Semilinear Elliptic Problems: Old and New

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Elliptic partial differential equations are a very import class of equations with obvious connections to applied sciences (e.g. physics, biology, chemistry and engineering) as well as to other fields of Mathematics such as Differential Geometry, Functional Analysis and Calculus of Variations. Because of these facts they are a quite fascinating topic and an increasingly active field of research. In this article we focus our attention on semilinear problems of type $\Delta u = f(u)$, more specifically on the Lane-Emden equation, as a mean to explain some of the tools and methods (mostly topological or variational) that are available to treat elliptic problems. The topics addressed concern existence and multiplicity of solutions, as well as their qualitative properties such as sign and symmetry. The goal is not to provide a complete state-of-the-art (which would not fit in a few pages), but rather to present some relevant and interesting questions and, whenever possible, to explain which ones we cannot answer yet.

1 Introduction

Many problems can be modelled with the aid of elliptic partial differential equations[1]. One of the most well known examples is the classical Poisson equation: given a bounded regular domain $\Omega \subset \mathbb{R}^n$, we take

$$-\Delta u = f \text{ in } \Omega.$$

Its solutions may represent the shape of an elastic membrane in equilibrium subject to a vertical load $f : \Omega \to \mathbb{R}$ ($u(x)$ corresponds to the vertical displacement at the point x); an electrostatic potential (for $f = \rho/\varepsilon$, where $\rho(x)$ is the volume charge density and ε the permittivity of the medium), a gravitational potential (for $f = -4\pi G\rho$, where ρ is the density of the object and G the gravitational constant), or the stationary solutions for the heat equation (in this case u represents a temperature, and f is a heat source or sink). Here $\Delta u = \sum_{i=1}^n \partial^2 u/\partial x_i^2$ is the Laplace operator (the trace of the Hessian matrix). To obtain existence and uniqueness of solution, one couples the equation with boundary conditions: *Dirichlet boundary conditions* ($u = g$ on $\partial\Omega$) or *Neumann boundary conditions* ($\partial u/\partial \nu := \nabla u \cdot \nu = g$ on $\partial\Omega$, where $\nu = \nu(x)$ is the exterior normal at $x \in \partial\Omega$) are typical examples arising in applications. Linear problems are very well understood and can be found in classical textbooks (see for instance [14, 26]), while current research aims at a good understanding of nonlinear problems. Among the wide class of possible nonlinear problems, the simplest to treat (although already quite rich mathematically, as we will see) are semilinear ones, where $f : \mathbb{R} \to \mathbb{R}$, $f = f(u)$, is a nonlinear function, that is, the nonlinearity occurs at the level of the zero order terms. Let us see some examples.

Example 1.— The stationary Fisher equation: $-\Delta u = au(b - u^2)$. Solutions are equilibrium points of the evolutionary equation

$$v_t - d\Delta_x v = v(a - bv), \qquad (x,t) \in \Omega \times \mathbb{R}^+,$$

where a, b are positive constants, and $v = v(x,t)$ represents the density of a population at the point x and time t, subject to diffusion (modelled by the Laplacian

[1] Is is out of the scope of this article to give a more general definition of what is an elliptic problem, but we can briefly explain the nomenclature. It comes from an analogy with the classification of conics, and from the classification of a general linear second order PDE in dimension 2: $a(\partial^2 u/\partial x_1^2) + 2b(\partial^2 u/\partial x_1 \partial x_2) + c(\partial^2 u/\partial x_2^2)$ + lower order terms $= f$ is called elliptic if $b^2 - ac < 0$. A typical example is the case $a = c = 1$, $b = 0$, which corresponds to having the Laplace operator in dimension 2.

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term) and following a logistic law of growth (compare with the well known ordinary differential equation version of it: $u' = u(a - bu)$).

Example 2.— The Lane-Emden equation: $-\Delta u = |u|^{p-2}u$, which appears in astrophysics. If $n = 3$, and u is *radially symmetric* and positive, then $\theta(|x|) = u(x)$ solves the *Lane-Emden equation of index* $p - 1$,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) + \theta^{p-1} = 0, \quad \theta(0) = 1, \; \theta'(0) = 0$$

which can be used to model self-gravitating spheres of plasma such as stars. Up to constants, we have that θ is the temperature, θ^p is the pressure, the first root r_1 of θ is the star's radius, and $\int_0^{r_1} \theta(r)^{p-1} r^2\, dr$ is the total mass [12]. We will meet again this equation in Section 3.

