GENERIC BEHAVIOURS OF CONSERVATIVE DYNAMICAL SYSTEMS

by **Maria Joana Torres***

Science treats only the general; there is no science of the individual.
—ARISTOTLE[1]

Dynamical systems is the study of the long-term behaviour of evolving systems. The foundations were set by the innovative work of Henri Poincaré (1854–1912), *Les Méthodes Nouvelles de la Mécanique Céleste* (1892–1899), with fundamental questions concerning the stability of the solar system. Attempts to answer those questions revealed the incapacity of solving *exactly* the mathematical questions arising from physical systems. It appeared that understanding *typical* systems, or systems *in general*, was mathematically more fruitful.

The aim of the modern theory of dynamical systems is thus to describe the behaviour of *typical* trajectories, for *typical* evolution laws. Furthermore, when dealing with real-world systems, neither the initial data nor the evolution law are known exactly. For these reasons, one is most interested in properties that are *stable*, i.e., that are persistent under small perturbations of the evolution law.

1 WHAT IS A GENERIC BEHAVIOUR?

Ce qui limite le vrai, ce n'est pas le faux, c'est l'insignifiant.
—RENÉ THOM[2]

One of the oldest aspirations of humanity is to understand the motion of celestial bodies–the sun, moons, planets and visible stars of the solar system. The first complete mathematical formulation of the classical n-Body Problem was presented by Isaac Newton (1643–1727) in his masterpiece *Philosophiae Naturalis Principia Mathematica*, first published in 1687. Informally, the n-Body Problem can be stated as (see [15]): *Given only the present positions and velocities of a group of n celestial bodies, predict their motions for all future time and deduce them for all past time.* The Two-Body Problem is known as the *Kepler problem*, in honor of Johannes Kepler (1571–1630), who provided inspiration for Newton's gravitational model, with his laws on planetary motions deduced from the astronomical observational data of Tycho Brahe (1546–1601). Unfortunately, the Kepler problem revealed to be the only easy case among the n-Body Problem.

Most of the great mathematicians of the eigh-

[1] Quoted in *Itinerary for a Science of the Detail*, René Thom.

[2] In *Prédire n'est pas expliquer.*

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teenth and nineteenth century tackled the equations of the Three-Body Problem but were unable to make much progress. Until the final decade of the nineteenth century, the goal was to obtain exact results, to integrate equations and obtain complete solutions. But physical phenomena are in general nonlinear and some are even chaotic. At the end of the nineteenth century, a new *qualitative* era was opened by Poincaré, who introduced geometric, topological and probabilistic methods in order to understand qualitatively the complex behaviour of most of the solutions of the Three-Body Problem (see [13]). Poincaré's work on this problem provided a glimpse of chaotic behaviour in a dynamical system, a feature entirely understood by the mathematical community only three quarters of a century later, after George D. Birkhoff (1884–1944) and then Stephen Smale (1930–) show its importance.

At the end of the fifties, the development of dynamical systems as a theory was greatly influenced by the program of classifying singularities of differential mappings. One of the main goals of this program was to classify functions from $\mathbb{R}^n$ to $\mathbb{R}^p$ by the kind of singularities exhibited. But since such a characterization for *all* such functions is impossible, mathematicians settle for a classification of *almost all* functions. The approach was then to derive the type of singularities a certain *prototype* function can have, and then prove that any other function could be approximated by the prototype. This property of prototypes was expressed by René Thom (1923–2002) in terms of a *generic property*. Thom picked up the term *generic* from Italian algebraic geometers, who had already defined a generic property as *a property that is satisfied for all points of a space except for the points of a thin submanifold of that particular space* (see [12]). As did Thom in [34, p. 357], consider the space $L^m_{n,p}$ of functions of class C^m from $\mathbb{R}^n$ to $\mathbb{R}^p$ equipped with the C^m topology, which is a Baire space. Thom defined a *generic property* (P) in the space $L^m_{n,p}$ as a property that holds for all functions belonging to this space, except for a *rare* subset of that space. A generic property was then known as a property that is satisfied by the elements that form an *open and dense* subspace of the domain, which is the complement of a closed subspace without interior points (see [12]).

In the context of dynamical systems theory, the term generic (borrowed from Thom) was used for the first time by Mauricio Peixoto (1921–2019) in [25], motivated by his interest on structural stability.

