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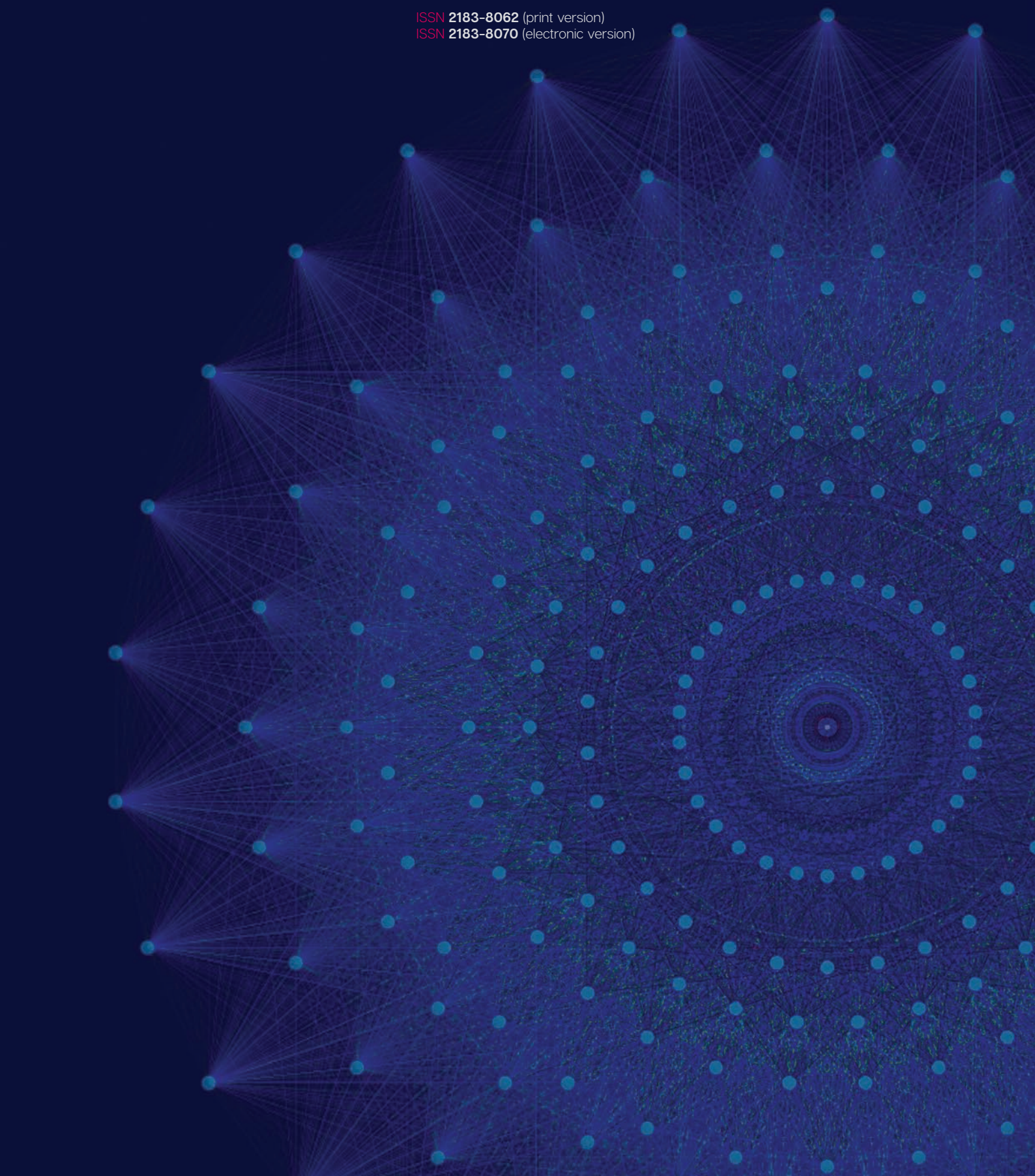

2020 December .42

CIM

Contents

Editorial

In March of 2020, a new direction board of CIM took office and the editorial board of the CIM bulletin was renewed. The editorial board is committed to continue the efforts of fulfilling the bulletin's main goals of promoting Mathematics and especially mathematical research.

The COVID-19 pandemic conditioned strongly CIM's usual activities, which means that, contrary to previous issues, only one report of a scientific meeting supported by CIM is presented in the current issue. Nonetheless, we compensate this shortcoming with three articles regarding advances on the cutting edge of the discipline. Namely, we include an article concerning the topological properties of configuration spaces of points; an article considering stochastic processes modelling an interacting particle system, whose algebraic structure helps to analyse its macroscopic dynamics; and an article with an insightful introduction to analysis on graphs and, in particular, quantum graphs.

Inserted in the cycle of historical articles, we feature an article dedicated to the work and life of António Aniceto Monteiro, focusing, in particular, on his modernist essay about General Analysis, from 1939, which anticipated Bourbaki's treaty published in the 1940s.

We present an interview honouring José Basto-Gonçalves, who belonged to the first scientific committee of CIM, for his role in disseminating and stimulating mathematical research in the University of Porto.

We also include an interview to André Neves, who was the distinguished mathematician invited to deliver this year's Pedro Nunes' lecture, which is an emblematic initiative of CIM, counting with the support of SPM.

We recall that the bulletin continues to welcome the submission of review, feature, outreach and research articles in Mathematics and its applications.

Ana Cristina Moreira Freitas

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https://www.fep.up.pt/docentes/amoreira/

António Monteiro and his Modernist Essay

by **José Francisco Rodrigues***

The 1939 "ensaio sobre os fundamentos da análise geral" (Essay on the foundations of General Analysis), by the Portuguese mathematician António Aniceto Monteiro (1907–1980), in spite of being awarded an important prize by the Lisbon Academy of Sciences, has been unknown and ignored until its recent rediscover and its facsimile publication [AM1939]. The ensaio is a 130 pages typed monograph that introduces mathematical modernism and prepares a turning point in the mathematical activities in Portugal, preceding the creation of the *Centro de Estudos Matemáticos de Lisboa* in 1940, the first Portuguese research centre, affiliated with the *Faculdade de Ciências* of the Lisbon University and independently supported by the *Instituto para a Alta Cultura*, the incipient national science foundation at the time [Ro].

António Monteiro, Modernist and Mathematician

On the occasion of the centenary of his birth, the Portuguese Mathematical Society (SPM) published in 2007 a remarkable photobiography [AM_Fb2007] and a special issue of its Bulletin [AM_B2007] with the proceedings of an International Colloquium at the University of Lisbon. In the presentation of his *Works* [AM_O2008],

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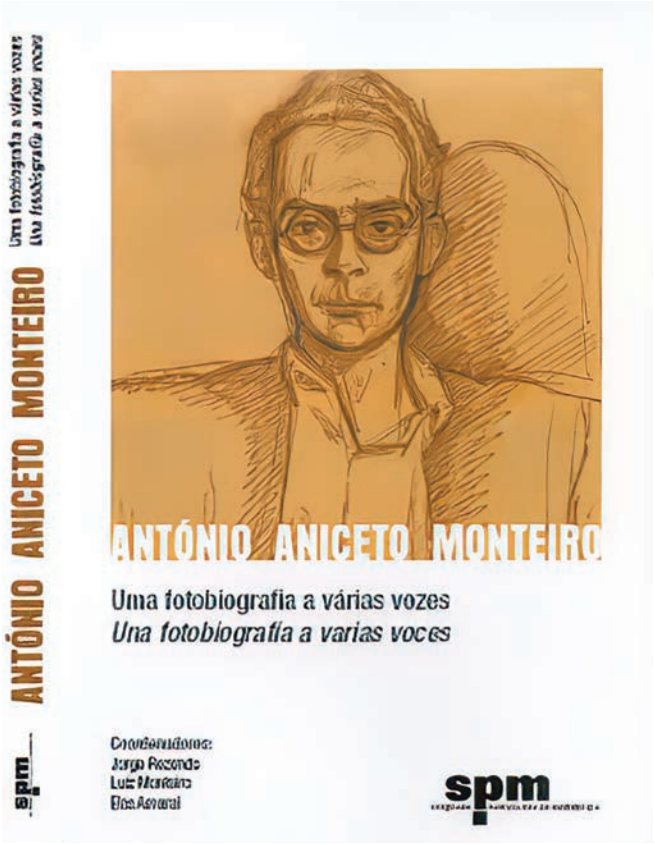

Figure 1. Cover of the photobiography [AM_Fb2007]

Figure 2. Cover of [AM_Bc2007]

consisting of eight volumes of about 2800 pages, which does not include his 1939 Essay, Jean-Pierre Kahane, from the Paris Academy of Sciences, wrote

> The works of António A. Monteiro belong to the world history of mathematics. They cover a large variety of topics from classical analysis to topology and from advanced algebra to logic in its more modern chapters. Some of them come from courses and synthetic presentations, but the majority of them are research papers. They are presented in different styles, occasionally handwritten, and also in different languages. Despite their intrinsic value, these works are a testimony of an age and of an exceptional life. They were written, in four different countries: France, Portugal, Brazil and Argentina. Monteiro was the founder of mathematical journals and various mathematical institutions, first in Portugal, then in Latin America. He had to emigrate from Portugal because of Salazar's regime and was also affected by the military dictatorship in Argentina. His life testifies the link between the struggle for science and the struggle for freedom.

Monteiro was born in Moçâmedes (Angola) in 1907, the son of a Portuguese colonial army officer, he came to Lisbon, already orphan, to attend the Military College in 1917 before his graduation in Mathematics in 1930, at the *Faculdade de Ciências*, and his departure to Paris with a fellowship, where he followed courses at the *Faculté des Sciences* and seminars at the *Institut Henri Poincaré*. In 1936, he presented his thesis, *Sur l'additivité des noyaux de Fredholm*, at the University of Paris under Maurice Fréchet (1878–1973) [AM_Fb2007] and [AM_O2008]. During his stay in Paris, Monteiro assumes also his mission of studying "the organization of a Centre for Mathematical Studies which would have, among others, the objective of achieving the complete resurgence of Portuguese mathematical traditions", and, in his correspondence, he even refers to the acquisition of books for the *Instituto de Matemática* [AM_Bc2007].

After his return to Lisbon, he refused to sign a compulsory political statement in order to be integrated at the University — Monteiro would have said: "I do not accept limitations on my intelligence" — and he was thus unable to pursue in Portugal the career as a mathematician he developed in his exile in 1945 in Brazil and in Argentina from 1950 until his jubilation and removal, also for political reasons, from the *Universidad Nacional del Sur* in 1975, in Bahia Blanca, where he had been Professor Emeritus since 1972 and where he died in 1980 [Re].

Between 1937 and 1943, Monteiro's scientific and academic activity in Lisbon was carried out as a precarious inventor of scientific libraries in Portugal. In spite of financial difficulties, he was a major participant of the brief decade of the Portuguese Mathematical Movement (1936–1946), which began with the activities of

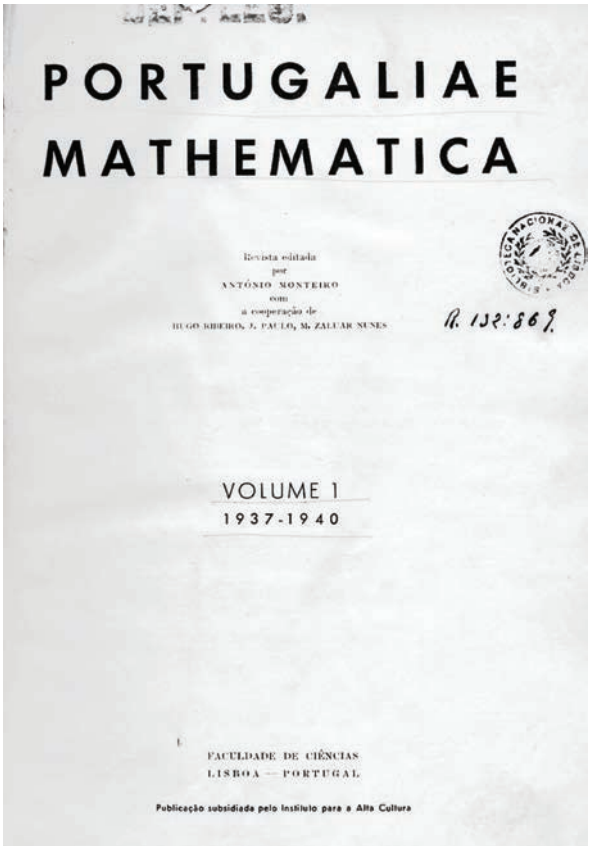
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Figure 3. Frontispiece of vol. 1 of Portugaliae Mathematica.

the *Núcleo of Mathematics, Physics and Chemistry* at the end of 1936, with the founding, by Monteiro, of the journal *Portugaliae Mathematica* in 1937 (Fig.3), with the beginning of the *Seminário de Análise Geral* in 1939, the creation of the *Centro de Estudos de Matemática de Lisboa* (CEML), under his scientific direction, the *Gazeta de Matemática* and the Portuguese Mathematical Society, all in 1940.

His remarkable qualities as researcher and professor were first developed in Lisbon with the orientation of young mathematicians in the first three years of the activity of the CEML, which were marked by the influence of the modernist ideas of the 1939's "Ensaio" and culminating in the visit of Maurice Fréchet in early 1942 (Fig.4), and the successive departure of the young grantees abroad, including the first two Portuguese disciples of Monteiro, Hugo Ribeiro for the ETH in Zurich and José Sebastião e Silva for the University of Rome [AM_B2007].

He continued his activities at the *Centro de Estudos de Matemática do Porto*, created in 1942 and affiliated with the *Faculdade de Ciências* of the University of Porto, where he had a third disciple, Alfredo Pereira Gomes who also began his academic career in France and Brazil, and where he was supported by the *Junta de Investigação Matemática* (JIM). This remarkable association was created in 1943 and was sponsored by private funds during some years. JIM aimed to bring together almost all the (few) Portuguese researchers in the country, had as its primary objective "to promote the development of mathematical research" and played a very important role in funding scientific publications, particularly, the journal *Portugaliae Mathematica*, after the government stopped the initial financial support. This support to Portugaliae Mathematica was accomplished in connection with the CEML, and was limited to the first three volumes.

Figure 4. M. Fréchet, P. J. da Cunha and A. Monteiro at Faculdade de Ciências de Lisboa in early 1942 ([AM_Fb2007]).

Between 1945 and 1948, Monteiro was professor of *Análise Superior* at the University of Brazil, now the *Universidade Federal de Rio de Janeiro*, where he had a strong influence on young mathematicians. These include Maurício Peixoto, who was his co-author, Maria Laura Lopes, whose thesis of 1949 solved a question raised by Monteiro, and also Leopoldo Nachbin, who succeeded him in directing the series of monographs *Notas de Matemática* and was the author of its No. 4, the influential lectures notes on *Espaços Vetoriais Topológicos* (1948). This series, which Monteiro had founded in 1948, was published in Rio de Janeiro until 1972 and included in its No. 5 the text on *Rings of Continuous Functions* by Marshal H. Stone. It was continued with volume 48 by North-Holland and reached in 2008, already with Elsevier, the number 208 of that well known collection of *Mathematics Studies* [Ro].

Appointed Professor at the *Universidad Nacional de Cuyo*, San Juan, Argentina, in 1950, Monteiro founded the *Revista Matemática Cuyana* with M. Cotlar and E. Zarantonello, in 1955. Although he was invited to the University of Buenos Aires, Monteiro, with some of his new Argentine disciples, moved in 1957 to the recently created *Universidad Nacional del Sur*, in Bahía Blanca, where he founded the Mathematical Institute, the Mathematical Library, new series of monographs and developed research on Algebraic Logic and Latices. In 1974 he was appointed an honorary member of *Unión Matemática Argentina*, due to his remarkable intellectual influence. According to the Argentine mathematician Eduardo Ortiz, Monteiro belongs "to an old tradition of Argentinian progressive and independent thought to which the country owes some of its most valuable achievements" [AM_PM1980].

Although he spent a sabbatical year in Europe, during 1969–1970, visiting several universities in France, Romania, Belgium, Italy and England, he did not return to Lisbon until March 1977, with a scholarship of the National Institute of Scientific Research. There he resumed his research for about two years at the Centre for Mathematics and Fundamental Applications (CMAF), the direct successor of the CEML he had directed thirty seven years ago. During this period, he supervised his fourth Portuguese disciple, M. Isabel Loureiro, and wrote the extensive work *Sur les Algèbres de Heyting Symétriques*, which, in 1979, was awarded the Gulbenkian Prize for Science and Technology and was published in *Portugaliae Mathematica* [AM_PM1980], in a volume posthumously dedicated to him. In a letter dated June 5, 1978 sent to Alfredo Pereira Gomes, his former disciple from Porto, then professor at the University of Lisbon, despite his state of health, Monteiro wrote "I am really satisfied with the results of my scientific activity in Portugal. This is mainly due to the Centro de Matemática (CMAF), which provided me with free time to study" [AM_PM1980].

On his return to Bahía Blanca, where he had residence with his family, Monteiro died on 29 October 1980 in the country of his second exile. In a letter to his Argentine friend he wrote: "That's life dear Ortiz. One uses and spends oneself on tasks that cannot be finished: and yet one begins with enthusiasm and dedication, because hopes and certainties are never lost. Sadnesses of Bahia Blanca! on the margins of the Napostá; between winds and storms in which the earth drowns us, I see Lisbon distant - memories of my childhood!" [AM_B2007].

The forerunner 1939's "Ensaio sobre os fundamentos da análise geral"

In the Foreword of his Ensaio Monteiro wrote:

> The General Analysis was founded at the beginning of this century by Maurice Fréchet, with the aim of generalising the differential and integral calculus for those functions where the independent variable – and possibly the function itself – are elements of any nature (. . .) having as goal the study of the correspondences between variables of any nature.

In fact, Fréchet was a pioneer in proposing, in his 1906 thesis under the guidance of Jacques Hadamard (1865–1963), an abstract approach to mathematical analysis based on general structures: *class* (L), spaces with convergence; *class* (E), spaces with *écart*, i.e., with distance, renamed metric spaces by F. Hausdorff, in 1914, as a subclass of topological spaces, and *class* (V), a generalization of spaces (E) provided with neighborhoods (*voisinages*). These notions were evolving and Fréchet set out his results in the influential book *Les Espaces Abstraits* [F], which strongly marked Monteiro's early mathematical activity and, in particular, his essay. Monteiro shared the view of Norbert Wiener (1894–1964), who visited Fréchet in Strasburg in 1920 and later wrote about him: "more than anyone else who had seen what was implied in the new mathematics of curves rather than points (…) One of the specific things which attracted me in Fréchet was that the spirit of his work [spirit of abstract formalism]".

The 1939 essay clearly reflects the "spirit of abstract formalism", which Monteiro absorbed from Fréchet and had anticipated Bourbaki in France, but unfortunately has remained unpublished and lost in the archives of the Lisbon Academy of Sciences until recently. It consists of four chapters: Abstract Set Theory (13 p.); Abstract Algebra (52 p.); Abstract Topology (26 p.) and Abstract Analysis or General Analysis (38 p.) and corresponds to

Figure 5. Announcements of the course and the seminar by A. Monteiro, already at the CEML, attached to the Faculdade de Ciências of Lisbon [AM_Fb2007].

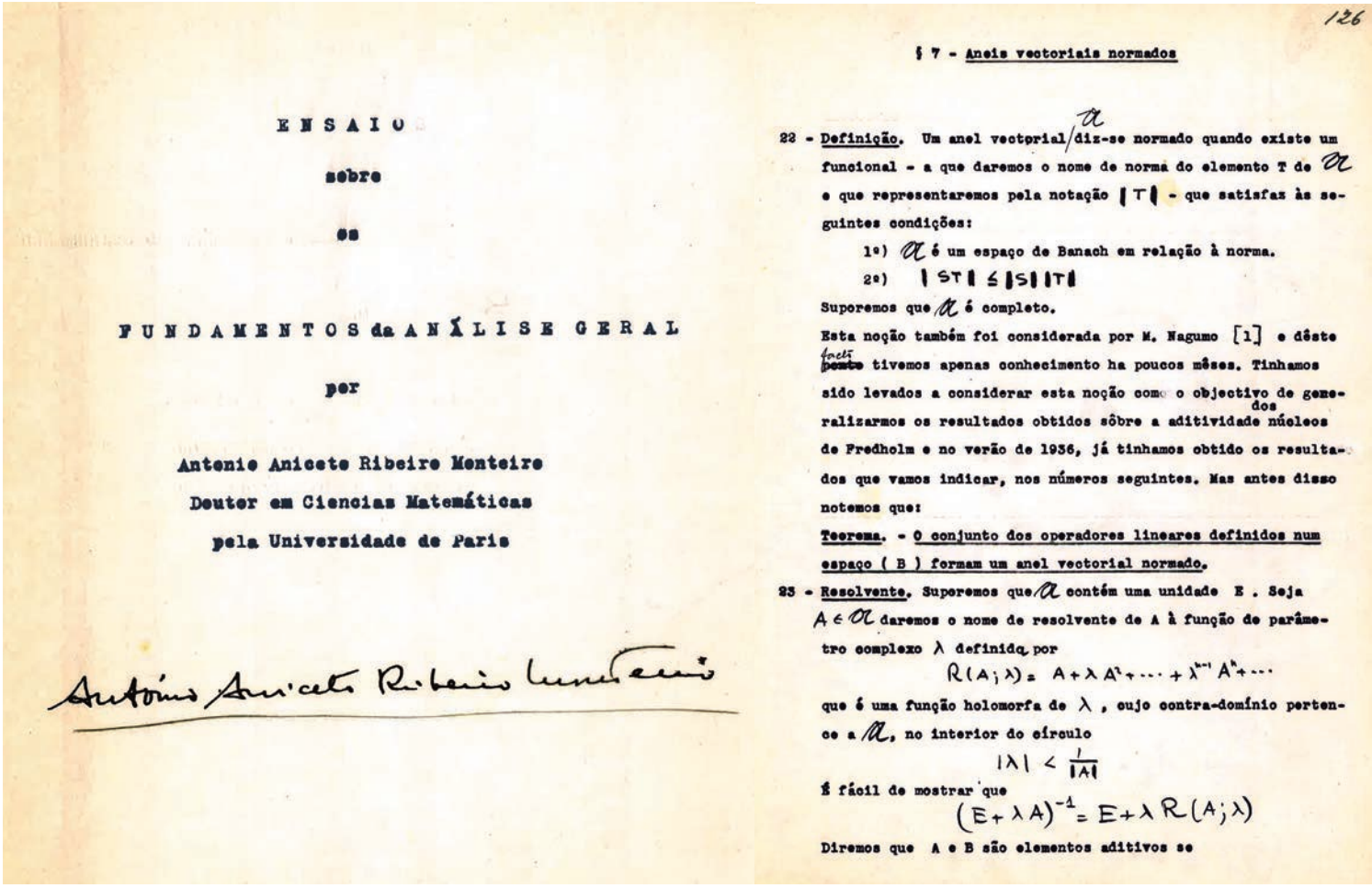
a programmatic plan, which was put into practice immediately by himself, with a Course in1939 and in the Seminar of General Analysis in 1940 (Fig. 5), already within the scope of the recently created *Centro de Estudos Matemáticos de Lisboa* [Ro].