Example 3.— The time independent nonlinear Schrödinger equation: $-\Delta u + (V(x) + \lambda)u = \mu |u|^2 u$ appears naturally when looking for standing wave solutions of the Schrödinger Equation $i\Phi_t = -\Delta\Phi + V(x)\Phi - \mu|\Phi|^2\Phi$, that is, solutions whose modulus is time-independent: $\Phi(x, t) = e^{i\lambda t}u(x)$. This evolutionary equation appears in quantum mechanics, nonlinear optics and in the study of Bose-Einstein condensation.

Example 4.— The Yamabe Problem (Differential Geometry):

$$-\frac{4(n-1)}{n-2}\Delta_{\mathscr{M}} u + S_g(x)u = S_{g_0} u^{2^*-1}, \qquad (1)$$

where $2^* = 2n/(n-2)$. Here $\mathscr{M}$ is a compact manifold of dimension $n \geq 3$ with a Riemannian metric g and scalar curvature S_g. Yamabe in 1960 made the conjecture that there always exist a conformal metric g_0 with constant scalar curvature (actually he *proved it*, but a mistake in the proof was found by Trudinger in 1968). The final proof was given in the 1984 after the contributions of Yamabe, Trudinger, Aubin and Schoen (see a detailed account in [21]). The previous equation has a positive solution u and, for the metric $g_0 := u^{4/(n-2)}g$, the manifold has constant scalar curvature S_{g_0}. The exponent 2^* is called the Sobolev exponent and plays a crucial role in the theory of Sobolev spaces and weak solutions (we will meet them briefly in Section 2). In local coordinates, (1) reads

$$-\frac{1}{a(x)}\operatorname{div}(A(x)\nabla u_i) + S_g(x)u = S_{g_0} u^{2^*-1}$$

in an open bounded domain Ω,

$$a(x) = \frac{n-2}{4(n-1)}\sqrt{|g(x)|} := \frac{n-2}{4(n-1)}\sqrt{\det g_{ij}(x)}$$

and

$$A(x) = \sqrt{g(x)}(g^{ij}(x))_{ij},$$

where $(g^{ij}(x))_{ij}$ is the inverse matrix of the metric $(g_{ij}(x))_{ij}$.

The purpose of this article is to briefly explain some of the questions that mathematicians working in this field try to answer. To fix ideas, we focus on the equation introduced in Example 2, as it is one of the simplest prototypical situations. Whenever is possible and not too complicated to do, we will leave some open problems for the interested reader. Before entering into more recent and sophisticated material, it is helpful to review some classical one for the Poisson equation. We do it in the next section.

2 Weak Solutions. The Variational Method

Let us deal with the Poisson equation with zero Dirichlet boundary conditions:

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \qquad (2)$$

where $f = f(x) : \overline{\Omega} \to \mathbb{R}$ is a regular function (observe that the function f depends only on the x variable, not on the solution itself like in Example 2). A classical solution is a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying the equalities in (2) pointwise. It is typically not easy to find classical solutions *directly*, being common to take the variational point of view: if u is a $C^2(\overline{\Omega})$ solution, then it is not hard to prove that u is a solution to the minimization problem

$$\inf\left\{\mathscr{E}(u) \;:\; u \in C^2(\overline{\Omega}),\; u = 0 \text{ on } \partial\Omega\right\},$$

where

$$\mathscr{E}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega fu. \qquad (3)$$

This is called the Dirichlet's Principle, and it is connected with one of the physical interpretations of the problem: for an elastic membrane, we are minimizing the total potential energy [26, §6.1]. This may look like a very good way of finding solutions, however it is not as simple as it sounds: there are situations where there are no minimizers (there are famous counterexamples by Weierstrass (1870) and F. Prym (1871)). In a nutshell, the problem is that there is a natural norm present:

$$u \mapsto \left(\int_\Omega |\nabla u|^2 + \int_\Omega u^2\right)^{1/2}, \qquad (4)$$

however the function space $C^2(\overline{\Omega})$ is not complete when equipped with it. This, among other things, led to Sobolev spaces (mid 1930's), defined as the closure of $C^2(\overline{\Omega})$ for (4) or, equivalently, as[2]

$$\begin{aligned} &H^1(\Omega) = W^{1,2}(\Omega) = \\ &= \left\{ u \in L^2(\Omega) \, : \, \frac{\partial u}{\partial x_i} \in L^2(\Omega) \right\} = \\ &= \left\{ u : \Omega \to \mathbb{R} \, : \, \int_\Omega u^2, \int_\Omega \left(\frac{\partial u}{\partial x_i} \right)^2 < \infty \; \forall i \right\}. \end{aligned} \tag{5}$$

We also denote by $H^1_0(\Omega)$ the set of functions in $H^1(\Omega)$ which are zero at the boundary (in the sense of traces). We advice the reader to check for instance [26, §7] or [14, §5] for the details. This leads to the minimization problem

$$\inf \left\{ \mathscr{E}(u) \, : \, u \in H^1_0(\Omega) \right\},$$

for $\mathscr{E} : H^1_0(\Omega) \to \mathbb{R}$, as defined in (3).