In 1967, Smale presented in [31] the advances of the theory of dynamical systems so far. At that moment, the mathematical definitions needed to be adapted to a new classification program. Smale proposes that the term *generic* should be associated with a behaviour that holds for a *residual* subset of systems. Given a topological space M, a subset of the space is called *residual* or *topologically generic* if it contains a countable intersection of open and dense sets. Note that if M is a Baire space—for example, if M is completely metrizable—a countable intersection of dense open subsets is still dense. A property (P) is generic if it is verified on a residual subset. We observe that a countable intersection of residual sets is a residual set.

2 Two main theories in dynamical systems

In the 1960s two main theories in dynamical systems were developed: the *hyperbolic* theory for general systems and the *KAM* (for Kolmogorov–Arnold–Moser) theory for a distinguished class of conservative systems.

In the study of physical systems which evolve in time as solutions of certain differential equations one is led naturally to the consideration of measure preserving—*conservative*—systems. These conservative (or incompressible) flows are associated to *divergence-free vector fields*, they preserve a volume form on the ambient manifold and thus come equipped with a natural invariant measure μ which we call Lebesgue measure.

The hyperbolic theory was initiated by Smale, in the west, and Dmitri Anosov (1936–2014), Yakov Sinai (1935–) and Vladimir Arnold (1937–2010), in the former Soviet Union. It was part of a revolution in our vision of determinism, providing a mathematical foundation for the fact that deterministic systems often present chaotic behaviour in a robust way. Let M be a compact smooth Riemannian manifold and $f : M \to M$ be a diffeomorphism. Recall that, an *invariant* (i.e. $f(\Lambda) = \Lambda$) compact set $\Lambda \subset M$ is a *uniformly hyperbolic set* for f if the tangent bundle over Λ admits a continuous decomposition $T_\Lambda M = E^u \oplus E^s$, invariant under the derivative, and for which there are constants $C > 0$ and $\lambda \in (0,1)$ so that $\|Df^{-n}(x)|_{E^u_x}\| \leq C\lambda^n$ and $\|Df^n(x)|_{E^s_x}\| \leq C\lambda^n$, for every $x \in \Lambda$ and $n \geq 1$. The diffeomorphism f is called *Anosov (or globally hyperbolic)* if the whole manifold M is a uniformly hyperbolic set. The definition of uniform hyperbolicity for a smooth flow $f^t : M \to M, t \in \mathbb{R}$, is analogous except that (unless

Λ consists of equilibria) the previous decomposition becomes $T_\Lambda M = E^u \oplus E^0 \oplus E^s$, where E^0 is a line bundle tangent to the flow lines.

3 Generic dynamical behaviours

The properties of generic dynamical systems depend mostly on the dimension of the manifold and of the C^r-topology considered, $r \geq 0$. We refer the reader to [11], for a recent overview on the subject.

3.1 Generic dynamics in low regularity

3.1.1 *Metric and topological transitivity*

Before Poincaré's work, the founders of statistical mechanics, James Maxwell (1831–1879) and Ludwig Boltzmann (1844–1906) tried to provide a rigorous formulation of the kinetic theory of gases and statistical mechanics. A key ingredient was Boltzmann's Fundamental Principle, which asserts that the time and space averages of an observable (a function on the phase space) can be set equal (see [22, 19]). The ergodic theorems of Birkhoff and John von Neumann (1903–1957) ("time averages exist a.e.") set the foundation for the current definition of *ergodicity*: *any invariant set has zero measure or full measure*. If this holds, then time averages coincide with space averages at least for typical points—Boltzmann's Fundamental Principle.

An important question in the 1930s was then: *Is ergodicity with respect to volume a typical property?* The question was first addressed by John Oxtoby (1910–1991) and Stanisław Ulam (1909–1984) in [24], who proved that a generic volume-preserving homeomorphism of a compact manifold is ergodic. A natural question, still open in general, is whether such a result extends to the space of volume-preserving C^1 diffeomorphisms. If one considers the other extreme of regularity, C^∞ diffeomorphisms, ergodicity is not a typical property at all: KAM theory assures that there are open sets of volume-preserving C^∞ diffeomorphisms that are not ergodic (see [35, 1]).