In the first chapter, considering the Set Theory as a chapter of General Analysis, Monteiro characterized it as the theory that "occupies itself with the properties of the sets of points that remain invariant in relation to the group of biunivocal transformations", such as Abstract Algebra, "one of the most recent chapters of modern mathematics", which deals with the properties "which remain invariant in relation to the group of isomorphisms (biunivocal correspondences which respect the operation considered)" and, in the third chapter, Topology, also a chapter of the General Analysis, as the theory "which studies the properties of the sets of points which remain invariant in relation to the group of bicontinuous transformations or homeomorphisms." Besides the influence of the French school, Monteiro, who had followed the Julia's seminars in Paris since 1933 until 1936 (on Groups and Algebras, Hilbert Spaces and Topology), was also well aware of the contemporary mathematical developments of the Polish and the Russian schools in Set Theory and Topology and of the German school in Algebra.

Finally in the fourth and main chapter, Monteiro introduced "the notion of algebraic-topological space — which we can define as a space where there is simultaneously an algebra and a topology", with the aim of dealing with the "study of invariant properties for a topo-isomorphism, that is, by a biunivocal correspondence that is simultaneously a homomorphism and an isomorphism", especially in what he called "analytical spaces", that is, those for which the algebraic operation is continuous. Among these, he introduced the perfectly decomposable abelian topological groups, for which he proved "a theorem of structure", which, being "analogous to Banach's and Cantor-Bernstein's theorems" (about the equivalence of two sets with the same power), establishes, in particular, that if two of those groups "have the same algebraic dimension they are topo-isomorphic". The new notion of algebraic dimension is a generalization of the "linear dimension of a (vector) space of type (F) recently introduced by Banach" in his classic 1932 book on *Théorie des opérations linéaires*. Culminating a modern synthesis of some algebraic-topological structures, including topological groups, normed rings and Banach spaces, Monteiro generalized the results of his thesis on the additivity of Fredholm kernels, obtaining necessary and sufficient conditions for the additivity of the resolvents within the rings of linear operators in Banach spaces (Fig. 6).

The clarity and novelty with which the new abstract ideas are described and put into practice by Monteiro in his *Ensaio* is remarkable and it represents a significant progress, certainly independent and unknown to the collective of mathematicians who, under the name of N. Bourbaki, were creating the *Éléments de Mathématique* which would only start publishing a year later in Paris. In his autobiography, André Weyl (1906–1998), one of the founders and most influential mathematicians of this collective, recorded the spirit of the time by writing [W, p.114]:

> In establishing the tasks to be undertaken by Bourbaki, significant progress was made with the adoption of the notion of structure, and of the related notion of isomorphism. Retrospectively these two concepts seem ordinary and rather short on mathematical content, unless the notions of morphism and category are added. At the time of our early work these notions cast new light upon subjects which were still shrouded in confusion: even the meaning of the term "isomorphism" varied from one theory to another. That there were simple structures of group, of topological space, etc., and then also more complex structures, from rings to fields, had not to my knowledge



ENSAIO

sobre

os

FUNDAMENTOS da ANÁLISE GERAL

por

Antonio Aniceto Ribeiro Monteiro

Douter em Ciencias Matemáticas

pela Universidade de Paris

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§ 7 - Aneis vectoriais normados

22 - Definição. Um anel vectorial $\mathfrak{A}$ diz-se normado quando existe um funcional - a que daremos o nome de norma do elemento T de $\mathfrak{A}$ e que representaremos pela notação $|T|$ - que satisfaz às seguintes condições:

1º) $\mathfrak{A}$ é um espaço de Banach em relação à norma.

2º) $|ST| \leq |S|\,|T|$

Suporemos que $\mathfrak{A}$ é completo.

Esta noção também foi considerada por M. Nagumo [1] e dêste facto tivemos apenas conhecimento ha poucos mêses. Tinhamos sido levados a considerar esta noção como o objectivo de generalizarmos os resultados obtidos sôbre a aditividade dos núcleos de Fredholm e no verão de 1936, já tinhamos obtido os resultados que vamos indicar, nos números seguintes. Mas antes disso notemos que:

Teorema. - O conjunto dos operadores lineares definidos num espaço (B) formam um anel vectorial normado.

23 - Resolvente. Suporemos que $\mathfrak{A}$ contém uma unidade E . Seja $A \in \mathfrak{A}$ daremos o nome de resolvente de A à função de parâmetro complexo λ definida por

$$R(A;\lambda) = A + \lambda A^2 + \cdots + \lambda^{n-1} A^n + \cdots$$

que é uma função holomorfa de λ , cujo contra-domínio pertence a $\mathfrak{A}$, no interior do círculo

$$|\lambda| < \frac{1}{|A|}$$

É fácil de mostrar que

$$(E + \lambda A)^{-1} = E + \lambda R(A;\lambda)$$

Diremos que A e B são elementos aditivos se

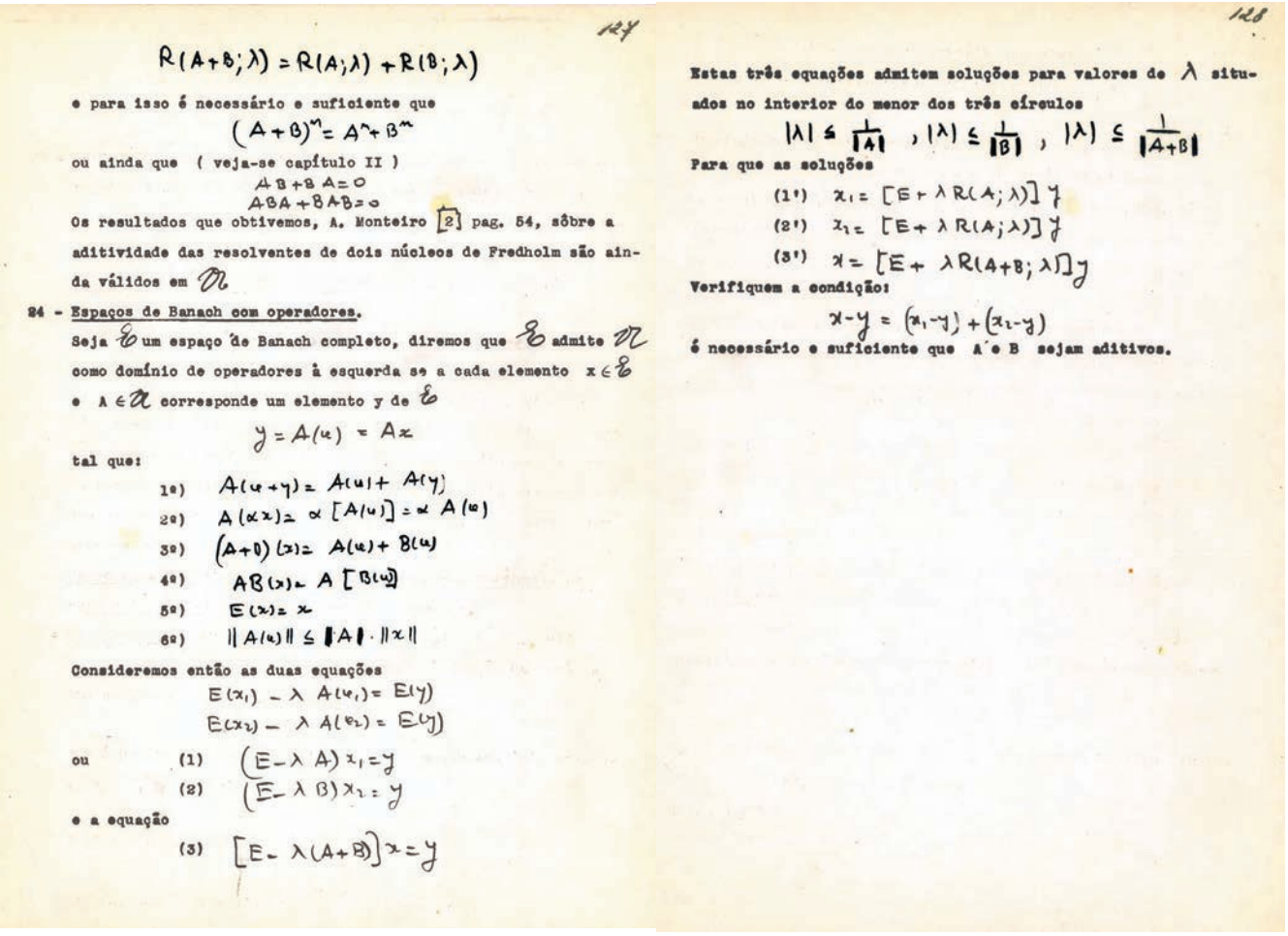
Figure 6. Pages of the Essay with Monteiro's original result [AM1939].

been said by anyone before Bourbaki, and it was something that needed to be said.

That was relevant to be said, and Monteiro also knew it and wrote it very clearly not only in the Preface of his *Ensaio*, that he delivered the 4th February 1939 at the Academy of Sciences of Lisbon, but also throughout its four chapters, which substantial contents coincide in great portions with those of the first four issues of Bourbaki's treaty, published in Paris in 1940 and 1942.

In fact, if the initial objective of those young mathematicians from the *École Normale Supérieur* of Paris, who founded the Bourbaki group in 1935, was to write a new course on Differential and Integral Calculus in the form of a modern treatise on Mathematical Analysis to replace the classical *Cours d'analyse* of the old French school, they evolved into an axiomatic and abstract presentation of "*les structures fondamentales de l'analyse*". The first four fascicules begin the Éléments de Mathématique: Livre I — Théorie des Ensembles (Fascicule de résultats), 1939; Livre II — Algèbre (Structures algèbriques), 1942; Livre III — Topologie Générale (Chap.I, Structures topologiques; Chap.II, Structures uniformes), 1940; Livre III — Topologie Générale (Chap.III, Groups topologiques; Chap.IV, Nombres réels), 1942.

The 45 pages booklet on Set Theory, although dated 1939, has the printing date of February 1940, and begins by explaining the "*mode d'emploi de ce traité*", which "takes the mathematics at the beginning, gives complete demonstrations and, in principle, does not suppose any particular mathematical knowledge, but only a certain habit of mathematical reasoning and a certain power of abstraction". In the English translation of the 1970 profound and enlarged revision of the Set Theory fascicule, one can read: "the axiomatic method allows us, when we are concerned with complex mathematical objects, to separate their properties and regroup them around a small number of concepts: that is to say, using a word which will receive a precise definition later, to classify them according to the *structures* to which they belong." The second book, on Algebra, published in 1942, has about 160 pages and contains the first chapter of algebraic structures. It is a synthesis of modern algebra which is considered as a result "above all of the work of the modern German school" and recognizes the 1930 book by van der Waerden, also used by Monteiro in his 1939 *Ensaio*, as a source of inspiration.



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$$R(A+B;\lambda) = R(A;\lambda) + R(B;\lambda)$$

e para isso é necessário e suficiente que

$$(A+B)^n = A^n + B^n$$

ou ainda que (veja-se capítulo II)

$$AB+BA=0$$
$$ABA+BAB=0$$

Os resultados que obtivemos, A. Monteiro [2] pag. 54, sôbre a aditividade das resolventes de dois núcleos de Fredholm são ainda válidos em $\mathfrak{R}$

24 - **Espaços de Banach com operadores.**

Seja $\mathfrak{E}$ um espaço de Banach completo, diremos que $\mathfrak{E}$ admite $\mathfrak{R}$ como domínio de operadores à esquerda se a cada elemento $x \in \mathfrak{E}$ e $A \in \mathfrak{R}$ corresponde um elemento y de $\mathfrak{E}$

$$y = A(x) = Ax$$

tal que:

1º) $A(x+y) = A(x) + A(y)$

2º) $A(\alpha x) = \alpha\,[A(x)] = \alpha\, A(x)$

3º) $(A+B)(x) = A(x) + B(x)$

4º) $AB(x) = A\,[B(x)]$

5º) $E(x) = x$

6º) $\|A(x)\| \leq \|A\| \cdot \|x\|$

Consideremos então as duas equações

$$E(x_1) - \lambda\, A(x_1) = E(y)$$
$$E(x_2) - \lambda\, A(x_2) = E(y)$$

ou

(1) $(E - \lambda A)\, x_1 = y$

(2) $(E - \lambda B)\, x_2 = y$

e a equação

(3) $[E - \lambda(A+B)]\, x = y$

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Estas três equações admitem soluções para valores de λ situados no interior do menor dos três círculos

$$|\lambda| \leq \frac{1}{|A|}\,,\ |\lambda| \leq \frac{1}{|B|}\,,\ |\lambda| \leq \frac{1}{|A+B|}$$

Para que as soluções

(1') $x_1 = [E + \lambda R(A;\lambda)]\, y$

(2') $x_2 = [E + \lambda R(A;\lambda)]\, y$

(3') $x = [E + \lambda R(A+B;\lambda)]\, y$

Verifiquem a condição:

$$x - y = (x_1 - y) + (x_2 - y)$$

é necessário e suficiente que A e B sejam aditivos.

Bourbaki's third book, dedicated to General Topology, is in fact the second to be published in 1940 and consists of two chapters dealing with structures of another kind which "give a mathematical sense to the intuitive notions of limit, continuity and neighbourhood". The Chapter I, on topological structures, begins with open sets, to define topological space, and bases the notion of convergence on the concept of filter, obtaining the complete equivalence between neighborhood, open set and the topology of convergence. The Chapter II deals with uniform structures, which makes it possible to extend the structure of the metric spaces introduced by Fréchet in 1906, and to generalise to the uniform spaces important results, in particular of compactness and completeness. Chapters III and IV of General Topology were published in 1942. Chapter III, starting with the definition of a topological group, develops the theory based on filters and their convergences and on the properties of uniform structures, and concludes with some topics on topological rings and fields. Chapter IV introduces the group of the real numbers, proves the usual topological properties and basic results on series and on numerical functions, ending with an extensive and fairly complete twelve-page historical note. Those three chapters are more innovative and have a broader scope then the corresponding two last chapters od Monteiro's *Ensaio*.

However, comparing the structure of the four chapters and the respective sections of António Monteiro's essay, delivered on February 4, 1939 at the Lisbon Academy of Sciences, with the contents of these first four fascicules by Bourbaki, which are a work of another dimension and with another ambition, we are surprised by the coincidence of their sequencing and even by the overlapping of many of their contents. Naturally Monteiro absorbed in Paris, during his stay between 1931 and 1936, the new ideas and the most recent results of modern mathematics. Monteiro´s objectives had, in a completely different scale and context, some parallelism with the ambitious programme of the Bourbaki collective, but he could not know either the plans or the contents of the *Éléments de Mathématiques*. However, although Monteiro had never been a "bourbakist" or revealed sympathies for the work of Bourbaki's disciples, we dare to consider that his remarkable and forgotten *Ensaio* is, in fact, a forerunner of the great project of that collective author, which is also characterised by a remarkable modernism and structuralism [Ro].

The influence of the contents of the *Ensaio* and its

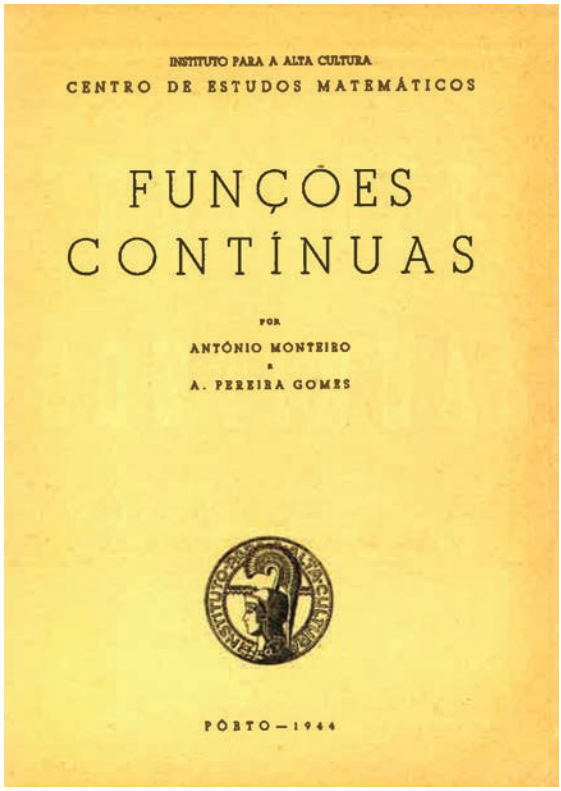
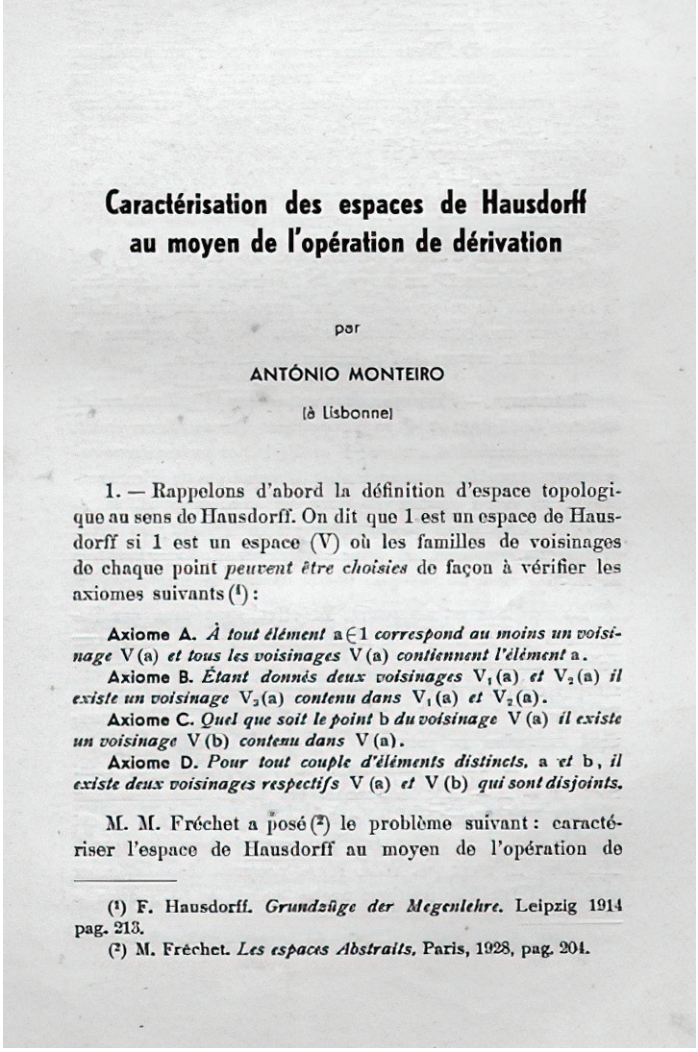


Figure 7. The study on Continuous Functions, published by the CEM of Porto in 1944.



Caractérisation des espaces de Hausdorff au moyen de l'opération de dérivation

par

ANTÓNIO MONTEIRO

(à Lisbonne)

1. — Rappelons d'abord la définition d'espace topologique au sens de Hausdorff. On dit que 1 est un espace de Hausdorff si 1 est un espace (V) où les familles de voisinages de chaque point *peuvent être choisies* de façon à vérifier les axiomes suivants (1) :

Axiome A. *À tout élément* $a \in 1$ *correspond au moins un voisinage* V(a) *et tous les voisinages* V(a) *contiennent l'élément* a.
Axiome B. *Étant donnés deux voisinages* $V_1(a)$ *et* $V_2(a)$ *il existe un voisinage* $V_3(a)$ *contenu dans* $V_1(a)$ *et* $V_2(a)$.
Axiome C. *Quel que soit le point* b *du voisinage* V(a) *il existe un voisinage* V(b) *contenu dans* V(a).
Axiome D. *Pour tout couple d'éléments distincts,* a *et* b, *il existe deux voisinages respectifs* V(a) *et* V(b) *qui sont disjoints.*

M. M. Fréchet a posé (2) le problème suivant : caractériser l'espace de Hausdorff au moyen de l'opération de

(1) F. Hausdorff. *Grundzüge der Megenlehre.* Leipzig 1914 pag. 213.
(2) M. Fréchet. *Les espaces Abstraits,* Paris, 1928, pag. 204.

Figure 8. A paper on general topology by A. Monteiro, published by CEM of Lisbon in *Portugaliae Mathematica*, **1** (1940), 333–339, and cited in the Kelley's book [K].

author is notorious. With the intense mathematical research activities with a small group of students during only three years at the CEML, in Lisbon, and about one year at the CEMP, in Porto, he started together an outline of an ephemeral Portuguese School of General Topology, which influence extended to Rio de Janeiro. In the classic 1955 book *General Topology* [K], the American mathematician J. L. Kelley, in his Foreword, not only thanks Hugo Ribeiro, but also cites in his Bibliography three articles by him and two by Monteiro, all published in *Portugaliae Mathematica* between 1940 and 1945, a note to the C.R. Acad. Sc. Paris by A. Pereira Gomes and the monograph in Portuguese by L. Nachbin on Topological Vector Spaces, both published in 1948 [Ro].

