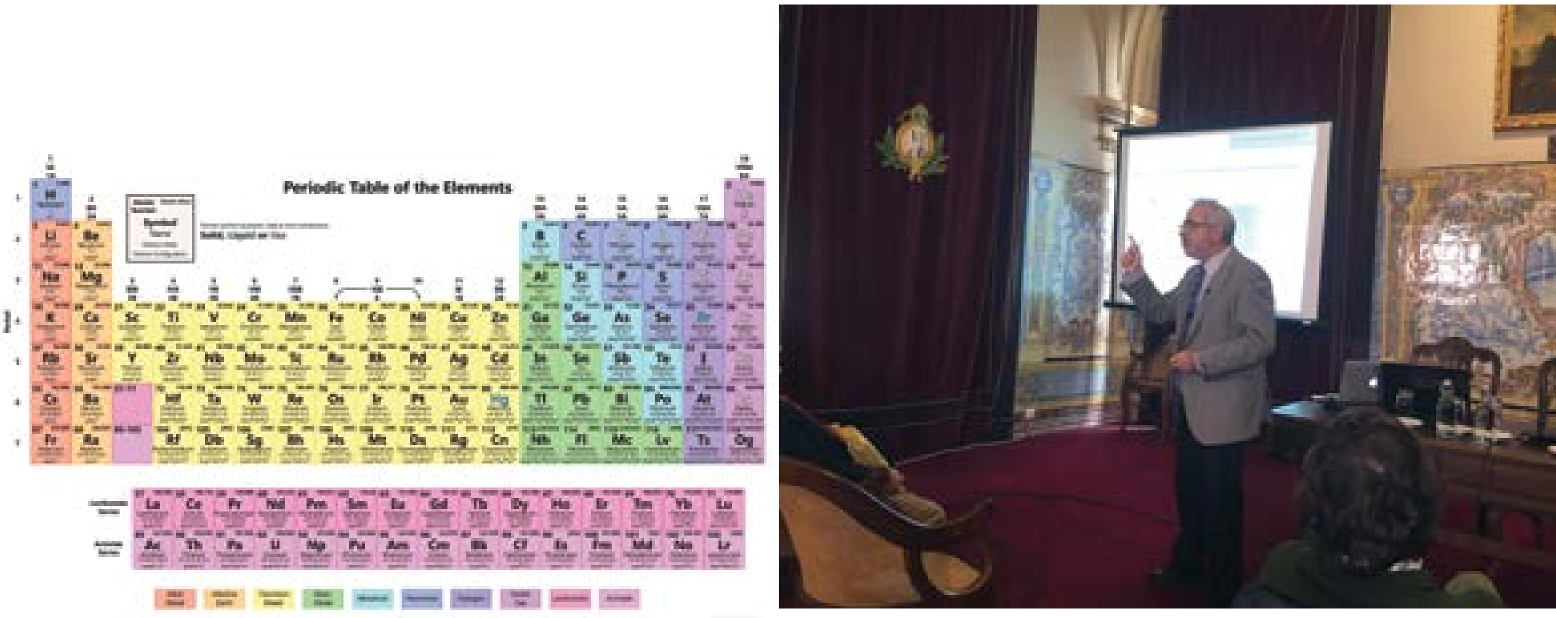


Figure 3—The contemporary Periodic Table and the presentation of Manuel Yáñez.

hundred and fifty years ago, "*the first important systematization in the realm of chemistry was done, ordering the elements in terms of its atomic mass. This first attempt was crucial even though not totally correct. A better knowledge of the atomic structure improved this initial systematization in terms of the atomic number; but a real understanding of the periodicity in the atomic properties was possible only when the mathematical functions describing the electrons within an atom were obtained*" [MY]. In fact, the initial criterion (atomic mass) was pure empiricism but the elements appeared grouped in families with common chemical properties, a question that had no answer at that moment. In the words of Niels Bohr: The Periodic Table was the guiding star for the exploration of the fields of chemistry and physics. But the ordering was not completely satisfactory. For instance, tellurium-iodine, argon-potassium and cobalt-nickel couples should be located in the reversed order. But some predictions were totally correct: for instance, when isolated and characterised, Eka-Boron was renamed Scandium (Sc), Eka-Aluminum turned into Gallium (Ga), Eka-Manganesse into Technetium (Tc), and Eka-silicon into Germanium (Ge). The final version of the table was achieved thanks to the periodic law presented by the British Henry Moseley at the beginning of the 20th century. Moseley verified that when representing the square root of the radiation frequency as a function of the order number in the periodic system, a straight line was obtained, a reflection of some property of the atomic structure. This property is described by an affine equation relating the square root of the frequency and the atomic number (Z) or number of positive charges of the nucleus. But still it was not evident which were the real reasons for this periodicity!

With an interesting quotation of the Belgian scientist A. Quetelet (1796-1874), "*The more the physical sciences progress, the more they tend to enter the domain of mathematics, which is like a center towards which everything converges. We can judge the degree of perfection achieved by a science by the ease with which it can be subjected to calculation*", Manuel Yáñez observed that the first clear explanation came from Mathematics starting with quantization in 1925, with $H\Psi = E\Psi$, after Schrödinger established and solved the partial derivatives equation that now has his name and was able to accurately describe the electronic properties and the spectra of hydrogen-like systems. But there is always a but — this equation cannot be easily solved for more than 1 electron, raising difficult questions of efficient approximations and, despite many impressive progresses, even today the placement of the elements has not settled after 150 years, starting from the very first element. Nevertheless, the use of the variational principle looking for the minimal energy of an atomic and molecular system was the engine behind this understanding, though soon it became also clear that, in some specific cases, second order properties could be not adequately described even if the precision got for the ener-

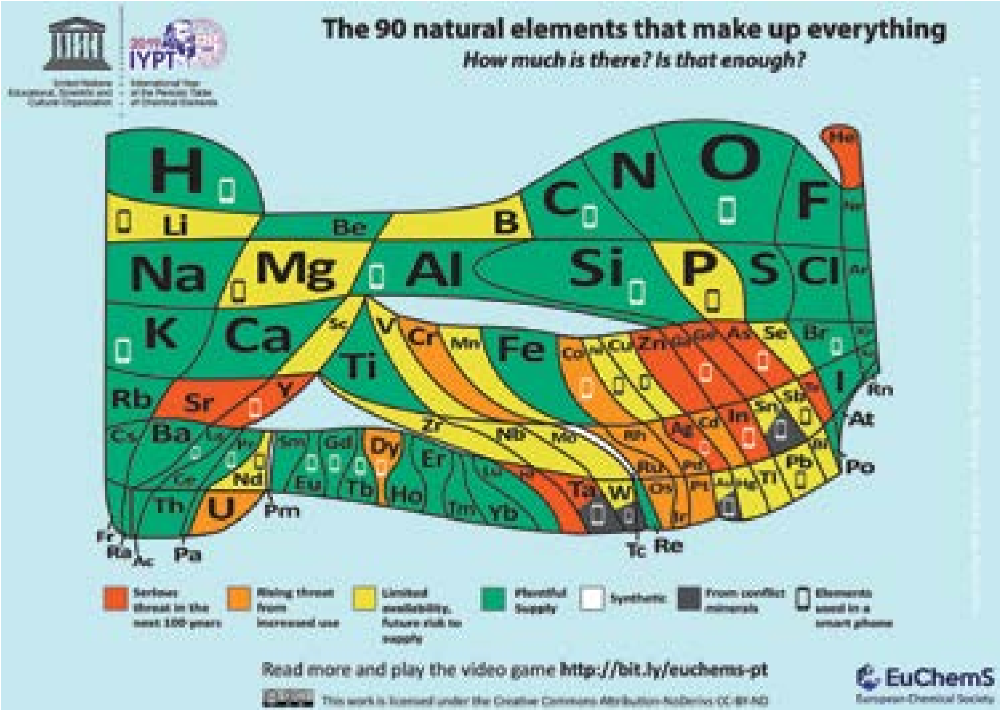


Figure 4—The visualisation of the 90 natural elements in this table is based in the proportion between the area occupied by each element and its approximate amount existing in the earth's crust and atmosphere.

gy was large. Magnetic properties are a good example, or the singularity of the elements of the first row of the periodic table with respect to the others within the same group. Then he presented recent results that show that the Periodic Table is still a living object with many surprises yet to be unravelled.

For example, after the inclusion in the theory of relativistic effects in the 1970's, Yáñez described the great difference between the nonrelativistic and the relativistic effects on the properties of gold. He concluded his presentation by showing the colourful table of the European Chemical Society showing an impressive visualisation of the 90 natural elements, that make everything, which is based in the proportion between the area occupied by each element and its approximate amount existing in the earth's crust and atmosphere, in particular referring the elements that are used in a smartphone.

Adelino Galvão proposed to answer the question he had formulated in the title of his talk — *Are atoms the bricks of molecules?* — from the chemists' point of view as architects of the electronic cloud to give it shape and function. Considering molecules as "Many-Body" systems with N nuclei and n electrons, under the Born-Oppenheimer approximation nuclei are mere artifacts to provide the external potential that holds and shapes the electronic cloud. As explained in his talk, atomic nuclei are bare necessities (like rebars in concrete) to provide the external potential that hold and shape electrons within the molecule, but otherwise useless in what concerns its chemical properties. Those are defined, in the Quantum formalism, by a statistical distribution function that it is believed to be expressed by a linear expansion in a basis of pre-existent atomic centered distribution functions. Time has proved that this approach must be completed with extra polarization and diffuse functions to get results within experimental accuracy.

The shape of the wave function resulting from the overlap of so many different basis functions has no resemblance with the original atoms that made the molecule. For instance, in the C-O molecule how can one divide between C and O the charge density in the middle of the bond? Then, Galvão observed that the traditional chemical concepts like bonds, rings, nonbonding pairs, electrophilic, nucleophilic, electronegativity, functional group, etc. are lost in the wave function formalism. Those can be recovered by proving that the wave function uniquely defines the charge density (and vice-versa) and both are uniquely defined by the external potential. In his exposition, Galvão has shown, by topological analysis of the charge density, how to recover the traditional chemical language, such as atoms, in a quantum formalism and precisely define what is an atom in a molecule. Most chemical properties can be defined by the critical points in the charge density maps (zero gradient) and the corresponding eigenvalues of the hessian: 3 negative eigenvalues (maxima) are atomic posi-

Figure 5.—Adelino Galvão explaining one of the slides of his presentation.

tions; two negative ones (saddle points) are bonds, and two positive ones are rings. The gradient relief maps provide dimension to atomic positions by means of its zero-flow surface under the Gauss theorem. Reactivity insight is given by the topology of the Laplacian whose signal maps areas of charge depletion (favourable to nucleophilic attacks) or charge concentration (susceptible to electrophilic attacks). As an example, the lone pairs of water are nothing more than a volume where the Laplacian is extremely negative. The eigenvectors of the hessian at these locations provide the orientation of the lone pairs. To finalize he has shown that moving from meaningless (in the physical sense) statistical distributions to charge densities reconciles classic and quantum chemistry and enables direct comparison with experimental quantities obtainable by diffraction techniques.

The initial success of Quantum Mechanics to understand the hydrogen atom raised the natural question of studying the larger atoms. Numerous problems were encountered and it has generated relevant mathematical research for simplified quantum atomic models, namely, semi classical asymptotics, field theories, potential theory, computational issues and analytic number theory. Successful developments were made in the framework of the Thomas-Fermi theory.

In his talk on *Counting lattice points and atomic energies oscillations*, Antonio Córdoba briefly reviewed one aspect of an ambitious plan to explain the periodic table from first principles of Quantum Mechanics, by reporting collaboration with Charles Fefferman (Princeton University) and Luis Seco (University of Toronto), in their joint work [CFT]. Although not a very accuracy determination of the ground state energies for larger atoms can be achieved today, there are nevertheless some interesting results that can be obtained involving almost periodic trigonometric sums which are reminiscent of those appearing in Number Theory for counting Lattice Points inside circles or spheres.

After introducing the ground state energy of a neutral atom $E(\mathcal{Z})$ with nuclear charge $\mathcal{Z}$, which is represented by the standard Schrödinger Hamiltonian with Coulomb potentials, and the semiclassical approximation, Córdoba described rigorous results for the expansion

$$E(\mathcal{Z}) = C_{TF}\mathcal{Z}^{7/3} + C_S\mathcal{Z}^2 + C_{SD}\mathcal{Z}^{5/3} + \Psi_Q(\mathcal{Z}) + \cdots.$$

The leading term on $\mathcal{Z}^{7/3}$ was conjectured by Thomas and Fermi in 1927 and proved rigorously by E. H. Lieb and B. Simon in 1977. The middle term in $\mathcal{Z}^2$ was explained in 1952 by J.M.C. Scott, through numerical computations, and proved in the works of K. H. Siedentop and R. Weikard in 1989 and W. Hughes in 1990. Finally, the second correction in $\mathcal{Z}^{5/3}$ was conjectured by P. Dirac in 1930 and proposed by J. Schinger in 1981, but was proved

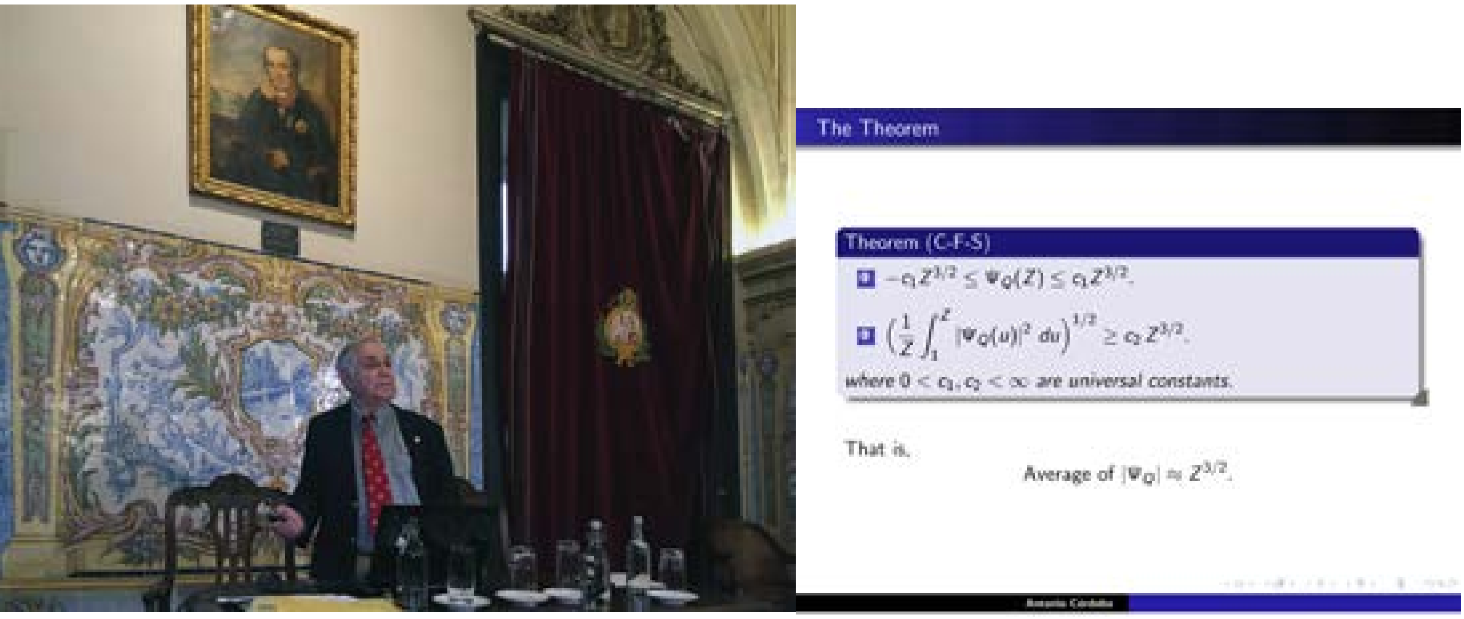


Figure 6.—Antonio Córdoba describing the main result on the fourth term of the ground state energy of a neutral atom in terms of its nuclear charge.

in 1990–94 in a series of papers by C. Fefferman and L. Seco (see [C] for references). The question of higher order expansions is still under investigation, but Córdoba presented a theorem of [CFT] on the next term, which is oscillatory, by stating that $|\Psi_Q(\mathscr{Z})| \le c_1 \mathscr{Z}^{3/2}$, where c_1 is a universal constant, and the average of $|\Psi_Q| \sim \mathscr{Z}^{3/2}$. This problem is strictly related to the basic number-theoretic problem of estimating the number of lattice points inside a domain enclosed by a given curve. That theorem of Córdoba-Fefferman-Seco is a relevant estimate to the bound of the size $\mathscr{Z}^{\gamma}$ with $\gamma < 3/2$ for the Thomas-Fermi density.

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REPORT

Nonlinear PDEs in Braga

by **Fernando Miranda*** and **Lisa Santos***

The workshop **Nonlinear PDEs in Braga** was a joint organization of the following Portuguese research centers of mathematics: CMAT (University of Minho), CMAFcIO (University of Lisbon) and CMUC (University of Coimbra), and also counted with the support of CIM and FACC-FCT.

This meeting was devoted to the analysis of various types of nonlinear PDEs (elliptic, parabolic, hyperbolic, dispersive, etc.) and applications. The scientific level of the workshop was very high, with the participation of top main speakers.

The event took place in the historical center of Braga, in the Congregados Building, which belongs to the University of Minho and was a former Convent.

The meeting had 40 participants from 9 countries and the scientific program counted with 20 lectures of which 12 were plenary and with a period for discussion.

More information is available in
http://w3.math.uminho.pt/NLPDEsBraga/

* Centro de Matemática, Universidade do Minho

AN INTERVIEW WITH

Graciano de Oliveira

by **Ana Paula Santana***

Graciano Neves de Oliveira was born on the 7th of May 1938, in Cabanas de Viriato, Portugal. He graduated in Mathematics at the University of Coimbra in 1961, and in 1969 he obtained his doctorate at the University of Coimbra with the thesis *On Stochastic and Double Stochastic Matrices*. Due to his political ideas, he had problems in keeping a job at Portuguese universities, until in 1976 he became a professor at the University of Coimbra. He retired from this university in 2002.

Graciano de Oliveira's main research work is in matrix theory and multilinear algebra, an area in which he published extensively and produced several seminal articles. He supervised 11 Ph. D. students and was the leader of the Coimbra school of matrix theory.

Graciano de Oliveira was President of the Portuguese Mathematical Society from 1986 to 1988 and from 1996 to 2000, and Vice-President of the International Linear Algebra Society from 1993 to 1995. He also served on the Editorial Boards of *Linear Algebra and its Applications* and *Portugaliae Mathematica*.

* CMUC and Department of Mathematics FCTUC

I start with three classical questions: When you were a child, did you show a special aptitude for Mathematics? In high school were you a good student in general? Was Maths your favourite subject?

Yes. I remember when I was at high school (Liceu), I mean when I was 11 or 12, I clearly distinguished Mathematics from the other courses. I felt as if what I had to learn was divided into two sets: one containing Mathematics only, the other the rest. In general I was a reasonably good student. But only in the last two years of the Liceu (after opting for Science) and at the University did I start getting very good grades in tests and examinations.

If you were not a mathematician, what do you think you might have been in your professional life?

A very difficult question. I do not know. I know that I preferred Science to Arts and that I was not good in sports. One or two years before entering the University I was hesitating very much mainly due to lack of information. There was no internet and my parents knew nothing about what a University could offer. However I felt I would like to do research in Science. Something theoretical with no laboratories.

How was your experience as a student at the University of Coimbra?

Good in general. Scientifically it was not so good. The curricula were, in general, very old fashioned. But in order to get a pass with good mark it was necessary to work very hard.

Were you already interested in politics?

No. At that time I did not understand what politics meant. I had always lived under a dictatorship, politics was something we did not discuss. I would say that for many people, politics was something that did not exist. However in the last 2 years or so at the University, I started to open my eyes. The beginning of the colonial war was crucial. I had lived in Angola when I was a child . . .

When you graduated you became a university assistant. I guess that was a very natural thing to do?

Yes. All the good students were recruited as University assistants. Except if they had a negative report from the political police or, in some Departments, if they were female.

Did your interest in Matrix Theory start at that time?

It started in my first year as University assistant. I wanted to study something and I was fascinated by research activities but I did not know how to start. It was Professor Luís Albuquerque who suggested some papers on Matrix Theory to start with. It is an interesting curiosity that Professor Albuquerque is better known as a historian rather than as a mathematician.

But then you were conscripted into the army and went to Mozambique.

That's right. And I spent more than three years in the Army, two of which were spent in Mozambique as a lieutenant. It was a great setback to my academic career and an important contribution to my understanding of politics including colonialism. I did not like being a soldier. But today I think military service should be mandatory even though unpleasant. Without it, the 1974 democratic and anti-colonialist revolution would not have happened as it did.

After the army you returned to the University of Coimbra. But you did not stay long. In 1966 you went to Oxford to study Mathematics, but things did not work out as planned . . .

I returned in August 1966 and in September I went to Oxford. Things did not work out as planned indeed. I learnt some Group Theory and Ring Theory but there was nobody working in what I was interested in. Moreover, at that time, a

Ph. D. obtained in another country would not be recognized in Portugal. Therefore it was not worth getting a Ph. D. in Oxford as I would have to get it again in Portugal and without it, it was not possible to continue an academic career. However while I was in Oxford I wrote what, in my opinion, is my best paper. This paper would lead, years later, to the Sá-Thompson interlacing inequalities. My paper was inspired on a paper by Leon Mirsky. He worked in Sheffield, we exchanged a few letters but I never met him. I had written to him when I decided to go to the United Kingdom and I asked him to be my thesis advisor. He said he could not accept because he was too busy. Bad luck for me. But his work had a great influence on me.