In their 1912 article, Paul (1880–1933) and Tatiana (1876–1964) Ehrenfest discussed questions related with the ergodic hypothesis and then proposed the alternative *quasi-ergodic hypothesis* (see [22]): *some orbit of the system will pass arbitrarily close to every point of the phase space*, i.e., the system is *(topologically) transitive*. It is the topological counterpart of an ergodic system.

Concerning the continuous-time counterpart of the Oxtoby-Ulam theorem, there is a lack in the literature. Motivated by this, the main goal in [10] was to study the abundance of transitivity-like properties of C^0 conservative flows generated by Lipschitz vector fields and to establish a weaker (topological) counterpart of the Oxtoby-Ulam theorem: C^0-generic flows are strongly transitive: *the shortest hitting time from a ball to any other ball of the same radius is uniformly bounded above by a constant depending only on the radius.* For Lipschitz divergence-free vector fields without singularities it was proved in [10, Theorem A] that:

Theorem 1.— C^0-generic non-singular Lipschitz divergence-free vector fields generate conservative flows that are strongly transitive.

It was also proved that C^0-generic Lipschitz divergence-free vector fields generate transitive flows (see [10, Theorem B]).

3.1.2 *Perturbation of orbits: Closing lemma*

The problem of closing a nonperiodic trajectory is a well-known problem in the theory of dynamical systems, whose origin remounts to Poincaré (see [26, vol 1, p.82]). We want to *close*, in the sense that we transform into a periodic orbit, a given orbit with some return property (for example, non-trivial recurrence or non-wandering) by performing a small perturbation on the original system.

Poincaré believed that such a closing could be done in quite general situations. However, until now, there are positive answers only if the perturbations are with respect to coarse topologies like e.g., C^0, Hölder, Sobolev-$(1,p)$ and C^1. The C^0 closing lemma can readily be proved, except perhaps for the geodesic flows. But the C^1 closing lemma reveals some fundamental difficulties. The C^1 closing lemma for non-conservative systems was first established by Pugh [27] in the late 1960s and for conservative systems was prove by Pugh and Robinson [28] in the early 1980s. A non-conservative version of the closing lemma for the Sobolev-$(1,p)$ topology was recently presented in [17]. In [2], it was given a simpler and different proof of the Sobolev closing lemma which also works in the conservative case.

More precisely, let U be an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary and let $1 \leq p, q \leq \infty$. A measurable map $f = (f_1, \dots, f_n) : U \to \mathbb{R}^n$ is in the Sobolev class $W^{1,p}(U, \mathbb{R}^n)$ if, for all $i = 1, \dots, n$,

f_i and all its distributional partial derivatives $\partial f_i / \partial x_j$ are in $L^p(U)$. We are interested only in Sobolev maps that are continuous up to the boundary, i.e., we consider the space $W^{1,p}(U, \mathbb{R}^n) \cap C^0(\overline{U}, \mathbb{R}^n)$. Finally we define $\mathbb{W}^{1,p}(U)$ as the set of all homeomorphisms $f : U \to U$ such that $f \in W^{1,p}(U, \mathbb{R}^n) \cap C^0(\overline{U}, \mathbb{R}^n)$. We also define $\mathbb{W}^{1,p,q}(U)$ as the set of all elements in $\mathbb{W}^{1,p}(U)$ whose inverse is in $\mathbb{W}^{1,q}(U)$. In $\mathbb{W}^{1,p}(U)$ and $\mathbb{W}^{1,p,q}(U)$ we consider the natural metrics defined by $d_{\infty,p}(f, g) = \|f - g\|_\infty + \|D(f - g)\|_p$ and $(f, g) \mapsto d_{\infty,p}(f, g) + d_{\infty,q}(f^{-1}, g^{-1})$, respectively. We consider also the subspaces $\mathbb{W}^{1,p}_\mu(U)$ and $\mathbb{W}^{1,p,q}_\mu(U)$ of volume preserving elements. The spaces $\mathbb{W}^{1,p}(U)$, $\mathbb{W}^{1,p}_\mu(U)$, $\mathbb{W}^{1,p,q}(U)$ and $\mathbb{W}^{1,p,q}_\mu(U)$ satisfy the Baire property.