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Where Lie Algebra meets Probability?!

by **Chiara Franceschini*** and **Patrícia Gonçalves****

One of the biggest challenges in statistical physics is to understand phenomena out of equilibrium. A common setting to model non-equilibrium dynamics is to consider stochastic processes of Markovian type with an open boundary acting on the system at different values, thus creating a flux. In these notes, we consider an interacting particle system known in the literature as the symmetric simple exclusion process (SSEP) which is connected to two reservoirs. We show how the algebraic construction of such Markov jump processes helps in analyzing microscopic quantities used to derive macroscopic universal laws. In particular, we will characterize through its moments the non-equilibrium stationary measure.

1 Introduction

Microscopic dynamics of random walks interacting on a discrete space under some stochastic rules are known as interacting particle systems (IPS) and were introduced in the mathematics community by Spitzer in 1970 (see [14]) but before were widely used by physicists, see [15]. The idea of introducing such systems is that, as it often happens in mathematics and physics, they can be used as toy models to describe complex stochastic phenomena involving a large number (typically of the order of the Avogadro's number) of interrelated components. Regardless their simple rules at the microscopic level, IPS are often remarkably suitable models capable of capturing the sort of phenomena one is interested at the macroscopic level. Mathematically speaking, they are treated as continuous time Markov processes with a finite or countable discrete state space. Typically, in the field of IPS one is interested in deriving the macroscopic laws of some thermodynamical quantities by means of a scaling limit procedure. The setting can be described as follows. One considers a continuous space, which is called the macroscopic space. This space is then discretized by a scaling parameter n and time is speeded up by a function of n. On the discrete space one considers a microscopic dynamics consisting of the infinitesimal evolution of particles according to some stochastic law. The dynamics conserves one (or more) thermodynamical quantity and its (their) space/time evolution is the object of our interest. The mathematical rigorous derivation of the macroscopic laws for such quantity, which can be a PDE or a stochastic PDE, depending on whether one is at the level of the Law of Large Numbers or at the level of the Central Limit theorem, is a central problem in the field of IPS. This derivation gives not only validity to the equations obtained but also some physical motivation for their study. In these notes we present, as toy model, the most classical IPS and our aim is, first, explain how to rewrite the Markovian generator of the process in terms of the generators of a Lie algebra, this is a known procedure in the literature. This technique allows to derive a dual process for our model, whose dynamics is simpler. It can be used to give relevant information about our original model; second, explain how to extract from our random dynamics a solution to a PDE, describing the space-time evolution of the density of our model.

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2 The model

To make our presentation simple, we consider as macroscopic space the interval $[0,1]$. Let n be a scaling parameter, we split that continuous space into intervals of size $1/n$.

To an interval of the form $[x/n,(x+1)/n)$ we associate the microscopic position x and then we have a discrete space which is the microscopic space. Now we define the random dynamics. Among the simplest and most widely studied IPS there is the SSEP, see e.g. [12], whose dynamics can be described as follows. After a certain random time, a particle decides to jump to a position of the microscopic space. In SSEP, particles jump among sites under the *exclusion* rule, namely each site can accommodate at most one particle, therefore, if a particle wants to jump to an occupied site, that jump is forbidden. Interactions are to nearest-neighbors (and this is why the process coins the name *simple*) and the jump rates to left and right are identical (*symmetric*). Our toy model is the SSEP with an open boundary, namely we attach two reservoirs that can inject or remove particles from their neighbor positions. The time between jumps is exponentially distributed, which guarantees that this process is Markovian, therefore its evolution can be entirely described via its Markov generator. In the next subsection we define it rigorously.

2.1 Probabilistic description

Consider the microscopic space $\Sigma_n := \{1,\dots,n-1\}$, called bulk, which corresponds to the macroscopic interval $[0,1]$. The construction of the SSEP evolving on Σ_n is done in the following way. To properly define the exchange dynamics, for each $x \in \Sigma_n$, we call $\eta(x)$ the occupation variable at site x: if $\eta(x) = 0$ (resp. $\eta(x) = 1$) it means that site x is empty (resp. occupied). With this restriction, the state space of our Markov process is $\Omega_n := \{0,1\}^{\Sigma_n}$. We denote by $\eta \in \Omega_n$ a configuration of particles. To each bond of the form $\{x,x+1\}$ with $x = 1,\dots,n-2$, we associate a Poisson process of parameter 1, that we denote by $N_{x,x+1}(t)$. Now we describe the boundary dynamics. We artificially add the sites $x = 0$ and $x = n$ that stand for the left and right reservoirs, respectively. We associate two independent Poisson Processes to each bond $\{0,1\}$ and $\{n-1,n\}$ in the following way: $N_{0,1}(t)$ (resp. $N_{n,n-1}(t)$) with parameter $\alpha n^{-\theta}$ (resp. $\delta n^{-\theta}$) and $N_{1,0}(t)$ (resp. $N_{n-1,n}(t)$) with parameter $\gamma n^{-\theta}$ (resp. $\beta n^{-\theta}$). All the Poisson processes described above are independent, so that the probability that two of them take the same value is equal to zero. This means that only one jump occurs whenever there is a possible transition. Before we proceed, we note that the role of the parameters $\alpha,\gamma,\beta,\delta \geq 0$ is to fix the reservoirs' density/current, while the role of $\theta \in \mathbb{R}$ is to tune the strength of the reservoirs' according to the scale parameter n. Taking, for example, θ negative the reservoirs are strong and for θ positive, they are weak and interactions between the boundary and the bulk is weaker as the value of θ increases.

We observe that given the initial configuration of the system plus the realization of all the Poisson processes, it is straightforward to obtain the whole evolution of the system. The role of the Poisson processes is to fix the random time between jumps. We show an example in figure 1 where we consider $n = 5$: an initial condition is given namely $\eta_0 = \delta_2$, ie the configuration with just a particle at site 2, together with all the realizations of the Poisson processes.

In figure 2 we exhibit all the configurations that we obtained from the initial configuration $\eta_0 = \delta_2$ and all the realizations of the Poisson processes given in figure 1.

We warn the reader that below we indexed the configurations in terms of the marks of the Poisson processes and not in time, since our Markov chain evolves in continuous time.

We denote by $\eta^{x,x+1}$ the configuration obtained from η by swapping the values $\eta(x)$ and $\eta(x+1)$, that is $\eta^{x,x+1}(z) = \mathbf{1}_{\Sigma_n\setminus\{x,x+1\}}(z)\eta(z) + \mathbf{1}_{\{x\}}(z)\eta(x+1) + \mathbf{1}_{\{x+1\}}(z)\eta(x)$. On the other hand, when we see a mark of a Poisson process from the boundary, for example, $N_{0,1}(t)$ (resp. $N_{1,0}$), this means that we inject (resp. remove) a particle at the position $x = 1$, if this site is empty (resp. occupied), otherwise nothing happens. More precisely, η^1 is the configuration obtained from η by flipping the occupation variable at 1, that is $\eta^1(z) = \mathbf{1}_{\Sigma_n\setminus\{1\}}(z)\eta(z) + \mathbf{1}_{\{1\}}(z)(1-\eta(1))$.

The exchange dynamics is described by the generator L_{ex}, which acts on functions $f : \Omega_n \to \mathbb{R}$ as $L_{ex}f(\eta) = \sum_{x=1}^{n-2} L_{x,x+1}f(\eta)$ where

$$L_{x,x+1}f(\eta) = c_{x,x+1}(\eta)\left[f(\eta^{x,x+1}) - f(\eta)\right] \qquad (2)$$

and the rates are

$$c_{x,x+1}(\eta) = \{\eta(x)(1-\eta(x+1)) + \eta(x+1)(1-\eta(x))\}.$$

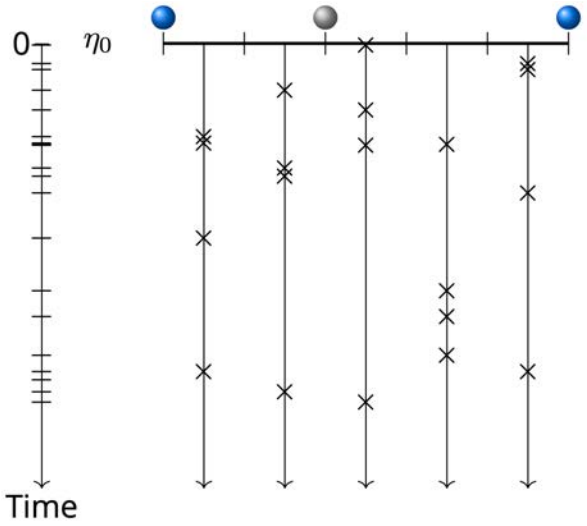

Figure 1.— An initial configuration and marks of the Poisson clocks between each bond.

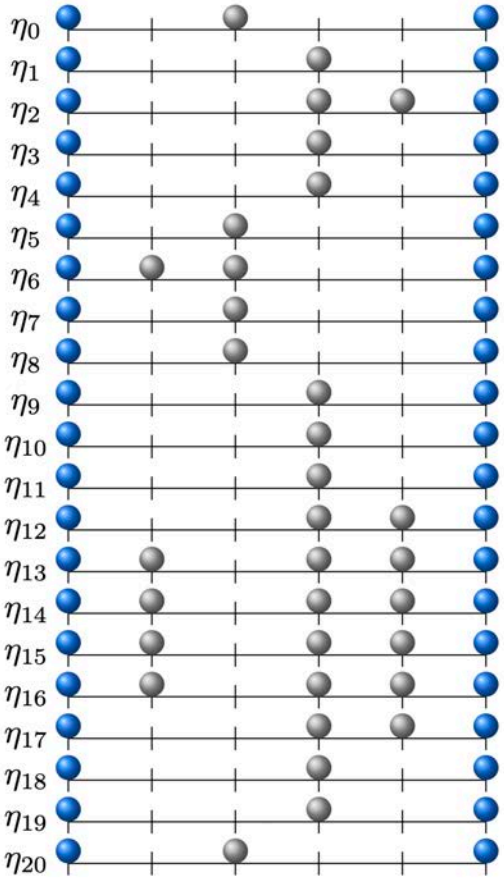

Figure 2.—Configurations evolving according to the marks of the Poisson processes.

The left reservoir generator acts on functions $f : \Omega_n \to \mathbb{R}$ as

$$L_\ell f(\eta) = \frac{1}{n^\theta}\{\alpha(1-\eta(1)) + \gamma\eta(1)\}\left[f(\eta^1) - f(\eta)\right]$$

and the right reservoir L_r is defined analogously, with 1 replaced by $n-1$, α by δ and γ by β. Finally, the open SSEP dynamics is described by a superposition of the two dynamics described above, the exchange and the flip dynamics, so that its full generator is given by

$$L_{\text{SSEP}} = L_\ell + L_{ex} + L_r. \tag{3}$$

Observe that the left and right reservoirs at different densities (respectively $\rho_a = \alpha/(\gamma+\alpha)$ and $\rho_b = \delta/(\beta+\delta)$) impose a flux of particles throughout the system. See a picture below for an illustration of the dynamics just defined.

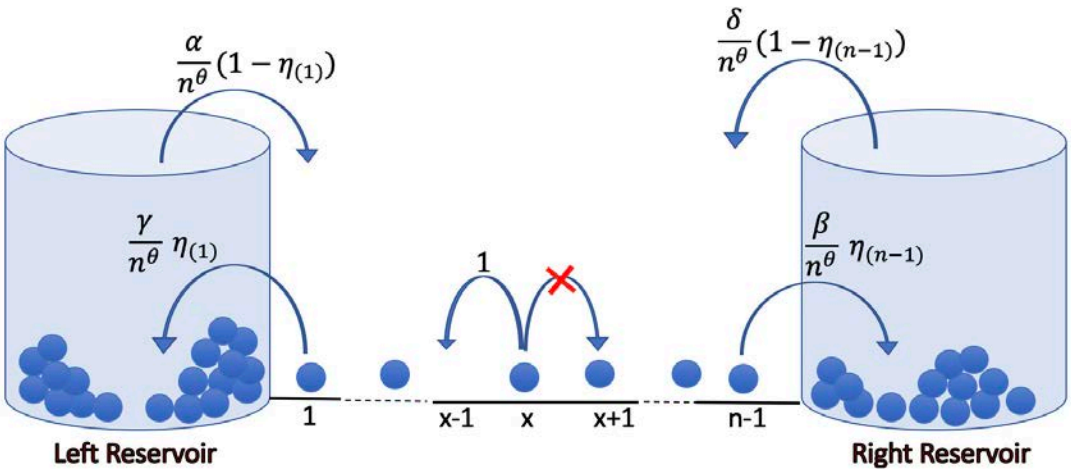

Figure 3.—Schematic description of dynamics of open SSEP.

2.2 Algebraic description

An interesting feature of some IPS is that their generators can be entirely described by the generators of a suitable algebra, for our toy model this will be the Lie algebra $\mathfrak{su}(2)$. More of such constructions were introduced in [10] and further developed in [4]. The Lie algebra $\mathfrak{su}(2)$ is a 3-dimensional vector space of traceless matrices together with the bilinear map $[\cdot,\cdot] : \mathfrak{su}(2) \times \mathfrak{su}(2) \to \mathfrak{su}(2)$ called Lie bracket, which is anti-symmetric, i.e. $[x,y] = -[y,x]$ and satisfies the Jacobi identity, i.e. $[x,[y,z]]+[y,[z,x]]+[z,[x,y]] = 0$ for all $x, y, z \in \mathfrak{su}(2)$. We equip $\mathfrak{su}(2)$ with the adjoint map $* : \mathfrak{su}(2) \to \mathfrak{su}(2)$, i.e. $x \to x^*$ such that $\left(x^*\right)^* = x$ and $\left[x^*, y^*\right] = [y,x]^*$. Usually a basis for $\mathfrak{su}(2)$ is given by the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which are hermitian and unitary. Such matrices satisfy the following commutator and adjoint relations $[\sigma_j, \sigma_{j+1}] = 2i\sigma_{j+2}$ and $\sigma_j^* = \sigma_j$ for $j \in \mathbb{N} \pmod 3$. We also introduce the quadratic element called Casimir (which does not belong to $\mathfrak{su}(2)$) as $C = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$; it is central, i.e. it commutes with all the elements of $\mathfrak{su}(2)$ and it is self-adjoint. For our purpose, it will be more convenient to introduce the basis of real operators J^0, J^+ and J^- which we call generators of the Lie algebra $\mathfrak{su}(2)$. They are given by

$$J^- := \frac{\sigma_1 - i\sigma_2}{2}, \quad J^+ := \frac{\sigma_1 + i\sigma_2}{2}, \quad J^0 := \frac{\sigma_3}{2},$$

and they satisfy the following commutation and adjoint relations $[J^0, J^\pm] = \pm J^\pm, [J^+, J^-] = 2J^0$ and $(J^0)^* = J^0, (J^+)^* = J^-$. The Casimir element in this setting is $\mathscr{C} = 2(J^0)^2 + J^+J^- + J^-J^+$. Besides the matrices representation, an equivalent representation is given by the action on functions $f : \{0,1\} \to \mathbb{R}$ as

$$\begin{cases} (J^- f)(\eta_x) = (1-\eta_x) f(\eta_x + 1) \\ (J^+ f)(\eta_x) = \eta_x f(\eta_x - 1) \\ (J^0 f)(\eta_x) = (\eta_x - 1/2) f(\eta_x) \end{cases}$$

where we made the convention $f(-1) = f(2) = 0$. The operators above are also known as angular momentum operators. We now show how to write the open SSEP dynamics in this context. In particular, it is verified that the exchange generator defined in (3) can be written as the tensor product of the Casimir element. For sites $x, x+1$ we get

$$L_{x,x+1} = J_x^+ J_{x+1}^- + J_x^- J_{x+1}^+ + 2J_x^0 J_{x+1}^0 - 1/2 \,, \quad (4)$$

and then summing over Σ_n we obtain L_{ex}.

A similar description holds true for the generators of the boundary reservoirs,

$$L_\ell = \frac{1}{n^\theta} \left\{ \alpha \left[J_1^- + J_1^0 - \frac{1}{2} \right] + \gamma \left[J_1^+ - J_1^0 - \frac{1}{2} \right] \right\}$$

and similarly L_r is obtained replacing the algebra generators acting on site $n-1$. Note that above the notation J_x^a for $a \in \{0,+,-\}$ means that the generator is acting on the occupation variable at site $x \in \Sigma_n$. Why is this algebraic description useful? In the next section we see that, whenever it is possible to describe a Markov generator using an algebra representation then a useful property, duality, can be derived.

3 Duality for Markov generators

The advantage of dealing with a stochastic evolution lies in the possibility to use probabilistic techniques which considerably simplify the analysis of the system. A powerful tool to deal with Markov processes is *duality theory*, see [13]. This theory allows several simplifications: in a nutshell, one can infer information on a given process by using a simpler one, its dual. For our toy model, we will see how to relate the open SSEP with a simpler system where the open boundary is turned into an absorbing boundary. Indeed, duality in the context of IPS allows "replacing" boundary reservoirs, modeling birth and death processes, with absorbing reservoirs which, as time goes to infinity, will eventually absorb all the particles in the system. It is due to this simplification that one can study properties such as the k–point correlation function of a non-equilibrium system using properties of a dual system consisting of only k dual particles. The link between these two processes, the original, denoted by η_t and with state space Ω, and its dual, denoted by $\hat{\eta}_t$ and with state space $\hat{\Omega}$, is provided by a set of so-called *duality functions* $D : \Omega \times \hat{\Omega} \to \mathbb{R}$, i.e. a set of observables that are functions of both processes and whose expectations, with respect to the two randomness, satisfy the following relationship for all $t \geq 0$

$$\mathbb{E}_\eta[D(\eta_t, \hat{\eta})] = \hat{\mathbb{E}}_{\hat{\eta}}[D(\eta, \hat{\eta}_t)] \,. \quad (5)$$

Above $\mathbb{E}_\eta$ (resp. $\hat{\mathbb{E}}_{\hat{\eta}}$) is the expectation with respect to the law of the η_t process initialized at η (resp. the $\hat{\eta}_t$ process initialized at $\hat{\eta}$). If the generators of the processes are explicit, denoting by $\mathscr{L}$ the generator of $\hat{\eta}_t$ and by $\hat{\mathscr{L}}$ the generator of its dual, $\hat{\eta}_t$, then a duality relation with duality function D translates in saying

that

$$\Big(\mathscr{L}\, D(\cdot,\hat{\eta})\Big)(\eta) = \Big(\hat{\mathscr{L}}\, D(\eta,\cdot)\Big)(\hat{\eta}). \qquad (6)$$

In other words, the action of $\mathscr{L}$ on the first variable of D is equivalent to the action of $\hat{\mathscr{L}}$ on the second variable of D. This is when the algebra comes in. Proving the above relation knowing just the definition of the original process η_t by its generator would be very complicated, however with the algebraic description of η_t one can have a feeling of what to look for. The idea is that we can decompose the Markov generator using the algebra generators as building blocks. This simplifies the analysis because instead of looking for a duality function, one looks for an intertwiner function between two representations of the Lie algebra $\mathfrak{su}(2)$. Such intertwiner function between the J^0, J^-, J^+ representations yields the duality function. Given the special features of our toy model, the duality function will turn out to be product of indicator functions, for which the direct computation is not hard. Nevertheless, for more general dynamics which, for example, allows more than a particle per site, the duality function has a more complicated form and the aforementioned decomposition brings advantages in the proof of the duality relationship.

3.1 Duality for open SSEP

In the following result we give all the ingredients to find a duality relation for our model: it states that, the open SSEP is dual, via a moment duality function D, to a Markov process with the same exclusion dynamics in Σ_n but with only absorbing reservoirs at the boundary.

Theorem 1 (Duality for open SSEP).– For the open SSEP with generator given in 3, the duality relation in (6) is verified for $D : \Omega_n \times \hat{\Omega}_n \to \mathbb{R}$ given by

$$D(\eta,\hat{\eta}) = \left(\rho_a\right)^{\hat{\eta}_0} \prod_{x\in\Sigma_n} \mathbf{1}_{\{\eta_x\geq\hat{\eta}_x\}} \left(\rho_b\right)^{\hat{\eta}_n} \qquad (7)$$

where $\hat{\Omega}_n := \mathbb{N}_0 \times \Omega_n \times \mathbb{N}_0$ and the dual generator $\hat{L} = \hat{L}_\ell + \hat{L}_0 + \hat{L}_r$ acts on functions $f : \hat{\Omega}_n \to \mathbb{R}$ as $\hat{L}_{ex} f(\hat{\eta}) = \sum_{x=1}^{n-2} L_{x,x+1} f(\hat{\eta})$ in the bulk, where $L_{x,x+1}$ is as in (2); and

$$\hat{L}_\ell f(\hat{\eta}) = \frac{1}{n^\theta}(\alpha+\gamma)\hat{\eta}(1)\left[f(\hat{\eta}^1) - f(\hat{\eta})\right]$$

and analogously for the right reservoir with $\alpha+\gamma$ replaced by $\beta+\delta$.

For the interested reader, a rigorous proof of this theorem (for $\theta = 0$) can be found in [5] and it is not hard to generalize it for any value of the parameter θ.