In 1969 you obtained your Ph.D. in Coimbra with your work on *Stochastic and double stochastic matrices*. But then you wrote a document criticizing the organization of scientific research in Portugal. This was not very well received at the university.

That's right. In 1969, June or so, the Minister of Education sent out a note asking for suggestions for a reform that was under preparation. I sent in my opinion. The Faculty of Science and Technology did not like it. Thus instead of forwarding it to the Minister of Education they preferred to sack me. In fact I was very critical and my opinion was considered offensive.

Following that you spent several years in Lisbon and Coimbra as a researcher with a grant from the Calouste Gulbenkian Foundation. Then you went to Brazil for a short period. Mathematically speaking that must have been a very interesting time. You published a lot both in national and international journals.

I was unemployed for a while. Then the Gulbenkian Foundation gave me a grant so that I could work at the Centro de Cálculo Científico in Lisbon.

Professor Ruy Luís Gomes (*émigré* in Brazil for political reasons) learnt of my situation and offered me a position in a Brazilian University in Recife. Since I was not happy with computing, I accepted. The idea was to move for good to Brazil with the whole family (I had 4 children the eldest of them being 7 years old). But since I had never been to Brazil, I and my wife thought it would be wiser going alone first to see how things were in Recife. There is another detail which is interesting to explain. Before I was sacked in Coimbra, I had been admitted as candidate to Associate Professor. I should have passed a sort of examination (public discussion of my CV, etc.) in order to be promoted. As a consequence of being sacked this was frozen. I think that was illegal but at that time the law did not prevail, we lived under a dictatorship! I went to Recife in January 1971. In April I received a letter from Coimbra informing me that if I returned, the examination for promotion would be unfrozen. A few months after returning I realized that the information in the letter was a lie. Unemployed again, it was the Gulbenkian Foundation that granted me a scholarship. No University in Portugal wanted me. Professor Sebastião e Silva tried to bring me to the University of Lisbon but did not manage as this might be considered as disrespect to the University that had sacked me!

Years later, in 1972, what happened in 1969 was falling into oblivion and in September I was employed at the University of Lisbon thanks to Professor Almeida Costa.

It was at that time that you started getting interested in problems in multilinear algebra. Your lectures at the Coimbra Ateneu, 1971-72, are considered the beginning of research in Portugal in that area.

I spent the year 1971/72 in Coimbra. Mainly at home thanks to a grant from the Gulbenkian Foundation.

As far as I can remember I worked mainly in Multilinear Algebra, wrote a monograph and I read many books not related to Mathematics. That was when I met J. Dias da Silva and Marques de Sá. I was not welcome at the University but I wanted to hold some seminars. They took place in the Ateneu de Coimbra. Dias da Silva, Marques de Sá and Alice Inácio attended regularly these seminars. They were still undergraduate students.

J. Dias da Silva and Marques de Sá became the first two of your 11 Ph.D. students. The usual practice at the time was to send university assistants abroad to prepare their Ph.D.'s. How did the decision of supervising your own Ph.D. students arise?

Dias da Silva was the first one to get a Ph. D. under my guidance. At that time practically anyone wishing to get a Ph. D. had to go abroad. I thought this was unacceptable and that our Professors and our Universities should be able to provide guidance to young students. There was very little tradition of research among us. I thought I should try to reverse the situation.

And you did manage to do it! Both of them became national and international recognized experts in their areas of work. You must be very proud of your descendants.

Yes, both of them produced important research work. Marques de Sá received the Householder prize. This was important as it showed that it was possible to produce high quality research that was internationally recognized. Dias da Silva went to the University of Lisbon where he created a school of Multilinear Algebra and Combinatorics.

In 1976, after the April 1974 revolution, you returned to the University of Coimbra. You organized a regular Algebra Seminar and regular visits of foreign mathematicians in the area of Matrix Theory to Coimbra.

That is right, after the 1974 revolution I could return to Coimbra and start regular seminars and we had a large number of foreign visitors. This was, at that time, not very usual. I was so lucky to have a lot of very good collaborators and that their work started to be internationally known.

In the academic year 1978-79 I was a student of yours at the Algebra I and Algebra II courses. Your teaching

assistants were J.F. Queiró and A. Leal Duarte. How I enjoyed those classes! Our exams used to last the whole afternoon! Do you remember that?

I also still remember very well when you were a talented undergraduate student of mine.

Thank you for your compliment!

At that time, shortly after the democratic revolution, many of us thought that students should not be stressed in the examinations for lack of time. And that they should be given an unlimited amount of time in exams. This turned out to be impossible.

At that time a second wave of students of yours was in full bloom: Maria Emília Miranda, Natália Bebiano, A. Leal Duarte, Ion Zaballa, . . . You were in constant activity and I heard your enthusiasm described as infectious . . .

In fact I do not know how my enthusiasm was described!

The constant questioning, love for conversation, for argument, even for provocation are strong features of yours.

Are they? Do you think I have these features? It makes me happy that you think that way.

Amongst your many papers, do you have a favorite one?

I think my best papers are the three papers on *Matrices with Prescribed Characteristic Polynomial and a Prescribed Submatrix*. Another one I like is a joint paper with Dias da Silva on *Conditions for Equality of Decomposable Symmetric Tensors*.

The famous Marcus-Oliveira conjecture — on the determinant of the sum of two normal matrices — keeps eluding those who work on it. It is amazing that a problem so easy to state seems to be so hard to prove.

About 35 years ago (I do not remember the exact date) I suspected that the determinant of the sum of two normal matrices with given eigenvalues should lie in a very simple region defined using those eigenvalues. As far as I can remember, I had this idea when I was reading a paper by M. Fiedler. I was unable to prove it in spite of all my efforts. So I publicized it as a conjecture. This triggered a number of papers with partial results. When I was in Macau, Emília Miranda sent me a paper by Marvin Marcus where this conjecture was stated. Marcus in fact was the first one to put forward this conjecture. Now it seems that some authors call it Marcus-Oliveira conjecture. To my knowledge it remains unsolved. It is frequent that simple statements lead to very hard proofs.

From 1989 to 1992 you were at the University of Macau. How was that experience?

In Macau I had a very happy time. It was a good chance to know many countries in the Far East where I had never gone. I could meet people and make friends from many nationalities. The cuisine in Macau was excellent. And I had the opportunity to deliver many talks on Mathematics in China and to establish mathematical connections. I organized the first Mathematical Olympiads in Macau as well as the first international participation. This gave me a great pleasure.

I have very good recollections of my experience in Macau.

Throughout your adult life you never stayed long in one place. Do you not like to get too used to people and places? Why are you so restless?

Do you think I am restless? In fact in my adolescence my parents lived in several places including Angola. I was used to moving about . . .

The Portuguese Mathematical Society, SPM, has been one of your great passions. You were president of SPM twice, and held several other posts through the years.

I always thought that the Portuguese Mathematical Society could play an important role in the development of mathematical activity. I was President twice and Secretary General once (later on, Secretary General was changed to President). Recently the Portuguese Mathematical Society honored me with the title of Honorary Member. I am very happy with that.

What, in your opinion, should be done to improve Maths literacy in our country?

What impresses me most is the beauty (and the mysteries) of Mathematics, not so much its applications. These come in second place. This is my point of view, I believe many people do not think this way. Nowadays my point of view is not very popular. However I think that in making propaganda of Mathematics, the beauty aspects should be emphasized. Many colleagues are already making efforts to popularize Mathematics among the youngsters. The Portuguese Mathematical Society has made, and will make I believe, an important contribution.

In 2003 you retired, and although you kept teaching, you stopped your scientific research. Was that a decision on your part, or it just happened?

I retired from Coimbra in 2002. This was my decision. Stopping research was not my decision, it took place due to the ageing process. And because Mathematics is a very difficult subject that needs deep concentration. After 2002, I taught in a private University and read Mathematics and published 2 or 3 papers. They were expository papers on subjects that I liked very much: noncommutative fields and p-adic fields. I did not have any original idea. I did this for pleasure only. I dislike the prevailing opinion of putting too much pressure on researchers to publish numberless papers. A result of this is that a high percentage of papers are trash, something close to pollution. This never happened in the past. It is difficult to write deep papers, they may need years to be thought out of the box. The pleasure of learning and of doing research is diminishing because of this.

Mathematicians tend to be very narrow minded in the sense that they specialize too much. This is imposed upon them by the urge to publish. If they learn too much Mathematics they are left behind. Acquiring a broad knowledge implies fewer papers . . . but what counts is how many . . .

Going back to your question on improving Mathematics literacy, I think students should not be told that understanding Mathematics is not a demanding job. The main reason is that it is not true and I see no virtue in telling lies. Understanding Mathematics needs many hours of work and concentration. Today there is an obsession for meaningless rankings. This is negative and hinders deep thought. In many cases instead of a linear ranking I think that a partially ordered set would be more appropriate. For example in the worldwide ranking of Universities: it is not difficult to find two Universities where it is unrealistic to say that one is better than the other. However a ranking shows that A is a little above B. Does it mean anything? No, with a very high probability. But what matters is that there are many people believing (or pretending!) that A is above B. There are practical consequences . . .

Since Mathematics is very difficult and beyond my skills at this age, I keep reading other things. There are many other interesting subjects to explore . . . .

Thank you for your words. It was a pleasure to have this talk with you.

Topological complexity of surfaces

by **Lucile Vandembroucq***

The notion of topological complexity of a space has been introduced by M. Farber in [F03] in order to give a topological measure of the complexity of the motion planning problem in robotics. Given a mechanical system, this problem consists of constructing an algorithm telling how to move from any initial state to any final state. If X is the space of all the possible states, which is called the configuration space of the system, then such an algorithm takes as input pairs of configurations $(A, B) \in X \times X$ and produces a continuous path $\gamma\colon\ I = [0, 1] \to X$ from the initial configuration $A = \gamma(0)$ to the terminal configuration $B = \gamma(1)$. In other words, a motion planning algorithm corresponds to a section $s : X \times X \to X^I$ of the evaluation map

$$\mathrm{ev} : X^I \to X \times X, \quad \gamma \mapsto (\gamma(0), \gamma(1))$$

where X^I is the space of continuous paths $\gamma\colon I = [0, 1] \to X$ (equipped with the compact-open topology). Such a section always exists when X is path-connected but is not continuous in general. For instance, for the circle $X = S^1 \subset \mathbb{R}^2$, one can obtain $s : X \times X \to X^I$ by defining $s(A, B)$ as the shortest path from A to B when A and B are not diametrically opposed ($A \neq -B$) and as the shortest path in the counterclockwise direction when they are diametrically opposed ($A \neq -B$). So defined, the function s is continuous on each piece but is not globally continuous. Actually, given a space X, it is easy to see that there exists a globally continuous section $s\colon X \times X \to X^I$ if and only if X is contractible, i.e., continuously deformable to a point (see Theorem 2 below). This means that we will need at least 2 continuous rules, or more precisely 2 continuous local sections of the evaluation map, to describe a complete motion planning algorithm on a non-contractible space. Roughly speaking, the topological complexity is an invariant which gives a lower bound for the number of continuous rules needed to describe such a complete algorithm. General references on this topic include [F03], [F08].

In this note, after some generalities about this invariant, we will survey the determination of the topological complexity when X is a (connected closed) surface, which has been initiated in [F03] and [FTY03] and recently completed in [D16] and [CV17].

1 Topological complexity

Let X be the configuration space of a mechanical system. In general, such a space (or some of its path components) can be identified to a nice topological space like a manifold, a polyhedron... For instance, the circle $X = S^1$ corresponds to the configuration space of a planar robotic arm revolving about one fixed extremity, the torus $X = S^1 \times S^1$ can be interpreted as the configuration space of an articulated arm with two bars, the projective plane $X = \mathbb{R}P^2$ corresponds to all the positions of a bar rotating about its center... Universality theorems (see [JS01], [KM02]) assert that any reasonable topological space (e.g. any smooth compact connected manifold) can be seen as (a path component of) the configuration space of a mechanical system. Examples of mechanisms having some surface as configuration space are for instance given in [JS01] and [H07].

For a general (non-empty) path-connected topological space X, Farber formalized the notion of topological complexity as follows. Note that we here consider the normalized version of this concept in the sense that the topological complexity of a point will be 0 (instead of 1 in the original definition of [F03]).

Definition 1.— The topological complexity of X, $\mathrm{TC}(X)$, is the least integer k such that there exists a cover of $X \times X$ by $k + 1$ open sets $U_0, \ldots, U_k$ on each of which the evaluation map

$$\mathrm{ev} : \mathrm{X}^{\mathrm{I}} \to \mathrm{X} \times \mathrm{X}, \quad \mapsto ((0), (1))$$

admits a local continuous section, that is, for any i, there exists a continuous map $s_i : U_i \to X^I$ satisfying $\mathrm{ev} \circ \mathrm{s_i} = \mathrm{id}$.

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Given such an open cover with $k+1$ local sections, we can construct a global non continuous section s of ev by (for instance) setting $s = s_i$ on $F_i = U_i \setminus (U_0 \cup \cdots \cup U_{i-1})$. This is well-defined since the sets F_i give a partition of $X \times X$. Actually, in [F04], Farber has shown that, if X is a manifold or a polyhedron, we can equivalently define $\mathrm{TC}(X)$ to be the least integer k such that there is a partition of $X \times X$ by $k+1$ sets F_i which are required to be ENR (Euclidian Neighborhood Retract) and equipped with local continuous sections of ev. The extra conditions on X and on the sets F_i are not really restrictive for practical purposes, however, the definition given here in terms of open cover is more convenient for the study of the topological properties of TC such as its invariance.

Recall that two spaces X and Y are homotopically equivalent ($X \simeq Y$) if there exist two (continuous) maps $f : X \to Y$ and $g : Y \to X$ such that the composites $g \circ f$ and $f \circ g$ are homotopic to the identity, that is, there exist homotopies $H : X \times I \to X$, $G : Y \times I \to Y$ satisfying $H(x, 0) = g \circ f(x)$, $H(x, 1) = x$, $G(y, 0) = f \circ g(y)$, and $G(y, 1) = y$. If $\mathrm{TC}(Y) \leq k$ and $V_0, \ldots, V_k$ is an open cover of $Y \times Y$ with local sections $\sigma_0, \ldots, \sigma_k$ of ev, then the $(k+1)$ sets $U_i = f^{-1}(V_i)$ form an open cover of $X \times X$ and, for each i, the continuous map s_i given by

$$s_i(A, B)(t) = \begin{cases} H(A, 1-3t) & 0 \leq t \leq 1/3 \\ g(\sigma_i(f(A), f(B))(3t-1)) & 1/3 \leq t \leq 2/3 \\ H(B, 3t-2) & 2/3 \leq t \leq 1 \end{cases}$$

takes $(A, B) \in U_i$ to a path in X from A to B and is hence a local section of ev on U_i. Consequently, $\mathrm{TC}(X) \leq k$. This shows that $\mathrm{TC}(X) \leq \mathrm{TC}(Y)$ whenever $X \simeq Y$. Similarly, we can see that $\mathrm{TC}(Y) \leq \mathrm{TC}(X)$ and conclude to the invariance of TC:

Theorem 2 ([F03]).— *The topological complexity has the following fundamental properties:*

- *TC is a homotopy invariant: if $X \simeq Y$ then $\mathrm{TC}(X) = \mathrm{TC}(Y)$.*
- *$\mathrm{TC}(X) = 0$ if and only if X is contractible (that is, X is homotopically equivalent to a point).*

In order to understand the second statement, observe that if $\mathrm{TC}(X) = 0$, then there is a global continuous section s of ev. Then, fixing a point $* \in X$, the map $(x, t) \mapsto s(x, *)(t)$ gives a homotopy between the identity and the inclusion $* \hookrightarrow X$. In other words, X is contractible. The other direction follows directly from the invariance and from the fact that the topological complexity of a point is 0.

Since a sphere is not contractible we always have $\mathrm{TC}(S^n) \geq 1$. Actually we have

Theorem 3 ([F03]).— *$\mathrm{TC}(S^n) = 1$ if n is odd and $\mathrm{TC}(S^n) = 2$ if n is even.*

We will see at the end of this section why $\mathrm{TC}(S^n)$ must be greater than 1 when n is even. The fact that $\mathrm{TC}(S^n) \leq 1$ for $n = 2k-1$ odd can be seen by adapting the construction given for S^1 in the introduction. Fix a nowhere zero continuous tangent vector field V on $S^{2k-1} \subset \mathbb{R}^{2k}$ (for instance $V(x, y) = (-y, x), x, y \in \mathbb{R}^k$) and consider the open cover of $S^{2k-1} \times S^{2k-1}$ given by $U_0 = \{(A, B) \mid A \neq -B\}$ and $U_1 = \{(A, B) \mid A \neq B\}$. Then define $s_0(A, B)$ to be the shortest path from A to B (with constant speed) and $s_1(A, B)$ to be the shortest path from A to $-B$ followed by the meridian from $-B$ to B in the direction of the (non-zero) tangent vector $V(-B)$. This construction can be adapted to the even dimensional case but with an additional open set (and local section) since any continuous tangent vector field vanishes on an even-dimensional sphere. Alternatively, one can deduce the general inequality $\mathrm{TC}(S^n) \leq 2$ from the link between the topological complexity and the classical Lusternik Schnirelmann category (which was introduced in the thirties in order to give a lower bound for the number of critical points of a differential map defined on a manifold, see [CLOT03] as a general reference).

Definition 4.— *The LS category of a space Y, $\mathrm{cat}(Y)$, is the least integer m such that Y can be covered by $m+1$ open sets U_i which are contractible in Y - that is, for which the inclusion $U_i \hookrightarrow Y$ is homotopic to a constant map.*

The condition for $U \hookrightarrow Y$ to be homotopic to a constant map means that we have a continuous way to associate with a point y of U a path in Y from y to a fixed point. This is then not difficult to see the following relations between TC and cat (see [F03]):

$$\mathrm{cat}(X) \leq \mathrm{TC}(X) \leq \mathrm{cat}(X \times X).$$

The LS-category is also a homotopy invariant which satisfies $\mathrm{cat}(Y) = 0$ if and only if Y is contractible. Independently on the dimension, we have $\mathrm{cat}(S^n) = 1$ since any sphere can be covered by 2 contractible open sets (in themselves and therefore in S^n). For manifolds, we have $\mathrm{cat}(Y) \leq \dim Y$ and we also always have $\mathrm{cat}(Y \times Z) \leq \mathrm{cat}(Y) + \mathrm{cat}(Z)$. Then the chain of inequalities above can be completed as

$$\mathrm{cat}(X) \leq \mathrm{TC}(X) \leq \mathrm{cat}(X \times X) \leq 2\mathrm{cat}(X) \leq 2\dim(X).$$

In particular $\mathrm{TC}(S^n) \leq 2\mathrm{cat}(S^n) = 2$.

We also note the following interesting case of equality:

Theorem 5 ([Fo4]).— *If G is a path-connected topological group then $\mathrm{TC}(G) = \mathrm{cat}(G)$.*

Let us see why $\mathrm{TC}(G) \leq \mathrm{cat}(G)$. Consider the continuous map $\mu : G \times G \to G$ given by $\mu(x, y) = xy^{-1}$ and suppose that $U \hookrightarrow G$ is homotopic to a constant map through a homotopy H. Since G is path-connected, we can suppose that the constant map is $u \mapsto e$ where e is the unit of G. Then, for $(x, y) \in V = \mu^{-1}(U) \subset G\times G$ we can consider the path from x to y given by $t \mapsto H(xy^{-1}, t)y$ and the associated local section of ev.

The direct determination of the LS-category and TC of a space is in general not easy. For instance, there are still Lie groups for which the LS-category is not known and one usually tries to use more calculable lower and upper bounds for estimating these invariants.

A very useful lower bound for the LS-category of a space Y is given by the cup-length of its cohomology. We here consider the cohomology of Y with coefficients in a field $\Bbbk$ and suppose that Y is a path-connected manifold. Recall that $H^*(Y;\Bbbk) = \bigoplus_{m\geq 0} H^m(Y;\Bbbk)$ is a graded $\Bbbk$-algebra satisfying $H^m(Y;\Bbbk) = 0$ for $m > \dim(Y)$ and $H^0(Y;\Bbbk) = \Bbbk \cdot 1 = \langle 1\rangle$ where 1 is the unit of the algebra, and that the graded multiplication (called the cup-product) $H^p(Y;\Bbbk)\otimes H^q(Y;\Bbbk) \to H^{p+q}(Y;\Bbbk)$ is commutative in the graded meaning: $ab = (-1)^{\deg(a)\deg(b)}ba$. The cup-length of $H^*(Y;\Bbbk)$ is defined by

$$\mathrm{cl}_\Bbbk(Y) = \max\{n \mid \exists\, a_1, \dots, a_n \in H^{>0}(Y;\Bbbk) \text{ s.t. } a_1 \cdots a_n \neq 0\}$$

and we have

$$\mathrm{cl}_\Bbbk(Y) \leq \mathrm{cat}(Y) \leq \dim(Y).$$

For instance the cohomology of the torus $T = S^1 \times S^1$ with coefficients in $\mathbb{Q}$ is given by $H^0(T;\mathbb{Q}) = \langle 1\rangle$, $H^1(T;\mathbb{Q}) = \langle a, b\rangle$ and $H^2(T;\mathbb{Q}) = \langle \omega\rangle$, with the multiplicative structure $a^2 = b^2 = 0$, $ab = -ba = \omega$. We then have $\mathrm{cl}_\mathbb{Q}(T) = 2$ and therefore $\mathrm{cat}(T) = 2$ since $\dim(T) = 2$. Since T is a topological group, we can also conclude that $\mathrm{TC}(T) = 2$.