Recall that a point $x \in U$ is said to be a *non-wandering point* for f if for any neighbourhood V of x there exists $n \in \mathbb{N}$ such that $f^n(V)$ intersects V. The set of non-wandering points $\Omega(f)$ contains the set $\mathrm{Per}(f)$ of periodic points. It was proved in [2, Theorem A] the Sobolev-$(1, p)$ closing lemma:

Theorem 2.— Let $n \geq 2$. Consider X to be any of the spaces $\mathbb{W}^{1,p}(U)$, $\mathbb{W}^{1,p}_\mu(U)$, $\mathbb{W}^{1,p,q}(U)$, $\mathbb{W}^{1,p,q}_\mu(U)$, where $p, q \in [1, \infty[$. Let $f \in X$ and $z \in \Omega(f)$. Then, for all $\varepsilon > 0$ there exists $y_\varepsilon \in U$ and $h_\varepsilon \in X$ such that $\lim_{\varepsilon \to 0} |y_\varepsilon - z| = 0$, $\lim_{\varepsilon \to 0} \|h_\varepsilon - f\|_X = 0$ and $y_\varepsilon \in \mathrm{Per}(h_\varepsilon)$.

3.1.3 *Density of periodic orbits*

Another fundamental result in the theory of dynamical systems is the *general density theorem*. It asserts that *generically the closure of the set of periodic orbits is the set where the dynamics is truly relevant*: in the non-conservative case this set is the non-wandering set and in the conservative case this set is the whole manifold. The general density theorem has been proved in many different settings and there is a vast literature on the subject (see [2] and references therein).

The general density theorem is a direct consequence of the combination of the closing lemma and the stability of the closed orbits. As a consequence, the general density theorem turns out to be easier in the C^1 case when compared to the C^0 one, because in the differentiable case the stability of periodic points can be expressed through hyperbolicity but in the topological case is more subtle. Within Sobolev homeomorphisms, in order to use hyperbolicity we had to request for differentiability at least for a map arbitrarily close from the Sobolev perspective. But this bypass through a differentiable map is very difficult to obtain. In fact, regularization of Sobolev-$(1, p)$ homeomorphisms is available only for planar domains. In [2, Theorem B], it was proved the planar general density theorem for Sobolev-$(1, p)$ maps:

Theorem 3.— Let $U \subset \mathbb{R}^2$. There exists a $\mathbb{W}^{1,p}$-residual subset $\mathscr{R} \subset \mathbb{W}^{1,p}(U)$ such that if $f \in \mathscr{R}$, then $\overline{Per(f)} = \Omega(f)$.

3.2 Smooth Generic Dynamics

3.2.1 *Palis-Smale stability conjecture*

Structural stability is one of the most fundamental topics in dynamical systems and contains some of the hardest conjectures in the area. The concept of *structural stability* was introduced in the mid 1930s by Andronov and Pontryagin. Roughly speaking it means that under small perturbations the whole orbit structure remains the same: there exists a homeomorphism of the ambient manifold mapping orbits of the initial system into orbits of the modified one. The aim of Smale's program in the early 1960s was to prove the genericity of structurally stable systems. Although Smale's program was proved to have serious flaws one decade later, it played a major role in the development of the theory of smooth dynamical systems. It led to the construction of hyperbolic theory, studying uniform hyperbolicity, and characterizing structural stability as being essentially equivalent to uniform hyperbolicity. Indeed, one of the high points in the development of smooth dynamics is the proof by Robbin, Robinson, Mañé and Hayashi [21, 29, 30, 20] that structural stability indeed characterizes hyperbolic dynamical systems. For C^1-diffeomorphisms this was achieved in the 1980s, for flows in the 1990s. The C^r structural stability conjecture for $r \geq 2$ remains wide open. In the conservative setting we highlight the seminal paper of Newhouse [23] where it was proved that *a symplectic diffeomorphisn of a compact manifold is structurally stable if and only if it is Anosov*. In the continuous-time setting, similar results were obtained for conservative flows in [7] and for Hamiltonian flows in [6], but in lower dimension (three and four, respectively). These results were generalized in [18] and [8] for arbitrary dimension, respectively. Let us describe the Hamiltonian framework.

A *Hamiltonian system* can be seen as the apotheosis of mathematical models of classical mechanics. The mathematician William R. Hamilton (1805–1865) developed a formalism for the equations of the dynamics, which played a major role in the development of the theory of classical dynamical systems. The *Hamiltonian formalism* was originally formulated combining the formulation of mechanics of Joseph-Louis Lagrange (1736–1813) (itself deduced from Newton's laws) with variational methods (see [16]). In the modern language, Hamiltonian systems are a part of *symplectic geometry*.