Note that the action in the bulk of the dual generator has the same dynamics of the original one, while the boundary generators only absorb particles from sites 1 and $n-1$. The plan is to describe the dual process with another $\mathfrak{su}(2)$ and find the duality function as intertwiner function of the algebra generators. This is given by the following representation which act on the same functions $f : \{0,1\} \to \mathbb{R}$ as

$$\begin{cases} (\hat{J}^- f)(\hat{\eta}_x) = \hat{\eta}_x f(\hat{\eta}_x - 1) \\ (\hat{J}^+ f)(\hat{\eta}_x) = (1-\hat{\eta}_x) f(\hat{\eta}_x + 1) + (2\hat{\eta}_x - 1) f(\hat{\eta}_x) \\ \qquad\qquad -\hat{\eta}_x f(\hat{\eta}_x - 1) \\ (\hat{J}^0 f)(\hat{\eta}_x) = (\hat{\eta}_x - 1/2) f(\hat{\eta}_x) - \hat{\eta}_x f(\hat{\eta}_x - 1) \end{cases}$$

where, again, $f(-1) = f(2) = 0$. One can check that the above generators are a representation for the Lie algebra $\mathfrak{su}(2)$. Moreover, since the Casimir is irreducible in any representation we can describe the dual process in the same way as in equation (4). At this point one can see that

$$g(\eta_x,\hat{\eta}_x) = \frac{\eta_x!}{(\eta_x-\hat{\eta}_x)!}\Gamma(2-\hat{\eta}_x)\mathbf{1}_{\{\eta_x\geq\hat{\eta}_x\}}$$

satisfy

$$J^a g(\cdot,\hat{\eta}_x)(\eta_x) = \hat{J}^a g(\eta_x,\cdot)(\hat{\eta}_x)$$

for $a \in \{+,-,0\}$, i.e. $g(\eta_x,\hat{\eta}_x)$ intertwines two representations of the $\mathfrak{su}(2)$ algebra. Note that since both $\eta_x, \hat{\eta}_x \in \{0,1\}$ the above function correspond to the duality function in Ω_n. For the left reservoir generator (the right is analogous) one has to check that

$$\left(L_\ell D(\cdot,\hat{\eta})\right)(\eta) = \left(\hat{L}_\ell D(\eta,\cdot)\right)(\hat{\eta}),$$

namely that

$$\begin{aligned} &\frac{\alpha(1-\eta_1)}{n^\theta}\rho_a^{\hat{\eta}_0}\left[\mathbf{1}_{\{\eta_1+1\geq\hat{\eta}_1\}} - \mathbf{1}_{\{\eta_1\geq\hat{\eta}_1\}}\right] \\ &-\frac{\gamma\eta_1}{n^\theta}\rho_a^{\hat{\eta}_0}\left[\mathbf{1}_{\{\eta_1-1\geq\hat{\eta}_1\}} - \mathbf{1}_{\{\eta_1\geq\hat{\eta}_1\}}\right] \\ &= \frac{(\alpha+\gamma)\hat{\eta}_1}{n^\theta}\left[\rho_a^{\hat{\eta}_0+1}\mathbf{1}_{\{\eta_1\geq\hat{\eta}_1-1\}} - \rho_a^{\hat{\eta}_0}\mathbf{1}_{\{\eta_1\geq\hat{\eta}_1\}}\right], \end{aligned}$$

which is verified since both η_1 and $\hat{\eta}_1$ are either 0 or 1.

4 Stationary probability measure and correlations via duality

The open SSEP is an irreducible continuous time Markov process with finite state space, therefore, by a classical theorem we know that there exists a unique stationary measure, that we denote by μ_{ss}. When $\rho = \rho_a = \rho_b$ the stationary measure of our process is an homogeneous product Bernoulli measure with parameter ρ. Moreover, this measure is also reversible. Nevertheless, when the equality $\rho = \rho_a = \rho_b$

fails, the invariant measure is no longer of product form. Heuristically speaking, the density/current at the reservoirs has a different intensity with respect to left/right reservoirs intensity and, therefore, there is an induction of a current flow of particles in the system. Take, for example, $\alpha = \beta = 0$ and $\gamma = \delta = 1$, so that particles are injected in the system from the right reservoir and only exit through the left one. Below we explain briefly how to get some information regarding this measure. Without loss of generality, in order to get information about the stationary measure it will be easier to consider the special case where the reservoirs' rates satisfy $\gamma = 1-\alpha$ and $\beta = 1-\delta$. From here we assume that this is the case. Under these conditions the density of the reservoirs coincide with their injection rate.

4.1 Application of duality

The peculiarity of having a dual process where the boundary becomes only absorbing relies on the fact that, even if two extra sites are considered, the total mass of the dual process can only decrease during the time evolution. As time increases, the bulk will become empty and all the dual particles will eventually stay either on the left or the right reservoirs. In particular, we now show how duality connects the moments of the initial process η with the absorption probabilities of the dual process $\hat{\eta}$. This is done via the following formula

$$\mathbb{E}_{\mu_{ss}}[D(\eta,\hat{\eta})] = \sum_{m=0}^{k} \rho_a^m \rho_b^{k-m} \mathbb{P}_{\hat{\eta}}(m)\,, \qquad (7)$$

where $k = \sum_{x=0}^{n} \hat{\eta}_x$ is the total number of dual particles and $\mathbb{P}_{\hat{\eta}}(m)$ is the probability that m particles are absorbed at the left reservoir (and the remaining $k-m$ go to the right reservoir) starting from the configuration $\hat{\eta}$. The proof relies on the fact that, as $t \to \infty$, all the dual particles will be at sites 0 or n. More details can be found in [5] and [6]. Indeed,

$$\mathbb{E}_{\mu_{ss}}[D(\eta,\hat{\eta})] = \lim_{t\to\infty} \mathbb{E}_{\eta}[D(\eta_t,\hat{\eta})] =$$

$$\lim_{t\to\infty} \mathbb{E}_{\hat{\eta}}[D(\eta,\hat{\eta}_t)] = \mathbb{E}_{\hat{\eta}} \rho_a^{\hat{\eta}_\infty(0)} \rho_b^{\hat{\eta}_\infty(n)} =$$

$$\sum_{m=0}^{k} \rho_a^m \rho_b^{k-m} \mathbb{P}_{\hat{\eta}}(\hat{\eta}_\infty(0) = m,\ \hat{\eta}_\infty(n) = k-m)\,.$$

Suppose we start with a dual configuration $\hat{\eta} = \delta_{x_1} + \delta_{x_2} + \dots \delta_{x_k}$, namely we choose to put a dual particle in each site $x_1, x_2, \dots, x_k$. In this case equation (7) reads as

$$D(\eta,\hat{\eta}) = \prod_{i=1}^{k} \eta_{x_i},$$

which is exactly the function of our interest for the initial process η_t. We now show how to find the 2-point correlation function via the absorption probabilities of two dual exclusion particles. Note that equation (7) specialized for $k = 2$ and $\hat{\eta} = \delta_x + \delta_y$ reads

$$\mathbb{E}_{\mu_{ss}}[\eta_x \eta_y] = \rho_b^2 \mathbb{P}_{x,y}(0) + \rho_a \rho_b \mathbb{P}_{x,y}(1) + \rho_a^2 \mathbb{P}_{x,y}(2) \quad (8)$$

where $\mathbb{P}_{x,y}(m)$ for $m = 0, 1, 2$ is the probability that m particles are absorbed on the left reservoir starting with a particle in x and a particle in y. Before going to the two particles' problem we show how to solve the absorption probabilities for just one particle in the same setting.

4.2 Absorption probability for one dual walker: drunkard's walk

This is a common exercise in probability, known as the drunkard's walk. Recall our dual process, imagine that site 0 is the drunk man's house and site n is a dangerous cliff. The man is at site $x \in \Sigma_n$ and he takes random steps to the left and to the right with the same probability: what is his chance of escaping the cliff? The house and the cliff are absorbing sites in the sense that once he reaches one of them, he will stay there forever. The jump rates are described by the dual generator $\hat{L}_{\text{SSEP}}$. Let us call $p_x := \mathbb{P}_x(1)$ the probability that he reaches home starting at x. Then obviously $p_0 = 1$ and $p_n = 0$. For $x \in \{2, \dots, n-2\}$ the probability of jumping right or left is the same, 1/2, while if he is in 1 (resp. $n-1$), goes to 0 (resp. n) with probability $1/(n^\theta+1)$ and to 2 (resp. $n-3$) with the complement probability, $n^\theta/(n^\theta+1)$. Mathematically we have to solve the following system, which is found by conditioning on the first possible jump of the drunk man:

$$\begin{cases} p_1 = \frac{1}{n^\theta+1} + \frac{n^\theta}{n^\theta+1} p_2 \\ p_x = \frac{p_{x-1}+p_{x+1}}{2} \qquad \text{for } x \in \{2, \dots, n-2\} \\ p_{n-1} = \frac{n^\theta}{n^\theta+1} p_{n-2}\,. \end{cases}$$

A simple computation shows that last identities can be rewritten in such a way that $p_x = p(x)$ is the solution of $(\mathscr{B}_n^\theta p)(x) = 0$, where the operator $\mathscr{B}_n^\theta$ acts on

functions $f : \{0, \cdots, n\} \to \mathbb{R}$ as

$$(\mathscr{B}_n^\theta f)(x) = \tfrac{1}{2}\Delta_n f(x), \quad \text{for } x \in \{2, \cdots, n-2\},$$

$$(\mathscr{B}_n^\theta f)(1) = n^2(f(2) - f(1)) + \frac{n^2}{n^\theta}(f(0) - f(1)),$$

$$(\mathscr{B}_n^\theta f)(n-1) = n^2(f(n-2) - f(n-1)) + \frac{n^2}{n^\theta}(f(n) - f(n-1)).$$

From this we know that, for $x \in \{2, \cdots, n-2\}$, we are looking for an harmonic function of the one dimensional discrete laplacian. Therefore p_x is a polynomial in x, i.e. $p_x = Ax + B$, for $A, B \in \mathbb{R}$ for $x \in \{2, \cdots, n-2\}$.

By using the boundary conditions, we find $A = -1/(n-2+2n^\theta)$ and $B = (n-1+n^\theta)/(n-2+2n^\theta)$, so that, for $x \in \{2, \cdots, n-2\}$ we have

$$p_x = -\frac{x}{n-2+2n^\theta} + \frac{n-1+n^\theta}{n-2+2n^\theta} \tag{9}$$

We now turn to the absorption probabilities of two exclusion processes.

4.3 Absorption probabilities for two dual walkers

The idea is the same as before, we condition on the first possible jump and we obtain a difference equation which is close to a two dimensional laplacian with some boundary conditions. Recall that $\mathbb{P}_{x,y}(m)$, for $m = 0, 1, 2$ is the probability that m particles are absorbed on the left boundary starting from the configuration with one particle in x and one particle in y. To simplify notation we use $p_{x,y} := \mathbb{P}_{x,y}(m)$ and we neglect the dependence on θ, n and m. Conditioning on the first jump we get the following identities

$$\begin{cases} p_{x,y} = \frac{1}{4}[p_{x-1,y} + p_{x+1,y} + p_{x,y-1} + p_{x,y+1}] \\ \qquad\qquad \text{for } 1 \neq x \sim y \neq n-1 \\ p_{x,x+1} = \frac{1}{2}[p_{x-1,x+1} + p_{x,x+2}] \\ \qquad\qquad \text{for } x \neq 1, n-2 \\ p_{1,y} = \frac{n^\theta}{1+3n^\theta}[p_{2,y} + p_{1,y-1} + p_{1,y+1}] \\ \qquad + \frac{1}{1+3n^\theta} p_{0,y} \quad \text{for } 2 < y < n-1 \\ p_{x,n-1} = \frac{n^\theta}{1+3n^\theta}[p_{x,n-2} + p_{x-1,n-1} + p_{x+1,n-1}] \\ \qquad + \frac{1}{1+3n^\theta} p_{x,n} \quad \text{for } 2 < y < n-1 \\ p_{1,n-1} = \frac{1}{2+2n^\theta}[p_{0,n-1} + p_{1,n}] \frac{n^\theta}{2+2n^\theta}[p_{2,n-1} + p_{1,n-2}] \\ p_{1,2} = \frac{1}{1+n^\theta} p_{0,2} + \frac{n^\theta}{1+n^\theta} p_{1,3} \\ p_{n-2,n-1} = \frac{1}{1+n^\theta} p_{n-2,n} + \frac{n^\theta}{1+n^\theta} p_{n-3,n-1}. \end{cases} \tag{10}$$

We observe that the above identities do not depend on the choice of m, nevertheless, as we will see below, the boundary conditions satisfied by $p_{x,y}$ do depend on m.

As above, by introducing the operator $\mathscr{O}_n^\theta$, that we define below, we can write the above system in a concise form. The operator acts on functions $f : \{0, \cdots, n\} \times \{0, \cdots, n\} \to \mathbb{R}$ in the following way: for $x \sim y \in \{1, \cdots, n-1\}$ we have

$$\begin{aligned} (\mathscr{O}_n^\theta f)(x, y) &= a_{x-1} f(x-1, y) + f(x+1, y) \\ &\quad + a_{y+1} f(x, y+1) + f(x, y-1) \\ &\quad - (a_{x-1} + 2 - a_{y+1}) f(x, y) \end{aligned}$$

and for $x \in \{1, \cdots, n-2\}$ we have

$$\begin{aligned} (\mathscr{O}_n^\theta f)(x, x+1) &= a_{x-1} f(x-1, x+1) \\ &\quad + a_{x+2} f(x-1, x+2) \\ &\quad - (a_{x-1} + a_{x+2}) f(x, x+1). \end{aligned}$$

The coefficients satisfy $a_0 = a_n = \frac{1}{n^\theta}$, otherwise $a_x = 1$. We know that, conditioning on the fist jump, $p_{x,y} = p(x, y)$ satisfies $(\mathscr{O}_n^\theta p)(x, y) = 0$. The above equation tells us that we are looking for the harmonic function of a two dimensional laplacian which is reflected if $x \sim y$ and deformed by a factor that depends on θ if we are close to the boundary. A general solution for $m = 0, 1, 2$ is of the form $p_{x,y} = p_{x,y}(m) = A_m x + B_m y + C_m xy + D_m$. The twelve unknown constants are found by using the boundary conditions, the law of total probability, some geometric symmetries because the walk gives symmetric jumps and also the previous result regarding the drunkard's walk. For $m = 2$, it is easy to deduce that $p_{x,y}(2)$ satisfies

$$\begin{cases} p_{0,y}(2) := p_y(1) = \frac{n-1-y+n^\theta}{n-2+2n^\theta} \\ p_{0,0}(2) := 1 \\ p_{x,n}(2) := 0 \end{cases}$$

For $m = 0$, it is easy to deduce that $p_{x,y}(0)$ satisfies

$$\begin{cases} p_{x,n}(0) := p_x(0) = \frac{x-1+n^\theta}{n-2+2n^\theta} \\ p_{n,n}(0) := 1 \\ p_{0,y}(0) := 0 \end{cases}$$

For $m = 1$, it is easy to deduce that $p_{x,y}(1)$ satisfies

$$\begin{cases} p_{0,y}(1) := p_y(0) = \frac{y-1+n^\theta}{n-2+2n^\theta} \\ p_{x,n}(1) := p_x(1) = \frac{n-1-x+n^\theta}{n-2+2n^\theta} \\ p_{0,n}(1) := 1 \\ p_{0,0}(1) = p_{n,n}(1) := 0 \end{cases}$$

Where the $p_x(1)$ is the probability found in the previous section and $p_x(0) = 1 - p_x(1)$, its complement. Using the last three equations of (10) together with the boundary conditions of each absorbed probability

we can write for all $m = 0, 1, 2$, the probability $p_{x,y}(m)$ in terms of the C_m's only. Now we consider the following identity given by symmetry arguments

$$p_{x,y}(2) = p_{n-y,n-x}(0)$$

which allows immediately to identify $C := C_2 = C_0$. Using the law of total probability:

$$p_{x,y}(2) + p_{x,y}(1) + p_{x,y}(0) = 1$$

we get $C_1 = -2C$. And, finally, using the condition for $p_{x,y}(1)$:

$$2p_{x,x+1}(1) = p_{x-1,x+1}(1) + p_{x,x+2}(1),$$

we find that $C = \frac{1}{(n-2+2n^\theta)(n-3+2n^\theta)}$. For sake of completeness we write below the explicit values of the three absorption probabilities.

$$\begin{aligned}
p_{x,y}(2) &= \frac{(n-x-2+n^\theta)(n-1-y+n^\theta)}{(n-2+2n^\theta)(n-3+2n^\theta)} \\
p_{x,y}(0) &= \frac{(x-1+n^\theta)(y-2+n^\theta)}{(n-2+2n^\theta)(n-3+2n^\theta)} \\
p_{x,y}(1) &= \frac{x(n+1)+y(n-1)-2xy}{(n-2+2n^\theta)(n-3+2n^\theta)} \\
&\quad + \frac{2(n^\theta-1)(1+n+n^\theta)}{(n-2+2n^\theta)(n-3+2n^\theta)}.
\end{aligned}$$

At this point one can easily find the stationary correlation by plugging the above result into equation (8). All the computations presented above are in agreement with those obtained by another method, called the matrix ansatz product [7], where the stationary correlations are also found, for details we refer the reader to [8]. An intuitive representation of the dynamics of two dual exclusion particles on $\{0, 1, \dots, n\}$ is given in the picture below. This dynamics can always be represented by the dynamics of a single particle which is performing a symmetric random walk but now evolving inside the two dimensional simplex. The red points are the traps where the random walk is absorbed forever. They represent the three possible ways that two dual exclusion particles can be absorbed in the boundary of the lattice $\{0, 1, \dots, n\}$. If the random walk reaches the vertical cathetus it means the leftmost exclusion particle has been absorbed in 0, while if it reaches the horizontal cathetus, the rightmost exclusion particle has been absorbed in n. Note that one of these two events has to happen in order that the random walk hits one of the three traps. Once the random walk reaches one of the two cathetus it cannot leave that cathetus, since in the dynamics of the two exclusion one particle is already absorbed. On the cathethus the dynamics of the two dimensional random walk is exactly the same as the one of the one dimensional random walk with absorbing boundary, whose absorption probability is given by the drunkard's walk. Note that since two exclusion particles cannot be on the same site, we removed the diagonal $y = x$, while the upper diagonal $y = x + 1$ represents the sites where the two exclusion particles are neighbors.

We observe that these arguments can be extended to higher point correlations functions like $\mathbb{E}_{\mu_{ss}}[\eta_{x_1} \cdots \eta_{x_k}]$ and also to higher dimensions, but for the purposes of this article we decide to present only the one-dimensional case and the two-point correlation function.

5 The evolution of density

The dynamics described above in different ways, if not in the presence of stochastic reservoirs, would conserve one quantity: *the number of particles*. More precisely, starting from a configuration η_0 with $k \leq n-1$ particles, at any time t we would see exactly the same number of particles on η_t. Adding the stochastic reservoirs, this conservation law is destroyed and the goal is to see the effect at the macroscopic level of adding reservoirs to the system. We define then a random measure π^n that gives weight $1/n$ to each particle as

$$\pi^n(\eta, du) = \frac{1}{n}\sum_{x=1}^{n-1} \eta(x)\delta_{x/n}(du)$$

which is a positive measure with total mass bounded by 1. We assume that we start our process η_t from a measure μ_n for which the following result is true: $\pi^n(\eta, du)$ converges, as $n \to +\infty$, to the measure $\pi(du) = g(u)du$ where $g : [0,1] \to \mathbb{R}$ is a measurable function. Observe that π^n is a random measure while π is deterministic. The above convergence is in the weak sense and, by the randomness of π^n, it is also in probability with respect to μ_n, more precisely, μ_n is such that, for any $\delta > 0$ and any function $f \in C([0,1])$ it holds

$$\lim_n \mu_n\Big(\eta : \Big|\pi^n(\eta, du)(f) - \langle f, g\rangle\Big| > \delta\Big) = 0. \quad (11)$$

Above $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2[0,1]$ and $\pi^n(\eta, du)(f)$ denotes the integral of f with respect to the measure $\pi^n(\eta, du)$. The goal is then to show that the same result holds true at any later time t, but the limit measure is given by $\pi_t(du) = \rho(t,u)du$, where the density $\rho(t,u)$ is the solution of a PDE. This result is known in the literature as hydrodynamic limit and the PDE is the hydrodynamic equation. In the case of

the open SEEP we have the following result.

THEOREM 2 (HYDRODYNAMICS FOR SSEP).– Starting from μ_n as described above i.e. satisfying (11) for a certain measurable function $g[0,1] \to \mathbb{R}$; the trajectory of random measures $\pi_t^n(\eta_{tn^2}, du)$ converges, as $n \to +\infty$, to the trajectory of deterministic measures given by $\pi_t(du) = \rho(t,u)du$, where $\rho(t,u)$ is the unique weak solution of the heat equation $\partial_t \rho(t,u) = \partial_u^2 \rho(t,u)$ starting from g and with:

- Dirichlet boundary conditions $\rho(t,0) = \rho_a$ and $\rho(t,1) = \rho_b$, for any $t > 0$, when $\theta < 1$;
- Robin boundary conditions $\partial_u \rho(t,0) = (\alpha + \gamma)(\rho(t,0) - \rho_a)$ and $\partial_u \rho(t,1) = (\beta + \delta)(\rho_b - \rho(t,1))$, for any $t > 0$, when $\theta = 1$;
- Neumann boundary conditions $\partial_u \rho(t,0) = \partial_u \rho(t,1) = 0$, for any $t > 0$, when $\theta > 1$.

The proof of the previous theorem, by using the entropy method developed in [11], can be seen in [1] for the regime $\theta \geq 0$ and in [2] for the regime $\theta < 0$. We observe that above the time scale has been re-scaled to tn^2, which is the time scale for which the evolution of the density is non- trivial, known as diffusive time scale. What if one takes shorter time scales of the form n^s with $s < 2$? Then we do not see any space/time evolution of $\rho(t,u)$. As a consequence of the previous result we see that on a strong action regime of the reservoir dynamics, the density profile is fixed at the boundary; while on the weak action regime, the space derivative (current) of the profile becomes fixed.

6 HYDROSTATICS AND CORRELATION FUNCTIONS

The reader now might ask about the stationary measure. Can we obtain the previous result starting from the measure μ_{ss}? This result is know in the literature as *hydrostatic limit* and to recover it from last theorem one just has to derive (11) for a certain function g. The candidate is exactly the stationary solution of the corresponding PDE, which in the cases above is of the form $\bar{\rho}(u) = au + b$, where a and b are fixed by the boundary conditions. To prove the result we need two things:

1. Define for $x \in \Sigma_n$ the discrete profile $\rho_t^n(x) = \mathbb{E}_{\mu_n}[\eta_{tn^2}(x)]$ and extend it to the boundary by setting $\rho_t^n(0) = \rho_a$, $\rho_t^n(n) = \rho_b$. Taking $\mu_n = \mu_{ss}$, we need to know that the stationary discrete profile $\rho^n(x)$ is close to $\bar{\rho}(\frac{x}{n})$. One way to do it is from Kolmogorov's equation, in which one finds that it solves the equation

$$\partial_t \rho_t^n(x) = (\mathscr{B}_n^\theta \rho_t^n)(x), \quad x \in \Sigma_n, \quad t \geq 0$$

where the operator $\mathscr{B}_n^\theta$ was defined in previously.