A similar lower bound for the topological complexity has been introduced by Farber. Call a *zero-divisor* of $H^*(X;\Bbbk)$ an element of the kernel of the cup-product $H^*(X;\Bbbk) \otimes H^*(X;\Bbbk) \to H^*(X;\Bbbk)$. This kernel is an ideal of the tensor algebra whose multiplication satisfies $(a \otimes b)(c \otimes d) = (-1)^{\deg(b)\deg(c)} ac \otimes bd$. The *zero-divisor cup-length* of $H^*(X;\Bbbk)$ is then defined by

$$\mathrm{zcl}_\Bbbk(X) = \max\{n \mid \exists\, n \text{ zero-divisors } z_i \text{ s.t. } z_1 \cdots z_n \neq 0\}$$

and we have

Theorem 6 ([Fo3]).— *$\mathrm{zcl}_\Bbbk(X) \leq \mathrm{TC}(X)$.*

We can now complete the calculation of $\mathrm{TC}(S^n)$ for n even. Recall that we already know that $1 \leq \mathrm{TC}(S^n) \leq 2$. Taking coefficients in $\mathbb{Q}$, we have only non-zero cohomology in degree 0 and n:

$$H^0(S^n;\mathbb{Q}) = \langle 1\rangle, \quad H^n(S^n;\mathbb{Q}) = \langle a\rangle.$$

The zero-divisors are given by

$$\mathbb{Q}(a \otimes 1 - 1 \otimes a) \oplus \mathbb{Q}(a \otimes a).$$

Since $(a\otimes 1 - 1\otimes a)^2 = (-1-(-1)^n)a\otimes a = -2a\otimes a \neq 0$ if n is even, we can conclude that $\mathrm{zcl}_\mathbb{Q}(S^n) = 2$, and therefore $\mathrm{TC}(S^n) = 2$, if n is even.

2 Topological complexity of surfaces

We first consider the orientable (closed connected) surfaces. We denote by Σ_g the orientable surface of genus g ($g \geq 0$) so that $\Sigma_0 = S^2$, Σ_1 is the torus $T = S^1 \times S^1$ and, for $g \geq 2$, Σ_g can be described as the connected sum of g tori T. The topological complexity of Σ_g has been determined by Farber in 2003:

Theorem 7 ([Fo3]).— *We have*

- *for $g \leq 1$, $\mathrm{TC}(\Sigma_g) = 2$;*
- *for $g \geq 2$, $\mathrm{TC}(\Sigma_g) = 4$.*

Since the cases $g = 0, 1$ have been discussed in the previous section, we now focus on the case $g \geq 2$. For dimensional reason, we have $\mathrm{TC}(\Sigma_g) \leq 2\dim(\Sigma_g) = 4$. The cohomology of Σ_g with coefficients in $\mathbb{Q}$ is given by $H^0(\Sigma_g;\mathbb{Q}) = \langle 1\rangle$, $H^2(\Sigma_g;\mathbb{Q}) = \langle\omega\rangle$, and

$$H^1(\Sigma_g;\mathbb{Q}) = \langle a_1, b_1, \dots, a_g, b_g\rangle,$$

with the multiplicative structure

$$a_i^2 = b_i^2 = 0, a_ib_i = -b_ia_i = \omega, a_ib_j = -b_ja_i = 0 \text{ for } i \neq j.$$

Note that we clearly have $\mathrm{cat}(\Sigma_g) = 2$. Considering the zero-divisors $\alpha_i = a_i \otimes 1 - 1 \otimes a_i$, $\beta_j = b_j \otimes 1 - 1 \otimes b_j$ in $H^*(\Sigma_g;\mathbb{Q}) \otimes H^*(\Sigma_g;\mathbb{Q})$, we can check that, for $i \neq j$,

$$\alpha_i\beta_i\alpha_j\beta_j = 2\omega \otimes \omega \neq 0.$$

As a consequence $\mathrm{zcl}_\mathbb{Q}(\Sigma_g) \geq 4$ for $g \geq 2$, which permits us to conclude that $\mathrm{zcl}_\mathbb{Q}(\Sigma_g) = \mathrm{TC}(\Sigma_g) = 4$.

We now turn to the non-orientable surfaces. For $g \geq 1$, we denote by N_g the non-orientable surface of genus g, which can be described as the connected sum of g copies of the real projective plane $\mathbb{R}P^2$. In particular, $N_1 = \mathbb{R}P^2$ and $N_2 = \mathbb{R}P^2\#\mathbb{R}P^2$ is the Klein bottle.

The cohomology of N_g with coefficients in $\mathbb{Z}_2$ is given by $H^0(N_g;\mathbb{Z}_2) = \langle 1\rangle$, $H^2(N_g;\mathbb{Z}_2) = \langle\omega\rangle$, and

$$H^1(N_g;\mathbb{Q}) = \langle a_1, \dots, a_g\rangle,$$

with the multiplicative structure

$$a_i^2 = \omega, \quad a_i a_j = a_j a_i = 0 \text{ for } i \neq j.$$

The LS-category is then easy to determine ($\mathrm{cat}(N_g) = 2$) since we have $\mathrm{cl}_{\mathbb{Z}_2}(N_g) = \dim(N_g) = 2$ for any $g \geq 1$. Regarding zero-divisors with coefficients in $\mathbb{Z}_2$ we can check that

$$(a_i \otimes 1 - 1 \otimes a_i)^3 = \omega \otimes a_i + a_i \otimes \omega \neq 0$$

and that any product of 4 zero-divisors vanishes. Consequently, $\mathrm{zcl}_{\mathbb{Z}_2}(N_g) = 3$ and

$$3 = \mathrm{zcl}_{\mathbb{Z}_2}(N_g) \leq \mathrm{TC}(N_g) \leq 2\dim(N_g) = 4.$$

It then follows that $\mathrm{TC}(N_g)$ is either 3 or 4. The topological complexity of $N_1 = \mathbb{R}P^2$ has been determined by Farber, Tabachnikov, and Yuzvinsky in 2003:

Theorem 8 ([FTY03]).— $\mathrm{TC}(\mathbb{R}P^2) = 3$.

The inequality $\mathrm{TC}(\mathbb{R}P^2) \leq 3$ can be obtained through an explicit open cover of $\mathbb{R}P^2 \times \mathbb{R}P^2$ with 4 local sections of the evaluation map as described in [FTY03]. However, it is worth noting the following more general result on the topological complexity of the n-dimensional real projective space $\mathbb{R}P^n$ which was established in [FTY03]:

Theorem 9 ([FTY03]).— For n distinct from 1, 3, 7, $\mathrm{TC}(\mathbb{R}P^n)$ is the least integer k such that there exists an immersion of $\mathbb{R}P^n$ in $\mathbb{R}^k$. For $n \in \{1,3,7\}$, $\mathrm{TC}(\mathbb{R}P^n) = \mathrm{cat}(\mathbb{R}P^n) = n$.

This remarkable result shows that $\mathrm{TC}(\mathbb{R}P^n)$ coincides with the so-called immersion dimension of $\mathbb{R}P^n$ and, as is well known, $\mathbb{R}P^2$ can be immersed in $\mathbb{R}^3$ but not in $\mathbb{R}^2$. Although many values of this immersion dimension are known, the complete determination of this number as a function of n is still an open problem. It then turns out that, while the LS-category of $\mathbb{R}P^n$ is easy to calculate ($\mathrm{cat}(\mathbb{R}P^n) = \mathrm{cl}_{\mathbb{Z}_2}(\mathbb{R}P^n) = \dim(\mathbb{R}P^n) = n$), the topological complexity of this space can be very difficult to determine. Surprisingly, the determination of $\mathrm{TC}(N_g)$ for $g \geq 2$ has also revealed to be less immediate than that of $\mathrm{cat}(N_g)$. In 2016, Dranishnikov established that $\mathrm{TC}(N_g) = 4$ for $g \geq 4$ and showed that his methods do not extend to the lower genus cases $g \in \{2,3\}$ ([D16], [D17]). The general case has been solved in 2017:

Theorem 10 ([CV17]).— For $g \geq 2$, $\mathrm{TC}(N_g) = 4$.

We briefly describe the method used in [CV17] where it is proved that the topological complexity of the Klein bottle N_2 is 4 with an argument which permits one to inductively prove that $\mathrm{TC}(N_g) = 4$ for any $g \geq 2$.

As the calculation mentioned above shows, the lower bound given by the zero-divisor cup-length of N_g with coefficients in $\mathbb{Z}_2$ does not permit one to complete the calculation of $\mathrm{TC}(N_g)$ for $g \geq 2$. However, a twisted coefficients version of the zero-divisor cup-length has revealed to be sufficient.

Recall that a system of twisted (or local) coefficients on a space Y is given by a module M over the group ring

$$\mathbb{Z}[\pi] = \{\sum_{finite} n_i a_i \mid n_i \in \mathbb{Z}, a_i \in \pi\}$$

where $\pi = \pi_1(Y)$ is the fundamental group of Y. In other words, M is an abelian group with an action of π.

Let M be a system of twisted coefficients on $X \times X$ and let $M|X$ be the system induced by the diagonal map $\Delta : X \to X \times X$, $x \mapsto (x,x)$. With such coefficients a cohomology class $u \in H^*(X \times X; M)$ is a zero-divisor if

$$\Delta^*(u) = 0 \in H^*(X; M|X)$$

where Δ^* denotes the morphism induced in cohomology by Δ. This is a generalization of the notion of zero-divisor considered above since, when $M = \Bbbk$ (with the trivial action of $\pi_1(X \times X) \cong \pi_1(X) \times \pi_1(X)$), the kernel of the cup-product $H^*(X;\Bbbk) \otimes H^*(X;\Bbbk) \to H^*(X;\Bbbk)$ can be identified with the kernel of Δ^* through the Künneth isomorphism $H^*(X \times X;\Bbbk) \cong H^*(X;\Bbbk) \otimes H^*(X;\Bbbk)$. Moreover, as shown in [F08], one has $\mathrm{TC}(X) \geq k$ whenever the cup-product of k zero-divisors $u_i \in H^*(X \times X; M_i)$ is non-zero.

In [CF10], Costa and Farber associate with a space X a zero-divisor $\mathfrak{v} = \mathfrak{v}_X \in H^1(X \times X; I(\pi))$ where $\pi = \pi_1(X)$ and $I(\pi) = \{\sum n_i a_i \in \mathbb{Z}[\pi] \mid \sum n_i = 0\}$ is the augmentation ideal, which is a $\mathbb{Z}[\pi \times \pi]$-module through the action given by:

$$(a,b) \cdot \sum n_i a_i = \sum n_i (a a_i b^{-1}).$$

Here $n_i \in \mathbb{Z}$ and $a, b, a_i \in \pi$.

Through a calculation at the chain/cochain level using the bar resolution associated with a discrete group, it is shown in [CV17] that the fourth power of $\mathfrak{v}$ is not zero when $X = N_2$ is the Klein bottle and that consequently $\mathrm{TC}(N_2) = 4$. Using next the map $N_g \to N_{g-1}$ which collapses the last summand of $N_g = \mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2$ and the associated morphisms (in cohomology, between the fundamental groups...), an inductive argument permits one to see that the fourth power of the class $\mathfrak{v}$ associated to N_g does not vanish for all $g \geq 2$. Consequently, $\mathrm{TC}(N_g) = 4$ for $g \geq 2$.

As a final remark, we note that except S^2 and $\mathbb{R}P^2$ all the surfaces above are aspherical, which means that their only possibly non-zero homotopy group is their fundamental group. Many interesting spaces are aspherical spaces and many works focus on the study of the topological complexity of such spaces, not only with the aim to calculate this invariant for specific examples but also with the general goal to better understand its properties. The homotopy type of an aspherical space, and therefore its LS-category and its topological complexity, is completely determined by its fundamental group. By a theorem of Eilenberg and Ganea [EG57], the LS-category of such a space is known to be equal to the cohomological dimension of its fundamental group. The problem, posed by Farber in [F06], of finding such an expression of the topological complexity of an aspherical space in terms of algebraic invariants of its fundamental group has become a challenging open problem on the invariant TC.

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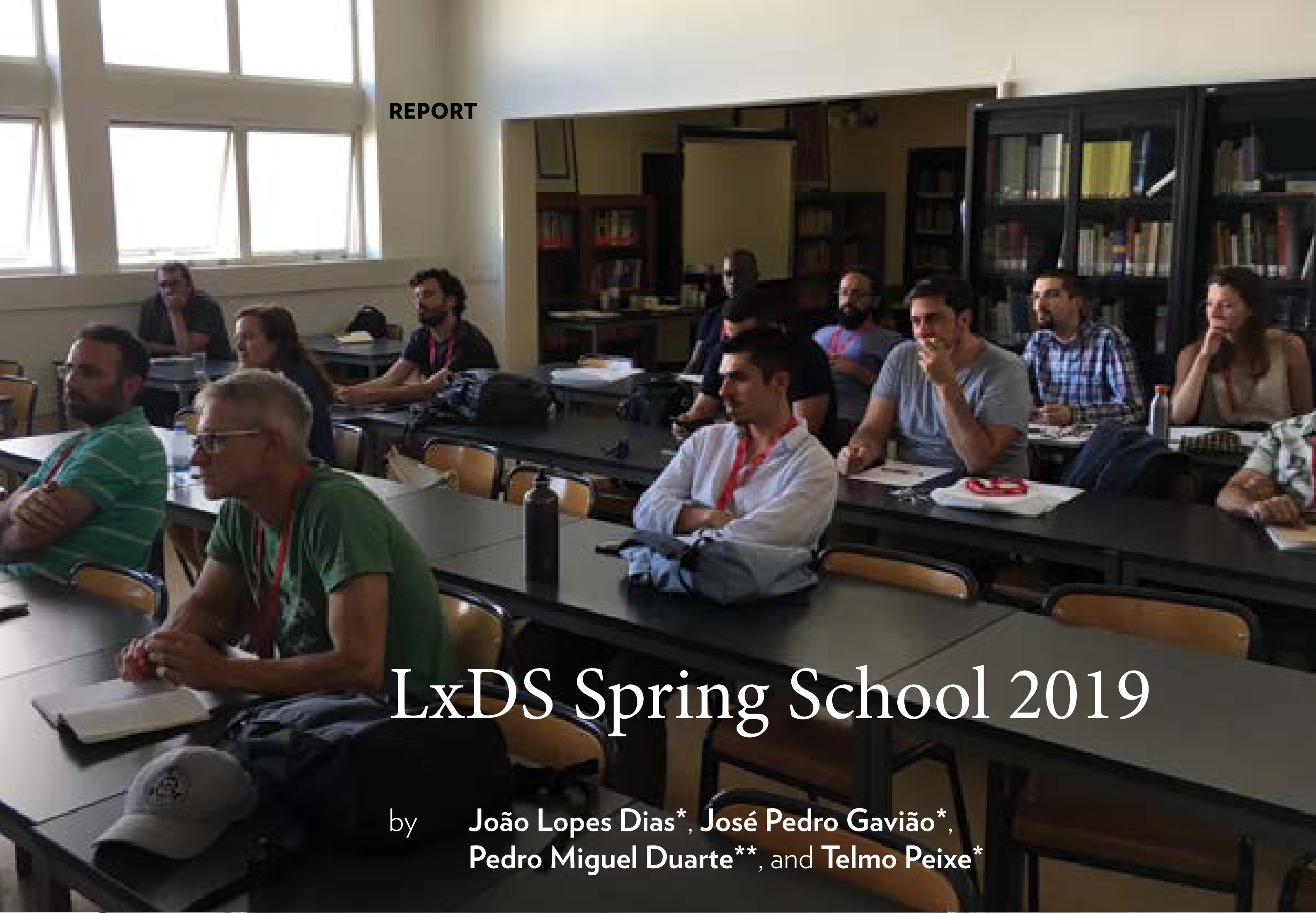
REPORT



LxDS Spring School 2019

by **João Lopes Dias***, **José Pedro Gavião***, **Pedro Miguel Duarte****, and **Telmo Peixe***

The LxDS-Lisbon Dynamical Systems Group, the Department of Mathematics at FCUL, CEMAPRE, REM and CMAF-CIO organized a spring school on the days 29-31 of May 2019 in dynamical systems, which took place at the Faculty of Sciences of the University of Lisbon (FCUL).

* Universidade de Lisboa, ISEG, REM, CEMAPRE
** Universidade de Lisboa, FCUL, CMAFcIO

The school consisted of three courses in dynamical systems, which were given by specialists of recognised international merit. Namely,

Transfer of energy in Hamiltonian PDEs
Marcel Guardia, Universitat Politècnica de Catalunya,

Multiplicative and subadditive ergodic theorems
Anders Karlsson, Université de Genève,

Extreme events for dynamic systems
Jorge Freitas, University of Porto.

The school had more than 20 participants, some of them international.

In addition to the courses the school also had a session of brief oral presentations, in which some of the PhD students and researchers presented their most recent work. There was also a poster session.

Due to financial support provided by CIM, the organisation managed to support the participation of two PhD students (one national and one international) covering their travel, lodging and meals during the school days.

Organising Committee

João Lopes Dias
Universidade de Lisboa, ISEG, REM, CEMAPRE

José Pedro Gaivão
Universidade de Lisboa, ISEG, REM, CEMAPRE

Pedro Miguel Duarte
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Luís Sampaio
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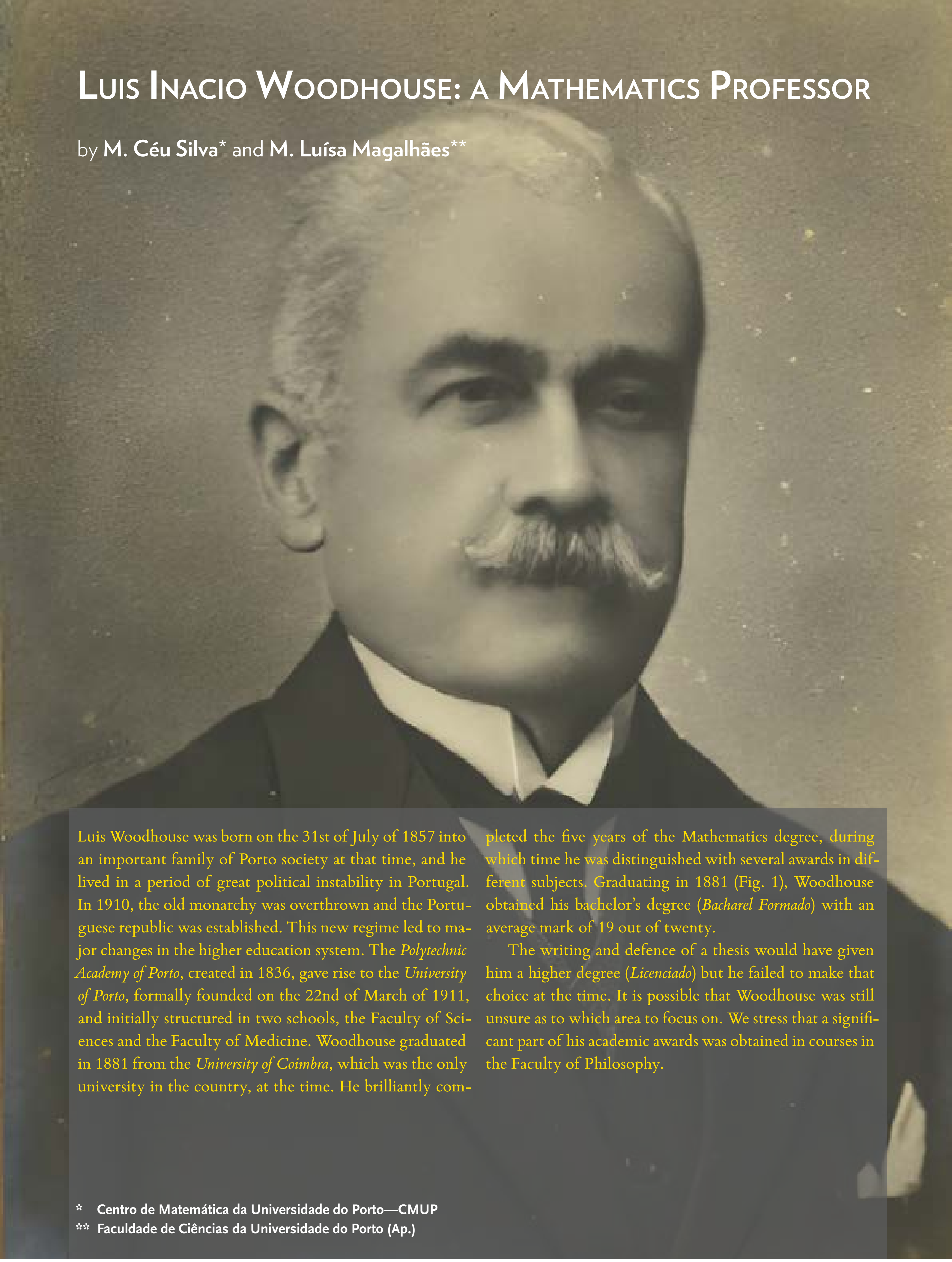
Filipe Santos
Universidade de Lisboa, ISEG, REM, CEMAPRE

Luis Inacio Woodhouse: a Mathematics Professor

by **M. Céu Silva*** and **M. Luísa Magalhães****



Luis Woodhouse was born on the 31st of July of 1857 into an important family of Porto society at that time, and he lived in a period of great political instability in Portugal. In 1910, the old monarchy was overthrown and the Portuguese republic was established. This new regime led to major changes in the higher education system. The *Polytechnic Academy of Porto*, created in 1836, gave rise to the *University of Porto*, formally founded on the 22nd of March of 1911, and initially structured in two schools, the Faculty of Sciences and the Faculty of Medicine. Woodhouse graduated in 1881 from the *University of Coimbra*, which was the only university in the country, at the time. He brilliantly completed the five years of the Mathematics degree, during which time he was distinguished with several awards in different subjects. Graduating in 1881 (Fig. 1), Woodhouse obtained his bachelor's degree (*Bacharel Formado*) with an average mark of 19 out of twenty.