Let (M, ω) be a symplectic manifold, where M is a $2n$-dimensional ($n \geq 2$), closed,[3] connected and smooth Riemannian manifold, endowed with a symplectic structure, i.e. a closed and nondegenerate 2-form ω. A *Hamiltonian* is a real-valued C^r function on M, $2 \leq r \leq \infty$. The associated *Hamiltonian vector field* X_H is defined by $\omega(X_H(p), u) = dH_p(u)$, for all $u \in T_pM$; this vector field generates the Hamiltonian flow X^t_H. From now on, we shall be restricted to the C^2-topology and thus we set $r = 2$. Observe that H is C^2 if and only if the associated Hamiltonian vector field X_H is C^1. A scalar $e \in H(M) \subset \mathbb{R}$ is called an *energy* of H. An *energy hypersurface* $\mathscr{E}_{H,e}$ is a connected component of $H^{-1}(\{e\})$ and it is *regular* if it does not contain singularities. Observe that a regular energy hypersurface is a X^t_H-invariant, compact and $(2n-1)$-dimensional manifold. A *Hamiltonian system* is a triple $(H, e, \mathscr{E}_{H,e})$, where H is a Hamiltonian, e is an energy and $\mathscr{E}_{H,e}$ is a regular connected component of $H^{-1}(\{e\})$. Fixing a small neighbourhood $\mathscr{W}$ of a regular $\mathscr{E}_{H,e}$, there exist a small neighbourhood $\mathscr{U}$ of H and $\epsilon > 0$ such that, for all $\tilde{H} \in \mathscr{U}$ and $\tilde{e} \in (e - \epsilon, e + \epsilon)$, $\tilde{H}^{-1}(\{\tilde{e}\}) \cap \mathscr{W} = \mathscr{E}_{\tilde{H},\tilde{e}}$. We call $\mathscr{E}_{\tilde{H},\tilde{e}}$ the *analytic continuation* of $\mathscr{E}_{H,e}$. In the space of Hamiltonian systems we consider the topology generated by a fundamental systems of neighbourhoods. Given a Hamiltonian system $(H, e, \mathscr{E}_{H,e})$ we say that $\mathscr{V}(\mathscr{U}, \epsilon)$ is a *neighbourhood* of $(H, e, \mathscr{E}_{H,e})$ if there exist a small neighbourhood $\mathscr{U}$ of H and $\epsilon > 0$ such that for all $\tilde{H} \in \mathscr{U}$ and $\tilde{e} \in (e - \epsilon, e + \epsilon)$ one has that the analytic continuation $\mathscr{E}_{\tilde{H},\tilde{e}}$ of $\mathscr{E}_{H,e}$ is well-defined. A Hamiltonian system $(H, e, \mathscr{E}_{H,e})$ is said to be *Anosov* if $\mathscr{E}_{H,e}$ is uniformly hyperbolic for the Hamiltonian flow X^t_H associated to H. We say that the Hamiltonian system $(H, e, \mathscr{E}_{H,e})$ is *structurally stable* if there exists a neighbourhood $\mathscr{V}$ of $(H, e, \mathscr{E}_{H,e})$ such that, for any $(\tilde{H}, \tilde{e}, \mathscr{E}_{\tilde{H},\tilde{e}}) \in \mathscr{V}$, there exists a homeomorphism $h_{\tilde{H},\tilde{e}}$ between $\mathscr{E}_{H,e}$ and $\mathscr{E}_{\tilde{H},\tilde{e}}$, preserving orbits and their orientations. Moreover, $h_{\tilde{H},\tilde{e}}$ is continuous on the parameters $\tilde{H}$ and $\tilde{e}$, and converges to id when $\tilde{H}$ C^2-converges to H and $\tilde{e}$ converges to e. The stability conjecture for Hamiltonians was established in [8, Theorem 2]:

Theorem 4.— If $(H, e, \mathscr{E}_{H,e})$ is a structurally stable Hamiltonian system, then $(H, e, \mathscr{E}_{H,e})$ is Anosov.