Observe that the above equation is closed in terms of $\rho_t^n(\cdot)$, this is a consequence of the fact that the generator of the dynamics does not increase the degree of functions. A simple computation allows to derive the stationary solution of the previous equation and to show that it is close to $\bar{\rho}(\cdot)$. Alternatively, we could use the results we obtained by duality which give

$$\mathbb{E}_{\mu_{ss}}[\eta(x)] = \rho_b \mathbb{P}_x(0) + \rho_a \mathbb{P}_x(1). \tag{12}$$

From (9) we conclude that

$$\rho_{ss}^n(x) = \frac{\beta - \alpha}{2n^\theta + n - 2} x + \frac{\beta - \alpha}{2n^\theta + n - 2}(n^\theta - 1) + \alpha, \tag{13}$$

from where we can easily check that

$$\lim_{n \to +\infty} \max_{x \in \Sigma_n} |\rho_{ss}^n(x) - \bar{\rho}(\tfrac{x}{n})| = 0.$$

2. We need to study the behavior of the two-point correlation function defined generally by

$$\varphi_t^n(x,y) = \mathbb{E}_{\mu_n}[\bar{\eta}_{tn^2}(x)\bar{\eta}_{tn^2}(y)],$$

where $\bar{\eta}_{tn^2}(x) = \eta_{tn^2}(x) - \rho_t^n(x)$ and show that, for $\mu_n = \mu_{ss}$, it vanishes as $n \to +\infty$. As for the discrete profile, we can also apply Kolmogorov's equation and derive a discrete equation for the evolution of this function and then obtain its stationary solution. Alternatively, we could use the results we obtained by duality from where we can get the explicit expression for the stationary correlations. A simple, but long, computation shows that

$$\varphi_{ss}^n(x,y) = \mathbb{E}_{\mu_{ss}}^n[\bar{\eta}(x)\bar{\eta}(y)] = -\frac{(\beta - \alpha)^2 (x + n^\theta - 1)(n - y + n^\theta - 1)}{(2n^\theta + n - 2)^2 (2n^\theta + n - 3)}$$

from where we conclude that

$$\max_{x,y} |\varphi_{ss}^n(x,y)| \to_{n \to +\infty} 0.$$

Even if for hydrostatics it is enough to know the order of decay of $\varphi_{ss}^n(x,y)$, thanks to the approach shown above we were able to write the actual form of the two point correlation function. We note that from the previous identity we can obtain the following relation-

ship

$$\varphi^n_{ss}(x,y) = -\frac{(\beta-\alpha)^2}{2n^\theta + n - 3} p_x(0) p_y(1)$$

where $p_x(1)$ is given in (9). We also note that for $\theta = 0$ the above identity becomes

$$\varphi^n_{ss}(x,y) = -\frac{(\beta-\alpha)^2}{n-1} G^{\mathrm{Dir}}\left(\frac{x}{n}, \frac{y}{n}\right)$$

where $G^{\mathrm{Dir}}(u,v) = u(1-v)$ is the Green function of the 2-dimensional laplacian on $\{(u,v) : 0 \le u \le v \le 1\}$, reflected on the line $u = v$ and with homogeneous Dirichlet boundary conditions, that is $G^{\mathrm{Dir}}(u,v)$ is the solution of

$$\Delta^R G^{\mathrm{Dir}}(u,v) = -\delta_{u=v}$$

where for $u \neq v$,

$$\Delta^R G^{\mathrm{Dir}}(u,v) = \partial^2_u G^{\mathrm{Dir}}(u,v) + \partial^2_v G^{\mathrm{Dir}}(u,v)$$

and for $u = v$,

$$\Delta^R G^{\mathrm{Dir}}(u,v) = \partial_v G^{\mathrm{Dir}}(u,v) - \partial_u G^{\mathrm{Dir}}(u,v)$$

and $G^{\mathrm{Dir}}(0,v) = G^{\mathrm{Dir}}(u,1) = 0$. We get the scaling form

$$\lim_{n\to+\infty} n\varphi^n_{ss}(x,y) = -(\beta-\alpha)^2 G^{\mathrm{Dir}}(u,v)$$

for the continuous correspondents $\frac{x}{n} \to u$ and $\frac{y}{n} \to v$. We now see what happens for the cases when $\theta \neq 0$. For $0 < \theta < 1$, a simple computation shows that the limit above also holds. For $\theta = 1$ we get

$$\lim_{n\to+\infty} n\varphi^n_{ss}(x,y) = -\frac{(\beta-\alpha)^2}{9} G^{\mathrm{Rob}}(u,v)$$

where $G^{\mathrm{Rob}}(u,v) = \frac{1}{3}(u+1)(2-v)$ and corresponds to the Green function of the 2-dimensional laplacian defined above, but with homogeneous Robin boundary conditions given by $\partial_u G^{\mathrm{Rob}}(0,v) = G^{\mathrm{Rob}}(0,v)$ and $\partial_v G^{\mathrm{Rob}}(u,1) = -G^{\mathrm{Rob}}(u,1)$. Finally, for $\theta > 1$, if we use the same scaling as above, we see that

$$\lim_{n\to+\infty} n\varphi^n_{ss}(x,y) = 0.$$

Nevertheless, a simple computation shows that

$$\lim_{n\to+\infty} n^\theta \varphi^n_{ss}(x,y) = -\frac{(\beta-\alpha)^2}{8}(u+1)(1-v)$$

for the continuous correspondents $x/n^\theta \to u$ and $y/n^\theta \to v$; and this is the correct order to see a non-trivial limit in the case of very slow boundary. For higher point correlation functions, we can use exactly the same argument as above in order to obtain the exact rates of convergence of the corresponding stationary correlations. Moreover, we conjecture that we can write the stationary k-th point correlation function $\varphi^n_{ss}(x_1,\dots,x_k)$ as a product between a scaling factor and the absorption probabilities of k independent one-dimensional random walks. We also believe that this argument could be extended to other models for which duality is known but all this is left for a future work.

Recently, it has been developed a method in [9] to derive the hydrodynamic and the hydrostatic limits in presence of duality for a similar model called the symmetric inclusion process, where many particles can occupy the same site and show a preference of laying together. The macroscopic behavior for this process is the same as the one described above, but the proof now boils down to the sole use of duality.

We conclude by saying that there are many other models for which one has to explore the notion of duality, specially for asymmetric models where the equations for correlation functions are no longer closed. There is a long and standing work to develop around these problems and here we just collected some nice and simple results for a toy model where Lie algebra and, consequently, duality allows getting a lot of relevant information about our model.

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AN INTERVIEW WITH

José Basto Gonçalves

by **Helena Reis***

José Agostinho Basto Gonçalves (born 28 January 1952) graduated in Mathematics at the University of Porto in 1975 and in 1981 he received his PhD degree in Mathematics from the University of Warwick. He returned to Porto and played a massive role in the creation of a scientific culture in the Math. Department, helping to instill a research-oriented mentality in several generations of students.

His main research work lies in the scope of control theory and of the geometric theory of differential equations. He became Full Professor of the University of Porto in 1991 and he retired in 2008. Over the course of his career, he has supervised two PhD theses and has mentored a number of Master and undergraduate students.

He was member of the first Scientific Committee of CIM (1996-2000) and member of the Statutory Audit Committee from 2000 to 2004. He has also been president of the northern regional direction of SPM.

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Can you tell us in which moment you realized that you wanted to be a Mathematician and also let us know how this happened?

I did not think about it, but had always assumed, even as a child, that my work would be computations or something similar, and without a fixed timetable. I was very fortunate to get all that!

I initially entered an Engineering course, thinking that the Maths course was only for high school teachers, and Engineering was the course with more Maths in it. But after two years I changed to Mathematics.

You have graduated from Warwick. What were your reasons to choose this university? Also, was Warwick your first choice or have you considered other universities as well?

I attended a course at ICTP (Trieste) organized by prof. Markus from Minnesota and Warwick and prof. Olec from the Academy of Sciences (I think) in Czechoslovakia, and met prof. Pritchard from Warwick. Also, my friends Luisa Magalhães and Eugenia Sá were doing their Ph.D. at Warwick so . . .

You were one of the first people in Porto to have gone to Warwick for graduate school and since then, many others have followed this path some of them under your recommendation. Do you somehow feel to have a *scouting job* for Warwick?

Perhaps I was enthusiastic about my time at Warwick, it had been extraordinary for me; also it was easier to recommend people, and I knew well the conditions there. I always thought that it would be better if people went to different universities, but the knowledge of previous students is very important for the decision to leave Porto and study abroad.

Did you return to Porto immediately after defending your Phd thesis in Warwick? Can you describe the situation of the research in Mathematics in Portugal — and more precisely in the north region — at that time?

I returned to Porto in 1981 after 4 years at Warwick. As far as I remember there were no scientific papers in Mathematics published in Porto before 74, but when I came back the level of teaching was very different from my experience before leaving (I was in engineering for two years and then did Applied Mathematics).

Before I left, the Applied Mathematics group was at best very old fashioned. At the end of the 70's, the presence of Ricardo Lima was very influential at the group, and later in the University, through new people that studied for a Ph.D. thanks to him. He taught interesting courses, talked about research and helped former students to obtain contacts abroad and financial support.

In 81 the ambient in Applied Mathematics was very good: Manuel Rogério Silva, Teresa e Pedro Lago had already returned, there was no great interest in proper Mathematics but things were moving, research was being done and there was enthusiasm.

In Pure Mathematics there was no published research in the beginning of the 80's, but the teaching was up to date. I think that this was already a fact even before 1974.

Certainly, anyone studying now (or since the end of the 80's) in Porto for a degree should be able to finish a Ph.D. anywhere.

I was very fortunate in many aspects: I was in engineering to begin with, I learned a lot more physics than is now common (I did not like that at the time but it was useful!), I studied topics In Mathematics courses that were old fashioned then, but were very considered later, and my final year in the Mathematics degree was 74/75 when the list of courses had a great change.

By the time you returned to Portugal, other young Portuguese mathematicians were also returning home from graduate school. How did you get organized to foster the creation of *culture of research* with high standards around this time? For example, you used to have local collaborators in research or have you actually put direct effort into interacting with other colleagues from Portuguese universities?

There was no organization but certainly a mutual interest: discovering the others and discussing the new results and topics.

Our research budget in 1982 was something like 150 euros, the Fundação Calouste Gulbenkian and the director Alberto Amaral were an important help, later JNICT and then INIC had a complete change: we had a project with the money to invite very good mathematicians to give two-week courses, had a very generous budget to get equipment and to face the daily expenses and a number of young beginner mathematicians.

At the University of Porto, mathematicians are spread around several faculties, how does this affect research and teaching, and how have you dealt with this?

I have studied Applied Mathematics and always worked in the department of Applied Math; it was much improved with time due to the efforts of its members. But the University

Photo by **Isabel Labouriau**

had and has more than a dozen departments (or similar structures) responsible for Mathematics classes, and the Science Faculty had a Pure Mathematics department as well.

I never liked this situation, even when personally very convenient. We have tried to encourage collaboration within the university and endeavoured to surpass the inconveniences this causes, first in CMAUP, then in CMUP, after the two centres merged, by having all mathematicians together at the same research centre; the two departments in the Science Faculty are now just one department, but the problem persists at the University and it should not.

How easy was it in the ,70s and ,80s to get a grant to study/travel abroad? When and how did this change?

In 1975 the number of grants was very small, INIC and Gulbenkian and NATO altogether had fewer than 100 for all sciences and humanities (or at least this was commonly said) at Ph.D. level. The situation with Mariano Gago as head of JNICT was completely changed: now FCT has more than 1500 grants for Ph.D.

How did you obtain funding for research through your career? How important was it for your research?

I studied for a Ph.D. in England with a Gulbenkian grant. The first budget I had from INIC after returning was less than 200 euros a year, that when everything was missing in the department (books, journals etc.); at that time Gulbenkian was a great help, with money for books, journals, for attending conferences. The community of active Portuguese mathematicians was very small, travelling was expensive and to stay abroad was even more expensive, all communications were done by letter through standard post service, there was still no e-mail or internet . . .

Young people could get a job and work with us (now is a lot harder) but then everything was missing: there were very few books or journals, first we shared a personal computer bought with University research money, then two

computers . . . We were a small group, six or seven, but we also shared with other people in the University.

Again this was completely changed still in the 80's, first with INIC and afterwards with Luis Magalhães at FCT. We were able to invite scientists for giving courses lasting one or two weeks, everybody could go to a conference per year, our library was quite good, we had computers and printers, and there was very little bureaucracy. A paradise!

Funding was a constant worry in the beginning of the 80's, but that did not affect research much: with money the effort was less, there were more people involved, for me it just was easier but for younger people good funding was fundamental.

You were member of the first Scientific Committee of CIM (1996-2000) whose goal was to develop and promote the mathematical research in Portugal. What was the role of CIM in those years and how were the measures implemented?

I had not a clear idea of what should be the role of CIM, but I thought it was a good idea and could be developed without a lot of money (that of course did not exist).

What sort of progress have you detected/felt? Were they clear right from the beginning or they gradually become clear in the years to come?

I am not a big believer in the power of an institution. Having a permanent teaching/research staff is of course not indispensable but not having it does not help.

Going back to the question about sending students abroad, I am aware that you consider important — if not absolutely indispensable — for young mathematicians to acquire international experience (by the way, as your former student, I remember to have my "wrists slapped" for staying in Portugal for graduate school).

The mathematical community was very small, the number of research papers was almost zero in Porto, it was important for the students to have a different view and contact with much better research environment.

Do you think nonetheless that acquiring this international experience used to be more important years ago and/or consider that significant changes have occurred and that nowadays this type of experience is somehow less relevant?

Now the number of people involved, in Portugal, is completely changed, the international relations exist and work, it is not as necessary to go abroad to change. However, doing everything, first degree to PhD in the same place is still not a good idea.

Besides the scientific connection with England, you also have many contacts in Brazil, where several Portuguese mathematicians, especially from Porto, have obtained their Phd degrees . . . Would you comment on the role the collaboration with Brazil played in your career as well as in the evolution of the research at CMUP?

I learned a lot in USP — S. Carlos, and (very) slowly moved from Control to more standard mathematics; this was only possible thanks to people from all the world I met there at the São Carlos Workshops on Singularities (every two years) and at the university. And I have made very good friends . . .

Today CMUP is a top center for Mathematics recognized both at international and national levels. What is the feeling that such an evolution brings to you and the colleagues from your generation given that you have been the initial promoters of the culture of research? Are you especially proud of the work accomplished?

I am very happy seeing that the new normal was almost unthinkable when I began. Like a coach, I expect the new ones to do better than I did!

Our research centre CMAUP went from *good* to *excellent*, many students finished their Ph.D. here or abroad, CMUP is now *excellent*, things are much better than they were.

Do you have hobbies or other regular interests outside the academic community?

I should have thought about that long ago . . .

I would like to close the interview with a comment rather than with a question. I would like to make clear that Professor José Basto played an important role in my Mathematical education. Namely, you were the instructor of 5 courses I have enrolled in over my undergraduate and Master courses. In addition, you have supervised my Master dissertation as well as my Phd thesis. I am very much indebted with you for everything I have learned and I also thank you for having persuaded me (finally after my thesis defenses) to go abroad for a post-doc in France . . . it certainly was very important in my life.

Dynamical Aspects of Pseudo-Riemannian Geometry

by **Ana Cristina Ferreira*** and **Helena Reis****

The conference Dynamical Aspects of Pseudo-Riemannian Geometry was held at the School of Sciences of the University of Minho from 2 through 6 March 2020. The event has received financial support from the following institutions: **Centro de Matemática da Universidade do Minho** (CMAT), **Centro de Matemática da Universidade do Porto** (CMUP), **Centro Internacional de Matemática** (CIM), **Fundação Luso-Americana para o Desenvolvimento** (FLAD), **Fundação para a Ciência e Tecnologia** (FCT), **Institut de Mathématiques de Toulouse** (IMT)and **Université of Luxembourg** (UNI.LU).

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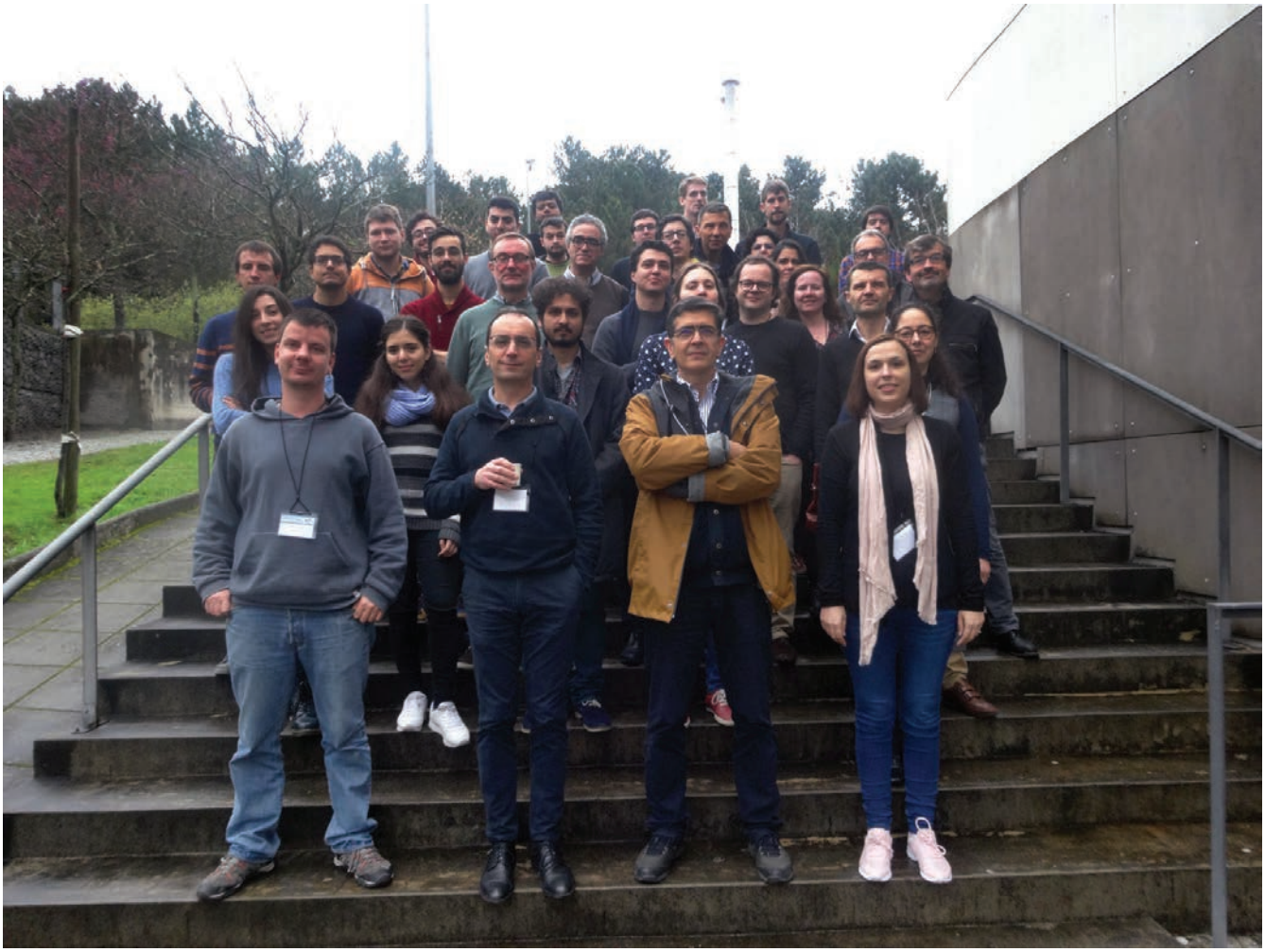

The event consisted of an international conference revolving around dynamical systems naturally arising in important problems belonging to the field of pseudo-Riemannian geometry and, in particular, of Lorentzian geometry. In this way, the conference has attracted interest from experts in dynamical systems and geometry as well as from certain physicists.

The conference has brought together more than 40 experts in the mentioned areas coming from various countries and including several field leaders for a program consisting of 3 advanced mini-courses, 12 invited talks and a poster session. The program also included some exercise sessions, related to the material covered in the mini-courses, and an open problem session. For further information check the link https://cmup.fc.up.pt/Pseudo-Riemannian-Geometry/

Thanks to the generous support provided by our sponsors, several graduate students and post-docs were able to attend the conference and the mini-courses.

Mini-Courses

François Béguin (Univ. Paris 13)
Geometry and dynamics of spatially homogeneous spacetimes

João Pimentel Nunes (IST)
Quantization and Kahler Geometry

Abdelghani Zeghib (CNRS - ENS Lyon)
Configuration spaces: Geometry, Topology, Dynamics, Physics and Technology

Invited Speakers

Ilka Agricola (Univ. Marburg)
Einstein manifolds with skew torsion

Thierry Barbot (Univ. Avignon)
Conformally flat Lorentzian spacetimes and Anosov representations

Alexey Bolsinov (Univ. Loughborough)
Integrable dynamical systems on Lie algebras and their applications in pseudo-Riemannian geometry

Marie-Amelie Lawn (Imperial College London)
Translating solitons in Lorentzian manifolds

Daniel Monclair (Univ. Paris-Sud)
Gromov-Thuston spacetimes

Vicent Pecastaing (Univ. Luxembourg)
Pseudo-Riemannian conformal dynamics of higher-rank lattices

Miguel Sanchez (Univ. Granada)
Lorentzian vs Riemannian completeness and Ehlers-Kundt conjecture

Andrea Seppi (Univ. Grenoble)
Examples of four-dimensional geometric transition

Rym Smai (Univ. Avignon)
Anosov representations and conformally flat spacetimes

Peter Smillie (Caltech PMA)
Hyperbolic planes in Minkowski 3-space

Andrea Tamburelli (Univ. Rice)
Polynomial maximal surfaces in pseudo-hyperbolic spaces

Jérémy Toulisse (Univ. Nice)
Maximal surfaces in the pseudo-hyperbolic space

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An introduction to Analysis on Graphs: what are Quantum Graphs and (why) are they Interesting?

by **James B. Kennedy***

We give a gentle introduction to analysis on graphs. We focus on the construction of prototypical difference operators on discrete graphs, differential operators on metric graphs, and the parallels between the two. The latter lead naturally to quantum graphs, metric graphs on which a Schrödinger-type differential operator acts, for which we finish by discussing a number of recent applications and ongoing areas of investigation. These are drawn mostly, but not exclusively, from mathematical physics.