The functional $\mathscr{E}$ is differentiable in $H^1_0(\Omega)$. It is standard (within the field of Calculus of Variations) to prove that minimizers exist; moreover, if u is a minimizer, then, for every $v \in H^1_0(\Omega)$,

$$\begin{aligned} \mathscr{E}'(u)[v] &:= \tfrac{d}{dt} \mathscr{E}(u + tv)|_{t=0} = \\ &= \textstyle\int_\Omega \nabla u \cdot \nabla v - \int_\Omega f v = 0. \end{aligned}$$

Definition 1.— We say $u \in H^1_0(\Omega)$ is a *weak solution* (or *variational solution*) of (2) if

$$\int_\Omega \nabla u \cdot \nabla v - \int_\Omega f v = 0 \qquad \forall v \in H^1_0(\Omega).$$

Therefore u is a weak solution of (2) if, and only if, u is a critical point of $\mathscr{E}$. We have also seen that minimizers (which exist) provide weak solutions. On the other hand, if u is sufficiently regular, then the notion of classical and weak solution coincide. This theory allows to treat separately the existence of solutions and their regularity, and the existence part is translated into finding critical points of a certain functional. This leads to the subject *Variational Methods/Critical Point Theory*, which can be used to tackle not only linear problems, but also semilinear problems like the ones presented in the Introduction.

3 Study of a model problem: the Lane-Emden equation

3.1 Statement of the problem and some technical background

Let $n \geq 3$, $p > 1$ and let Ω be a bounded, regular, connected open set. We work from now on with the prototypical problem considered in Example 2, under Dirichlet boundary conditions

$$-\Delta u = |u|^{p-2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{6}$$

Clearly $u \equiv 0$ is always a solution, but we are interested in nontrivial ones. Based on what we have seen in the previous section, a natural definition of weak solution is

$$\int_\Omega \nabla u \cdot \nabla v - \int_\Omega |u|^{p-2} uv = 0 \qquad \forall v \in H^1_0(\Omega),$$

and (at least formally, for now) weak solutions correspond to critical points of the functional

$$\begin{aligned} &\mathscr{I} : H^1_0(\Omega) \to \mathbb{R}, \\ &\mathscr{I}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{1}{p} \int_\Omega |u|^p \end{aligned} \tag{7}$$

(observe that $(|t|^p)' = p|t|^{p-2} t$ for every $t \in \mathbb{R}$, $p > 1$). To make these statements precise and correct, actually we need a restriction on the exponent p: the integral $\int_\Omega |u|^p$ is not always finite for $u \in H^1_0(\Omega)$. One needs to recall *Sobolev inequalities*: for $1 \leq 2^* = 2n/(n-2)$ there exists $C_{n,p} > 0$ such that

$$\left(\int_\Omega |u|^p \right)^{1/p} \leq C_{n,p} \left(\int_\Omega |\nabla u|^2 \right)^{1/2}$$

for every $u \in H^1_0(\Omega)$, which amounts to say that the embedding $H^1_0(\Omega) \hookrightarrow L^p(\Omega)$ is continuous. The number 2^* is the critical Sobolev exponent, which we already met in Example 4. Therefore, in conclusion, (7) is defined only for $p \leq 2^*$. In this case, in order to look for weak solutions of the problem (6), we may try to find critical points of $\mathscr{I}$.

Now the question is: how can we find a critical point of $\mathscr{I}$? And how many of them are there? The answer depends on p: not only the geometry of $\mathscr{I}$ changes from $p < 2$ to $p > 2$, but also the situations

[2] Observe (or recall) that the meaning of $(\partial u / \partial x_i) \in L^2(\Omega)$ is not obvious at all for a function u in $L^2(\Omega)$. It means that the first order *weak derivatives* of u are $L^2(\Omega)$ — functions; in other words, for every $i = 1, \dots, N$, there exists $g_i \in L^2(\Omega)$ (which we call $\partial u / \partial x_i$) such that

$$\int_\Omega g_i \varphi = - \int_\Omega u \frac{\partial \varphi}{\partial x_i} \qquad \forall \varphi \in C^\infty(\Omega) \text{ with compact support.}$$