The writing and defence of a thesis would have given him a higher degree (*Licenciado*) but he failed to make that choice at the time. It is possible that Woodhouse was still unsure as to which area to focus on. We stress that a significant part of his academic awards was obtained in courses in the Faculty of Philosophy.

* Centro de Matemática da Universidade do Porto—CMUP
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EM NOME DE DEUS, AMEN.

O DOUTOR ANTONIO BERNARDINO DE MENEZES,

Figure 1.—Certificate of Woodhouse's degree.

Furthermore, as a student he became interested in various scientific areas, in which he published four papers. The solution of a problem on number theory proposed by Gomes Teixeira in the *Jornal de Sciencias Mathematicas e Astronomicas* [1877, p. 96]. It was published with the title On the question Proposed in nº 6 in the *Jornal de Sciencias Mathematicas e Astronomicas* [Woodhouse 1877]. The proposal of a new proof to a geometrical question — the location of the centres of the so-called "Villarceau circles" — which had been solved two years earlier by Pedro Amorim Viana, a professor of Mathematics at the Polytechnic Academy of Porto. Woodhouse's proof with the title *Proof of a theorem of geometry* (*Demonstração d'um theorema de geometria*) was published in *O Instituto of Coimbra* [Woodhouse 1878]). A paper on the dating of the earth's geological past, entitled *Paleontological Chronology (General aspects)* (*Chronologia paleontológica (Traços geraes)*), published in the *Revista Scientifica e Litteraria* [Woodhouse 1880] and another paper *Astronomy (Hypothese Cosmogonica)* (*Astronomia (Hypothese Cosmogonica)*) published in the same journal [Woodhouse 1881]), in which Woodhouse sought explanations for the formation and evolution of the solar system and the movement of the planets. The choice of these last two themes was likely influenced by two of the teachers who most inspired him: Julio Augusto Henriques and João José Dantas Souto Rodrigues. It was also then that his friendship with Francisco Gomes Teixeira began. This friendship endured and strengthened until Woodhouse's death on the 13th of March of 1927.[1]

Woodhouse started his teaching career at the Polytechnic Academy of Porto in the academic year 1883/84. A year later, he was appointed as the professor of *Differential and Integral Calculus and Descriptive Geometry*. In 1885 he submitted the dissertation *On the Integration of Differential Equations of Dynamics* (*Da Integração das Equações Differenciaes da Dynamica*) [Woodhouse 1883] previously published in his application for full professor. On the 23rd of September of 1885, he became the holder of the 1st chair of the *Polytechnic Academy of Porto — Analytical geometry, higher algebra and spherical trigonometry*. Woodhouse was appointed as *Professor Ordinário* of the 1st group (Analysis and Geometry) of the 1st section (Mathematical Sciences) when the Faculty of Sciences was created. He was also awarded the degree of Doctor in Mathematical Sciences by the Academic Council, by proposal of the Faculty Director on the 30th of November of 1918. For over 40 years, Woodhouse taught *Analytical geometry, higher algebra and spherical trigonometry*, having adapted the program to teaching requirements over the years.[2] In 1922, he proposed doubling the number of lectures to the academic council because he considered that one year had become insufficient to teach the main subjects. As his proposal was rejected, he decided to create a free, extracurricular, complementary algebra degree. Unfortunately, the course did not have the partic-

[1] Beires 1951, pp. 190–191.

[2] Carvalho 1934, p. 196.

ipation he expected, so he interrupted it after several lectures.[3] Sometimes, either for convenience of distributing his teaching duties or filling his teaching schedule, Woodhouse taught other courses, such as *Astronomy and Geodesy*, *Celestial Mechanics*, and *Probabilities*.

Due to his personality traits — impartiality and rigor —, and to his versatility, he was often chosen for several different administrative positions in the institutions where he taught: the *Polytechnic Academy of Porto*, the *Faculty of Sciences*, the *Industrial and Commercial Institute*, and the *Higher Trade Institute*. He was part of administrative committees: the treasurer and Pro vice chancellor of the *University of Porto*; the 1st president of the Mathematics Section when the *Faculty of Sciences* was created in 1911; elected by his colleagues to the University Senate; director of the *Faculty of Sciences of Porto*. Woodhouse accumulated these activities with others that went beyond the scope of the Academy. In the City Council of Porto, he was a substitute member of the Executive Committee, a member of the council, and chair of the commission of studies on reorganizing the *Municipal Museum of Porto*. He was also vice president of the *Portuguese Association for the Progress of Sciences*, and a member of the drafting committee of the journal *O Instituto*.

Despite his many duties, Woodhouse's main contribution was to high school and higher education in its pedagogical and didactic aspects. His mediation in connection with these two levels of education was very important. In high school, Woodhouse took part in several committees which drafted proposals and suggested changes to the existing curricula. He presided over the final exams of the general course in the *Central High School of Porto* (*Liceu Central do Porto*), and was a member of the School Board, an interlocutor between the school inspectorate and the City Council. Among its tasks, the School Board was dedicated to preparing Pedagogical Conferences for training high school teachers. At the *Polytechnic Academy* and the *Faculty of Sciences of Porto*, Woodhouse actively participated in meetings of Academic Councils suggesting changes in the programs, redistribution of subjects, and reorganization of courses. Also worthy of note was his effort to create the Geographer Engineer course in the University of Porto.

At the closing session of the 1st Luso-Spanish Congress for the Progress of Sciences held in Porto in 1921, Woodhouse's talk was on *Teaching Mathematics in Portuguese Universities* (O *Ensino Matemático nas Universidades Portuguesas* [Woodhouse 1921]). In this talk, he clearly showed his scientific, pedagogic and didactic concerns regarding education, especially concerning Mathematics. Three measures proposed by Woodhouse should be mentioned. In order to minimize students' mathematical shortcomings at the beginning of higher education, he proposed creating a transition year between high school and college, in which high school curricula were expanded and deepened. He argued that a course in general mathematics being compulsory for students from the three sections of the Faculty of Sciences would alleviate the problem until a major reform of high school programs was made. On the other hand, he proposed a new plan for mathematical studies at the *Faculty of Sciences*, specially focused on differentiating curricula according to each student's degree. Woodhouse emphasized the challenging career of a teacher in the classroom, as a driver of learning and student interest, but was always aware of the demand and rigor.

We stress that according to Woodhouse the *Faculty of Sciences* should offer the course of History of Mathematics. He defended this position at academic council meetings in 1913, but this was not implemented because Gomes Teixeira was not available to do so. Recognizing the importance of research in History of Mathematics, in 1924 he proposed creating a Scientific Research Institute in the History of Portuguese Mathematics to the School Board, an institute for which Gomes Teixeira would be the director. The proposal was accepted by unanimous vote, although the institute was to be created only two years later.

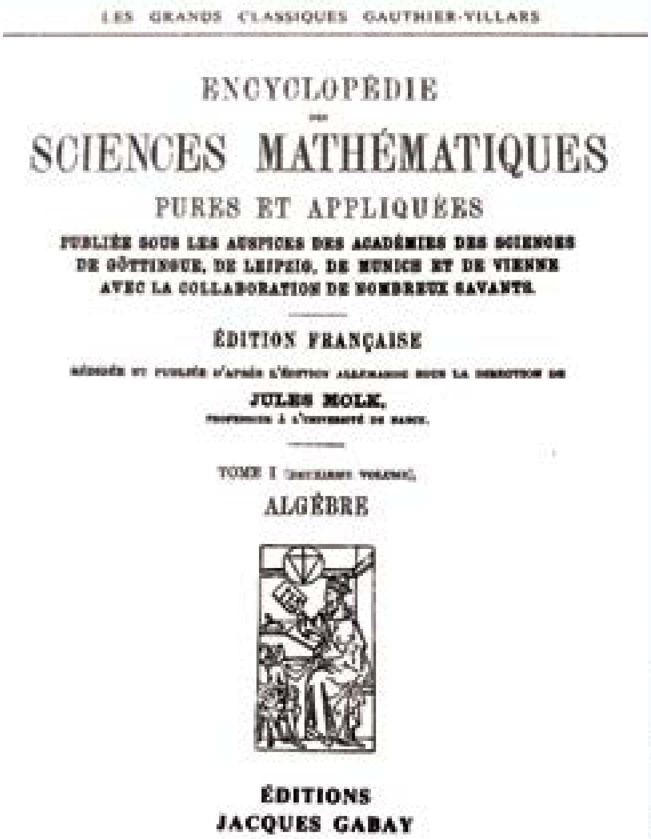
Contribution to the history of algebra

Woodhouse's scientific interest focused mostly on algebra. He published his first work in algebra in 1885, Fundamental *Principle of the theory of algebraic equations* (*Princípio Fundamental da Teoria das equações algébricas*), in the *Jornal de Sciencias Mathematicas e Astronomicas* [Woodhouse 1885]. This work provided a non-constructive proof of the fundamental theorem of Algebra (it does not give a procedure to determine the solution, merely showing its existence). The work gave Woodhouse international recognition, resulting in his name being mentioned in the *Encyclopédie des Sciences Mathématiques Pures et Appliquées* (Fig. 2 and Fig. 3).[4]

Woodhouse starts off with the polynomial function $F(z) = \sum_{j=0}^{n} A_j z^j$ where the complex coefficients A_j are written in the polar form, and the variable z is represented both in polar and Cartesian coordinates. The work itself lacks figures although the solution presented presumes drawing one. He considered a plane coordinate system with origin O. Let $A_j = \rho_j(\cos\omega_j + i\sin\omega_j)$, $j = 0, 1, \ldots, n$, $z = r(\cos\theta + i\sin\theta) = x + iy$, and then $P_0 = O + A_0$ and

[3] Carvalho1934, p. 196.

[4] Neto, E., Vavasseur, R. Le 1907, pp. 203–204.



LES GRANDS CLASSIQUES GAUTHIER-VILLARS

ENCYCLOPÉDIE
DES
SCIENCES MATHÉMATIQUES
PURES ET APPLIQUÉES

PUBLIÉE SOUS LES AUSPICES DES ACADÉMIES DES SCIENCES
DE GÖTTINGUE, DE LEIPZIG, DE MUNICH ET DE VIENNE
AVEC LA COLLABORATION DE NOMBREUX SAVANTS.

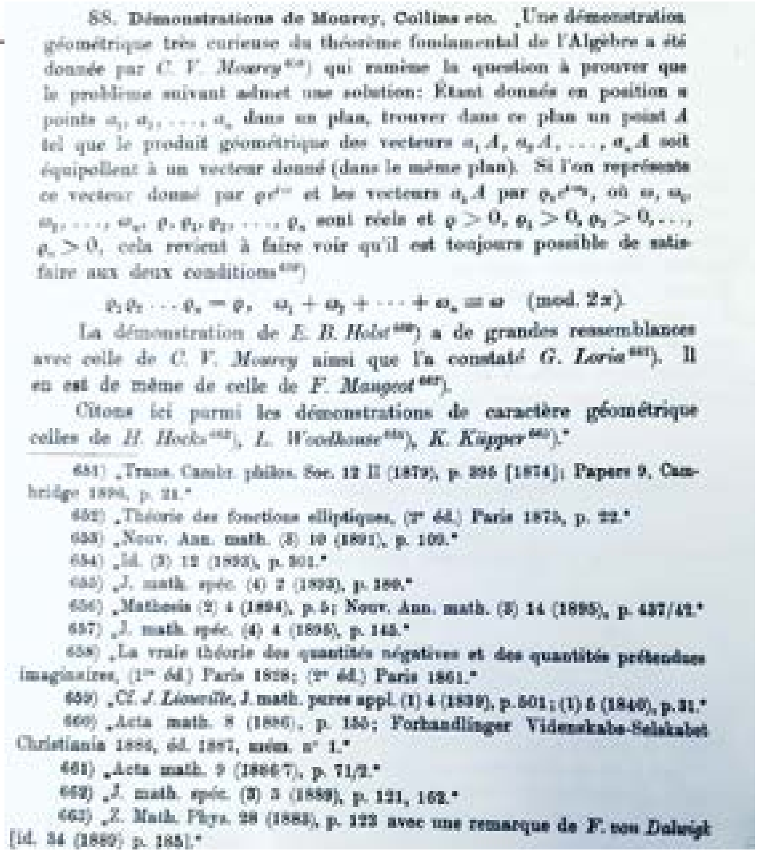
ÉDITION FRANÇAISE

JULES MOLK,

TOME I (DEUXIÈME VOLUME)
ALGÈBRE

ÉDITIONS
JACQUES GABAY

Figure 2.—First page of the Tome 1 Volume 2 of the Encyclopédie des Sciences Mathématiques Pures et Appliquées.



88. **Démonstrations de Mourey, Collins etc.** „Une démonstration géométrique très curieuse du théorème fondamental de l'Algèbre a été donnée par *C. V. Mourey*[648]) qui ramène la question à prouver que le problème suivant admet une solution: Étant donnés en position n points $a_1, a_2, \dots, a_n$ dans un plan, trouver dans ce plan un point A tel que le produit géométrique des vecteurs $a_1A, a_2A, \dots, a_nA$ soit équipollent à un vecteur donné (dans le même plan). Si l'on représente ce vecteur donné par $\varrho e^{i\omega}$ et les vecteurs a_kA par $\varrho_k e^{i\omega_k}$, où $\omega, \omega_1, \omega_2, \dots, \omega_n, \varrho, \varrho_1, \varrho_2, \dots, \varrho_n$ sont réels et $\varrho > 0, \varrho_1 > 0, \varrho_2 > 0, \dots, \varrho_n > 0$, cela revient à faire voir qu'il est toujours possible de satisfaire aux deux conditions[649])

$$\varrho_1\varrho_2\cdots\varrho_n = \varrho, \quad \omega_1 + \omega_2 + \cdots + \omega_n \equiv \omega \quad (\text{mod. } 2\pi).$$

La démonstration de *E. B. Holst*[650]) a de grandes ressemblances avec celle de *C. V. Mourey* ainsi que l'a constaté *G. Loria*[651]). Il en est de même de celle de *F. Maugeot*[652]).

Citons ici parmi les démonstrations de caractère géométrique celles de *H. Hocks*[652]), *L. Woodhouse*[653]), *K. Küpper*[654])."

651) „Trans. Cambr. philos. Soc. 12 II (1879), p. 395 [1874]; Papers 9, Cambridge 1896, p. 21."

652) „Théorie des fonctions elliptiques, (2e éd.) Paris 1875, p. 22."

653) „Nouv. Ann. math. (3) 10 (1891), p. 109."

654) „Id. (3) 12 (1893), p. 301."

655) „J. math. spéc. (4) 2 (1893), p. 180."

656) „Mathesis (2) 4 (1894), p. 5; Nouv. Ann. math. (3) 14 (1895), p. 437/42."

657) „J. math. spéc. (4) 4 (1895), p. 145."

658) „La vraie théorie des quantités négatives et des quantités prétendues imaginaires, (1re éd.) Paris 1828; (2e éd.) Paris 1861."

659) „Cf. *J. Liouville*, J. math. pures appl. (1) 4 (1839), p. 501; (1) 5 (1840), p. 31."

660) „Acta math. 8 (1886), p. 155; Forhandlinger Videnskabs-Selskabet Christiania 1886, éd. 1887, mém. n° 1."

661) „Acta math. 9 (1886/7), p. 71/2."

662) „J. math. spéc. (3) 3 (1889), p. 121, 162."

663) „Z. Math. Phys. 28 (1883), p. 123 avec une remarque de *F. von Dalwigk* [id. 34 (1889) p. 185]."

Figure 3.—Page of the Encyclopédie where the name of Woodhouse is mentioned.

$P_j = P_{j-1} + A_j z^j$, $j = 0, 1, \dots, n$. Note that P_n represents $F(z)$. Woodhouse fixes r and considers the closed curves Q_j, $j = 1, \dots, n$ described by P_j with $0 \le \theta \le 2\pi$ (in particular Q_1 is a circle with centre P_0 and radius $\rho_1 r$) and the circles C_j with centre O and radius $= \sum_{k=0}^{j} \rho_k r^k$. The circle C_j contains the curve Q_j and the circle C_{j-1}, $j \ge 1$. The idea of the proof is to show that there are r, θ such that $|F(z)| = 0$. Woodhouse shows that the Cartesian coordinates of z vary continuously with r and θ, and that there exist r', r'' $(0 < r'' < r')$ such that: the point O lies outside of Q_n if $r = r''$, and the point O lies inside of Q_n if $r = r'$. By continuity, he concludes that there exists $r \in [r'', r']$ such that Q_n crosses the origin O. We stress that, in the particular case $n = 2$, we may just take r' equal to the positive root of $\rho_2 r^2 - 2(\rho_0 + \rho_1 r) = 0$, and r'' equal to the positive root of $\rho_2 r^2 + \rho_1 r + \rho_0 = 0$.

At the 1st Luso-Spanish Congress held in Porto in 1921, Woodhouse presented the talk *Francisco Simões Margiochi's contribution to the problem of algebraically solving equations* (*Contribuição de Francisco Simões Margiochi para o problema da resolução algébrica das equações*). Woodhouse was referring to the *Memoire in order to prove that literal and complete equations of degree greater than four cannot have the root form* (*Memoria com o fim de provar que não podem ter fórma de raizes, as equações literaes e completas de graus superiores ao quarto*) presented by Margiochi[5] to the *Academy of Sciences of Lisbon*, and published in the *Memórias da Academia de Sciencias de Lisboa* [Margiochi 1821]. In this work, Margiochi intended to prove that it is not possible to find a general algebraic expression to solve all polynomial equations with degree greater than four. Woodhouse explained Margiochi's procedure in detail and he showed that although the conclusion was correct, the method used did not allow him to prove it.

In 1924 Woodhouse published the work *The Horner Method and a forgotten Portuguese work* (1794) (*O Método de Horner e um trabalho português esquecido* (1794) [Woodhouse 1924]) in the *Imprensa Nacional* (Fig. 4).[6] This work was intended as a tribute to the Portuguese mathematician José Maria Dantas Pereira (1772–1836) who, in Woodhouse's opinion, should have been acknowledged for the method

[5] Fracisco Simões Margiochi was a Mathematics Professor at the *Royal Navy Academy of Lisbon*, and a member of the *Academy of Sciences of Lisbon*.

[6] Also in Journal of Mathematical Physics and Natural Sciences [Vol. 5 of 3th Series, No. 94 (1924–1927) pp. 53–68].

known as Horner's method. By providing greater visibility of these works, he showed that algebraic studies in Portugal were in line with the cutting edge of the discipline. Essentially, Woodhouse's argument developed in three parts. Part 1 is a commented summary of the method that Horner used to calculate the real roots of an algebraic equation with real coefficients.

The method is contained in the memoire entitled *The New Method of Solving Numerical Equations of All Orders* presented by the English geometer William George Horner (1786–1837) to the *Royal Society* in 1819, and published in the *Philosophical Transactions of the Royal Society* [Horner 1819]. It describes a method to approximate the roots of a polynomial equation using Ruffini's rule. As Woodhouse pointed out, the method was still unknown in 1794 when Dantas Pereira presented his paper *Reflections on certain successive summations of the terms of the arithmetic series, applied to the solutions of various algebraic questions* (*Reflexões sobre certas sommações sucessivas dos termos das series aritmeticas, applicadas ás soluções de diversas questões algébricas*) at the Academy of Sciences. Dantas Pereira's work, referred to by Woodhouse, was published in *Memorias de Mathematica e Phisica da Academia Real das Sciencias de Lisboa* [Pereira 1799]. In part 2, Woodhouse briefly described the content of Dantas Pereira's memoire, rewriting it with slight changes in formulation and notation. He pointed out that while the author's goal was not to solve equations, the computation of the value of a polynomial at a given point is present in his work. Part 3 deals with the comparison between Dantas Pereira's method and Horner's method in order to compute the values of a polynomial with integer coefficients for integer values of the variable. Woodhouse showed that their structures are essentially analogous, and furthermore, he recalled that though Dantas Pereira's method does not use Ruffini's rule, it is more efficient because it reduces the number of calculations.

The 3rd Luso-Spanish Congress for the Progress of Sciences took place in Coimbra in 1925. Woodhouse delivered the first talk in the Mathematics Section — *Mathematics in Portugal in the early nineteenth century* (*A Matemática em Portugal no principio do século XIX* [Woodhouse 1925]) (Fig. 5)— once again turning his attention to the algebraic questions in which the Portuguese mathematicians of the time were interested. And he recalled that the dominant themes were astronomy and algebraic analysis.

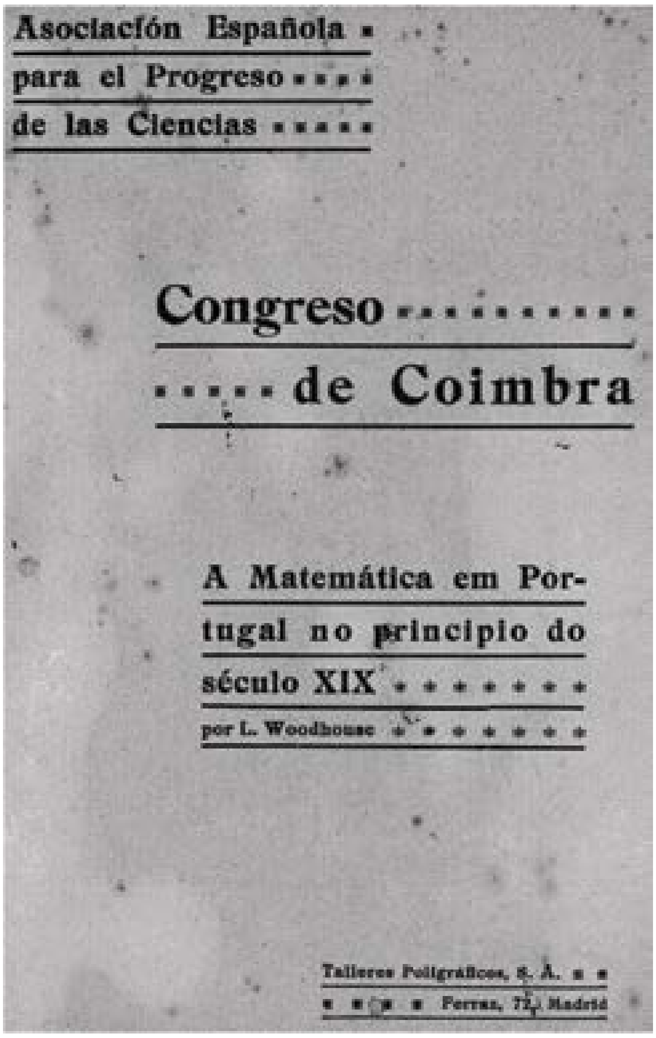
Figure 4.—Cover of the extract of the Method of Horner published in Journal of Mathematical Physics and Natural Sciences.