1 Introduction: two complementary types of graphs

Probably everyone has at least an intuitive notion of what a graph is: a collection of vertices, or nodes, joined by edges. Most mathematicians, perhaps even some non-mathematicians, probably have some idea of the role that graphs play in modelling phenomena as diverse as fine structures such as crystals and carbon nanostructures, social networks, the PageRank algorithm, data processing and machine learning, …, but may not be so familiar with the details.

Generally speaking, at a mathematical level, we are interested in some process taking place on the graph, such as described by a difference or differential equation. The mathematics behind such equations combines ideas from graph theory (obviously), linear algebra, functional analysis and the theory of differential equations, operator theory, and mathematical physics; yet many of the details seem to be largely unknown to the wider mathematical community. As a test: do you know what quantum graphs are?

Our goal here is to give somewhat uneven introduction to analysis on graphs: we first describe, in hopefully accessible terms, what this is: how to define functions and difference and differential operators on graphs, and study them – and in particular what are quantum graphs. Our starting point is that there are (at least) two natural, somewhat parallel, notions of graphs: discrete and metric graphs; the former give rise to difference operators, the latter to differential operators. We will first discuss the construction of these graphs, and then introduce prototypical difference and differential operators, principally realisations of the Laplacian, on each.

But our second goal is to highlight some of the parallels between the two kinds of graphs: indeed, one speaks of Laplacians in both the discrete and the metric case, nomenclature which is justified for various reasons, as we shall see. Finally, we will turn to quantum graphs, which in simple terms are metric graphs equipped with differential operators. We will describe a number of areas of current interest, especially within (parts of) the mathematical physics community. The list of topics we have selected is somewhat idiosyncratic; we include a brief mention of, and references to the literature for, a variety of others. The reader interested in discovering more is referred to the book [BK13], considered a standard reference in the area, the recent survey paper [BK20], the elementary introduction [Ber17], and the somewhat older volume [EKKST08], which contains a large number of still useful review articles.

Part of the work of the author relevant to this topic was supported by the Fundação para a Ciência e a Tecnologia, Portugal, via the program *Investigador FCT*, reference IF/01461/2015, and project PTDC/MAT-CAL/4334/2014, as well as by the Center for Interdisciplinary Research (ZiF) in Bielefeld, Germany, in the framework of the cooperation group on *Discrete and continuous models in the theory of networks*.

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1.1 Discrete graphs

In the case of *discrete graphs* we are more interested in the vertices and consider the edges as relations between the vertices, without necessarily having any direct physical meaning. More formally, a discrete graph G is a pair (V, E), where the vertex set V is any countable (in practice usually finite) set and each edge e in the edge set E may be regarded as a pair of vertices, that is, E may be identified with a subset of $\mathsf{V}\times\mathsf{V}$. Already here we have a further decision to make: whether to treat the edges $\mathsf{e} = (\mathsf{v}, \mathsf{w})$, $\mathsf{v}, \mathsf{w} \in \mathsf{V}$ as *ordered* or *unordered* pairs; we speak of *directed edges* (also called *bonds* in some circles) and *undirected edges*, respectively. In the case of directed edges $\mathsf{e} = (\mathsf{v}, \mathsf{w})$, we may distinguish between the *initial vertex* v and the *terminal vertex* w.

Many social networks may be modelled in this framework; for example, Facebook is a network in which each person (or entity) represents a vertex, and being (Facebook) friends corresponds to an undirected edge between the two vertices. Twitter, on the other hand, is directed, if one considers the edge (v, w) to mean v *is a follower of* w – as is the internet itself with links being edges between the pages represented by vertices. More generally, any model of a network in which there is no natural *distance* between vertices, nor physical bond linking them, is likely to fit into the framework of discrete graphs. This is of course a considerable simplification; for example, one may assign a weight function to the edges of a discrete graph to give a notion of the proximity of the respective vertices.

To do any sort of analysis, of course we need to define functions on our graph. In the case of discrete graphs, this is easy: if functions live on the vertices, then the space of all functions may be identified with $\mathbb{R}^{|\mathsf{V}|}$ or $\mathbb{C}^{|\mathsf{V}|}$. Some care must be taken if the vertex set V is infinite; it becomes natural to work with ℓ^p-spaces.

1.2 Metric graphs

Metric graphs, on the other hand, focus attention on the edges, and are thus more suited to modelling actual physical networks, or fine ramified structures such as nanostructures. We will write $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ for a metric graph, where now each edge $e \in \mathscr{E}$ is identified with a closed interval which may be finite, of some given length $\ell(e) > 0$, i.e. $e \simeq [0, \ell(e)] \subset \mathbb{R}$, or a half-line, $[0, +\infty)$. Care must obviously be taken with the latter, so here we will restrict ourselves to compact intervals.

In order to encode the topological structure of the graph, or equivalently to create a metric, one identifies all interval endpoints which correspond to a given vertex. While intuitively this is very simple, formally it is somewhat fiddly and may be done in a number of ways: for example:

- identify equivalence classes of endpoints, or
- define the underlying metric directly by declaring that the distance between two different interval endpoints corresponding to the same vertex is zero, thus allowing the construction paths between any two points on different edges, or alternatively
- work directly at the level of continuous functions.

For more details we refer to [BK13, Section 1.3], [Mug19] and [KKLM20, Section 2].

At any rate, this gives rise naturally to a metric space; the distance between two given points is the (Euclidean) distance of the shortest path between them. Technically the metric is a pseudometric, as it may take the value $+\infty$ if there is no path between a given pair of points, but it becomes a metric if and only if the graph is connected.

When it comes to defining spaces of functions, metric graphs are, unsurprisingly, more interesting than their discrete counterparts, albeit not yet at the level of L^p-spaces: we may simply define, for a graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with a finite edge set $\mathscr{E}$,

$$L^p(\mathscr{G}) = \bigoplus_{e\in\mathscr{E}} L^p(e) \simeq \bigoplus_{e\in\mathscr{E}} L^p(0, \ell(e))$$

(each edge being prototypically equipped with Lebesgue measure on the interval $[0, \ell(e)]$); indeed, L^p-functions will never see the vertices as the latter form a set of measure zero. Correspondingly, to integrate a function over the graph we integrate over each edge and sum the result. The structure of the graph is only encoded at the level of continuous functions: $C(\mathscr{G})$ will consist of those functions which are continuous on every edge, such that their values at all endpoints meeting at a vertex should agree. These are of course exactly the functions which are continuous with respect to the metric.

To define differentiable functions becomes more challenging because of the issue of defining the derivative across the vertices; instead, it becomes more nat-

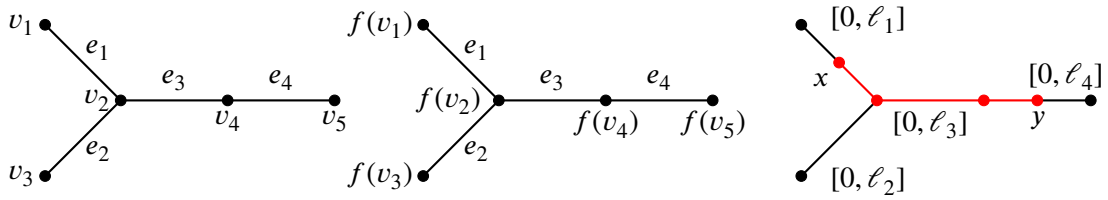

Figure 1. A simple graph on 5 vertices and 4 edges (left); to define a function f on a discrete realisation of the graph we specify the values at the vertices — this also gives rise naturally to difference operators (centre); one may instead identify each edge e_i with an interval $[0,\ell_i] \subset \mathbb{R}$ (or a half-line $[0,+\infty)$), and *glue together* the intervals at their endpoints in the right way, to form a metric graph (right). Here a path between two points x and y is marked in red.

ural to speak of vertex conditions which the functions should satisfy (such as continuity at the vertices, as imposed in $C(\mathcal{G})$). Two of the most natural such conditions to satisfy are the Dirichlet, or zero, condition, and the Kirchhoff condition, where the sum of inward-pointing derivatives at a vertex equals zero.[1]

In practice, one usually works with Sobolev spaces of weakly differentiable functions; for example, $H^1(\mathcal{G})$ is defined as those functions which are edgewise L^2-integrable with edgewise L^2-integrable weak derivative, and which are continuous across the vertices. This makes sense since by standard Sobolev embedding theorems one-dimensional H^1-functions are continuous and thus, up to choosing the correct representative, defined pointwise.

Three final observations are in order: firstly, thus defined, our graphs are not considered to be embedded in Euclidean space; there is no curvature of the edges or angle between them. Secondly, by labelling one vertex of an edge $e \simeq [0, \ell(e)]$ as 0 and the other as $\ell(e) > 0$, we are implicitly (or explicitly) imposing an orientation. However, for many practical purposes this orientation is irrelevant; the differential operators we shall define in the sequel are independent of this choice up to unitary equivalence. Thirdly, in the case of metric graphs it is easy to allow multiple edges between vertices, as well as loops (edges which begin and end at the same vertex); in the case of discrete graphs this is a bit more complicated and we will tacitly assume that our graphs are free of such features, even though most of what we discuss will remain true even with multiple parallel edges and loops.

2 Difference and differential operators

We see immediately that on discrete graphs, since the functions are identifiable with vectors, difference operators (or more generally matrices) will arise; while on metric graphs we may define (ordinary) differential expressions on the edges. In the latter case the point of interest becomes specifying the vertex conditions, or equivalently the domain of definition of the differential operator; a metric graph is essentially a smooth one-dimensional manifold with isolated singularities (the vertices). In both cases we will illustrate this via a prototypical operator, the Laplacian; note that here, in both cases, our edges will be undirected.

Let us start with metric graphs, as here we are closer to the *traditional* Laplacian from the theory of PDEs. In fact, we start with the differential expression $-f''$ on each edge. It is natural to impose continuity at all vertices, as this is essentially the minimal requirement for the functions to *see* the graph. Additionally imposing the Kirchhoff condition, which we may write as

$$\sum_{e \text{ adjacent to } v} \frac{\partial f}{\partial v}(v) = 0,$$

that is, the sum of the derivatives of f at the endpoint of each edge e directed into the vertex v, gives rise to the Laplacian with vertex conditions variously known as standard, natural, continuity-Kirchhoff, and even Neumann-Kirchhoff (if the vertex has *degree one*, i.e.,

[1] This condition is sometimes loosely called a *flow in equals flow out* condition, although this expression must obviously be interpreted with care, depending on the kind of *flow* one is imagining.

only one edge attached, then this reduces to the Neumann condition. Roughly speaking, in many ways the Laplacian with standard conditions behaves somewhat like the Neumann Laplacian on domains, or the Laplace-Beltrami operator on manifolds without boundary). For the Dirichlet condition at a vertex v, instead of the Kirchhoff condition we require that $f(v) = 0$.

If the graph $\mathcal{G}$ has finite total length, then such operators are self-adjoint, semi-bounded from below, and have compact resolvent; thus they behave exactly like Laplacians or Schrödinger operators on bounded domains and manifolds. Generalisations, such as adding a potential to each edge, are easy to incorporate in this framework.

All this is perhaps more naturally seen at the level of forms/weak solutions: the associated positive, symmetric sesquilinear form reads

$$a(f, g) = \int_{\mathcal{G}} f' \cdot \overline{g}' \, \mathrm{d}x$$

with form domain exactly $H^1(\mathcal{G})$ in the case of the standard Laplacian; if Dirichlet conditions are imposed at one or more vertices then the functions should additionally take on the value 0 there. (All this is a short exercise in integration by parts.) The eigenvalues and eigenfunctions of the Laplacian admit the usual min-max variational characterisation; for example, the smallest eigenvalue can be obtained by minimising $a(f, f)$ among all $f \in H^1(\mathcal{G})$ whose L^2-norm is 1.

On discrete graphs, the (discrete or combinatorial) Laplacian is defined purely in terms of the graph structure. We suppose $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ to be a discrete graph with finite vertex set $\mathsf{V} = \{\mathsf{v}_1, \dots, \mathsf{v}_n\}$ and finite edge set $\mathsf{E} = \{\mathsf{e}_1, \dots, \mathsf{e}_m\}$. We take as a starting point the following matrices:

- the *adjacency matrix*, the symmetric matrix whose (i, j)-entry is 1 if v_i and v_j share an edge, or 0 otherwise (in the case of directed edges this matrix can still be defined but will no longer be symmetric);

- the *degree matrix*, the diagonal matrix whose (i, i)-entry is the degree of v_i, i.e., the number of edges emanating from v_i.

The *(discrete)* Laplacian is the difference operator corresponding to the symmetric, positive semidefinite matrix $L := D - A$. For example, for the graph depicted in Figure 1, with the order of vertices as specified there, the Laplacian would be

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

The fact that this is a plausible discrete version of the Laplacian may be recognised in (at least) two ways:

- vectors x satisfying $Lx = 0$ have the *mean value property*, as can be checked with the above example: since the sum of each row of L is zero, the value of x at a vertex is equal to the sum of the values at the surrounding vertices – just as harmonic functions in $\mathbb{R}^d$, solutions of $\Delta f = 0$, satisfy the (continuous) mean value property;

- at the level of forms: L is associated with the positive, symmetric sesquilinear form

$$a(x, y) = \sum_{\mathsf{e} \in \mathsf{E}} (\mathcal{J}^T x)(\mathsf{e}) \overline{(\mathcal{J}^T y)(\mathsf{e})},$$

  where $\mathcal{J} \in \mathbb{R}^{n \times m}$ is the so-called (signed) *incidence matrix* encoding which vertices are the terminal and initial endpoints of which edges; in fact L may also be represented as $L = \mathcal{J}\mathcal{J}^T$, and we may intuitively think of $\mathcal{J}$ as a discrete counterpart of the divergence operator.

A word of caution: there is a common, *normalised* variant, namely

$$L_{\mathrm{norm}} := \mathrm{Id} - D^{-1/2} A D^{-1/2}$$

(that is, we normalise the operator by the degree of the vertices); its $|\mathsf{V}|$ eigenvalues (counting multiplicities) always lie in the interval $[0, 2]$. The standard reference on the topic is [Chu97]; see also [Mug14, Chapter 2] for the construction of all these matrices as well as their directed counterparts.

A strong mathematical parallel between the standard Laplacian on a metric graph and the normalised Laplacian on the corresponding discrete graph was established by von Below in 1985 [Bel85]. Namely, if all the edges of the metric graph have length 1, and we denote by λ_k the ordered eigenvalues of the standard Laplacian, then, up to certain special cases (corresponding to the normalised Laplacian eigenvalues 0 and 2) the eigenvalues of L_{norm} are given by $1 - \cos(\sqrt{\lambda_k})$; the values of the eigenvectors correspond to the values of the eigenfunctions at the vertices. Thus, at least for this special class of *equilateral* metric graphs, the standard Laplacian as a differential operator is *essentially determined* by the corresponding

discrete (normalised) Laplacian. Such connections between discrete and continuous versions have naturally been explored further over the last 30-odd years; see [LP16] and the references therein.

2.1 So what are quantum graphs?

We can now finally answer the first question posed in the title. By a quantum graph we understand a metric graph on which acts a differential operator, most often (but not necessarily) some kind of Schrödinger operator or *Hamiltonian* [Ber17, BK13]; of course this includes various realisations of the Laplacian.

While differential operators on metric graphs have been studied for a long time – they were actively studied in the 1980s, often under the name c^2*-networks* (e.g., [Bel85, Nic87]), and there are applications going back much further [RS53] – the name *quantum graph* is generally considered [Ber17] to trace back to the article *Quantum Chaos on Graphs* [KS97] from 1997, possibly as a contraction of the title.

3 A cornucopia of applications

We finish with a discussion of some current topics of interest in the community, as an answer to the second question posed in the title: we wish to give some idea of the variety of problems and applications which arise in the context of quantum graphs. One may broadly and imperfectly group the applications into models where it is intrinsically sensible to consider ramified structures (atomic or crystalline structures, honeycombs, ramified traps,…) and those where the graph represents a toy model used to study mathematical or quantum physical phenomena: graphs are simple one-dimensional objects which often display complex behaviour typical of higher-dimensional problems.

In the following list no claim is made to completeness, either in the list of topics or in the references given, which largely reflect the author's personal taste and prejudices. Where possible we have tried to provide some of the most recent references available to act as a starting point for a further literature search.

In keeping with these prejudices, as well as the general focus of the quantum graph community, we will mostly be interested in differential operators such as the Laplacian and Schrödinger operators, and their spectra. This is natural since by the spectral theorem the spectrum completely determines such self-adjoint operators.

3.1 Approximation of, or by, higher-dimensional objects

There are two senses in which graphs, be they discrete or metric, can be related to higher-dimensional domains or manifolds: one can consider a (metric) graph as the limit of a sequence of thin branching domains (*shrinking tubes*, or *fattened graphs*), or one can try and approximate a domain or manifold as the limit of as sequence of graphs. In the latter case one usually takes discrete graphs as the approximating objects, as a kind of discretisation of the domain or manifold.

Needless to say, there is an extensive literature on both. The latter is sometimes used to extend results from discrete graphs to manifolds (as in [LLPO15], see also Section 3.3). The former provides a justification for using quantum graphs to study phenomena like waveguides, be they acoustic, quantum or electromagnetic, thin super-conducting structures and so on; here we will follow, and refer to, [BK13, Section 7.5]. Another standard reference for shrinking tubes is the review paper [Gri08] contained in the volume [EKKST08]. Typical questions include whether the solutions of differential equations in the thin domains converge to the solution of some differential equation on the graph, and if so, what vertex conditions the problem in the limit satisfies. (More technically, we are interested in convergence of the resolvents of the operators in various norms, as well as of the operator eigenvalues and eigenfunctions.)

The precise results depend very much on the nature of the approximation, but in perhaps the simplest and most important case of Schrödinger operators in *Neumann tubes* (thin perfectly insulated tubes) shrinking uniformly, one does at least have convergence of the eigenvalues to the eigenvalues, in the correct order, of the Schrödinger operator on the graph with standard vertex conditions, and where the electric potential is, roughly speaking, the restriction of the potential on the thin domain to the graph it contains. Work is still ongoing to establish other kinds of convergence, in particular under different kinds of domain convergence.

In this case the limit object, the quantum graph with its Schrödinger operator, *forgets* many geometric features of the domains, such as angles between branches, curvature of edges and so on. If one allows the Neumann tubes to shrink in a *wilder*, non-

uniform fashion, then one may obtain more interesting limit quantum graphs, including where the operators satisfy other vertex conditions than the standard/Kirchhoff ones.

3.2 Spectral geometry

We have seen that discrete Laplacians may be defined directly in terms of the structure (topology) of a discrete graph, and that at least at the spectral level this can be transferred to equilateral metric graphs via von Below's formula. Can we say something about (non-equilateral) metric graphs, where we have to contend with both the topology and the edge length?

In the case of domains and manifolds a group of questions revolves around understanding how the eigenvalues and eigenfunctions depend on the geometry of the underlying domain or manifold. The classical example is the theorem of Faber-Krahn from the 1920s, based on an earlier conjecture of Lord Rayleigh, that among all domains of given volume the ball is the one whose first Dirichlet Laplacian eigenvalue is smallest. This is an analytic translation of the geometric isoperimetric inequality, that the ball minimises surface area for given volume; the first (nonzero) eigenvalue is of particular interest because it controls the rate of heat loss in the heat equation, the lowest frequency of the object, and so on. We refer to [Pay67] for a (classical) introduction and [Heno6, Hen17] for more modern surveys of the area of *shape optimisation and spectral theory*. The corresponding inverse problem, determining the domain/manifold based on the spectrum of a differential operator on it, corresponds to the question made famous by Mark Kac, "can one hear the shape of a drum?"; see [LR15].

On metric graphs the equivalent of the theorem of Faber-Krahn states that the smallest nonzero eigenvalue of the Laplacian with standard vertex conditions is minimised when the graph is an interval of the same length; this theorem first appeared around 30 years ago [Nic87]. It turns out that graphs are far more amenable to this kind of analysis than domains; see [BL17, BKKM19, KKMM16]. A surprisingly subtle question is which (geometric or topological) properties of a graph are sufficient to bound its eigenvalues and which are not. For example, fixing the diameter D (length of the longest path within the graph) alone places no control on the smallest nonzero standard Laplacian eigenvalue: it may be arbitrarily large or small [KKMM16]; however, if we restrict to *trees*, graphs without cycles, then it cannot exceed π^2/D^2, the corresponding eigenvalue of an interval of length D.

Work has also been done on *isospectral* graphs, quantum graphs which are different but have the same Laplacian spectra. On graphs the problem can be given a new twist since one has more chance of describing the corresponding eigenfunctions: one considers the so-called *nodal count*, the number of *nodal domains*, which are by definition the connected components of the set where the eigenfunction is nonzero. See [BK13, Section 7.1].

3.3 Clustering and partitions

A major preoccupation in applied graph theory is to detect the presence of clusters in a (usually discrete) graph. One might ask whether a given social network such as Facebook tends to be divided into groups of hig hly interconnected individuals with few links between the groups, thus creating the infamous *echo chambers*. Alternatively, one might wish to identify, say, weaknesses in a road network or an electricity grid: if the power lines here go down, does half the country lose power?

There are various ways to measure this. One natural way is the notion of *Cheeger constants* and *Cheeger cuts* borrowed from geometric analysis, originally introduced for manifolds. Say we wish to cut the graph G into two pieces S and $S^c = \mathsf{G} \setminus S$, which we do by cutting through edges. Then for each possible cut we look at the ratio

$$\frac{|\partial S|}{\min\{|S|, |S^c|\}}$$

of edges cut $|\partial S|$ to the smaller of the two sets S e S^c, as measured by the number of vertices in the set.

The infimum of this quotient over all possible cuts is the *Cheeger constant*; the smaller the constant, the easier it is to cut the graph into two (the traditional image for this is the dumbbell manifold, cut through its thin handle).