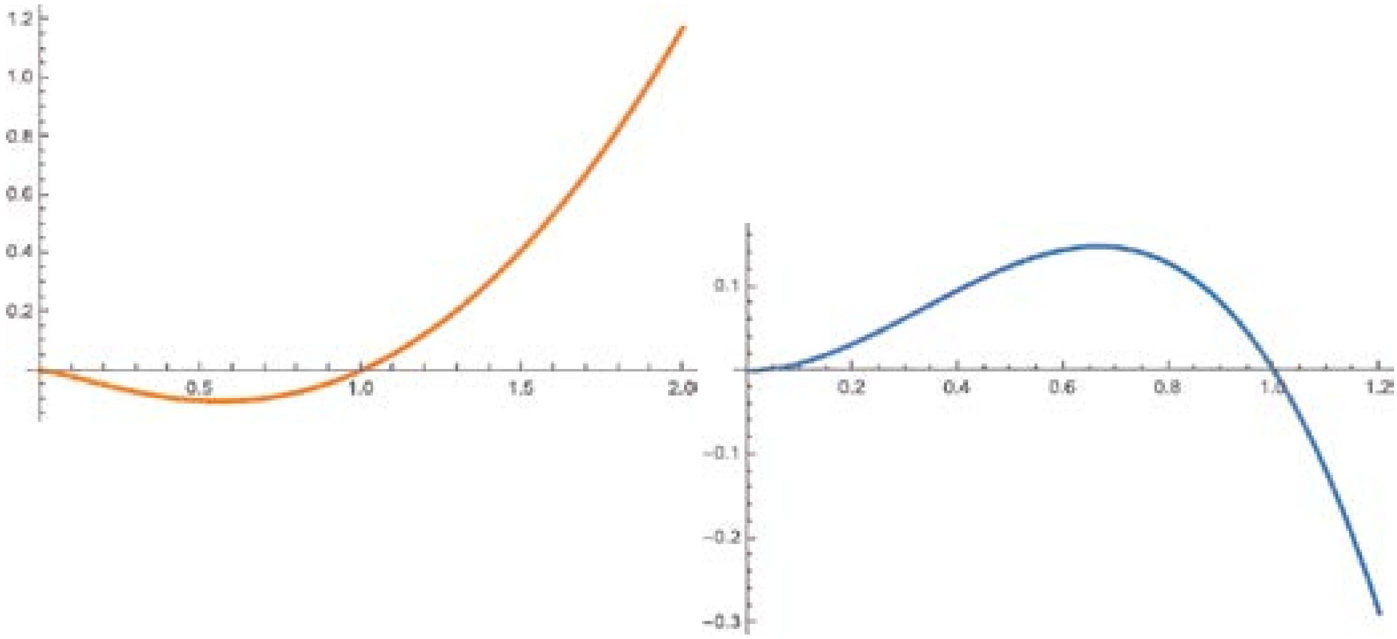

Figure 1.—Given $u \in H_0^1(\Omega)$, the shape of the map $t \in \mathbb{R}_0^+ \mapsto \mathcal{J}(tu)$ for $p < 2$ and $p > 2$ is represented on the left and right figures respectively.

$p < 2^*$ and $p = 2^*$ are very different: the embedding of $H_0^1(\Omega)$ in $L^p(\Omega)$ is compact only for $1 \le p < 2^*$. The discussion of the case $p > 2^*$ is much harder and less is known, being out of the scope of this article.

3.2 The linear case $p = 2$.

Before going nonlinear, let us analyse what happens in the linear case $p = 2$, that is: $-\Delta u = u$ in Ω, $u = 0$ on $\partial\Omega$. This problem may or may not have a solution; what we are asking in other words is if $\lambda = 1$ is an eigenvalue of the operator $A := -\Delta$ with Dirichlet boundary conditions. In this context, indeed, $\lambda \in \mathbb{R}$ is called an eigenvalue whenever $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$ admits a nontrivial (weak) solution. From the spectral theory of compact operators (using the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$), we deduce that the eigenvalues of $-\Delta$ (counting multiplicities) form a nondecreasing sequence

$$0 < \lambda_1(\Omega) < \lambda_2(\Omega) \le \lambda_3(\Omega) \le \cdots \to \infty$$

with associated eigenfunctions $(v_n)_{n\in\mathbb{N}}$ which form a Hilbert base of $H_0^1(\Omega)$. Exactly as for eigenvalues of a matrix, the eigenvalues admit a variational formulation, namely

$$\begin{aligned} &\lambda_1(\Omega) = \min\left\{\mathcal{R}(u) : u \in H_0^1(\Omega) \setminus \{0\}\right\}, \\ &\lambda_k(\Omega) = \min_{V \subset H_0^1(\Omega), \dim V = k} \max_{u \in V\setminus\{0\}} \mathcal{R}(u), \quad (k \ge 2) \end{aligned} \tag{8}$$

where $\mathcal{R}(u) = \int_\Omega |\nabla u|^2 / \int_\Omega u^2$ is called the Rayleigh quotient. The details can be found in [26, §6], for instance. Therefore, the question of whether (6) in the case $p = 2$ admits a nontrivial solution or not depends on the domain: the answer is affirmative only for domains for which $1 = \lambda_i(\Omega)$ for some i.