Woodhouse highlighted the importance of two memoires of João Evangelista Torriani[7] published in the *Memorias de Mathematica e Phisica da Academia Real das Sciencias de Lisboa* (*Memórias dos Correspondentes*): *Deduction of a general formula which comprises Newton's theorems as to the powers of the roots of equations* (*Dedução de huma fórmula geral que compreende os Theoremas de Newton sobre as potencias das raizes das equações* [Torriani 1812]) and P*roving the formulas proposed by Wronski for the general solution of equations* (*Dar a demonstração das formulas propostas por Wronski para a resolução geral das equações* [Torriani 1819]). In the latter of these works, Torriani studied Wronski's memoire regarding the general solution of algebraic equations, which had been discovered a few years earlier. Furthermore, he proves that the solution given by the Polish mathematician is false for equations of degree greater than three. For this work Torriani was given

[7] He was a mathematics professor at the *Royal Navy Academy of Lisbon*, and a member of the *Lisbon Academy of Sciences*.

[8] See *Memorias da Academia Real das Sciencias de Lisboa*, Tomo VI, 1819, pp. CXXIII–CXXVIII.

[9] Echols 1893, p. 180. More recently, Roy Wagner published in *HOPOS: Journal of the International Society for the History of Philosophy of Science* the paper *Wronski's Infinities* in which he refers to Torriani's proof. Unfortunately, Wagner dated Torriani's work from 1819, although it is from 1818 [Wagner 2014, p. 57].



Asociación Española para el Progreso de las Ciencias

Congreso de Coimbra

A Matemática em Portugal no principio do século XIX

por L. Woodhouse

Talleres Poligráficos, S. A. Ferraz, 72, Madrid

Figure 5—Front page of Woddhouse's talk at the Congress for the Progress of Sciences, in Coimbra in 1925.

an award by the *Royal Academy of Sciences of Lisbon*.[8] Moreover, he was internationally acknowledged in the *Bulletin of the New York Mathematical Society*.[9]

Woodhouse applied for membership to the *Royal Academy of Sciences of Lisbon* in 1925 with the following works: *Portuguese contribution to a famous algebra problem* (*Contribuição portuguesa para um celebre problema de algebra*[10] [Woodhouse 1921a]), *The Fundamental Principle of the theory of algebraic equations* (*Princípio Fundamental da teoria das equações algébricas* [Woodhouse 1885]), *The Horner Method and a forgotten Portuguese work (1794)* (*O Método de Horner e um trabalho português esquecido (1749)* [Woodhouse 1924]), *On the Integration of Differential Equations of Dynamics* (*Da integração das equações differenciaes da dynamica* [Woodhouse 1883]), *The Mathematical Renaissance in Portugal in the late 18th century and the Royal Academy of Sciences of Lisbon* (*O Renascimento Matemático em Portugal no fim do século XVIII e a Academia Real das Sciencias de Lisboa*[11] [Woodhouse 1923]), and *The Mathematical Teaching in Portuguese Universities* (*O Ensino Matemático nas Universidades Portuguesas* [Woodhouse 1921]). On the 4th of February of 1926, he was elected corresponding member of the *Lisbon Academy of Sciences*.

Final Notes

Luis Woodhouse was highly regarded as a Professor of Mathematics in Porto. He was remembered by his students for his qualities of rigor, austerity and integrity. In his classes he practiced what he understood to be the teacher's mission: "he has an unquestionable devotion to teach, attracting and not coercing", and he defended a "closer relationship between teachers and students" [Woodhouse 1921]. With a conciliatory character, he was always ready to intervene in unfair situations with students or colleagues. Woodhouse took a critical stance on using experimental methods in teaching Mathematics, stressing that: "Mathematical knowledge systematically acquired by experimental methods or even obtained without the precision of rigor... what a fragile research tool it will be in the hands of those who try to use it! What a restriction, what educational benefits will those who use them get in this way?" [Woodhouse 1921]. Although Woodhouse devoted his main effort to teaching in its various aspects, he played an important role in the dissemination of algebra in Portugal, in the 18th and 19th centuries. At the Luso-Spanish Congresses for the Progress of Sciences, which he attended regularly, he discussed scientific papers presented to the *Academy of Sciences of Lisbon* in the first decades of its creation, which had since been forgotten for many years. Additionally, he stressed that they addressed current issues among the foreign scientific community of the time, sometimes anticipating methods and results. Thus, he intended to show that, contrary to popular belief, interest in mathematics remained alive in the period following Anastacio da Cunha and Monteiro da Rocha, and prior to Daniel Augusto da Silva.

[10] It is the text of the conference Francisco Simões *Margiochi's contribution to the problem of algebraic solving of equations* presented by Woodhouse at the 1st Luso-Spanish Congress held in Porto in 1921.

[11] Woodhouse presented it in the Luso-Spanish Congress held in Coimbra in 1923.

[12] On Anastácio da Cunha see Luís Saraiva in *CIM Bulletin 38–39*, December 2017, pp. 40–45.

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For more information on Luis Woodhouse's life and works, the reader can consult

Luis Inacio Woodhouse (1857–1927). O Professor e a sua Obra. Maria do Céu Silva and Maria Luísa Magalhães. U.Porto Editorial, 2018.

Particle Systems and PDE's – VIII

by Patrícia Gonçalves* and Ana Jacinta Soares**

The International meeting Particle Systems and Partial Differential Equations VIII took place at Instituto Superior Técnico, University of Lisbon, from the 2nd to the 6th December 2019.

This was the eighth edition of the meeting, after five editions at the University of Minho, from 2012 to 2016, one edition at the University of Nice Sophia Antipolis, France, in 2017, and another one at the University of Palermo, Italy, in 2018.

The main idea of the meeting was to bring together active researchers working in the fields of particle systems and partial differential equations to present their latest scientific findings in both areas and to promote discussion on some of their areas of expertise.

The scientific program of this edition included two mini-courses, one by Martin Hairer (Imperial College, London, UK) and another one by Laure Saint-Raymond (ENS Lyon, France), twenty-two talks by invited speakers and two poster sessions. The meeting attracted a vast audience including young researchers and PhD students.

This was a joint organization by the University of Lisbon (CAMGSD, CMAFcIO), the University of Minho (CMAT) and the University of Nice Sophia Antipolis, and had the financial support from ERC, CAMGSD, CMAFcIO, CMAT, CIM, BPI, as well as the assistance by SPM.

For more information, please visit the meeting's website at https://sites.google.com/view/pspdeviii

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Representations of commutative rings via the prime spectrum

by **Jorge Vitória***

Given a commutative noetherian ring R we discuss its representations, i.e. its R-modules. The prime spectrum of R plays a fundamental role, controlling much of the structure of the category of R-modules. We illustrate this in two instances, surveying a parametrisation of localising subcategories and a parametrisation of flat ring epimorphisms.

1 Introduction

Representation theory studies actions of rings on abelian groups or vector spaces. The objects of study are, therefore, R-modules, where R is a ring. Here are some examples of representations.

1. Let $(X, +)$ be an abelian group. Then X admits a natural $\mathbb{Z}$-action, setting an integer n to act on $x \in X$ by iterating $|n|$ times the operation $+$ on x if $n \geq 0$ or on $-x$ if $n < 0$.

2. Let V be a vector space over a field $\mathbb{K}$. The field $\mathbb{K}$ naturally acts on V through the multiplication of vectors by scalars. This turns V into a $\mathbb{K}$-module. If, furthermore, we choose a linear endomorphism $f\colon V \longrightarrow V$, then we can define an action of the polynomial ring $\mathbb{K}[x]$ on V, setting the action of x^n on a vector v as $f^n(v)$. This yields a $\mathbb{K}[x]$-module structure on V.

3. Let G be a finite group acting on a $\mathbb{K}$-vector space V. Then V can be regarded as a $\mathbb{K}G$-module, where $\mathbb{K}G$ is the ring whose elements are formal linear combinations of elements of G over $\mathbb{K}$ (multiplication is defined by the operation in G). For example, if G is the (dihedral) group D_4 of symmetries of a square and $V = \mathbb{R}^2$, then D_4 acts naturally on $\mathbb{R}^2$. This turns V into a module over $\mathbb{R}D_4$, where the multiplication extends linearly the composition of symmetries in D_4.

4. Let $\mathfrak{g}$ be a complex Lie algebra with an action on a complex vector space V. Then V is naturally a module over $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra. For example, if $\mathfrak{g}$ is $\mathfrak{sl}(2,\mathbb{C})$, the Lie algebra of 2×2 matrices with zero trace, then $\mathfrak{g}$ acts naturally on $V = \mathbb{C}^2$. This action turns V into an $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$-module, where $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$ can be shown to be isomorphic to the ring $\mathbb{C}\langle x, y, z\rangle/(xy - yx + 2y, xz - zx - 2z, yz - zy + x)$ with non-commuting variables x, y and z.

A traditional aim of representation theory is to classify all representations of a given ring. This is, in general, a very difficult, if not impossible, task. One might, however, try to classify families of modules satisfying common properties. This macroscopic approach to representation theory has been very successful over the past decades, shifting the focus from individual modules to collections (or *subcategories*) of modules, with the help of category theory and homological algebra.

2 Categories of modules and some special subcategories

Given a ring R, consider the category $\mathsf{Mod}(R)$ whose objects are all (right) R-modules and whose morphisms are the R-linear homomorphisms of abelian groups, i.e. those that preserve the R-action.

Example 1.— The category $\mathsf{Mod}(\mathbb{Z})$ is (equivalent to) the category of abelian groups. Similarly, the category $\mathsf{Mod}(\mathbb{K})$ for a field $\mathbb{K}$ is (equivalent to) the category of $\mathbb{K}$-vector spaces. The category $\mathsf{Mod}(\mathbb{K}[x])$ is (equivalent to)

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the category whose objects are pairs (V, f), where V is a $\mathbb{K}$-vector space and f is a $\mathbb{K}$-linear endomorphism of V. The morphisms in this category between two pairs (V, f) and (W, g) are those linear maps $\alpha\colon V \longrightarrow W$ such that $g \circ \alpha = \alpha \circ f$.

In this note, by subcategory of $\mathsf{Mod}(R)$ we mean a strict and full subcategory (see [9] for basic terminology). This means that a *subcategory* is completely described by its collection of objects. We aim to classify some subcategories of $\mathsf{Mod}(R)$, and the ones we are particularly interested in are determined by *closure conditions*. A subcategory $\mathcal{U}$ of $\mathsf{Mod}(R)$ is said to be closed under

- (co)products if for any (set-indexed) family of R-modules lying in $\mathcal{U}$, its (co)product lies in $\mathcal{U}$.
- (co)kernels if for any map of R-modules $f\colon M \longrightarrow N$, we have that the (co)kernel of f lies in $\mathcal{U}$.
- extensions if for any short exact sequence of R-modules
$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$
if X and Z lie in $\mathcal{U}$, then so does Y.

The following definition sets up the kind of subcategories we will be interested in.

Definition 1.— Let R be a ring. A subcategory $\mathcal{X}$ of $\mathsf{Mod}(R)$ is said to be a **Serre subcategory** if for any short exact sequence of R-modules

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

X and Z lie in $\mathcal{U}$ if and only if Y lies in $\mathcal{U}$. Moreover, we say that $\mathcal{X}$ is **localising** if it is a Serre subcategory which is closed under coproducts and we say that $\mathcal{X}$ is **bireflective** if $\mathcal{X}$ is closed under products and coproducts, kernels and cokernels.

Example 2.— In the category of abelian groups, $\mathsf{Mod}(\mathbb{Z})$, consider the subcategory Tors, formed by all abelian groups for which every element has finite order, and the subcategories $\mathcal{U}_n$ ($n > 1$), formed by all abelian groups annihilated by n (i.e. abelian groups for which the order of every element divides n).

1. Given an abelian group M in Tors, every subgroup of M and every quotient group of M also lies in Tors. The same conclusion can easily be reached for extensions between two groups in Tors, and for coproducts of such groups. Therefore, Tors is a localising subcategory of $\mathsf{Mod}(\mathbb{Z})$.
2. The subcategory Tors is not, however, closed under products. Let M be the product indexed by the natural numbers of the abelian groups $\mathbb{Z}/n\mathbb{Z}$. The elements of M are the sequences $(a_n)_{n\in\mathbb{N}}$ with each a_n being an element in $\mathbb{Z}/n\mathbb{Z}$. It is easy to see that the sequence given by $a_n = 1$ does not have finite order and, thus, M does not lie in Tors.
3. It is easy to see that $\mathcal{U}_n$ is closed under kernels and cokernels and given any family of abelian groups annihilated by n, both its product and coproduct will be annihilated by n. As such, $\mathcal{U}_n$ is a bireflective subcategory of $\mathsf{Mod}(\mathbb{Z})$. However, the short exact sequence
$$0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \xrightarrow{1\mapsto n} \mathbb{Z}/n^2\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$
shows that it is not extension-closed (since $\mathbb{Z}/n^2\mathbb{Z}$ does not lie in $\mathcal{U}_n$). Hence, $\mathcal{U}_n$ is not a Serre subcategory and, as such, it is also not a localising subcategory.

In this note we will provide a way of classifying all localising subcategories and all extension-closed bireflective subcategories of $\mathsf{Mod}(R)$, when R is a commutative noetherian ring. In the remainder of this section, we want to explain why these subcategories are relevant in representation theory.

2.1 Categorical localisations

In order to study the structure of a category such as $\mathsf{Mod}(R)$, one technique is to consider its localisations, or Serre quotient categories. It follows from [4] that for any Serre subcategory $\mathcal{S}$ of $\mathsf{Mod}(R)$, there is an abelian category $\mathsf{Mod}(R)/\mathcal{S}$ and an exact functor

$$q_{\mathcal{S}}\colon \mathsf{Mod}(R) \longrightarrow \mathsf{Mod}(R)/\mathcal{S}$$

that sends all objects of $\mathcal{S}$ to the zero object and that, moreover, is universal with respect to this property. It is then to be expected that by studying the abelian category $\mathsf{Mod}(R)/\mathcal{S}$ together with the associated Serre subcategory $\mathcal{S}$, one might be able to glue data to the larger category $\mathsf{Mod}(R)$. While this may, in general, still be quite difficult, the task becomes easier if we require that $\mathcal{S}$ is also closed under coproducts, i.e. if $\mathcal{S}$ is localising. Indeed, given a localising subcategory $\mathcal{S}$ of $\mathsf{Mod}(R)$, both the inclusion functor of $\mathcal{S}$ into $\mathsf{Mod}(R)$ and the quotient functor $q_{\mathcal{S}}$ admit right adjoints ([4]). These adjoints can then be used to better relate the structures of these three categories.

EXAMPLE 3.— Consider again the category $\mathsf{Mod}(\mathbb{Z})$ and the localising subcategory Tors from Example 2. It turns out that the categorical quotient $\mathsf{Mod}(\mathbb{Z})/\mathsf{Tors}$ is equivalent to $\mathsf{Mod}(\mathbb{Q})$, i.e. the category of $\mathbb{Q}$-vector spaces. Moreover, the quotient functor $q_{\mathsf{Tors}}\colon \mathsf{Mod}(\mathbb{Z}) \longrightarrow \mathsf{Mod}(\mathbb{Z})/\mathsf{Tors}$ can be shown to be naturally equivalent to the tensor product $-\otimes_{\mathbb{Z}}\mathbb{Q}$, and its right adjoint identifies $\mathsf{Mod}(\mathbb{Z})/\mathsf{Tors}$ with the torsionfree and divisible abelian groups.

2.2 Ring epimorphisms

Epimorphisms in the category of (unital) rings are not just, as one naively could expect, surjective ring homomorphisms. In fact, it is an easy exercise to check that the embeeding of $\mathbb{Z}$ into $\mathbb{Q}$ is a ring epimorphism or, in other words, that any ring homomorphism from $\mathbb{Q}$ to a ring C is uniquely determined by the image of the integers. As it turns out, any ring of fractions of a ring R (where one formally inverts a suitably selected subset of R) yields a ring epimorphism from R.

Ring epimorphisms from a ring R are relevant in the representation theory of R because they correspond bijectively (up to a suitable notion of equivalence) to bireflective subcategories of $\mathsf{Mod}(R)$ ([5]). Note that any ring homomorphism $f\colon R \longrightarrow S$ induces an R-module structure on any (right) S-module M: just set the action of $r \in R$ on $m \in M$ by $m \cdot r := mf(r)$. This defines a faithful functor

$$f_*\colon \mathsf{Mod}(S) \longrightarrow \mathsf{Mod}(R)$$

which is called *restriction of scalars*. Moreover, it turns out that f is an epimorphism if and only if every R-linear map between S-modules is also S-linear (or, in other words, f_* is full). The assignment can now be defined by associating to a ring epimorphism $f\colon R \longrightarrow S$ the subcategory of $\mathsf{Mod}(R)$ obtained as the essential image of the functor f_* (which is naturally equivalent to $\mathsf{Mod}(S)$).

Classifying bireflective subcategories of $\mathsf{Mod}(R)$ then amounts to classifying families of modules that share the property of being modules over some *epimorphic image* of R (in a compatible way). In this note we restrict our aim to classifying those bireflective subcategories which are also extension-closed. This is because the associated ring epimorphisms exhibit a better homological behaviour (see [6] for details).

EXAMPLE 4.— Consider yet again the category $\mathsf{Mod}(\mathbb{Z})$ and the bireflective subcategories $\mathcal{U}_n$ $(n > 1)$ from Example 2. The ring epimorphism associated to $\mathcal{U}_n$ turns out to be $f_n\colon \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$. This is because the abelian groups admitting a natural $\mathbb{Z}/n\mathbb{Z}$-module structure are precisely those annihilated by n. These bireflective subcategories are not, however, extension-closed (see Example 2).

Consider now the ring epimorphism $f\colon \mathbb{Z} \longrightarrow \mathbb{Q}$. The restriction functor $f_*\colon \mathsf{Mod}(\mathbb{Q}) \longrightarrow \mathsf{Mod}(\mathbb{Z})$ is naturally equivalent to the right adjoint of q_{Tors} (see Example 3). One can then conclude that indeed the essential image of f_* (or, in other words, the bireflective subcategory associated to f) consists of the torsionfree divisible abelian groups and it is, therefore, extension-closed.

3 The prime spectrum

We turn our focus to the study of commutative noetherian rings. The structure of these rings is rather well-understood, and a key part of that understanding comes from their prime ideals. It is therefore not surprising that prime ideals also play an important role in the classification results we aim to survey.

Let R be a commutative ring. Recall that an ideal $\mathfrak{p}$ of R is said to be **prime** if $\mathfrak{p} \neq R$ and whenever ab lies in $\mathfrak{p}$ for two elements a and b in R, then at least one of them must lie in $\mathfrak{p}$. The set of prime ideals of R is called the **spectrum of** R and is denoted by $\mathsf{Spec}(R)$. The spectrum of R admits a natural partial order induced by inclusion of ideals and, moreover, it can also be endowed with an interesting topology, the **Zariski topology**, by declaring the closed subsets to be the ones of the form

$$V(I) := \{\mathfrak{p} \in \mathsf{Spec}(R)\colon I \subseteq \mathfrak{p}\}$$

for some ideal I of R. It is an easy exercise to check that this indeed yields a topology. Note that the closed points of $\mathsf{Spec}(R)$ are then precisely the maximal ideals of R. There is a full characterisation of the topological spaces arising as Zariski spectra of commutative rings. These are precisely the compact spaces which are T_0[1] and for which the compact open subsets form a basis for the topology and are closed under finite intersections ([7]). These are the so-called **spectral spaces**.

EXAMPLE 5.— Let us go back to $R = \mathbb{Z}$. The prime ideals are precisely those generated by a prime natural number and, in addition, the zero ideal (because $\mathbb{Z}$ is an integral domain!). Every non-zero prime ideal is maximal, and thus, we have countably many closed points, and a point (the zero ideal) whose closure is the whole spectrum. The poset of prime ideals can be depicted as in

[1] For every pair of distinct points there is an open subset containing one and not the other

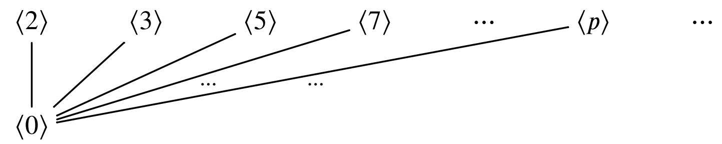

Figure 1: $\mathsf{Spec}(\mathbb{Z})$.

Figure 1 where the convention is that an edge represents an inclusion from the lower row to the upper row.

Given a commutative ring R and a prime ideal $\mathfrak{p}$ of R, we define the **height** of $\mathfrak{p}$ (denoted by $\mathsf{ht}(\mathfrak{p})$) as the largest integer n for which there is a chain of prime ideals of R of the form

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}.$$

The **Krull dimension** of R (denoted by $\mathsf{Kdim}(R)$) is then defined to be the supremum of the heights of its prime ideals, i.e. $\mathsf{Kdim}(R) := \sup\{\mathsf{ht}(\mathfrak{p}) : \mathfrak{p} \in \mathsf{Spec}(R)\}$.