3.2.2 *Shades of hyperbolicity*

The characterization of structurally stable systems using topological and geometrical dynamical properties has been one of the main objects of interest in the global qualitative theory of dynamical systems in the last 40 years. Here, we shall focus on *tracing orbit properties*, namely, the *shadowing* and *specification* properties (see [9] where the topological stability and expansiveness properties were also considered).

A dynamical system has the *shadowing property* if for any *almost orbit* (obtained, for example, by a numerical method with good accuracy) there is a close *true orbit*. In this case, an approximate pattern of trajectories given by numerical modeling reflects the exact structure of trajectories. A dynamical system has the *specification property* if one can shadow distinct n pieces of orbits, which are sufficiently time-spaced, by a single orbit. We say that the specification property is *weak* if $n = 2$.

Let (M, d) be a compact metric space and let $(X^t)_{t\in\mathbb{R}}$ be a continuous flow on M. Fix real numbers $\delta, T > 0$. We say that a pair of sequences $((x_i), (t_i))_{i\in\mathbb{Z}}$ ($x_i \in M, t_i \in \mathbb{R}, t_i \geq T$) is a (δ, T)-*pseudo-orbit* if $d(X^{t_i}(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. For the sequence $(t_i)_{i\in\mathbb{Z}}$ we write $\sigma(n) = t_0 + t_1 + \ldots + t_{n-1}$ if $n > 0$, $\sigma(n) = -(t_n + \ldots + t_{-2} + t_{-1})$ if $n < 0$, and $\sigma(0) = 0$. Let $x_0 \star t$ denote a point on a (δ, T)-chain t units time from x_0. More precisely, for $t \in \mathbb{R}$, $x_0 \star t = X^{t-\sigma(i)}(x_i)$ if $\sigma(i) \leq t < \sigma(i+1)$. In continuous-time setting the shadowing property should reflect the speed at which different points travel in their trajectories. For that reason we need to consider orbits up to reparametrization. By Rep we denote the set of all increasing homeomorphisms $\alpha : \mathbb{R} \to \mathbb{R}$, such that $\alpha(0) = 0$, called *(time) reparameterizations*. Fixing $\varepsilon > 0$, we define the set

$$\mathrm{Rep}(\varepsilon) = \left\{\alpha \in \mathrm{Rep}\colon \left|\frac{\alpha(t) - \alpha(s)}{t - s} - 1\right| < \varepsilon, s, t \in \mathbb{R}\right\},$$

of the reparameterizations ε-close to the identity. The flow $(X^t)_t$ satisfies the *shadowing property* if for any $\varepsilon > 0$ there exist $\delta, T > 0$ such that for any

[3] That is, compact and without boundary.

(δ, T)-pseudo-orbit $((x_i),(t_i))_{i\in\mathbb{Z}}$ there exist a point $z \in M$ and a reparametrization $\alpha \in \mathrm{Rep}(\varepsilon)$ such that $d(X^{\alpha(t)}(z), x_0 \star t) < \varepsilon$, for every $t \in \mathbb{R}$.

The flow $(X^t)_{t\in\mathbb{R}}$ has the *specification property* if for any $\varepsilon > 0$ there exists a $T = T(\varepsilon) > 0$ such that: given any finite collection τ of intervals $I_i = [a_i, b_i]$ $(i = 1 \dots m)$ of the real line satisfying $a_{i+1} - b_i \geq T(\varepsilon)$ for every i and every map $P : \bigcup_{I_i\in\tau} I_i \to M$ such that $X^{t_2}(P(t_1)) = X^{t_1}(P(t_2))$ for any $t_1, t_2 \in I_i$ there exists $z \in M$ so that $d(X^t(z), P(t)) < \varepsilon$ for all $t \in \bigcup_i I_i$.

It is well-known that Anosov systems, and thus, structurally stable Hamiltonian systems, satisfy the shadowing property. Moreover, mixing Anosov systems satisfy the specification property. In the context of Hamiltonian systems, it was proved in [9, Theorem 1] that if one requires the stability of shadowing (or weak specification) property under perturbation, then the Hamiltonian system is Anosov. Thus, we call these properties *shades of hyperbolicity*. We say that a property *stably holds* for some system if it holds for any system in some neighbourhood of that system.