3.3 The sublinear case $0 < p < 2$.

This is called the *sublinear* case. It is quite easy to see that $\mathcal{J}$ has a minimum in each direction: for a fixed $w \in H_0^1(\Omega)$, this corresponds to study the real function $f(t) = \mathcal{J}(tw)$, which has the form $at^2 - b|t|^p$ for some $a, b > 0$ (see the left picture on Figure 1). Using Sobolev inequalities and the direct method of Calculus of Variations [14, §8.2], one shows that $\mathcal{J}$ admits a negative global minimum in $H_0^1(\Omega)$: the level

$$\inf\{\mathcal{J}(u) : u \in H_0^1(\Omega)\} < 0$$

is achieved, providing a nontrivial solution (which is called a least energy solution). We know a lot about minimizers. First of all, they are signed: either $u > 0$ in Ω or $u < 0$ in Ω (consequence of the inequality $\mathcal{J}(|u|) \le \mathcal{J}(u)$ and the strong maximum principle [17, §2.2]). Positive solutions are unique [20]. This uniqueness property also implies symmetry properties in symmetric domains: for instance if the domain is radially symmetric (ball or annulus), the solution is radially symmetric (working in the space $\{u \in H_0^1(\Omega) : u(x) = u(|x|)\ \forall x \in \Omega\}$ provides a positive solution). More generally, we can consider the situation of a domain Ω which is invariant for a subgroup G of $O(N)$.

In the previous paragraph we described properties of minimizers. Does $\mathcal{J}$ admit more critical points (i.e., solutions of the problem (6))? The answer is affirmative (see e.g. [7]): there exists a sequence of critical points $(v_k)_k$, which satisfy

$$\mathcal{J}(v_k) < 0, \qquad \mathcal{J}(v_k) \to 0.$$

This is a consequence of the $\mathbb{Z}_2$–symmetry of the problem (the functional is invariant under the map $u \mapsto -u$); solutions can be found as saddle points of $\mathcal{J}$, characterized via min-max methods in an analogous way to what happens for higher eigenvalues (recall (8)). Observe that, since positive (and negative) solutions are unique, the previous multiplicity result yields the existence of infinitely many *sign-changing* solutions. The next step is then to understand them as better as possible. The study of the zero-set of sign-changing solutions (the *free-boundary* set $\Gamma = \{x \in \Omega : u(x) = 0\}$) has been done recently: up to a set with small Hausdorff dimension, Γ is a regular hypersurface [27, 28]. Moreover, one may also ask if, among all sign-changing solutions, there is one that minimizes the energy functional $\mathcal{J}$, that is, if the level

$$c_{nod} = \inf\{\mathcal{J}(u) : u \text{ is a sign-changing critical point of } \mathcal{J}\}$$

is achieved. The answer is affirmative, as shown recently in [9]. On radial domains this solution is *not* radially symmetric, but only axially symmetric. In this last paper it is also shown, quite remarkably, that the type of critical point we find depends on the domain: there are domains where the least energy nodal solution is a local minimizer of $\mathcal{J}$, and others where it is a saddle point. A complete understanding of how the domain influences the type of critical point is open.

3.4 The superlinear–subcritical case $2 < p < 2^*$.

For the case $p > 2$, in each direction the functional looks like the picture on the right in Figure 1. Using Sobolev inequalities, one can show that:

- the origin $u = 0$ is a strict local minimum;
- $\mathcal{J}$ is unbounded from below and from above.

In this case, to obtain solutions we cannot simply minimize (nor maximize) the functional in the whole $H_0^1(\Omega)$. Based on the geometry of the functional, we can use the following version of the celebrated result by Ambrosetti and Rabinowitz [3].

Theorem 2 (Mountain Pass Theorem).— Let H be a Hilbert space and let $\mathcal{J} : H \to \mathbb{R}$ be a $\mathscr{C}^{1,1}$ functional satisfying

- $\mathcal{J}(0) = 0$;
- there exists $r > 0$ such that $\mathcal{J}(0) \le \mathcal{J}(u)$ for every $\|u\| \le r$ and $\inf\{\mathcal{J}(u) : \|u\| = r\} > 0$;
- there exists v such that $\mathcal{J}(v) < 0$.