We will also restrict ourselves to the class of commutative noetherian rings. Recall that a commutative ring is said to be **noetherian** if there are no infinite strictly ascending chains of ideals. Examples of such rings are $\mathbb{Z}$ or quotients of polynomial rings by any ideal: $\mathbb{K}[x_1, ..., x_n]/I$ ($\mathbb{K}$ is a field). It is an easy exercise to check that the spectrum of a commutative noetherian ring R is a noetherian space (any ascending chain of open subsets stabilises). It can also be shown that, for every ideal I of such a ring R, there is a finite set of prime ideals $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$ such that $V(I) = \cup_{i=1}^{k} V(\mathfrak{p}_i)$. This implies that the Zariski topology on $\mathsf{Spec}(R)$ can be easily recovered from the poset of prime ideals. Moreover, we also have that the poset of prime ideals over a commutative noetherian ring satisfies the following theorem.

Theorem 2.— [8, Theorem 144] Let R be a commutative noetherian ring. Whenever there are prime ideals $\mathfrak{p}, \mathfrak{t}$ and $\mathfrak{q}$ such that $\mathfrak{p} \subsetneq \mathfrak{t} \subsetneq \mathfrak{q}$, then there are also infinitely many prime ideals $\mathfrak{s}$ such that $\mathfrak{p} \subsetneq \mathfrak{s} \subsetneq \mathfrak{q}$.

In particular, any commutative noetherian ring R which contains a prime ideal which is neither a maximal nor a minimal prime ideal has the property that $\mathsf{Spec}(R)$ is infinite. As a consequence, if a commutative noetherian ring R has only finitely many prime ideals, then R must have Krull dimension at most 1.

By definition, the complement of a prime ideal is closed under multiplication (and contains the unit of the ring). As such, it is a good candidate for a set of denominators in a ring of fractions. Given a prime ideal $\mathfrak{p}$ in a commutative ring R, we may consider the ring of fractions obtained by adding to R formal inverses to the elements in the complement $S = R \setminus \mathfrak{p}$, and we denote it by $R_\mathfrak{p}$. Moreover, there is a natural ring homomorphism $\psi_S : R \longrightarrow R_\mathfrak{p}$ sending an element r to the fraction $\frac{r}{1}$. This ring homomorphism is, in fact, a ring epimorphism and its associated bireflective subcategory is extension-closed. The following proposition recalls how $\mathsf{Spec}(R)$, $\mathsf{Spec}(R/\mathfrak{p})$ and $\mathsf{Spec}(R_\mathfrak{p})$ are related.

Proposition 3.— Let R be a commutative noetherian ring and $\mathfrak{p}$ a prime ideal of R. Then we have that

1. $\mathsf{Spec}(R/\mathfrak{p})$ is homeomorphic to the subspace $V(\mathfrak{p})$ of $\mathsf{Spec}(R)$;
2. $\mathsf{Spec}(R_\mathfrak{p})$ is homeomorphic to the subspace $\Lambda(\mathfrak{p}) := \{\mathfrak{q} \in \mathsf{Spec}(R) : \mathfrak{q} \subseteq \mathfrak{p}\}$ of $\mathsf{Spec}(R)$;
3. $\mathsf{Kdim}(R_\mathfrak{p}) = \mathsf{ht}(\mathfrak{p})$ and $\mathsf{Kdim}(R/\mathfrak{p}) \leq \mathsf{Kdim}(R) = \mathsf{ht}(\mathfrak{p})$.

Example 6.— Let R be again the ring of integers $\mathbb{Z}$, and let $\mathfrak{p}$ be the ideal generated by the prime 2. Note that $\mathbb{Z}$ is a ring of Krull dimension 1 and $\langle 2 \rangle$ is a prime ideal of height 1. Then $R/\mathfrak{p}$ is the field $\mathbb{Z}/2\mathbb{Z}$ which has a unique prime ideal: the zero ideal. This fits with the fact that $\mathsf{Spec}(\mathbb{Z}/2\mathbb{Z})$ ought to be homeomorphic to the subspace of $\mathsf{Spec}(\mathbb{Z})$ given by $V(\langle 2 \rangle) = \{\langle 2 \rangle\}$. A similar check can be done for the localisation of $\mathbb{Z}$ at $\langle 2 \rangle$. Indeed, the ring of fractions $\mathbb{Z}_{\langle 2 \rangle}$ can be described as a subring of $\mathbb{Q}$ as

$$\mathbb{Z}_{\langle 2 \rangle} = \{\frac{a}{b} \in \mathbb{Q} : \mathsf{gcd}(b, 2) = 1\}$$

where gcd denotes de greatest common divisor. The spectrum of $\mathbb{Z}_{\langle 2 \rangle}$ is indeed homeomorphic to $\Lambda(\langle 2 \rangle)$ and consists of two prime ideals: the zero ideal and the maximal ideal $2\mathbb{Z}_{\langle 2 \rangle}$.

4 Classification results

We want to illustrate the idea that for a commutative noetherian ring R, the structure of $\mathsf{Spec}(R)$ controls much of the representation theory of R, i.e. the structure of $\mathsf{Mod}(R)$. For this purpose we introduce the following notion of support. Given an R-module M, consider the set of primes

$$\mathsf{Supp}(M) = \{\mathfrak{p} \in \mathsf{Spec}(R) : \exists i \geq 0 : \mathsf{Tor}_i^R(M, k(\mathfrak{p})) \neq 0\}$$

where $k(\mathfrak{p}) := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is the residue field of R at $\mathfrak{p}$ and $\mathrm{Tor}_i^R(-,k(\mathfrak{p}))$ is the i-th derived functor of $-\otimes_R k(\mathfrak{p})$. We refer to [2] for further details.

If M is a finitely generated R-module, $\mathrm{Supp}(M)$ is easier to calculate: it coincides with $\{\mathfrak{p}\in\mathrm{Spec}(R) : M\otimes_R R_\mathfrak{p}\neq 0\}$ ([2, Lemma 2.2]). From a geometric standpoint this is the support of the associated coherent sheaf $\widetilde{M}$ over $\mathrm{Spec}(R)$, i.e. the set of points p in $\mathrm{Spec}(R)$ where the stalk $\widetilde{M}_p$ does not vanish.

We can also consider support of subcategories of R-modules: we define $\mathrm{Supp}(\mathcal{U})$, for $\mathcal{U}$ a subcategory of $\mathrm{Mod}(R)$, to be the union of $\mathrm{Supp}(U)$ where U runs over all R-modules in $\mathcal{U}$. This provides us with a way to assign a subset of $\mathrm{Spec}(R)$ to any subcategory of $\mathrm{Mod}(R)$, and this is the key tool to the classification results we will discuss next.

4.1 Localising subcategories

The first theorem we want to present is a well-known classification by support of localising subcategories of modules over a commutative noetherian ring. Moreover, it turns out that the subsets of the spectrum that arise as support of localising subcategories are arbitrary unions of closed sets. Such subsets are called **specialisation-closed subsets**. Equivalently, a subset V of the spectrum of a commutative noetherian ring R is specialisation-closed if for any $\mathfrak{p}$ in V, any prime $\mathfrak{q}$ containing $\mathfrak{p}$ must also lie in V.

It can be shown that a specialisation-closed subset V has minimal elements (for the partial order induced by inclusion) and that these completely determine V. Indeed, if V is specialisation-closed and $(\mathfrak{p}_i)_{i\in I}$ is the collection of the minimal elements of V, then $V = \cup_{i\in I} V(\mathfrak{p}_i)$.

Theorem 4.— [9, Ch.VI,§5, §6] For a commutative noetherian ring R, the assignment of support yields a bijection between

1. localising subcategories of $\mathrm{Mod}(R)$;
2. specialisation-closed subsets of $\mathrm{Spec}(R)$.

Note that the inverse assignment sends a specialisation-closed subset V of $\mathrm{Spec}(R)$ to the subcategory of all modules whose support is contained in V.

Remark 1.— It can be shown that for a commutative noetherian ring R, there is a topology on $\mathrm{Spec}(R)$ for which the open sets are the (Zariski) specialisation-closed subsets above described. This is called the Hochster dual topology. For more details on this duality, we refer to [7].

Recall that an R-module M is said to be **flat** if $-\otimes_R M$ is an exact functor or, equivalently, if the derived functors $\mathrm{Tor}_i^{\mathbb{R}}(-,\mathbb{M})$ are identically zero.

Example 7.— Consider again the localising subcategory Tors of $\mathrm{Mod}(\mathbb{Z})$ (see Example 2). We show that $\mathrm{Supp}(\mathrm{Tors}) = \mathrm{Spec}(\mathbb{Z})\setminus\{\langle 0\rangle\}$ (this is specialisation-closed: it is the set of all maximal ideals of $\mathbb{Z}$).

First observe that $\mathbb{Z}_{\langle 0\rangle} = \mathbb{Q}$ and, therefore, $k(\langle 0\rangle) = \mathbb{Q}$. Since $\mathbb{Q}$ is flat over $\mathbb{Z}$, we have that $\mathrm{Tor}_i^{\mathbb{Z}}(-,\mathbb{Q}) = 0$ for all $i\geq 1$. Hence $\langle 0\rangle$ lies in $\mathrm{Supp}(M)$ (for some abelian group M) if and only if $M\otimes_\mathbb{Z}\mathbb{Q}\neq 0$. Observe, however, that if M is an abelian group where every element has finite order, $M\otimes_\mathbb{Z}\mathbb{Q} = 0$. Indeed, if m is an element of M of order $n\geq 1$, then for any $q\in\mathbb{Q}$, we have that $m\otimes q = mn\otimes\frac{q}{n} = 0$. This shows that $\mathrm{Supp}(\mathrm{Tors})\subseteq\mathrm{Spec}(\mathbb{Z})\setminus\{\langle 0\rangle\}$.

To prove the converse we show that $\mathrm{Supp}(\mathbb{Z}/p\mathbb{Z}) = \{\langle p\rangle\}$ for any prime p in $\mathbb{Z}$. Since $\mathbb{Z}/p\mathbb{Z}$ is finitely generated, then it is enough to compute $\mathbb{Z}/p\mathbb{Z}\otimes_\mathbb{Z}\mathbb{Z}_{\langle q\rangle}$ for any prime q. Since p is invertible in $\mathbb{Z}_{\langle q\rangle}$ whenever $p\neq q$, a similar argument to the one above shows that $\mathbb{Z}/p\mathbb{Z}\otimes_\mathbb{Z}\mathbb{Z}_{\langle q\rangle} = 0$ for all $q\neq p$. Finally, it can be shown that $\mathbb{Z}/p\mathbb{Z}\otimes_\mathbb{Z}\mathbb{Z}_{\langle p\rangle}\cong k(\mathfrak{p})\neq 0$. This proves the claim.

4.2 Extension-closed bireflective subcategories

We now turn our attention to extension-closed bireflective subcategories of $\mathrm{Mod}(R)$, where R is commutative and noetherian as before. As mentioned earlier, the condition of extension-closure imposes some *nice* homological behaviour on the associated ring epimorphism. It turns out that, in the context of commutative noetherian rings, the extension-closure requirement gives us an *extremely nice* homological behaviour: flatness.

Theorem 5.— [1] Let R be a commutative noetherian ring and let $f\colon R\longrightarrow S$ be a ring epimorphism. Then the bireflective subcategory associated to f is extension-closed if and only if S is flat as an R-module.

Ring epimorphisms as those in the theorem above are called **flat ring epimorphisms**. The theorem is far from being true outside the commutative noetherian setting.

A classification of extension-closed bireflective subcategories therefore amounts to a classification of flat ring epimorphisms (up to equivalence). Observe moreover that, given a flat ring epimorphism $f\colon R\longrightarrow S$, the R-modules M for which $M\otimes_R S = 0$ form a localising subcategory of $\mathrm{Mod}(R)$ which is supported, by Theorem 4, on a specialisation-closed subset V. This gives us a way of assigning a specialisation-closed subset in $\mathrm{Spec}(R)$ to any given equivalence class of flat ring epimorphisms of R. What are the properties of this assignment?

Theorem 6.— [1] Let R be a commutative noetherian ring. Let Ψ be the assignment sending the equivalence class of a flat ring epimorphism $f\colon R \longrightarrow S$ to the support of the subcategory of R-modules M such that $M \otimes_R S = 0$. Then:

1. Ψ is an injective assignment;
2. The image of Ψ is contained in the set of specialisation-closed subsets of $\mathsf{Spec}(R)$ whose minimal elements have height at most 1;
3. If $\mathsf{Kdim}(R) \leq 1$ or if R is regular, then Ψ induces a bijection between flat ring epimorphisms up to equivalence and specialisation-closed subsets of $\mathsf{Spec}(R)$ whose minimal elements have height at most 1.

Recall that the commutative noetherian regular rings are precisely those of finite global dimension. Geometrically, regularity can be interpreted as the smoothness of the corresponding affine scheme. It is always possible to describe the image of Ψ, and the assumptions in point (3) of the theorem can be significantly relaxed - all of this at the expense of requiring some more technical tools ([1]). The following corollary characterises completely the support of extension-closed bireflective subcategories under the same assumptions as in (3) above. This follows from the fact that this support will be the complement of the specialisation-closed subset arising through Ψ.

Corollary 7.— [1] Let R be a commutative noetherian ring and suppose that R either has Krull dimension at most one or is a regular ring. Then the assignment of support establishes a bijection between

1. extension-closed bireflective subcategories of $\mathsf{Mod}(R)$;
2. subsets of $\mathsf{Spec}(R)$ whose complement is specialisation-closed and the minimal primes of the complement have height at most one.

Remark 2.— Note that for commutative noetherian rings of Krull dimension at most one, the assumption on the height of minimal primes in the specialisation-closed subsets is automatically fulfilled. Therefore, if R has Krull dimension at most one, there is a bijection between flat ring epimorphisms up to equivalence (or extension-closed bireflective subcategories) and specialisation-closed subsets of $\mathsf{Spec}(R)$.

Example 8.— We give a complete list of flat ring epimorphisms (up to equivalence) starting in $\mathbb{Z}$. Such ring epimorphisms are classified by specialisation-closed subsets of $\mathsf{Spec}(\mathbb{Z})$, and these are: $\varnothing$, elements of $\mathscr{P}(\mathsf{Max}(\mathbb{Z}))$ (the power set of maximal ideals of $\mathbb{Z}$) and $\mathsf{Spec}(\mathbb{Z})$.

1. Let us consider the first specialisation-closed subset $V = \varnothing$. The corresponding extension-closed bireflective subcategory $\mathscr{B}$ must be supported on $\mathsf{Spec}(\mathbb{Z}) \setminus V = \mathsf{Spec}(\mathbb{Z})$. Therefore, we have $\mathscr{B} = \mathsf{Mod}(\mathbb{Z})$ and the associated flat ring epimorphism (up to equivalence) is the identity on $\mathbb{Z}$.
2. Let V be the set of prime ideals determined by a subset of prime natural numbers P. The associated flat ring epimorphism can be checked to be (up to equivalence) the map
$$f_P\colon \mathbb{Z} \longrightarrow \mathbb{Z}[P^{-1}]$$
where $\mathbb{Z}[P^{-1}]$ can be identified with the subring of $\mathbb{Q}$ consisting of the fractions a/b such that b has no prime factors which are not in the set P and $f_P(r) = r/1$.
3. Finally, let V be $\mathsf{Spec}(\mathbb{Z})$. Since its complement is empty, the flat ring epimorphism (up to equivalence) that we are looking for is the trivial one: $f\colon \mathbb{Z} \longrightarrow 0$.

The situation described in the example above is extremely nice and not at all typical. In this case we were able to describe all flat ring epimorphisms as rings of fractions, but in general we may need more robust localisation techniques. This is explored in detail in [1].

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REPORT

Global Portuguese Mathematicians

by Samuel Lopes*, Jorge Milhazes de Freitas** and Diogo Oliveira e Silva***

Global Portuguese Mathematicians is a biennial conference series bringing together Portuguese mathematicians working worldwide. It aims to disseminate high-level research, promoting new collaborations as well as connecting the Portuguese-speaking mathematical community.

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In 2019, the second edition of the conference *Global Portuguese Mathematicians* took place at the Department of Mathematics of the University of Porto on June 24–26, 2019. With speakers from a broad range of experiences and stages in their careers, they came from all corners of the world and work in a wide variety of mathematical fields, including algebra, analysis, dynamical systems, geometry, number theory and statistics.

Further information is available on the conference website https://cmup.fc.up.pt/matematicos/.

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All invited speakers from the first and second editions of the conference *Global Portuguese Mathematicians* have been invited to submit a 10-page manuscript with a summary of their talks. These proceedings will be published as a special issue of the Boletim da Sociedade Portuguesa de Matemática in early 2020.

The organizers Samuel Lopes, Jorge Milhazes de Freitas and Diogo Oliveira e Silva have been invited to act as *Guest Editors* for this special issue.

The third edition of the conference *Global Portuguese Mathematicians* is tentatively scheduled to take place at the Department of Mathematics of the University of Coimbra in the Summer of 2021.

Solid n-twisted Möbius strips as real algebraic surfaces

by **Stephan Klaus***

We construct explicit polynomials p_n in three real variables x, y and z such that the associated affine variety $p_n^{-1}(0)$ gives a small tubular neighborhood of the n-twisted Möbius strips. The degree of p_n is given by $4+2n$. We give visualizations up to twisting number $n=6$ using the free software *surfer* of the open source platform Imaginary.

1 Introduction

It is a well-known fact from differential topology (e.g., see [1], chapter 1, §4) that for a smooth function $f : \mathbb{R}^m \to \mathbb{R}$ and a regular value $y \in \mathbb{R}$, the level set $M := f^{-1}(y)$ is a smooth $(m-1)$-dimensional hypersurface. Moreover, M has no boundary and is orientable. In case that M is compact and connected, it separates $\mathbb{R}^m$ in two regions, the *inside* and the *outside*, by the generalized Jordan-Brouwer separation theorem ([1], chapter 2, §5).

Hence, it is not possible to construct the Möbius strip as a smooth level set in this way as it has a 1-dimensional boundary and as it is non-orientable.

However, in [2] we have given polynomials of degree 6, 8 and 10 such that the visualization of the level sets $f^{-1}(0)$ (using the surfer-software [5] of the open source platform Imaginary) give Möbius strips with 1, 2 and 3 twists, respectively.

This apparent contradiction can be easily explained: These surfaces are not Möbius strips on the nose but thickened versions, i.e. boundaries of small tubular neighborhoods. We call them *solid* Möbius strips. Our method of construction (by *rotation with twisting*) will be explained in the next section. We remark that a similar method was used in [3] to construct the solid trefoil knot with a polynomial of order 14. An overview over these and other constructions of interesting surfaces can be found in [4].

The reason that we come back to the construction of Möbius strips is that we will present here a simplified construction which works for any number of twists and gives an explicit polynomial, whereas the method in [2] was ad-hoc in the degrees considered.

As a last remark we mention a theorem of Whitney [6]: For any closed subset $A \subset \mathbb{R}^m$ there exists a smooth function $f : \mathbb{R}^m \to \mathbb{R}$ such that $A := f^{-1}(0)$. Of course, in general 0 is then not a regular value of f.

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2 Rotation with n-twisting

We start with an affine real algebraic curve given as the level set of a polynomial $f(t,z) \in \mathbb{R}[t,z]$ for the value 0. In our case,

$$f(t,z) = \left(\frac{t}{a}\right)^2 + \left(\frac{z}{b}\right)^2 - 1$$

is an ellipse with semiaxes $0 < a < 1$ and $0 < b < 1$. We denote the eccentricity (with $a \leq b$) as the geometrically relevant parameter by

$$e := \frac{1}{b}\sqrt{b^2 - a^2}$$

and we are mainly interested in big eccentricity, e.g. $e \approx 1$.

Now, rotation with twisting denotes a mixture of two rotation motions. The first movement concerns the t-axis which we let rotate around the z-axis such that it

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spans the (x, y)-plane:

$$x = Ct, \qquad C := \cos\phi$$
$$y = St, \qquad S := \sin\phi$$

Here, ϕ denotes the angle between the t- and the x-axis. Note that $C^2+S^2 = 1$. At the same time we impose a second rotation ('twisting') within the (t, z)-plane around a center at $(1, 0)$ with angle ψ:

$$t' - 1 = c(t-1) + sz$$
$$z' = -s(t-1) + cz$$

Here, $c := \cos(\psi)$, $s := \sin(\psi)$ and $c^2 + s^2 = 1$. This variable transformation leads to the *master equation*

$$f(c(t-1) + sz, -s(t-1) + cz) = 0$$

where both rotation motions are coupled as an *n-twisted rotation*, i.e.

$$\psi = \frac{n}{2}\phi$$

because a twist is a *half* rotation.