Theorem 5.— Let $(H, e, \mathscr{E}_{H,e})$ be a Hamiltonian system. If any of the following statements hold:

(1) $(H, e, \mathscr{E}_{H,e})$ is stably shadowable;

(2) $(H, e, \mathscr{E}_{H,e})$ has the stable weak specification property,

then the Hamiltonian system $(H, e, \mathscr{E}_{H,e})$ is Anosov.

A natural question is whether these results are extensible to the subclass of Hamiltonians formed by the geodesic flows. Let (M, g) be a Riemannian manifold, where M is a closed, connected, Riemannian manifold of dimension ≥ 2 and $g \in \mathscr{R}^r$. Here $\mathscr{R}^r$ stands for the set of C^r Riemannian metrics, $2 \leq r \leq \infty$. Given a tangent vector $v \in T_xM$ at a point $x \in M$, denote by $\gamma_{x,v} : \mathbb{R} \to M$ the geodesic such that $\gamma_{x,v}(0) = x$ and $\dot{\gamma}_{x,v}(0) = v$. The geodesic flow of g is the flow on TM defined by $\varphi^t_g(x, v) = (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))$. Since geodesics travel with constant speed, the unit tangent bundle $S_gM = \{(x, v) \in TM : g_x(v, v) = 1\}$ is preserved by φ^t_g. It is widely known that the geodesic flow is a Hamiltonian flow given by $(x, v) \mapsto \frac{1}{2}g_x(v, v)$ on TM for a symplectic form depending on g. The perturbation tools for geodesic flows are very delicate as opposed to the general Hamiltonian case. We can only perturb the metric, hence the perturbation is never a local issue in phase space. This is the main reason why it is not known a closing lemma and a general density theorem for geodesic flows in the C^1 topology, i.e. C^2 in the metric. As a consequence, the hyperbolic structure of the closure of the periodic orbits cannot be extrapolated to the whole energy level and we are not able to assure global hyperbolicity. It was proved in [4] that:

Theorem 6.— There is a set $\mathscr{G}_1 \subset \mathscr{R}^2$ where $\mathscr{G}_1$ is C^2-open in $\mathscr{R}^2$ and $\mathscr{G}_1 \cap \mathbb{R}^\infty$ is C^∞-dense in $\mathscr{R}^\infty$ such that if $g \in \mathscr{G}_1$ and the geodesic flow satisfies any of the properties:

(1) is stably shadowable;

(2) has the stable weak specification property,

then $\overline{\mathrm{Per}(g)}$ is a uniformly hyperbolic set.

Notice that we were only able to show the result for a residual set of metrics, in contrast to the Hamiltonian case and also in contrast to the 2-dimensional case (for geodesic flows) previously considered in [5].

3.2.3 Topological entropy

The complexity of a dynamical system can be measured by the *topological entropy*. The topological entropy is a nonnegative real number that, roughly speaking, measures the rate of exponential growth of the number of distinguishable orbits with finite but arbitrary precision as time advances.

Let (M, d) be a compact metric space and $f : M \to M$ be a continuous map. For each $n \geq 1$, let $d_n(x, y) = \max\{d(f^i(x), f^i(y)) : 0 \leq i < n - 1\}$. A subset F of the phase space M is said to be (n, ϵ)-*spanning* if M is covered by the union of the dynamical balls $\{y : d_n(x, y) < \epsilon\}$ centered at the points $x \in F$. Denote by $N(n, \epsilon)$ the minimal cardinality of a (n, ϵ)-spanning set. Roughly, this gives the number of orbit segments that one can distinguish up to some precision. The *topological entropy* is then the exponential growth rate of this number as the precision increases,

$$h_{top}(f) = \lim_{\epsilon\to 0}\left(\limsup_{n\to\infty}\frac{1}{n}\log N(n, \epsilon)\right).$$

If a manifold has negative sectional curvature, its geodesic flow is Anosov and hence it has positive topological entropy. On manifolds with non negative curvature it is not so clear that one can perturb the metric to obtain positive topological entropy. Recently, Contreras proved in [14] that positive topological entropy is a generic property among geodesic flows on any closed manifold with dimension ≥ 2.

Very recently, it was obtained in [3] a similar result for billiards in generic convex bodies.

Tell me these things, Olympian Muses, tell
From the beginning, which came first to be?
Chaos was first of all
—Hesiod, Theogeny, II, 114-116[4]

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