Figure 2 gives an example on graphs: on this graph of 20 vertices, there is a way to make just two cuts to separate the graph into two groups of 10 vertices each, labelled as blue and red; this is in fact the optimal cut. One might imagine a social network where the vertices represent users; the blue users tend only to have friends with other blue users, while the red users likewise stay amongst themselves. In this case the Cheeger constant will be $2/\min\{10, 10\} = 1/5$, which may be considered small (the number has no absolute meaning but should be viewed in conjunc-

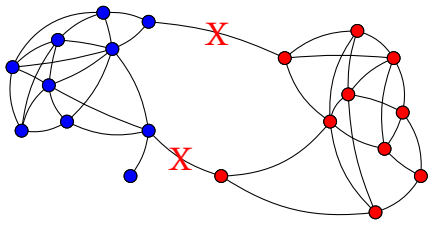

Figure 2. The Cheeger cut of a graph whith 20 vertices.

tion with the total number of vertices and edges).

One can also consider higher order Cheeger constants, which partition the graph into more than two pieces, and in fact estimating these constants, in particular in terms of Laplacian eigenvalues, is one instance where results were first proved on discrete graphs and then transferred to manifolds [LLPO15]. Cheeger constants have also been introduced on metric graphs, with the numerator remaining the same and the denominator becoming the total length (sum of edge lengths) of each piece [KM16, Nic87].

The Cheeger constant corresponds to the first eigenvalue of the so-called 1-Laplacian, i.e., the p-Laplacian when $p = 1$; the eigenvector/eigenfunction has two nodal domains which correspond to the Cheeger cut. This operator is singular, its eigenvalues lack the easy L^p-variational characterisation of the p-Laplacian eigenvalues, and actually calculating the constant of a large graph becomes a computationally hard problem.

Thus one can use the (2-)Laplacian (say, with standard vertex conditions) as a natural proxy, as its variational structure makes determining the eigenvalues and eigenfunctions much easier, both analytically and computationally. Ideally one would use the nodal domains of the k-th eigenvalue as an "optimal" partition into k pieces, but in general there is no simple relationship between the number of nodal domains of an eigenfunction and its number in the sequence. A natural alternative is to consider *spectral minimal partitions*, whereby one looks to minimise a functional of the eigenvalues over all partitions; prototypically this problem might take the form

$$\inf_{\mathscr{P}} \max_{i=1,\dots,k} \lambda_1(\Omega_i),$$

where the infimum is taken over all partitions $\mathscr{P}$ of the object (domain, graph, …) into k pieces $\Omega_1, \dots, \Omega_k$, and $\lambda_1(\Omega_i)$ is the first nontrivial eigenvalue of a suitable Laplacian on Ω_i; one could equally take a p-norm of the eigenvalues in place of the ∞-norm. Such problems were originally considered, and have been studied intensively, on domains and some manifolds; see [Hen17, Chapter 10] for a survey.

On metric graphs this topic is new: the first systematic study of spectral minimal partitions was undertaken in [KKLM20]. As is the case for spectral geometry, and actually for many problems considered here, one can say far more on metric graphs than on domains. Here, far more functionals can be meaningfully defined on the former than the latter, including more exotic combinations of eigenvalues (such as max-min rather than min-max problems). Understanding how these optimal partitions differ and what they reveal about the structure of the graph will be a topic of interest in the next few years.

3.4 Nonlinear Schrödinger equations

Until now we have always considered linear differential operators, as has historically usually been the case on metric graphs. There is, however, a notable family of exceptions, first considered just a few years ago [AST15a]. This principally involves studying existence, or nonexistence, of certain solutions of stationary nonlinear Schrödinger equations on metric graphs (NLSE for short). A stationary NLSE typically takes the form

$$-\Delta u + f(u) = \lambda u, \tag{1}$$

where in place of the usual potential term Vu a nonlinearity $f(u)$ is introduced; here, as in the literature, we will consider the prototypical power nonlinearity $f(u) = |u|^{p-1}u$. These equations, a bedrock of the Calculus of Variations literature, are most commonly studied in d-dimensional space, see [Caz03] for an introduction, but a number of applications, such as Bose-Einstein condensates in traps or optical fibres [AST15a], make it reasonable to consider NLSE in ramified structures, that is, on metric graphs, most commonly and naturally with standard vertex condi-

tions. Of most interest are the ground states, minimisers of the energy functional for which 1 is the Euler-Lagrange equation:

$$E(u) = \frac{1}{2}\|u'\|_2^2 - \frac{1}{p}\|u\|_p^p,$$

where $\|\cdot\|_2$ and $\|\cdot\|_p$ are, respectively, the L^2- and L^p-norms, here on some graph $\mathcal{G}$. Here one usually considers *unbounded* graphs, with a finite number of edges but where some of them are half-lines ($\mathcal{G} = \mathbb{R}$ itself is a prototype, being two half-lines glued together at the origin), as well as the *subcritical case* $2 < p < 6$, which guarantees the Sobolev embedding $H^1 \hookrightarrow L^p$ in dimension 1.

It turns out that the existence or non-existence of ground states on such graphs depends heavily on the topology of the graph, as shown in a series of landmark papers [AST15a, AST15b, AST16, AST17]. Further research, including into stability of solutions and standing waves, other types of metric graphs, other restrictions on the parameters, and other equations is ongoing; see, for example, [DST20, Hof19, NP20] and the references therein.

3.5 Final remarks

The above list excludes a huge and growing number of topics from various areas of mathematics. We could mention quantum chaos (the presumable source of the name *quantum graph*, as discussed in Section 2.1; see also [BK13, Chapter 6]), as well as various other applications in mathematical physics such as scattering and inverse scattering, the Bethe-Sommerfeld property on the gap structure of the spectrum of periodic objects [ET17], Anderson localisation [DFS, DS19], the spectra of graphene and carbon nanotubes, Bose-Einstein condensates, and the quantum Hall effect. Differential equations on metric graphs also feature in other areas of mathematics as diverse as neural networks and models of population dynamics [DLPZ20, SCA14]. Surveys of many of these and further applications in mathematical physics may be found in [BK20], [BK13, Chapter 7] and the collection [EKKST08].

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AN INTERVIEW WITH

André Neves

by **Carlos Florentino***

André Neves is an outstanding portuguese mathematician working in the U.S.A. He has held research and professorship positions at Princeton University, at the Imperial College of London and at the University of Chicago, where he is based now. He obtained his undergraduate degree in 1999, at Instituto Superior Técnico, in Lisbon, and the Ph.D. in 2005 at Stanford University, under the supervision of Richard Schoen.

He works on Differential Geometry and Analysis of Partial Differential Equations. Among several distinctions and awards, he was invited speaker at the International Congress of Mathematicians in Seoul (2014) and was awarded a New Horizons in Mathematics Prize in 2015 for "outstanding contributions to several areas of differential geometry, including work on scalar curvature, geometric flows, and his solution (with Codá· Marques) of the 50-year-old Willmore Conjecture." He also received the Oswald Veblen Prize in Geometry in 2016 (conferred by the American Mathematical Society), and was recently elected to the American Academy of Arts and Sciences.

André Neves was the lecturer of this year's Pedro Nunes Lectures, with a seminar entitled: *Counting minimal surfaces in negatively curved 3-manifolds*, the first of this series which took place online. As in previous editions, we took this opportunity for a short interview.

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Let us start by talking about your early years. In School/ High School, was Mathematics your favorite subject? What other subjects captured your interest?

I always loved Mathematics. My fondest memories from High School are of being alone in my room studying Mathematics, while listening to music.

Can you tell us about your decision to study at Instituto Superior Técnico (IST)? What made you choose the undergraduate degree in Mathematics and Computation?

It was non-linear, I am afraid. I first chose to pursue an Engineering degree because I could not imagine that one could be a mathematician. I literally thought that all Mathematics had been done by Cauchy, Bolzano, and Weierstrass. Only when I took an ODE's class with Professor José Sousa Ramos I realized that there was a whole world out there to be explored. I then changed my major from Engineering to Mathematics.

Can you describe the study environment at IST during your time there, in particular the role of your professors and colleagues in your Mathematics background, and in your path to become a researcher?

I loved it! It was a very small group and I had the good fortune of being taught by several young mathematicians that had just finished their Ph.D.'s and returned from abroad (mostly from the United States). They would teach Mathematics that seemed very original, sophisticated, and it was truly inspiring.

It is clear that your Ph.D. at Stanford University was a fundamental step in your career. Can you also describe the academic environment there, in particular how you came to work on the interface between Geometry and Analysis, and the role of Prof. Richard Schoen, your advisor?

On my first year, I took a course in Riemannian Geometry that was taught by Rick Schoen. I loved the class and the subject and decided to pursue my Ph.D. in that area. Rick was the first outstanding mathematician I met, and he has served as a role model since.

He was not afraid of pursuing hard questions and if he felt he had a good idea, he would fiercely pursue it. Most importantly, I understood that there is always a good reason for an idea to work, and that trying things just for the sake of it rarely works. He shaped my mathematical career.

Other key moments in your career were the Postdoc at Princeton, the position at Imperial College and the award of an ERC (European Research Council) grant. Can you tell us about those periods?

The time at Princeton was a bit stressful because one of the hardest periods in a mathematics career is the transition from Postdoc to mathematician. There is a tension that comes from the fact that, on one hand, we have to come up with our own problems and carve our own way of doing Mathematics, but on the other hand we also have the pressure to have papers published, because we will be applying for tenure-track jobs soon.

At Imperial College, I had a tenured job and so the pressure of publishing decreased. I was freer to pursue other directions in Mathematics and to tackle problems in which I had to start from scratch.

The ERC grant helped me because it reduced my administrative duties and allowed me to hire postdocs, some of which became good collaborators of mine.

You have established many research collaborations, but the one with Fernando Codá Marques is noteworthy. Can you summarize the story of how you became collaborators and friends?

We met at Princeton and became friends as we were one of the few Portuguese speaking people at the Mathematics Department. We were working on the same field and so we kept discussing mathematics problems on a regular basis. For the first years not much happened, but after a while we were able to look at some old problems with a new point of view and then our mathematics collaborations started.

Some mathematicians follow mostly one problem or guiding principle in their research, others keep changing fields and exploring different topics. Do you have a main philosophical guiding principle?

It is hard for me to answer because I literally pursue the questions that interest me at any given time. As I have become more confident, I have learned that if there is some phenomenon I don't understand, then it is probably worthwhile to pursue that.

You are probably the portuguese mathematician that received more international prizes and awards. What do you think are the most important qualities a researcher must have to achieve such success at the international level?

Being courageous and bold in the sense of not being afraid of addressing problems that are perceived as being hard is a good quality to have, in my opinion.

You have worked both in Europe and in the United States. Do you think there are key differences in the ways Mathematics is viewed by academic departments, and their approaches to research training and funding?

In the U.S. the grants tend to be smaller, but more mathematicians are funded. In Europe, the grants are higher but less mathematicians are funded. I think that is partly because the University Departments in the U.S. fund the postdocs and the students, and so less funding is needed at the individual level. That being said, I am not sure that has

any effect in the quality of research. Both continents have stellar mathematicians.

Even though the World today is facing many crisis, and the areas of research and development are constantly evolving, many European countries have recently invested in Pure Mathematics, by creating Research Institutes in which Fundamental Mathematics form a central part, such as the ICMAT (Instituto de Ciencias Matemáticas, Madrid, Spain) or the IST Austria (Institute of Science and Technology Austria). Do you feel that the establishment of such a Research Institute in Portugal is crucial to promote research in Fundamental Science?

Of course! It would be wonderful if Portugal had a Research Institute in Mathematics like it has in other fields, for instance the *Champalimaud Centre for the Unknown* or the *Gulbenkian Institute of Science*.

Research today in Mathematics is being influenced and shifted by the rise of neighbouring disciplines, such as Data Science, Machine Learning, Quantum Computing, Mathematical Biology, etc, and there is a big pressure to give most of the funds to *Applicable Mathematics*. Do you think Pure Mathematics can continue it's path as before or, in order to strive and be funded, has to stay in close daily contact with Applied Sciences?

I think it is important for Mathematics to be in contact with Applied Sciences for two reasons. One, of course, is the funding issue. The other is more philosophical and is related with the fact that if we work with physical quantities that are governed by fundamental principles, then the mathematical research arising from that tends to reach several fields of Mathematics and Applied Sciences.

For instance, the fact that minimal surfaces are physical objects (i.e., they can be observed) is one of the reasons that minimal surfaces are found across several fields of Mathematics (Geometry, Relativity, Algebraic Geometry, Dynamical Systems, etc).

For instance, I ask questions and talk to colleagues of mine working in Materials Science to have a better idea of how minimal surfaces should distribute themselves in space (they call them gyroids).

What excites you most in mathematics research, and what makes you pursue problem after problem, even after solving already many famous ones? Do you want to tell us about your future projects?

I have always been attracted by simple questions that an undergraduate can understand, but that in order to be solved one needs sophisticated mathematics.

As for the problems that I pursue, it is a mix. Some of them I am motivated because the problems are spin-offs from a larger question that I cannot answer. Other times, I hear about some theorem or conjecture which I find fascinating and so I try to see if I can explain that to myself using the tools that I know. Most of the times I don't, and that means I now have a new direction of research.

You often mention your interest in following portuguese mathematics. Do you see yourself returning to Portugal and establishing a new research group here?

Of course. I would love to have the opportunity to establish a research group in Portugal and help the young undergraduates pursue a career in Mathematics, in the same way that people helped me when I was an undergraduate.

Configuration Spaces of Points

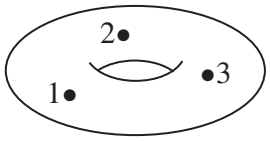
by **Ricardo Campos***

Given a topological space, one can consider its configuration space of n pairwise distinct points. We study the topological properties of such configuration spaces and address question of homotopy invariance.

1 Configuration spaces and definitions

Let X be a topological space and $n \geq 0$ be an integer. The *configuration space of n (non-overlapping) points on* X is the set

$$\mathrm{Conf}_n(X) = \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\}.$$



$\in \mathrm{Conf}_3(\text{Torus})$

Notice that being a subset of X^n, the configuration space $\mathrm{Conf}_n(X)$ is itself a topological space. Configuration spaces describe the state of an entire system as a single point in a *higher dimensional space*.

It is clear that many situations can be expressed in terms of configuration spaces. For instance, in mechanics, where objects can often be assumed to be points and are not allowed to take the same place, the configuration of the system is a point on the configuration space.

What might be less clear is why we should be interested in $\mathrm{Conf}_n(X)$ not just as a set, but also as a topological space, and in its homotopical properties.

Here are some examples where the topology of configuration spaces appear:

- Imagine you have n small robots on a plane with obstacles. The surface of movement X can typically be represented as the complement of the obstacles. If we approximate the robots by points, their movement corresponds to a path in $\mathrm{Conf}_n(X)$. Turning the problem around, we can consider the path space on $\mathrm{Conf}_n(X)$, $\mathrm{Map}([0,1], \mathrm{Conf}_n(X))$ with the two projections

$$p : \mathrm{Map}([0,1], \mathrm{Conf}_n(X)) \to \\ \to \mathrm{Conf}_n(X) \times \mathrm{Conf}_n(X)$$

given by the initial and final configuration. A motion planning algorithm is essentially a section of the map p. Unless the configuration space is contractible (which is almost never the case) such a section does not exist globally. The topological complexity (surveyed in last years Bulletin [11]) is a homotopy invariant that allows us to construct not-very-discontinuous sections.

- In knot theory one wishes to classify all knots up to ambient isotopy, which corresponds to the connected components of the space of smooth embeddings $\mathrm{Emb}(S^1, \mathbb{R}^3)$. As a first approximation of the knot one can discretise the knot into many points, which gives a particular kind of configuration on $\mathbb{R}^3$. In fact, a drastic generalisation of this problem is the goal of understanding the homotopy type of the embedding space $\mathrm{Emb}(M, N)$ between two smooth

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manifolds. Notice that such an embedding $f : M \to N$ induces a map at the level of configuration spaces $f_n : \mathrm{Conf}_n(M) \to \mathrm{Conf}_n(N)$ by evaluation pointwise. Under good conditions, the Goodwillie-Weiss embedding calculus [2] tells us that we can recover (up to homotopy) the embedding space $\mathrm{Emb}(M, N)$ from the data of all the f_n together with some additional algebraic structure.

- The pure braid group on n strands, denoted PB_n is the group whose elements are n braids (up to ambient isotopy), and whose group operation is composition of braids.

  The pure braid group PB_n is isomorphic to the fundamental group of the configuration space of n points on the plane $\pi_1(\mathrm{Conf}_n(\mathbb{R}^2))$.

- In quantum field theory, namely in Chern-Simons theory, one can construct invariants of framed smooth manifolds via integrals over configuration spaces [3].

We point out that some authors would call this the space of *ordered* configurations. The *unordered configuration space* (or configurations of indistinguishable points) can be seen as the quotient space of $\mathrm{Conf}_n(X)$ by the action of the symmetric group S_n which acts by permuting the x_i's.

In some sense unordered configuration spaces contain less information than ordered configuration spaces: as long as we can keep track of the symmetric group action, we can always recover the first from the latter.

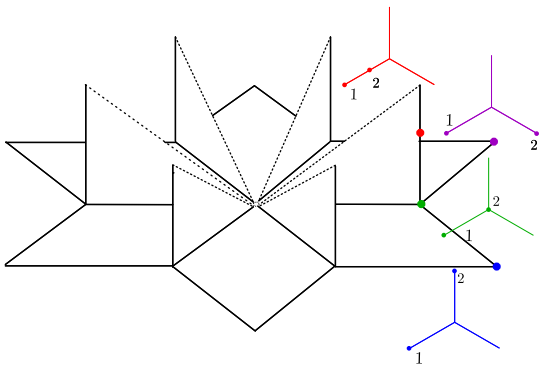
2 Examples

Configuration spaces are very simple to define, but surprisingly hard to understand. Compare them with X^n which we could call the *configuration space of n possibly overlapping points on* X. In practice, most invariants (such as the Euler characteristic, fundamental group, cohomology over a field) of X^n can be computed from the same invariant on X.

It is very instructive to see some examples to get a feel on configuration spaces. Notice that the baby cases $n = 0, 1$ give us in all generality $\mathrm{Conf}_0(X) = \varnothing$ and $\mathrm{Conf}_1(X) = X$, but this is essentially all we can say for an arbitrary topological space.

1. For the configuration of two points in the euclidean space, there is an identification $\mathrm{Conf}_2(\mathbb{R}^k) = \mathbb{R}^k \times S^{k-1} \times \mathbb{R}_{>0}$. This follows from the fact the position of the two points can be fully determined by first giving the position of the first point, then determining the vector the first point makes with the second one. One can picture the $(k-1)$-dimensional sphere S^{k-1} as the angle the points make with one another. Under this identification, the S_2 action maps a point in S^{k-1} to its antipode. Notice that in particular the unordered configuration space will be non-orientable for odd k.

2. If we consider the graph given by connecting three vertices to a fourth vertex, its configuration space $\mathrm{Conf}_2(\curlyvee)$ is the following space



   Notice that in dimension 1 there is a phenomenon of non-locality, in which even if two of the points are close, it might be difficult for them to exchange positions.

3. On the interval $I = (0, 1)$, the configuration space of n points has $n!$ connected components, corresponding to all possible ways to order n points. All of these components are homeomorphic to the open n-simplex

$$\{(x_1, \dots, x_n) \mid 0 < x_1 < \cdots < x_n < 1\}.$$

3 Homotopy type

This last example 3 is the simplest example of a topological invariant (connected components) on the configuration space that cannot be deduced just from the invariant on the base space. While one could be tempted to dismiss it as trivial, since it can only happen in dimension 1, it is actually a shadow of a more general problem.

Indeed, these issues are related with the non-functoriality of

$$\mathrm{Conf}_n : \text{Top. Spaces} \longrightarrow \text{Top. Spaces}.$$

If a map $f: X \to Y$ is not injective, the induced map $f^n: X^n \to Y^n$ will not restrict to the respective configuration spaces.

Recall that two spaces X and Y are said to be *homotopy equivalent*, denoted $X \sim Y$, if there are maps $f: X \to Y$ and $g: Y \to X$ such that $f \circ g$ is homotopic to the identity id_Y in the sense that there exists a map $h: Y \times [0,1] \to Y$ such that $h(y,0) = y$ and $h(y,1) = f \circ g(y)$, and similarly for $g \circ f$. When we talk about the *homotopy type* of X, we mean the equivalence class of all spaces homotopy equivalent to X.

Typically, invariants we study (and all those mentioned in this survey) of topological spaces depend only on the homotopy type. The natural question that one is led to ask is whether the homotopy type is preserved by taking configuration spaces, i.e., whether $X \sim Y$ will guarantee that $\mathrm{Conf}_n(X) \sim \mathrm{Conf}_n(Y)$. This sounds very implausible given the preceding discussion. Also, out of a homotopy equivalence between X and Y, there seems to be no way to construct a single map relating their configuration spaces.

In fact, the very first example 1 provides already a plethora of counter-examples, since regardless of the dimension, all Euclidean spaces are contractible (and hence homotopy equivalent), but $\mathrm{Conf}_2(\mathbb{R}^k) \sim S^{k-1}$ and no two different dimensional spheres have the same homology, cohomology or homotopy groups.

Given this, one might be surprised that the next open question is believed to be true.

Conjecture 1.– For simply connected compact manifolds without boundary, the homotopy type of $\mathrm{Conf}_n(M)$ only depends on the homotopy type of M.

The more general conjecture for non-simply connected spaces was only disproved in 2005, when Longoni and Salvatore were able to show that the lens spaces $L_{7,2}$ and $L_{7,1}$ provide a counter-example by computing the Massey products on the universal covers of the respective configuration spaces of 2 points [8].

In the last section I will try to provide some evidence for this conjecture in the form of Theorem 7. From now on, we will restrict our study to the case of smooth manifolds.

4 Compact version of configuration spaces

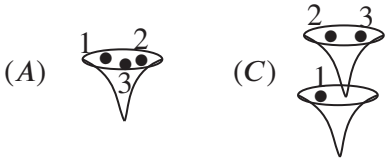
Suppose that M is a compact smooth manifold. Even if M is compact, the configuration space $\mathrm{Conf}_n(M)$ is not compact when $n \geq 2$, since a sequence of two points moving into the same place does not converge. This is an unfortunate property to lose: For instance, in situations where one wishes to consider integrals over configuration spaces (as in quantum field theory), one has to deal with issues of convergence.