Let

$$\Gamma := \{\gamma \in C^1([0,1]; H_0^1(\Omega)) : \gamma(0) = 0,\ \mathcal{J}(\gamma(v)) < 0\},$$

and

$$c := \inf_{\gamma\in\Gamma} \sup_{u\in\gamma} \mathcal{J}(u).$$

Then there exists a sequence $(u_k) \subset H$ such that $\mathcal{J}(u_k) \to c$ and $\mathcal{J}'(u_k) \to 0$.

The proof of this result uses deformation lemmas and the study of steepest descending flows (a simple proof can be found in [14, §8]). The existence of a sequence (u_k) such that $\mathcal{J}(u_k) \to c$ and $\mathcal{J}'(u_k) \to 0$, by itself, does not imply the existence of a critical point (take the counterexample $H = \mathbb{R}$, $u_k = -k$ and $\mathcal{J}(x) = e^x$). A new concept regarding compactness is needed:

> A functional $\mathcal{J} \in C^1(H, \mathbb{R})$ satisfies the *Palais-Smale condition at c* if, whenever we have a sequence (u_k) such that $\mathcal{J}(u_k) \to c$ and $\mathcal{J}'(u_k) \to 0$, then there exists a subsequence (u_{k_j}) of (u_k) and $u \in H$ such that $u_{k_j} \to u$ in H. In particular, $\mathcal{J}'(u) = 0$.

Using the compactness of the Sobolev embeddings for $p < 2^*$, one proves that $\mathcal{J}$ defined before in (7) satisfies this condition, and the Mountain-Pass theorem provides the existence of a critical point of $\mathcal{J}$, hence a solution of (6). What can we now say about this solution? It is also a *least energy solution* (a.k.a. ground state), in the sense that

$$c = \inf\{\mathcal{J}(u) : u \in H \setminus \{0\},\ \mathcal{J}'(u) = 0\}.$$

Exactly as in the sublinear case, the solution can be shown to be signed: it is either stricly positive or strictly negative in Ω. However, uniqueness of positive solutions does not hold in general, as an effect of the topology of the domain (there are multiplicity results in annular domains) or of the geometry (dumbbell shaped domains). There is a long standing conjecture by Kawohl (1985) and Dancer (1988) stating that, if the domain is convex, then there is uniqueness of positive solution of (6) for $2 < p < 2^*$. A good review of the state-of-the-art regarding this can be found in the introduction of [16]. What about the symmetry in radial domains? When the domain is a ball, positive solutions are radially symmetric (consequence of a the so called Moving Plane Method[3], which uses many types of maximum principles, see [15] or [17, §2.6]). However, if Ω is an annulus, the solutions (at least for large p) loose one axis of symmetry, being just axially symmetric [6]. As we can see, there are some key changes between the cases $p < 2$ and $p > 2$.

Regarding multiplicity of solutions, again by the $\mathbb{Z}_2$—invariance of the functional, there exist infinitely many (sign-changing) solutions; however, unlike the sublinear case, this time we can find a sequence of solutions (u_k) such that $\mathcal{I}(u_k) \to \infty$. A long standing open question is whether the symmetry is necessary to obtain multiplicity results, with partial results obtained over the years by Bahri, Beresticky, Struwe, Rabinowitz, Bolle, Ghoussoub, Tehrani, Lions, Ramos, T., Zou, among many others. The study of the regularity of the zero-set of sign changing solutions is actually simpler in the superlinear case $p > 2$ than in the sublinear one $p < 2$ (although in any case it is not at all simple); this is as a consequence of the map $f(t) = |t|^{p-2}u$ being of class C^1 for $p > 2$ [19, 22].

The critical case $p = 2^* = 2n/(n-2)$

In this case, we are dealing with

$$-\Delta u = |u|^{2^*-2}u \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

and the associated functional $\mathcal{I}$ does not satisfy the Palais-Smale condition for all levels c. The question of whether there are (nontrivial) solutions or not for $p = 2^*$ or $p > 2^*$ depends strongly on the domain. When Ω is star-shaped, for instance, there are no solutions (by the Pohozaev identity, see for instance [4, Theorem 3.4.26]); however there are examples of contractible domains where solutions do exist. This shows that the topology of the domain is not enough to characterize the situation, although it has influence: if, for some positive d, the homotopy group of Ω with $\mathbb{Z}_2$ coefficients is non trivial, $\mathcal{H}_d(\Omega, \mathbb{Z}_2) \neq \{0\}$, then we have a positive solution [5]. Multiplicity results are much more recent (and challenging); recent contributions are due to Clapp, Ge, Musso, Pistoia, Weth, among others.