In our case of an ellipse we get (after multiplication with a^2b^2 in order to get rid of denominators)

$$\begin{aligned} a^2b^2 &= b^2(c(t-1)+sz)^2 + a^2(-s(t-1)+cz)^2 \\ &= c^2(b^2(t-1)^2 + a^2z^2) + 2cs(b^2-a^2)(t-1)z+ \\ &\quad +s^2(a^2(t-1)^2 + b^2z^2). \end{aligned}$$

3 Elimination of the rotation and twisting variables

Now we need to eliminate the variables C, S, c, s and t from the master equation in the section above in order to get a single equation $p(x, y, z) = 0$. At a first glance one could think that this is not possible with a polynomial p because of the transcendental functions cos and sin. However, with the de Moivre formula

$$\exp(\phi i)^n = \exp(n\phi i) = \exp(2\psi i) = \exp(\psi i)^2$$

we get for the real and the imaginary parts:

$$\begin{aligned} p_n(C,S) &:= \Re((C+iS)^n) = C^n - \binom{n}{2}C^{n-2}S^2+ \\ &\quad +\binom{n}{4}C^{n-4}S^4 \mp \cdots = c^2 - s^2 \\ q_n(C,S) &:= \Im((C+iS)^n) = \binom{n}{1}C^{n-1}S- \\ &\quad -\binom{n}{3}C^{n-3}S^3 \pm \cdots = 2cs \end{aligned}$$

with homogeneous polynomials p_n and q_n of order n. Because of $c^2 + s^2 = 1$ we obtain

$$c^2 = \frac{1}{2}(1 + p_n(C,S)), \qquad s^2 = \frac{1}{2}(1 - p_n(C,S)).$$

Now we use $C = x/t$ and $S = y/t$ and we insert the expressions for c^2, $2cs$ and s^2 into our master equation. Using the homogeneity of p_n and q_n we multiply the equation with t^n. This yields

$$(*) \qquad \begin{aligned} &\frac{1}{2}(t^n + p_n(x,y))(b^2(t-1)^2 + a^2z^2)+ \\ &\qquad +q_n(x,y)(b^2-a^2)(t-1)z+ \\ &+\frac{1}{2}(t^n - p_n(x,y))(a^2(t-1)^2 + b^2z^2) - a^2b^2t^n = 0. \end{aligned}$$

Hence we have eliminated the rotation and twisting variables C, S, c and s. The last step is the algebraic elimination of the variable t. (Of course, we could just replace t by $\sqrt{x^2+y^2}$ but this would not give a polynomial equation.)

4 Elimination of the variable t

This last step can be achieved in a more general context. Suppose we have given a polynomial $g(x, y, z, t) \in \mathbb{R}[x, y, z, t]$ and a polynomial $h(x, y, z) \in \mathbb{R}[x, y, z]$ and we want to eliminate t from the system

$$g(x,y,z,t) = 0$$
$$h(x,y,z) = t^2$$

We are in particular interested in the case of $h(x, y, z) = x^2 + y^2$. The algebraic elimination can be achieved by splitting g in even and odd powers of t:

$$g(x,y,z,t) = g_0(x,y,z,t^2) + tg_1(x,y,z,t^2)$$

Then from $g = 0$ we get $g_0 = -tg_1$ and squaring this equations yields

$$g_0(x,y,z,h(x,y,z))^2 = h(x,y,z)g_1(x,y,z,h(x,y,z))^2$$

which is the final solution of our elimination problem above.

In order to apply this procedure to equation $(*)$, we sort the terms according to powers of t. From the structure of the equation with its 4 terms, there appear only the powers t^{n+2}, t^{n+1}, t^n, t^2, $t^1 = t$ and $t^0 = 1$, and $(*)$ is equivalent to the equation in figure 1.

The matrix-like shape with 4 rows reflects the origin of each entry in the bracket from one the 4 terms of $(*)$. Of course, we use here the abbreviations $p_n = p_n(x, y)$ and $q_n = q_n(x, y)$. A further simplification with

$$A := a^2 + b^2 = b^2(1+\epsilon), \qquad B := b^2 - a^2 = b^2(1-\epsilon),$$
$$D := a^2b^2 = b^4\epsilon,$$

where e denotes the eccentricity and $\epsilon := 1-e^2$ (i.e., $\epsilon \approx 0$

$$
\begin{array}{rcccccc}
t^{n+2} & (& \frac{1}{2}b^2 & & +\frac{1}{2}a^2 & &) \\
+t^{n+1} & (& -b^2 & & -a^2 & &) \\
+t^n & (& \frac{1}{2}(b^2+a^2z^2) & & +\frac{1}{2}(a^2+b^2z^2) & -a^2b^2 &) \\
+t^2 & (& \frac{1}{2}p_nb^2 & & -\frac{1}{2}p_na^2 & &) \\
+t & (& -p_nb^2 & q_n(b^2-a^2)z & +p_na^2 & &) \\
+1 & (& \frac{1}{2}p_n(b^2+a^2z^2) & -q_n(b^2-a^2)z & -\frac{1}{2}p_n(a^2+b^2z^2) & &) \\
=0 & & & & & &
\end{array}
$$

Figure 1

is a thin ellipse and $\epsilon = 1$ gives a round torus), yields:

$$
(**) \qquad \begin{aligned} &\frac{A}{2}t^{n+2} - At^{n+1} + (\frac{A}{2}(1+z^2) - D)t^n + \frac{B}{2}p_nt^2 \\ &-B(p_n - q_nz)t + \frac{B}{2}p_n(1-z^2) - Bq_nz = 0 \end{aligned}
$$

Now, in order to apply the above method of t-elimination, we have to distinguish the two cases of even and odd twisting numbers n.

4.1 Even twisting number $n = 2m$

By sorting the odd t-powers to the right side we get from $(**)$ that

$$
\begin{aligned} &\left(\frac{A}{2}\left(1+t^2+z^2\right) - D\right)t^n + \frac{B}{2}p_n\left(1+t^2-z^2\right) - Bq_nz = \\ &\qquad = t[At^n + B(p_n - q_nz)]. \end{aligned}
$$

Thus we have proved the following result by applying t-elimination:

Theorem 1.— For an even twisting number $n = 2m$, the n-twisted solid Möbius strip is given as an affine real algebraic surface for the following polynomial equation in x, y and z of degree $4+2n$:

$$
\begin{aligned} &\Big[\left(\frac{A}{2}\left(1+x^2+y^2+z^2\right) - D\right)\left(x^2+y^2\right)^m + \\ &\quad \frac{B}{2}p_n\left(1+x^2+y^2-z^2\right) - Bq_nz\Big]^2 = \\ &\qquad = (x^2+y^2)[A(x^2+y^2)^m + B(p_n-q_nz)]^2. \end{aligned}
$$

4.2 Odd twisting number $n = 2m+1$

By sorting the odd t-powers to the right side we get from $(**)$ that

$$
\begin{aligned} &-At^{n+1} + \frac{B}{2}p_n\left(1+t^2-z^2\right) - Bq_nz = \\ &\qquad = t\left[-\left(\frac{A}{2}\left(1+t^2+z^2\right) - D\right)t^{n-1} + B(p_n - q_nz)\right]. \end{aligned}
$$

Thus we have proved the following result by applying t-elimination:

Theorem 2.— For an odd twisting number $n = 2m+1$, the n-twisted solid Möbius strip is given as an affine real algebraic surface for the following polynomial equation in x, y and z of degree $4+2n$:

$$
\begin{aligned} &\left[-A\left(x^2+y^2\right)^{m+1} + \frac{B}{2}p_n\left(1+x^2+y^2-z^2\right) - Bq_nz\right]^2 = \\ &= (x^2+y^2)\Big[-\left(\frac{A}{2}\left(1+x^2+y^2+z^2\right) - D\right)(x^2+y^2)^m + \\ &\qquad +B(p_n - q_nz)\Big]^2. \end{aligned}
$$

5 Visualization for small values of n

The first three cases $n = 1, 2$ or 3 were already considered in our paper [2] with more clumsy computations. Our new general formula recovers our former results.

5.1 Twisting number $n = 0$

Our formula also works in the untwisted case $n = 0$. Then we have $p_0 = 1$ and $q_0 = 0$ and we get the following polynomial equation of order 4:

$$
\begin{aligned} &\left[\frac{A}{2}\left(1+x^2+y^2+z^2\right) - D + \frac{B}{2}\left(1+x^2+y^2-z^2\right)\right]^2 = \\ &\qquad = (x^2+y^2)[A+B]^2. \end{aligned}
$$

This gives not only the usual torus, but also for small a a surface of shape of a solid (finite) cylinder barrel and for small b a surface of shape of a solid annulus. See figure 2 (*surfer* code included).

5.2 Twisting number $n = 1$

This is the classical Möbius strip. We have $m = 0$, $p_1 = x$, $q_1 = y$ and we get the following polynomial equation of order 6:

$$
\begin{aligned} &\left[-A\left(x^2+y^2\right) + \frac{B}{2}x\left(1+x^2+y^2-z^2\right) - Byz\right]^2 = \\ &= \left(x^2+y^2\right)\left[-\frac{A}{2}\left(1+x^2+y^2+z^2\right) + D + B(x-yz)\right]^2. \end{aligned}
$$

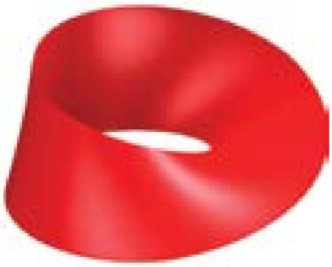
See figure 3 (*surfer* code included).

Note the small term $c^5 * 0.0001$. The reason for this modification is that the polynomial $(*)$ was constructed by multiplication of the preceding equation with t^n.

Figure 2

(0.5*(a^2+b^2)*(1+x^2+y^2+z^2)-a^2*b^2+0.5*(b^2-a^2)
*(1+x^2+y^2-z^2))^2-4*b^4*(x^2+y^2)

Figure 3



(-(a^2+b^2)*(x^2+y^2)+0.5
*(b^2-a^2)*x*(1+x^2+y^2-z^2)
-(b^2-a^2)*y*z)^2-(x^2+y^2)
(-0.5(a^2+b^2)
*(1+x^2+y^2+z^2)+a^2*b^2
+(b^2-a^2)*(x-y*z))^2
+c^5*0.0001

Figure 4

((0.5*(a^2+b^2)
*(1+x^2+y^2+z^2)-a^2*b^2)
(x^2+y^2)+0.5(b^2-a^2)
(x^2-y^2)(1+x^2+y^2-z^2)
-2*(b^2-a^2)*x*y*z)^2
-(x^2+y^2)*((a^2+b^2)
*(x^2+y^2)+(b^2-a^2)
*(x^2-y^2-2*x*y*z))^2
+c^7*0.0001

This adds the z-axis as a singular 1-dimensional set to the smooth 2-dimensional level let. Because this introduces a numerically critical behavior in a small neighborhood of the z-axis, the surfer software produces a *ghost image* there. Note that this effect becomes more dominant with larger twisting numbers n. Now, the small extra term allows a smoothing of the level set. With the right sign of c, the smoothing eliminates the z-axis as a singular set.

5.3 Twisting number $n = 2$

Thus we have $m = 1$, $p_2 = x^2 - y^2$, $q_2 = 2xy$ and we get the following polynomial equation of order 8:

$$\Big[\Big(\frac{A}{2}\big(1+x^2+y^2+z^2\big)-D\Big)\big(x^2+y^2\big)+ \frac{B}{2}\big(x^2-y^2\big)\big(1+x^2+y^2-z^2\big)-2Bxyz\Big]^2 = $$
$$= \big(x^2+y^2\big)\Big[A\big(x^2+y^2\big)+B\big(x^2-y^2-2xyz\big)\Big]^2.$$

See figure 4 (*surfer* code included).

5.4 Twisting number $n = 3$

Thus we have $m = 1$, $p_3 = x^3 - 3xy^2$, $q_3 = 3x^2y - y^3$ and we get the following polynomial equation of order 10:

$$\Big[-A\big(x^2+y^2\big)^2+\frac{B}{2}\big(x^3-3xy^2\big)\big(1+x^2+y^2-z^2\big)- B\big(3x^2y-y^3\big)z\Big]^2 =$$
$$= \big(x^2+y^2\big)\Big[-\Big(\frac{A}{2}\big(1+x^2+y^2+z^2\big)-D\Big)\big(x^2+y^2\big)+ B\big(x^3-3xy^2-\big(3x^2y-y^3\big)z\big)\Big]^2.$$

See figure 5 (*surfer* code included).

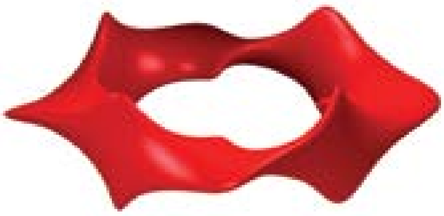
5.5 Twisting number $n = 4$

Thus we have $m = 2$, $p_4 = x^4 - 6x^2y^2 + y^4$, $q_4 = 3x^3y - 3xy^3$ and we get the following polynomial

Figure 5

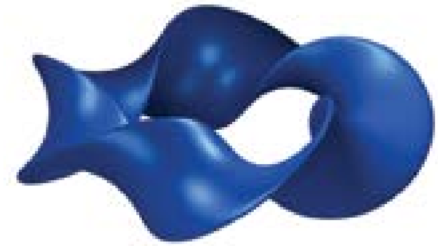
```
(-(a^2+b^2)*(x^2+y^2)^2+0.5
*(b^2-a^2)*(x^3-3xy^2)
*(1+x^2+y^2-z^2)
-(b^2-a^2)*(3x^2y-y^3)*z)^2
-(x^2+y^2)*(-(0.5*(a^2+b^2)
*(1+x^2+y^2+z^2)-a^2*b^2)
*(x^2+y^2)+(b^2-a^2)
*(x^3-3x*y^2
-(3*x^2*y-y^3)*z))^2
+c^9*0.0001
```

Figure 6



```
((0.5*(a^2+b^2)
*(1+x^2+y^2+z^2)-a^2*b^2)
*(x^2+y^2)^2+0.5
*(b^2-a^2)
*(x^4-6x^2y^2+y^4)
*(1+x^2+y^2-z^2)-3
*(b^2-a^2)
*(x^2-y^2)*x*y*z)^2
-(x^2+y^2)*((a^2+b^2)
*(x^2+y^2)^2+(b^2-a^2)
*(x^4-6*x^2*y^2+y^4-3
*(x^3*y-x*y^3)*z))^2
+c^9*0.0001
```

Figure 7



```
(-(a^2+b^2)
*(x^2+y^2)^3+0.5
*(b^2-a^2)
*(x^5-10x^3y^2+5xy^4)
*(1+x^2+y^2-z^2)-(b^2-a^2)
*(5x^4y-10x^2y^3+y^5)*z)^2
-(x^2+y^2)*(-(0.5*(a^2+b^2)
*(1+x^2+y^2+z^2)-a^2*b^2)
*(x^2+y^2)^2+(b^2-a^2)
*(x^5-10x^3y^2+5xy^4
-(5x^4y-10x^2y^3+y^5)*z))^2
+c^13*0.0001
```

equation of order 12:

$$\begin{aligned}&\Big[\Big(\frac{A}{2}\big(1+x^2+y^2+z^2\big)-D\Big)\big(x^2+y^2\big)^2+\\&\quad\frac{B}{2}\big(x^4-6x^2y^2+y^4\big)\big(1+x^2+y^2-z^2\big)\\&\qquad-3B\big(x^3y-xy^3\big)z\Big]^2=\\&=\big(x^2+y^2\big)\Big[A\big(x^2+y^2\big)^2+\\&\quad B\big(x^4-6x^2y^2+y^4-3\big(x^3y-xy^3\big)z\big)\Big]^2.\end{aligned}$$

See figure 6 (*surfer* code included).

5.6 Twisting number $n = 5$

Thus we have $m = 2$, $p_5 = x^5 - 10x^3y^2 + 5xy^4$, $q_5 = 5x^4y - 10x^2y^3 + y^5$ and we get the following polynomial equation of order 14:

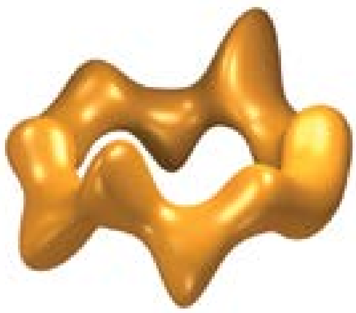
$$\begin{aligned}&\Big[-A\big(x^2+y^2\big)^3+\\&\quad+\frac{B}{2}\big(x^5-10x^3y^2+5xy^4\big)\big(1+x^2+y^2-z^2\big)-\\&\qquad-Bq_nz\Big]^2=\\&=\big(x^2+y^2\big)\Big[-\Big(\frac{A}{2}\big(1+x^2+y^2+z^2\big)-D\Big)\big(x^2+y^2\big)^2+\\&\quad+B\big(x^5-10x^3y^2+5xy^4-\big(5x^4y-10x^2y^3+y^5\big)z\big)\Big]^2.\end{aligned}$$

See figure 7 (*surfer* code included).

5.7 Twisting number $n = 6$

Thus we have $m = 3$, $p_6 = x^6 - 15x^4y^2 + 15x^2y^4 - y^6$, $q_6 = 6x^5y - 20x^3y^3 + 6xy^5$ and we get the following

Figure 8



((0.5*(a^2+b^2)*(1+x^2+y^2+z^2)-a^2*b^2)*(x^2+y^2)^3+0.5*(b^2-a^2)
(x^6-15x^4y^2+15x^2y^4-y^6)(1+x^2+y^2-z^2)-(b^2-a^2)*(6x^5y-20x^3y^3+6xy^5)
*x*y*z)^2-(x^2+y^2)*((a^2+b^2)*(x^2+y^2)^3+(b^2-a^2)*(x^6-15x^4y^2+15x^2y^4-y^6
-(6x^5y-20x^3y^3+6xy^5)*z))^2+c^15*0.0001

polynomial equation of order 16:

$$\Big[\Big(\frac{A}{2}(1+x^2+y^2+z^2)-D\Big)(x^2+y^2)^3+ \\ +\frac{B}{2}(x^6-15x^4y^2+15x^2y^4-y^6)(1+x^2+y^2-z^2)- \\ -B(6x^5y-20x^3y^3+6xy^5)z\Big]^2= \\ =(x^2+y^2)\Big[A(x^2+y^2)^3+B(x^6-15x^4y^2+15x^2y^4- \\ -y^6-(6x^5y-20x^3y^3+6xy^5)z)\Big]^2.$$

See figure 8 (*surfer* code included).

Note that for a twisting number larger than 6 the z-axis as a singular set is a numerically very unstable region such that a necessary correction strongly deforms the whole surface. Already for $n=6$ the deformation of the surface is quite strong.

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ECAS2019 Statistical Analysis for Space-Time Data

by **Giovani Silva* and Isabel Natário****

From July 15 to 17, 2019, *ECAS2019 on Statistical Analysis for Space-Time Data* was held at the Faculty of Science of the University of Lisbon, organized by the Portuguese Statistical Society (SPE) and the Spanish Society of Statistics and Operational Research (SEIO).

* Instituto Superior Técnico – Universidade de Lisboa
** Faculdade de Ciências e Tecnologia – Universidade Nova de Lisboa

Photos by **Paula Simões**

Patrick Brown

Hävard Rue

Liliane Bel

Four excellent courses on *Spatiotemporal Data Analysis* were presented at this event by well-known international experts in the field. Ege Rukak discussed the methodology of *spatial point patterns* and their applications; Patrick Brown talked about statistical models and inference for *spatiotemporal area data*; Hävard Rue and Haakon Bakka presented *spatial and spatiotemporal models* using the *SPDE approach* in the INLA frame; Liliane Bel showed the new trends in *spatiotemporal geostatistics*. All courses were based on R software and area-specific packages.

The participants, *mostly PhD students*, were able to present their research in a pleasant poster session where, in a relaxed, convivial atmosphere, they exchanged ideas and presented their work with other participants and lecturers.

A total of 61 participants attended, of which *26 were students*, exceeding expectations for the expected capacity of the event. About half of the participants were from Portugal, 6 came from Germany, 5 from Spain and the rest came from places as varied as Sweden, Italy, Belgium, France, the United Kingdom, Ireland, Mozambique, Canada, Colombia, Mexico and Australia.

It was an excellent opportunity to learn cutting-edge knowledge in this area of space-time Statistics and *networking*, provided by the many pleasant moments of pause prepared by the organization.

Committees and Lecturers

Photo by **Paula Simões**

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Algebraic Analysis and Geometry with a view on Higgs bundles and *D*-modules

by **André Oliveira***, **Carlos Florentino**** and **Teresa Monteiro Fernandes*****

The international conference on *Algebraic Analysis and Geometry with a view on Higgs bundles and D-modules* took place last June, from 3 to 7, at the Mathematics Department of the Sciences Faculty of the University of Porto.

It was organized by Carlos Florentino (FCUL, CMAFcIO), Teresa Monteiro Fernandes (FCUL, CMAFcIO), André Oliveira (CMUP, UTAD) and Luca Prelli (University of Padova, Italy), with the support of *Fundação para a Ciência e Tecnologia* (FCT), *Centro de Matemática da Universidade do Porto* (CMUP), *Centro de Matemática, Aplicações Fundamentais e Investigação Operacional* (CMAFcIO), *Dipartimento Matemática Università di Padova* (UniPD) and the *Centro Internacional de Matemática* (CIM).

Higgs bundles and *D*-modules are two extremely successful theories, touching an impressive wide variety of mathematical subjects (such as Algebraic and Differential Geometry and Topology or Partial Differential Equations), and which have substantial intersections and interactions. Thus, the main purpose of the conference was to bring together some of the world's leading researchers on both topics to further enhance their interactions and, of course, present their recent work on these topics.

In the particular case of Portugal, there are two main research hubs in these two topics, mostly located and in Lisbon (FCUL) and in Porto (FCUP). Despite the presence of researchers in Portugal in these two topics, the interaction between them has been residual, so this conference was certainly a first step towards a *scientific approach* between these two groups.

The conference consisted of a total 19 talks, of one hour each. The schedule was made in such a way that, on every day, there were basically the same number of talks focused on each of the two main subjects. There was also free time for participants to discuss and freely interact about their research.