A neat way to address this issue is to work instead with a suitable compactification of $\mathrm{Conf}_n(M)$. The most natural one is perhaps to consider M^n, but this could of course lose all homotopy information as in the case of $\mathbb{R}^k$. The strategy is instead to embed $\mathrm{Conf}_n(M) \hookrightarrow K$ in some compact manifold with boundary K, such that $\mathrm{Conf}_n(M)$ sits in K as its interior, since manifolds with boundary are homotopy equivalent to their interiors.

The construction of such manifold is due to Axelrod and Singer but is usually called the *Fulton-MacPherson compactification of* $\mathrm{Conf}_n(M)$, as it is a real analog of an iterated blow-up construction from algebraic geometry. This manifold is denoted $\mathsf{FM}_n(M)$ and admits a very visual description. Intuitively, instead of allowing two points moving towards each other to meet, we allow them to be *infinitesimally close together*, but still retaining the information of the direction in which they collided (i.e. they can still move around each other along a sphere of dimension $\dim M - 1$).

In the case where there are only two points, indeed $\mathsf{FM}_2(M)$ will be a manifold with boundary, but otherwise one needs to distinguish different situations when more than two points collide, so in general $\mathsf{FM}_n(M)$ will be a *compact smooth manifold with corners* whose interior is $\mathrm{Conf}_n(M)$.

In the figure above, fixing points 1 and 2, there are three possible cases when moving point 3 close to 1 and 2: (A) Point 3 stays at the same "infinitesimality stratum"; (B) Point 3 furthermore approaches infinitesimally point 1, even from the perspective of point 2; (C) Similar but point 3 approaches point 2.



For details and a nice exposition see [9].

5 The cohomology of $\mathrm{Conf}_n(\mathbb{R}^k)$

So far we have considered configuration spaces of a fixed number of points n, but given the same base manifold M, there are obvious relations between configurations of different number of points. Namely, given $1 \leq i \neq j \leq n$, there are *projection* maps

$$p_{ij} : \mathrm{Conf}_n(M) \longrightarrow \mathrm{Conf}_2(M), \qquad (1)$$

given by forgetting the position of all points but the ith and jth one.

These projection maps provide us with an attempt to inductively try to understand configurations of a large number of points from a smaller one. To illustrate this idea, let us consider in more detail the case of configurations of points in $\mathbb{R}^k$, where we can get a full description of the cohomology ring of $\mathrm{Conf}_n(\mathbb{R}^k)$. For concreteness, let us denote by $H^\bullet(M)$ the cohomology ring of M with real coefficients. Since $\mathrm{Conf}_n(M)$ is a smooth manifold, we will interpret this graded commutative $\mathbb{R}$-algebra as the cohomology of de Rham algebra of differential forms, denoted $\Omega^\bullet(\mathrm{Conf}_n(M))$.

From our previous example 1 we deduce that $H^\bullet(\mathrm{Conf}_2(\mathbb{R}^k)) = H^\bullet(S^{k-1}) = \mathbb{R}1 \oplus \mathbb{R}\omega$, where ω is a degree $k-1$ element representing the cohomology class of the volume form on S^{k-1}.

Theorem 2 (Arnold [1] and Cohen [5]).— The cohomology $\mathbb{R}$-algebra $H^\bullet(\mathrm{Conf}_n(\mathbb{R}^k))$ is given by

$$A_{n,k} = \frac{\mathrm{Sym}(\omega_{ij})_{1\leq i\neq j\leq n}}{\left(\omega_{ij} = (-1)^k\omega_{ji}, \omega_{ij}^2 = 0, \text{Arnold}\right)} \qquad (2)$$

where ω_{ij} are elements of degree $k-1$, Sym denotes the symmetric algebra and the Arnold relation is $\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0$.

Heuristically, ω_{ij} represents the interaction between the points i and j, while the Arnold relations represent three-point interactions. The theorem states that these relations generate all existing relations.

Proof (Sketch).— The first step is to construct the map $A_{n,k} \to H^\bullet(\mathrm{Conf}_n(\mathbb{R}^k))$. We interpret ω_{ij} as an element of $H^\bullet(\mathrm{Conf}_n(\mathbb{R}^k))$ by pulling back along the projection p_{ij} from equation (1) into the $(k-1)$-sphere: $\omega_{ij} \coloneqq p_{ij}^*\omega$.

The relation $\omega_{ij}^2 = 0$ holds since it holds for $\omega \in \Omega^{k-1}(S^{k-1})$. Switching the indices in ω_{ij} corresponds to applying the antipodal map in S^{k-1}, which has a degree opposite to the dimension of the sphere, from where it follows that $\omega_{ij} = (-1)^k\omega_{ji}$.

There are various short proofs of the Arnold relation, none of them completely immediate. It can be deduced by analysing the fibration $p_{12} : \mathrm{Conf}_3(\mathbb{R}^k) \to \mathrm{Conf}_2(\mathbb{R}^k)$. Alternatively, one can explicitly find a form η such that $d\eta = \mathit{Arnold}$ constructed as a fiber integral $\eta = \int_4 \omega_{14}\omega_{24}\omega_{34}$, as we will see in the final section.

Now that we have established maps

$$A_{n,k} \to H^\bullet(\mathrm{Conf}_n(\mathbb{R}^k)),$$

we need to show that they are isomorphisms, which can be done by induction on n. For this, we observe that the map $\mathrm{Conf}_n(\mathbb{R}^k) \to \mathrm{Conf}_{n-1}(\mathbb{R}^k)$ forgetting the last point is a fibration whose fiber is homotopy equivalent to a wedge sum of spheres $\bigvee_{i=1}^{n-1} S^{k-1}$ and then apply the Serre spectral sequence. ∎

6 Rational homotopy theory

Let us step back from configuration spaces for a moment to consider the general problem of understanding the homotopy type of spaces via some algebraic invariant.

The main issue is that invariants such as the cohomology ring of a space do not capture the homotopy type sufficiently faithfully. A potentially stronger invariant is given by the higher homotopy groups π_n, since Whitehead theorem guarantees that a map of CW-complexes $X \to Y$ inducing isomorphisms $\pi_n(X) \to \pi_n(Y)$ is a homotopy equivalence. This not only does not completely solve our problem, since CW-complexes might still have the same homotopy groups without having a map inducing an isomorphism, but it also has the additional issue that higher homotopy groups are extremely difficult to compute due to their torsion parts. Rational homotopy theory provides a good way to address both issues, if we are willing to work modulo torsion:

Definition 3.— We say that a map of simply connected spaces $X \to Y$ is a *rational homotopy equivalence* if the induced map

$$\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \pi_n(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism for all n. Equivalently, $X \to Y$ is a rational homotopy equivalence if $H^n(Y;\mathbb{Q}) \to H^n(X;\mathbb{Q})$ is an isomorphism for all n.

Sullivan [10] associated to a space X a differential graded (dg) commutative algebra $A_{PL}^\bullet(X)$ of *piecewise linear differential forms* X, which the reader can think of as the de Rham algebra for non-manifolds (and

over $\mathbb{Q}$ instead of $\mathbb{R}$), or alternatively as the singular $\mathbb{Q}$-cochains on X, $C^\bullet(X;\mathbb{Q})$ (except that the cup product is not commutative before passing to cohomology). The cohomology of the dg algebra $A^\bullet_{PL}(X)$ is the graded algebra $H^\bullet(X;\mathbb{Q})$.

In the category of rational dg commutative algebras, one considers the corresponding notion of homotopy equivalence, which is the one of a *quasi-isomorphism*. These are morphisms of dg commutative algebras $A^\bullet \to B^\bullet$ such that the induced map in cohomology $H^\bullet(A) \to H^\bullet(B)$ is an isomorphism.

The main result of Sullivan is that this construction captures faithfully the rational homotopy type of spaces.

CAVEAT 4.– Unlike ordinary homotopy equivalences, having a rational homotopy equivalence $X \to Y$ does not imply the existence of a rational homotopy equivalence $Y \to X$. We say that X and Y are rational homotopy equivalent and we write $X \sim_{\mathbb{Q}} Y$ if there is a zig-zag of rational homotopy equivalences

$$X \xleftarrow{\sim} X_1 \xrightarrow{\sim} \cdots \xleftarrow{\sim} X_n \xrightarrow{\sim} Y$$

Similarly, quasi-isomorphisms of dg commutative algebras are not *invertible* so the same remark holds.

There is a general construction in category theory: Given a category $\mathscr{C}$ with some set of *homotopy equivalences* $H \subset \text{Morphisms}(\mathscr{C})$, one can construct the homotopy category of $\mathscr{C}$, denoted $\mathscr{C}[H^{-1}]$, which possesses the same objects as $\mathscr{C}$, but where we formally invert the maps in H, such that they become isomorphisms in $\mathscr{C}[H^{-1}]$.

THEOREM 5 ([10]).– The construction A_{PL} establishes an equivalence of categories

$$A_{PL} : \text{scSpaces}[\text{r.h.e.}^{-1}] \to \mathbb{Q} - \text{DGCA}_{>1}[\text{q.i.}^{-1}]$$

from the category of simply connected topological spaces of finite type up to rational homotopy equivalence, to the category of dg commutative $\mathbb{Q}$-algebras of finite type concentrated in degrees > 1 up to quasi-isomorphism.

In practice, this result allows us to study topology completely via (differential graded) algebraic methods. Any dg commutative algebra quasi-isomorphic to $A_{PL}(X)$ is therefore called a *rational model of* X. A classical question in rational homotopy theory is whether one can find a small model for X. The smallest possible candidate to be a model of X would be its cohomology $\mathbb{Q}$-algebra, but in general it is not true that $H^\bullet(X;\mathbb{Q}) \sim_{\mathbb{Q}} A_{PL}(X)$. If that happens to be the case, we say that X is *formal*.

It should be pointed out that in the context of Sullivan's theorem there is nothing special about $\mathbb{Q}$ except that it is a field of characteristic 0. Replacing it with $\mathbb{R}$ we would talk about the *real homotopy type of* X instead.

7 MODELS FOR CONFIGURATION SPACES

In this final section we will see how one can construct a nice model of the real homotopy type of configuration spaces using graphs. This will in particular allow us to prove the real version of Conjecture 1, see Theorem 7.

THEOREM 6 ([4]).– Let M be a compact smooth manifold without boundary and $n \in \mathbb{N}$. There exists a nice dg commutative $\mathbb{R}$-algebra spanned by a certain type of graphs $\mathsf{Graphs}_n(M)$ modeling the real homotopy type of $\text{Conf}_n(M)$. This is expressed by a direct quasi-isomorphism of algebras into the algebra of semi-algebraic forms[1] of $\mathsf{FM}_n(M)$:

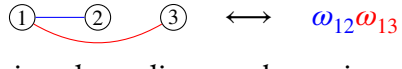
$$\mathsf{Graphs}_n(M) \longrightarrow \Omega(\mathsf{FM}_n(M)). \qquad (4)$$

Even though $\mathbb{R}^k$ is not a compact manifold, it is still instructive to go back to Theorem 2 and start by understanding how one could try to obtain a model of $\text{Conf}_n(\mathbb{R}^k)$ out of the computation of its cohomology.

The only reasonably natural attempt of establishing a quasi-isomorphism $H^\bullet(\text{Conf}_n(\mathbb{R}^k))$ into $\Omega(\mathsf{FM}_n(\mathbb{R}^k))$ would involve mapping $\omega_{ij} \in A_{n,k}$ into the volume form of the spheres S^{k-1}. However, since the Arnold relations are not satisfied at the level of differential forms this cannot produce a map compatible with the product.

While such a quasi-isomorphism of algebras cannot be directly constructed, Kontsevich [7] showed that configuration spaces in $\mathbb{R}^k$ are formal by establishing a zig-zag passing by an algebra of graphs.

Notice that one can identify the algebra $A_{n,k}$ with a graded vector space given by $\mathbb{R}$-linear combinations of graphs on n vertices, where an edge between the vertices i and j corresponds to ω_{ij}.



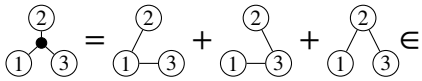
To be precise, depending on the parity of k, edges should be oriented or ordered, and changing orientation or order (by an odd permutation) produces a minus sign, but from now on we will work up to sign.

[1] Pretend it is the de Rham complex. This is a minor technicality due to the lack of smoothness of forgetting points near the corners.

Under this identification, edges have degree $k-1$ and the commutative product on graphs is given by superposition of vertices and taking the union of edges.

The idea now is to attempt to resolve $A_{n,k}$ by adding a new kind of vertices, that would mimic phantom points moving freely in the configuration space. Concretely, one can define a dg commutative algebra $\mathsf{Graphs}_n(\mathbb{R}^k)$, spanned by graphs with n labeled vertices as before and a arbitrary number of unlabeled vertices which now are of degree $-k$.

The differential of a graph $\Gamma \in \mathsf{Graphs}_n(\mathbb{R}^k)$ is given by summing all ways of contracting an unlabeled vertex in Γ along an edge. Here is an example exhibiting the Arnold relation as a coboundary:

d  $\in \mathsf{Graphs}_3(\mathbb{R}^k)$.

Notice that the differential kills an unlabeled vertex and an edge so it is indeed of degree $+1$ and (being careful with signs) it squares to zero. The product is still given by superposition of labeled vertices (in particular it adds the number of unlabeled vertices).

We can now produce a map into the algebra of forms

$$\mathsf{Graphs}_n(\mathbb{R}^k) \longrightarrow \Omega(\mathsf{FM}_n(\mathbb{R}^k))$$

given by "mapping edges i—j to the volume form of the sphere ω_{ij}[2] and integrating out the unlabeled vertices". In particular, the graph above yielding the Arnold relation is mapped to

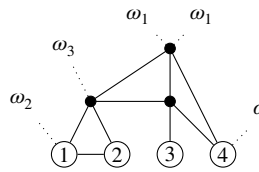
$$\eta = \int_{\mathsf{FM}_4 \to \mathsf{FM}_3} \omega_{14}\omega_{24}\omega_{34}.$$

The only non-immediate thing that needs to be checked is the compatibility with the differential, which follows mostly from the Stokes theorem for manifolds with corners.

In fact, to prove the formality of configuration spaces in $\mathbb{R}^k$, one just needs to show that the projection into graphs with no unlabeled vertices is a quasi-isomorphism, which can be achieved by a spectral sequence inductive argument.

While other configuration spaces over a compact manifold M will not be formal (and there is no analog of Theorem 2), stretching a bit the notion of graphs one can use similar ideas to construct the dg commutative $\mathbb{R}$-algebra $\mathsf{Graphs}_n(M)$ as follows:

As a vector space, $\mathsf{Graphs}_n(M)$ is spanned by graphs with n labelled vertices, some unlabeled vertices and vertices can be decorated by (possibly repeating) reduced cohomology classes in $\tilde{H}^\bullet(M)$.

 $\in \mathsf{Graphs}_4(M)$,

The product is still given by superposition of labeled vertices. Following heuristically Kontsevich, we wish to send such graphs to differential forms in $\mathsf{FM}_n(M)$ in a way that integrates out unlabeled vertices and sending cohomology classes in $H^\bullet(M)$ to representatives in $\Omega(\mathsf{FM}_1(M))$. To establish the map in (4), there are three main pieces:

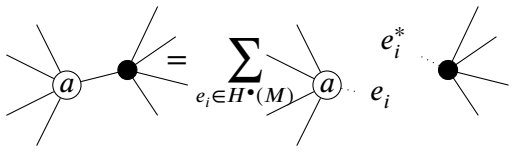
(i) In the case of $\mathbb{R}^k$, $\mathsf{FM}_2(\mathbb{R}^k)$ is essentially a sphere, so edges can be sent to volume forms. To which form in $\Omega^{\dim M-1}(\mathsf{FM}_2(M))$ will edges be sent to?

(ii) What kind of differential must $\mathsf{Graphs}_n(M)$ have such that (4) is compatible with differentials?

(iii) How to make this map a quasi-isomorphism?

We will not address the third point and without getting into details, let us just say that the first point is addressed by mapping edges to what in mathematical physics is called a *propagator* [3].

The second point is the trickiest: As far as $\mathsf{Graphs}_n(M)$ has been described, it only depends on the cohomology of M, so it has no chance of even capturing the real homotopy type of M, let alone the configuration space. All this is hidden in the differential, which splits into three pieces $d = d_{\text{contr.}} + d_{\text{Poinc.}} + d_{Z_M}$. A first piece $d_{\text{contr.}}$ which contracts edges as in the $\mathbb{R}^k$ case, a second piece $d_{\text{Poinc.}}$ which uses the Poincaré duality pairing on $H^\bullet(M)$ to split edges into two decorations

Δ 

and a third piece d_{Z_M} acting only on subgraphs consisting of unlabeled vertices which depends on the partition function Z_M of the *universal* perturbative AKSZ topological field theory on M. We interpret Z_M as a map from vacuum graphs (fully unlabeled graphs in $\mathsf{Graphs}_0(M)$) into real numbers.

THEOREM 7 ([4, 6]).— Let M be a simply connected smooth compact manifold without boundary. The

[2] Abusing the notation denoting by ω_{ij} both the form and its class in cohomology.

real homotopy type of M determines the real homotopy type of its configuration space of points.

$$M \sim_{\mathbb{R}} N \Rightarrow \mathrm{Conf}_n(M) \sim_{\mathbb{R}} \mathrm{Conf}_n(N), \quad \forall n \in \mathbb{N}.$$

Proof (Sketch).– Notice that since M is simply connected $H^1(M) = 0$ and therefore decorations of graphs in $\mathsf{Graphs}_n(M)$ have degree at least 2. Furthermore we can assume that $D = \dim M \geq 4$ by the Poincaré conjecture.

All pieces of the defining data of $\mathsf{Graphs}_n(M)$ depend only on the real homotopy type of M, with the possible exception of d_{Z_M}.

One can show that d_{Z_M} depends only on the value of Z_M on graphs consisting only of degree 0 unlabeled vertices with valence ≥ 3 (decorations count as valence).

The proof of the theorem now follows from the following purely combinatorial statement: Using that decorations have degree at least 2, vertices have degree $-D$ and edges have degree $D-1$, the only ≥ 3-valent graphs of degree 0 are trees.

It turns out that the values of Z_M on trees depend only on the real homotopy type of M. It follows that $\mathsf{Graphs}_n(M)$ and therefore $\mathrm{Conf}_n(M)$ also depends only on the real homotopy type of M. ∎

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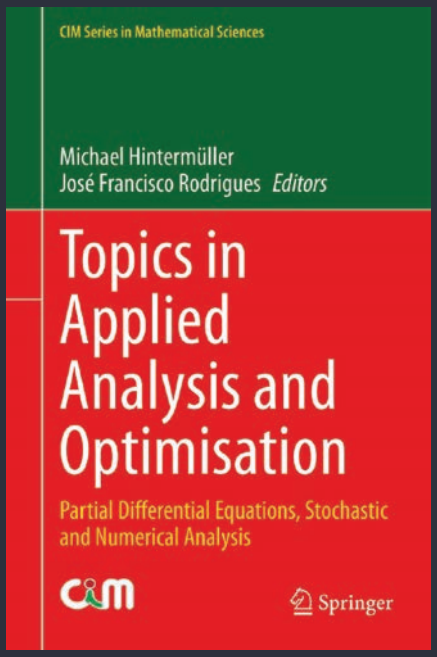

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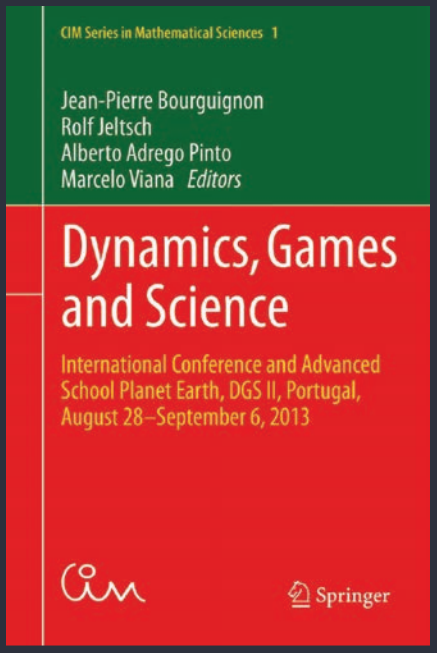

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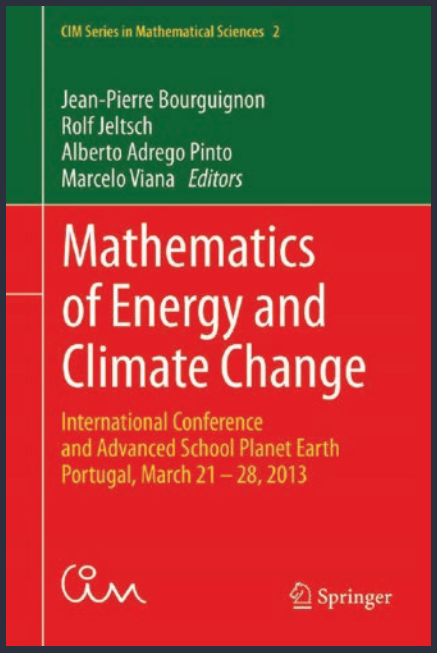

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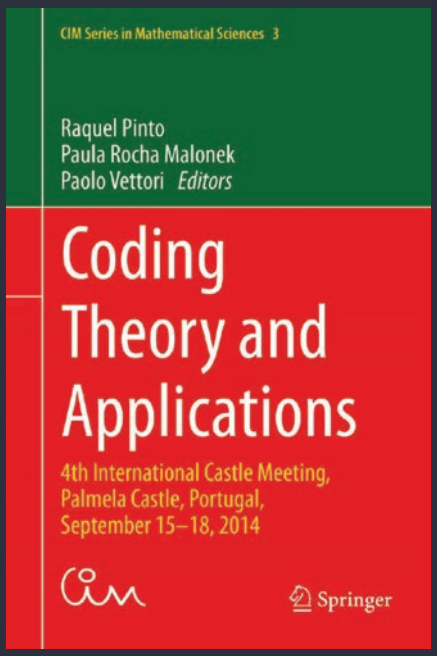

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