In order to show how delicate the situation is in the critical case, we make two remarks:

1. If the domain is not bounded but instead the whole $\mathbb{R}^n$, then we have (explicit!) solutions:
$$U_{\delta,\xi} = (n(n-2))^{(n-2)/4} \frac{\delta^{\frac{n-2}{2}}}{(\delta^2 + |x-\xi|^2)^{\frac{n-2}{2}}},$$
for $\delta > 0$, $\xi \in \mathbb{R}^n$.

2. If we consider a linear perturbation of the problem:
$$-\Delta u = \lambda u + |u|^{2^*-2}u \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega$$
the situation changes: this problem has positive solutions for $\lambda \in (0, \lambda_1(\Omega))$ and $n \geq 4$ (the problem is commonly known as the Brezis-Nirenberg problem [11]). The topology of the domain, in this situation, also influences multiplicity results: there exist at least $\text{cat}_\Omega(\Omega)$ solutions, where the (Lyusternik-Schnirelmann) category of Ω is the least integer d such that there exists a covering of Ω by d closed contratible sets. As $\lambda \to 0$, the solutions tends to concentrate and blowup at certain points which depend on geometric properties of Ω [18, 24].

4 Recent directions of Research

In the previous section we reviewed some old and new results regarding the Lane-Emden equation with Dirichlet boundary conditions. This is still a very active field of research and there are still many interesting questions left open. Although we described quite a few results, there would clearly be a lot more to be said. In this section, instead, we point out new directions of research that poped up more recently. One is the case of other boundary conditions such as the Neumann problem:

$$-\Delta u = |u|^{p-2}u \text{ in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

The difficulty of the problem is related with the fact that, in the associated functional $\mathcal{I}$, we have the presence of only $\|\nabla u\|_{L^2}$, which is a norm in $H_0^1(\Omega)$ but not in $H^1(\Omega)$. The study of least-energy solutions for

[3] The argument is based on a method developed by Alexandrov circa 1958 to establish that spheres are the only embedded compact hypersurfaces of R^n with constant mean curvature.

$p < 2^*$ (existence, study of its symmetry in symmetric domains, ...) is mostly recent [23, 25]. Quite remarkably, for $p = 2^*$, there are solutions in all regular domains, one of the few things that has been known for a while [13]. An open question is whether there is a solution for every $p > 2^*$ (or at least for $p \in (2^*, 2^* + \varepsilon)$ for sufficiently small ε).

Another direction of research is the case of systems. Here one might consider

$$-\Delta u_i = f_i(u_i) + u_i \sum_{j \neq i} \beta_{ij} u_j^2 \text{ in } \Omega$$

under a symmetric interaction $\beta_{ij} = \beta_{ji}$. From a physical point of view, this is connected with the search of standing wave solutions in systems of nonlinear Schrödinger type equations (coming from Bose-Einstein condensation and nonlinear optics). Mathematically speaking, this is a good prototype of a gradient system (the interaction term is the gradient of the potential $H(u_1, \dots, u_k) = (\sum_{j<i} \beta_{ij} u_i^2 u_j^2)/2$). Again, one might study existence, multiplicity and classification of solutions, concentration results in the critical case and symmetry questions. These issues are more complex for systems due to the possibility of different types of interaction between components, see for instance [30] and references.

Other variational systems (not of gradient type) are *Lane-Emden systems*:

$$-\Delta u = |v|^{q-2} v, \qquad -\Delta v = |u|^{p-2} u \text{ in } \Omega$$

(the reader might take a look at [8] for a recent survey).

5 Conclusion and recommended readings

In this short article we motived the study of some elliptic problems, starting from some classical material, introducing the concept of weak solutions and speaking about some variational methods, concluding with recent directions of research. With the exception of the fourth section and part of the third, everything is by now already included in introductory books. For Sobolev spaces, weak Solutions, and the linear theory of elliptic equations, the recommendation is [10, 14, 26] (the author of these lines uses a combination of these three books whenever he teaches a PDE course at the master level). For a gentle introduction to semilinear theory and the use of variational methods, we recommend [4], while [1, 2, 29, 31] contains more advanced material.

References

[1] Antonio Ambrosetti and Andrea Malchiodi. *Perturbation methods and semilinear elliptic problems on* $\mathbf{R}^n$, volume 240 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2006.

[2] Antonio Ambrosetti and Andrea Malchiodi. *Nonlinear analysis and semilinear elliptic problems*, volume 104 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2007.

[3] Antonio Ambrosetti and Paul H. Rabinowitz. Dual variational methods in critical point theory and applications. *J. Functional Analysis*, 14:349–381, 1973.