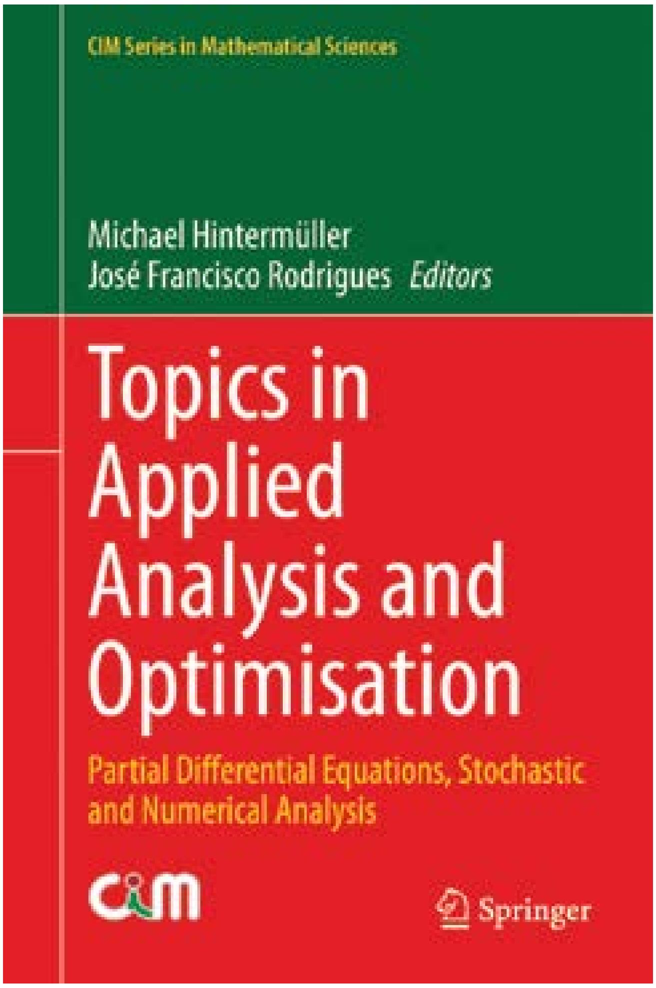
The invited speakers came from many different origins: USA, Japan, Italy, France, Spain, Austria, Germany and, of course, Portugal. Here is their list, together with the corresponding talk title:

* FCUP e UTAD [andre.oliveira@fc.up.pt]
** FCUL [caflorentino@fc.ul.pt]
*** FCUL [tmf@ptmat.fc.ul.pt]

D. Alfaya	Automorphism group of the moduli space of parabolic vector bundles
P. Boalch	Stokes decompositions and wild monodromy
A. Castaño Domínguez	Hypergeometric irregular mixed Hodge modules
A. D'Agnolo	Enhanced nearby and vanishing cycles
O. Dumitrescu	Interplay between Higgs bundles and opers
L. Fiorot	Relative regular Riemann-Hilbert correspondence
E. Franco	Torsion line bundles and branes on the Hitchin system
O. García-Prada	Prehomogeneous vector spaces and Higgs bundles
P. Gothen	Examples of higher Teichmüller components via G-Higgs bundles
T. Hausel	Very stable Higgs bundles, the nilpotent cone and mirror symmetry
L. Migliorini	Supports of the Hitchin fibration on the reduced locus
T. Mochizuki	Kobayashi-Hitchin correspondences for monopoles with periodicity
T. Pantev	Shifted symplectic structures on moduli of local systems
T. Reichelt	Hodge theory of GKZ systems
C. Sabbah	Good lattices of algebraic connections
P. Schapira	Towards a functorial filtration on holonomic D-modules
C. Simpson	Higher direct images of parabolic Higgs bundles and Hecke correspondences
K. Takeuchi	Exponential factors and Fourier transforms of D-modules
J.-B. Teyssier	Higher dimensional wild character varieties

There was no registration fee to participants of the event, but there was also no possibility of providing funding to help participants attend the conference. Even so, there were in total around 40 mathematicians attending, coming from many different countries. Among them, there were around 10 PhD students.

The web address of the conference website is https://cmup.fc.up.pt/aga-porto-2019, where more information can be found.



Topics in Applied Analysis and Optimisation

Partial Differential Equations, Stochastic and Numerical Analysis

Editors

Michael Hintermüller
Weierstrass Institute for Applied Analysis and Stochastics, Berlin, Germany

José Francisco Rodrigues
CMAF&IO, Faculdade de Ciências, Universidade de Lisboa, Lisboa, Portugal

This volume comprises selected, revised papers from the Joint CIM-WIAS Workshop, TAAO 2017, held in Lisbon, Portugal, in December 2017. The workshop brought together experts from research groups at the Weierstrass Institute in Berlin and mathematics centres in Portugal to present and discuss current scientific topics and to promote existing and future collaborations. The papers include the following topics: PDEs with applications to material sciences, thermodynamics and laser dynamics, scientific computing, nonlinear optimization and stochastic analysis.

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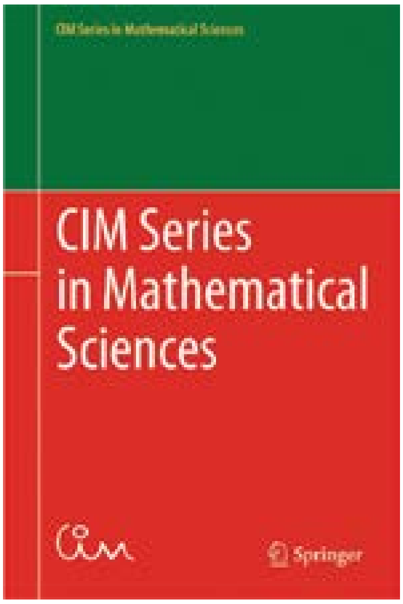

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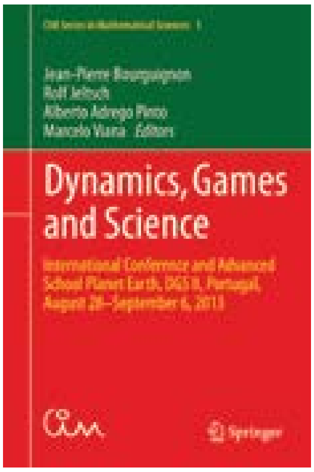

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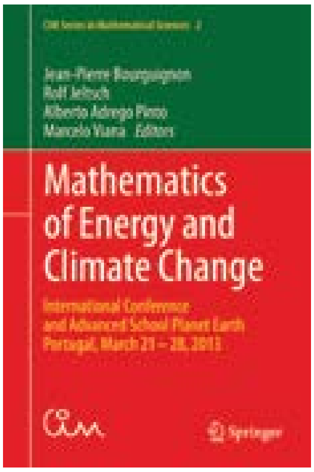

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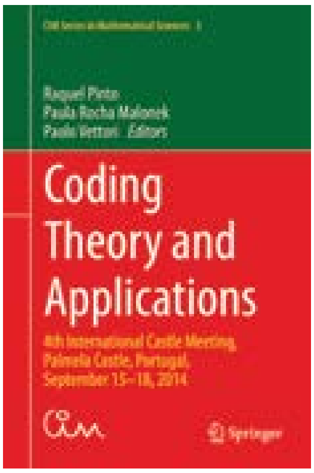

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IO 2013 - XVI Congress of APDIO, Bragança, Portugal, June 3–5, 2013

Mathematics and Cultural Heritage

by José Francisco Rodrigues*

To celebrate the 2018 European Year of Cultural Heritage, the Lisbon Academy of Sciences hosted on 19 December 2018 the Colloquium on Mathematics and Cultural Heritage. This was an initiative of the International Center for Mathematics (CIM) and the Portuguese Society of Mathematics (SPM), under the program of the Planet Earth Mathematics Committee, which has the support of the National Commission of UNESCO Portugal. The program was also supported by the University of Coimbra Science Museum (MCUC) and the University of Lisbon National Museum of Natural History and Science (MUHNAC).

The eight videos (in Portuguese) of the program are available at http://www.cim.pt/agenda/event/197 and were edited and published by the CMAFcIO/Ciências/ULisboa and supported by the Foundation for Science and Technology (FCT)

* CMAFcIO, Faculdade de Ciências, Universidade de Lisboa

PROGRAMA

15.00 **Palestras sobre matemática e património cultural**

15.00 **Azulejos e ciência**, *Henrique Leitão* (Universidade de Lisboa, CIUHCT)

15.15 **Os azulejos euclideanos jesuítas**, *António Leal Duarte* (Universidade de Coimbra) e *Carlota Simões* (Universidade de Coimbra)

15.30 **Relógios de Sol**, *Suzana Nápoles* (Universidade de Lisboa)

15.45 **Calçadas portuguesas**, *Ana Cannas da Silva* (ETH Zurich) e *Suzana Nápoles* (Universidade de Lisboa)

16.20 **Arquitetura e Matemática** *João Pedro Xavier* (Universidade do Porto, Faculdade de Arquitectura)

16.35 **A arte e a geometria de Almada Negreiros** *Pedro Jorge Freitas* (Universidade de Lisboa, DHFC e CIUHCT) e *Simão Palmeirim Costa* (Universidade de Lisboa, CIEBA)

16.50 **Literatura e Matemática**
António Machiavelo (Universidade do Porto)

17.10 **Concerto comentado—Matemática, processos composicionais e estratégias de preparação para a performance em obras para piano de Jaime Reis, João Madureira e Christopher Bochmann**, *Ana Telles* (Universidade de Évora, CESEM-UÉ)

AN INTERVIEW WITH

Martin Hairer

by **Chiara Franceschini*** and **Patrícia Gonçalves***

Photo by **Débora Rodrigues** • IST Lisbon

Martin Hairer is a British-Austrian mathematician working in the field of stochastic analysis, who was awarded a Fields medal in 2014, the most prestigious prize in the career of a mathematician. He holds a chair in Probability and Stochastic Analysis at Imperial College London and is considered to be one of the world's foremost leaders in the field of stochastic partial differential equations (SPDEs) in particular, and in stochastic analysis and stochastic dynamics, in general. By bringing new ideas to the subject he made fundamental advances in many important directions such as the study of variants of Hörmander's theorem, the systematization of the construction of Lyapunov functions for stochastic systems, the development of a general theory of ergodicity for non-Markovian systems, multiscale analysis techniques, the theory of homogenization, the theory of path sampling and, most recently, the theory of rough paths and the newly introduced theory of regularity structures. Besides the Fields medal, he was awarded several highly reputed prizes and distinguished with several honors and distinctions. Among those, we mention the LMS Whitehead prize, the Philip Leverhulme Prize, the Fermat prize, the Fröhlich prize. He was distinguished as a fellow of the Royal Society, the American Mathematical Society, the Austrian Academy of Sciences, the Berlin-Brandenburg Academy of Sciences and Humanities, the German National Academy of Sciences Leopoldina and was also distinguished as an Honorary Knight Commander of the British Empire. He was awarded several fellowships and grants including an ERC, a Leverhulme Leadership Award, EPSRC; he is editor of several leading journals and delivered talks in many highly reputed institutions worldwide.

* Instituto Superior Técnico, University of Lisbon

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We have interviewed Martin on the first week of December 2019 while he was participating in the conference "Particle Systems and Partial Differential Equations XIII", that was held at Instituto Superior Técnico of the University of Lisbon.

How did you end up studying mathematics? How much do you think you father's job influenced you?

I have certainly been influenced by my dad. Thanks to him I was exposed to maths in an informal way at an early age: I knew what a differential equation is, in some sense, when I was about 12 years old, and this is something that normally you do not learn at school. My dad would explain to me if I asked about his work, but he did not impose anything on me. On the other hand, my mother was a primary teacher for a couple of years, and then decided to work in a toy's library, and actually she is still doing that today.

Did you also know that you wanted to become a mathematician? Can you tell us your path into math?

It wasn't always clear to me that I would do mathematics, I was interested in programming as well, so it could have been a possibility for me to become a computer scientist or an independent developer. I didn't study mathematics for my PhD, I actually have a physics diploma from the University of Genève, but both maths and physics backgrounds overlap for most of the courses of the first years. The reason I have switched from physics to maths is because I was never fond of the experimental/laboratory works, nor the data analysis coming from there, I was more interested in the theoretical aspects, so it was a natural path to go to the theoretical physics rather than the experimental physics side; and even there, I sometimes found the arguments too handwaving. I didn't feel sufficiently secure to make such arguments, so I felt that, if I wanted to publish something, I wanted to reason with arguments that are rigorous: this translates into a future in mathematics.

How did you get the idea of singular SPDEs and regularity structures for such equations that lead to the Fields Medal in 2014?

The first work I did in this direction is an older paper I have with Andrew Stuart. He was interested in path sampling and wondering how to simulate in a computer a stochastic differential equation (SDE) conditioned on hitting a specific point. It is not easy to do this because it might never hit the point, so we thought of finding an SPDE such that its invariant measure is the bridge of that equation. If you do that for a diffusion with additive noise and gradient drift, then the SPDE that you get is a reaction-diffusion equation. It was natural to start

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wondering what happens in higher dimension by taking an arbitrary drift that is not necessarily gradient. In this case what you find is a kind of a reaction-diffusion equation but with a Burger's type drift. We wanted to give a notion of solution to those SPDEs, but it was not clear what would be the interpretation of this drift in terms of the solution. The article solving this problem was my first work in the area of singular SPDEs. Later, while I was working at the Courant Institute in New York, Gérard Ben Arous suggested to me to actually look into the Kardar-Parisi-Zhang (KPZ) equation and to try to give some meaning to its solutions.

Is there a point in your life when you realized that you actually had good chances to win the Fields medal? How did things change for you after?

The first time someone mentioned this to me was after my paper on the KPZ equation where I prove uniqueness of solutions, but to be honest I didn't think it was possible at that stage to get the medal. When I got the idea of regularity structures, I thought that it might be possible, but that work came out when I was almost forty, so it seemed pretty unlikely, since there was not much time left. After winning it, my life didn't change all that much: clearly now I do more public initiatives (public lectures, interviews like this), but it is not all that much, maybe three or four times per year now. I already received many invitations after my original paper on regularity structures came out, so the medal didn't really change the amount of travel I do.

You won the Fields medal in 2014, together with the first Latin-American, the first woman and first Iranian. Nowadays there is a lot of talking about minorities in our community and in the world in general. How do you think minorities should be regarded?

Well, I was the first Austrian (laughing). This is a tricky question. Thanks to my wife, Xue-Mei, who is also a mathematician, I can have some idea of what it means to be a woman in mathematics. I can see she gets a bit annoyed if she has the impression of being asked to do something or to be part of something only because she is a woman. Of course, you want to be asked/invited because of your mathematics, not because you are a woman. Nowadays, there is a lot of pressure on having more women in scientific committees as this would increase female participation. At the level of committees like workshop organisation, I can see how this helps to improve women's representation which is a good thing, but in some other cases it can also be counterproductive. For example, at the senior level, the proportion of women in the maths community is lower that the proportion of women you would like to have in committees, which translates into extra administrative work for women. This is also unfair, maybe they'd rather spend that time doing research. I am not sure there is a good solution, I hope at some point things, hopefully, will become more balanced.

The History of Mathematics has a lot of mathematicians that we admire. Do you have a particular admiration for one that you would like to mention? Do you have a professor that is was/is an idol for you or that inspired you, by the time when you were at the university?

I think that one of the most impressive mathematicians is Von Neumann, he was one of the last mathematicians who knew everything: both on the analytic side, the algebraic side, also in computer science, he was extremely broad in many things. In terms of professors' influence, probably the strongest influence is from my PhD advisor, Jean-Pierre Eckmann, I like his way of thinking about problems: he is fascinated by problems regardless of the mathematics he knows. In fact, he would learn new mathematics to understand and solve new problems. Moreover, he has 'good taste' in the sense that he can recognize what makes a problem interesting.

Among your many results and achievements, is there one that you are particularly proud of? And is there any other problem that you tried to attack in the past, but it is still open?

In terms of achievements, the paper I am most proud of is the one regarding regularity structures, I would also mention my paper regarding the Navier-Stokes equation, which was the complete answer to my PhD problem and came out few years after I defended my PhD thesis. In terms of open problems, one thing that is embarrassing is that I still haven't managed to solve the problem Jean-Pierre gave me for the diploma thesis! (Laughs) I was able to prove that the question he asked was wrong, but I still don't know how to find a proper answer. The setting regards a finite chain of anharmonic oscillators with nearest-neighbor coupling where the first and the last oscillators are coupled with two thermostats at different temperatures. Even to show that there exists an invariant measure is not obvious and it can be done only in specific situations, but not in general.

Have you ever visited Portugal before? What is your impression about the country?

My first visit to Portugal was in 2003 for a conference, but it was a short visit and I did not have enough time to form any impression on the country, this time is also short, so maybe on the next visit!

Is there one thing that you would have liked to know when you were younger, that now you would say to a younger Martin Hairer? Any advice for the young mathematicians who have just finished their PhD?

After the PhD I wasn't completely sure to continue in academia. I believe Jean-Pierre wanted me to go to Courant Institute for a post-doc position, but I went to England to conjugate my personal and my professional life. I decided to give a chance to academia, I had a fellowship from the Science Foundation for two years and after that I would have seen if I could get something decent. If not, I would have probably continued to work as a freelancer for software development, which is something I still do in my spare time. It took some year for the academic salary to be comparable to the one of a software developer!

What do you think is the biggest problem of our academic system (mathematics) and what would be the first thing that you do to improve it?

For sure the issue that you are forced to move quite a bit before finding a permanent position. This is related to what we have already discussed regarding women in mathematics. Naturally, people drop out of going into academia, because it is difficult to combine both family and work. Another problem regards the funding mechanism. You try to pick some winners in a competition and you give them more money than what they can actually spend. In our research fields, we don't need so much money, all the equipment we need usually is just a laptop and some money to travel, with additional funds going into hiring postdocs. My impression, is that it would be better to split some of the big grants among several people. This would be relatively inexpensive and would make more people happy. Clearly you give money to someone who has a sort of masterplan, with a big global vision. However, there are many researchers that don't have it, but who nevertheless produce very good mathematics. For this reason, I think there should be more money available in small grants. For example, the money of one large grant (say €300k/year) would be sufficient to provide 50 mathematicians with a travel grant of €6k/year. As a consequence of the current system, most post-doctoral are funded through these grants. At least in the UK, there are relatively few positions for researchers that are independent of such grants, although in the USA they are more common, an example is what they call instructor positions.

How do you see your research field in 20 years from now? What do you think will be the hottest problems for the community? Would universality classes' problem be solved?

I am not very optimistic regarding the last question, I don't think all the questions regarding universality will be addressed, but, on the other hand, I believe people will still be interested in such problems because many of the probability questions tend to have this flavor, i.e. to produce scaling limits of some objects of interest. It is hard to make specific predictions regarding which results are going to be cracked: if you do, it probably means you are about to crack it.

In the recent years, the Mathematical and the scientific community in general have been overwhelmed with the use of bibliometric data to assess and evaluate individu-

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als and institutions. What do you think about that?

I think in general it is very bad to have specific metrics. Or at least it wouldn't be as bad if you don't say which bibliometric data you are using for evaluating. If you use an unknown metric, people cannot game it; however, as soon as you publish which metric you are using to assess people, they are going to crack it. Of course, the aim is to design the metric in such a way that people will produce better research in gaming it, but unfortunately no one has figured out which sort of metric has this feature. You will always find an easier way of gaming it, instead of doing the things you are supposed to. There is a funny example regarding this in the UK. The government wanted to have trains running on time, so it started to impose fees on companies which had more than 5% of the trains running late. They defined what it means for a train to be on time, namely that it has to reach the final destination within 5 minutes of the scheduled time. This produced several consequences: the train companies rewrote the timetables, so that the journey would take 10 extra minutes. That's not too bad, as it would make the timetable more realistic. However, another thing that happened is that, since the rule for being on time just regarded the last station, they started to jump stations as this would take several minutes for people to get in and out. Clearly, in this way companies reached the target, but not in the way it was intended to be. Another funny story on how metrics produce unintended results regards the league tables for best university. One publication in the UK produces a ranking every year and some time ago they changed the criteria for what they call "research impact" to make the evaluation more objective. The outcome of that was that University of Alexandria in Egypt turned out to be number four worldwide. How is that? It was because of one single guy. He was a mathematician who produced a lot of papers with tons of self-citations published in a journal where he was editor-in-chief. I sent an email to the guy who came up with the methodology to point this out and I noticed that the year after they basically changed the methodology in a way to just rule out University of Alexandria with an ad hoc procedure.

PEDRO NUNES LECTURES

MARTIN HAIRER

RANDOM LOOPS

2ND OF DECEMBER—17:30

ANFITEATRO ABREU FARO · COMPLEXO INTERDISCIPLINAR
INSTITUTO SUPERIOR TÉCNICO · UNIVERSITY OF LISBON

Martin Hairer is an Austrian mathematician working in the field of stochastic analysis, who was awarded a Fields medal, in 2014.

He holds a chair in Probability and Stochastic Analysis at Imperial College London.

He is considered one of the world's foremost leaders in the field of stochastic partial differential equations in particular, and in stochastic analysis and stochastic dynamics in general. By bringing new ideas to the subject he made fundamental advances in many important directions such as the study of variants of Hörmander's theorem, systematisation of the construction of Lyapunov functions for stochastic systems, development of a general theory of ergodicity for non-Markovian systems, multiscale analysis techniques, theory of homogenisation, theory of path sampling and, most recently, theory of rough paths and the newly introduced theory of regularity structures.

Besides the Fields medal, he was awarded several highly reputed prizes and distinguished with several honours and distinctions. Among those, we mention the LMS Whitehead prize, the Philip Leverhulme Prize, the Fermat prize, the Fröhlich prize. He was distinguished as a fellow of the Royal Society, the American Mathematical Society, the Austrian Academy of Sciences, the Berlin-Brandenburg Academy of Sciences and Humanities, the German National Academy of Sciences Leopoldina. He was also distinguished as an Honorary Knight Commander of the British Empire.

He was awarded several fellowships and grants including an ERC, a Leverhulme Leadership Award, EPSRC. He is editor of several leading journals and delivered talks in many highly reputed institutions worldwide.

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