[4] Marino Badiale and Enrico Serra. *Semilinear elliptic equations for beginners*. Universitext. Springer, London, 2011. Existence results via the variational approach.

[5] A. Bahri and J.-M. Coron. On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain. *Comm. Pure Appl. Math.*, 41(3):253–294, 1988.

[6] Thomas Bartsch, Tobias Weth, and Michel Willem. Partial symmetry of least energy nodal solutions to some variational problems. *J. Anal. Math.*, 96:1–18, 2005.

[7] Thomas Bartsch and Michel Willem. On an elliptic equation with concave and convex nonlinearities. *Proc. Amer. Math. Soc.*, 123(11):3555–3561, 1995.

[8] D. Bonheure, E. Moreira dos Santos, and H. Tavares. Hamiltonian elliptic systems: a guide to variational frameworks. *Port. Math.*, 71(3-4):301–395, 2014.

[9] D. Bonheure, E. Moreira dos Santos, E. Parini, H. Tavares, and T. Weth. Nodal solutions for sublinear-type problems with dirichlet boundary conditions. *International Mathematics Research Notices*, in press, https://doi.org/10.1093/imrn/rnaa233.

[10] Haïm Brézis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[11] Haïm Brézis and Louis Nirenberg. Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.*, 36(4):437–477, 1983.

[12] S. Chandrasekhar. *An introduction to the study of stellar structure*. Dover Publications, Inc., New York, N. Y., 1957.

[13] Myriam Comte and Mariette C. Knaap. Solutions of elliptic equations involving critical Sobolev exponents with Neumann boundary conditions. *Manuscripta Math.*, 69(1):43–70, 1990.

[14] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2010.

[15] B. Gidas, Wei Ming Ni, and L. Nirenberg. Symmetry and related properties via the maximum principle. *Comm. Math. Phys.*, 68(3):209–243, 1979.

[16] Massimo Grossi, Isabella Ianni, Peng Luo, and Shusen Yan. Non-degeneracy and local uniqueness of positive solutions to the lane-emden problem in dimension two. arXiv:2102.09523 (2021).

[17] Qing Han and Fanghua Lin. *Elliptic partial differential equations*, volume 1 of *Courant Lecture Notes in Mathematics*. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, second edition, 2011.

[18] Zheng-Chao Han. Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 8(2):159–174, 1991.

[19] R. Hardt, M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and N. Nadirashvili. Critical sets of solutions to elliptic equations. *J. Differential Geom.*, 51(2):359–373, 1999.

[20] M. A. Krasnosel'skii. *Positive solutions of operator equations*. P. Noordhoff Ltd. Groningen, 1964. Translated from the Russian by Richard E. Flaherty; edited by Leo F. Boron.

[21] John M. Lee and Thomas H. Parker. The Yamabe problem. *Bull. Amer. Math. Soc. (N.S.)*, 17(1):37–91, 1987.

[22] Fang-Hua Lin. Nodal sets of solutions of elliptic and parabolic equations. *Comm. Pure Appl. Math.*, 44(3):287–308, 1991.

[23] Enea Parini and Tobias Weth. Existence, unique continuation and symmetry of least energy nodal solutions to sublinear Neumann problems. *Math. Z.*, 280(3-4):707–732, 2015.

[24] Olivier Rey. Proof of two conjectures of H. Brézis and L. A. Peletier. *Manuscripta Math.*, 65(1):19–37, 1989.

[25] Alberto Saldaña and Hugo Tavares. Least energy nodal solutions of Hamiltonian elliptic systems with Neumann boundary conditions. *J. Differential Equations*, 265(12):6127–6165, 2018.

[26] Sandro Salsa. *Partial differential equations in action*, volume 99 of *Unitext*. Springer, [Cham], third edition, 2016. From modelling to theory, La Matematica per il 3+2.

[27] Nicola Soave and Susanna Terracini. The nodal set of solutions to some elliptic problems: singular nonlinearities. *J. Math. Pures Appl. (9)*, 128:264–296, 2019.

[28] Nicola Soave and Tobias Weth. The unique continuation property of sublinear equations. *SIAM J. Math. Anal.*, 50(4):3919–3938, 2018.

[29] Michael Struwe. *Variational methods*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, fourth edition, 2008. Applications to nonlinear partial differential equations and Hamiltonian systems.

[30] Hugo Tavares, Song You, and Wenming Zou. Least energy positive solutions of critical schrödinger systems with mixed competition and cooperation terms: the higher dimensional case. arXiv:2109.14753 (2021).

[31] Michel Willem. *Minimax theorems*, volume 24 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 1996.