# Travelling waves and their speeds for FKPP equations — an overview in the framework of ODEs

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Abstract.—We review the basics on admissible speeds of travelling waves to FKPP (Fisher-Kolmogorov-Petrovski-Piskounov) equations, starting from the classical setting and pointing out some changes needed to deal with nonlinear diffusion. Most of the material is based on elementary theory of ordinary differential equations (ODEs).

Consider the following curious (and simple) problem concerning a class of first order ODEs: given a function $f : [0,1] \to \mathbb{R}$ of **type A**, i.e. continuous, $f(0) = f(1) = 0$ and $f(s) > 0\ \forall s \in ]0,1[$, *find the values of the parameter $c > 0$ so that the problem*

$$y' = 2(c\sqrt{y} - f(u)), \quad y(0) = y(1) = 0 \tag{1}$$

*has a solution $y(u) \geq 0$, $0 \leq u \leq 1$.*

More generally, we shall look also at the problem where the equation is instead

$$y' = q(c\, {y_+}^{1/p} - f(u)), \tag{2}$$

($p$, $q$ being positive conjugate, i.e. $1/p + 1/q = 1$.)

This problem turns out to be useful in the fields of Applied Mathematics where the equations known as *FKPP (Fisher-Kolmogorov-Petrovski-Piskounov) equations* provide relevant models. Unsurprisingly, the elementary problem is interesting in itself, since it carries a lot of important information and features from its motivating source.

In 1937 R. Fisher [12] proposed a partial differential equation model for the propagation of an advantageous gene in a one dimensional spatial setting. In the same year, Kolmogorov, Petrovsky and Piskunov [15] obtained significant properties of the PDE model. The simplest protopype is given by the PDE

$$u_t = u_{xx} + u(1-u). \tag{3}$$

$u$ denoting the frequence of the gene, taking values from 0 to 1.

Other problems in the applied sciences lead to (FKPP) equations

$$u_t = u_{xx} + f(u) \tag{4}$$

where $f$ is a function of type A. An example is the Zeldovich equation, $u_t = u_{xx} + u^2(1-u)$, from the theory of combustion, where $u$ means temperature and $u^2(1-u)$ represents the generated heat.

Equation (4) has two equilibria $u = 0$ e $u = 1$. Meaningful solutions $u(x,t)$ take values between 0 and 1. Let us look for **travelling waves** with **speed** $c$: $u(x,t) = u(\xi)$, $\xi = x + ct$, whose profile $u$ is increasing and connects the equilibria: $u(-\infty) = 0$, $u(\infty) = 1$. Substitution in $u_t = u_{xx} + f(u)$ leads to

$$u''(\xi) - cu'(\xi) + f(u(\xi)) = 0, \quad \xi \in \mathbb{R}. \tag{5}$$

Therefore we look for increasing solutions of (5) such that

$$u(-\infty) = 0, \quad u(+\infty) = 1. \tag{6}$$

The study of the FKPP equations has generated a rich literature. The reader is referred to the (somewhat arbitrary) short selection [12, 15, 3, 13, 16, 5] and to their references for an account of the mathematical development of the subject.

Travelling wave solutions $u(x+ct)$ and their speeds are of great interest, because under certain conditions they shape the behaviour of solutions $u(x,t)$ as $t \to +\infty$. As we shall recall in a moment, the admissible speeds form a half-line $c \geq c^*$, where the minimum $c^* > 0$ (called **critical speed**) has a special

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role. For example, as we show below, $c^* = 2$ for the simple model where $f(u) = u(1-u)^\alpha$, $\alpha \geq 1$; and (see [15]) if $u(x,t)$ denotes the solution such that $u(x,0)$ is the Heaviside function, then $u(x + a(t), t) \to u(x)$ for every $x$, as $t \to +\infty$, where $a(t) = 2t + o(t)$ at $\infty$ and $u(x)$ is the profile of the travelling wave with speed 2.

This article reviews, in an elementary setting and in as a self-contained manner as possible, the basics about some classic and more recent results on travelling waves for FKPP. In the final sections we sketch the corresponding results for the analogous model with nonlinear diffusion.

In what follows, we consider equation (5) where $f$ is continuous, of type A in $[0,1]$, and in addition

$$(H) \qquad \exists k > 0 \,:\, f(u) \leq \min(ku,\ k(1-u)),$$

for every $u \in [0,1]$.

A number $c \in \mathbb{R}$ is called an *admissible speed* to (5), or with respect to $f$, if there exists an increasing solution of (5)-(6), that is, a monotone heteroclinic connecting the equilibria 0 and 1.

Remark 1.— When $f$ is differentiable, linearization about the equilibrium at the origin yields immediately that *increasing* travelling waves can exist only if $c \geq 2\sqrt{f'(0)}$.

## 1 Reduction to a first order problem.

Let $u = u(\xi)$ be an increasing solution of

$$u''(\xi) - cu'(\xi) + f(u(\xi)) = 0$$

in $\mathbb{R}$. Then $u'(\xi) > 0, \xi \in \mathbb{R}$, there exists $\xi(u)$, the inverse function of $u = u(\xi)$, and $u'$ may be given as a function of $u$: $\phi(u) = u'(\xi(u))$. Therefore $\phi \ : ]0,1[ \to \mathbb{R}^+$ is $C^1$, and may be extended to $[0,1]$ with $\phi(0) = \phi(1) = 0$. $\phi$ is a solution of $\phi(u)\phi'(u) - c\phi(u) + f(u) = 0$. Hence, setting $\psi(u) := \phi(u)^2$, $\psi$ solves the problem (1) (the boundary conditions come from the fact that $u'(\pm\infty) = 0$).

Conversely, if $\psi$ solves (1) and we consider the Cauchy problem $u' = \sqrt{\psi(u)}$, $u(0) = 1/2$, it can be shown that its solution is defined in the whole real line $(-\infty, \infty)$ (using assumption $(H)$). That solution $u(t)$ satisfies (5–6) and $u'(t) > 0$, for every $t$.

In summary, $u = u(\xi)$ is an increasing solution connecting the two equilibria if and only if $\phi(u)^2$ solves (1). Hence, *the square root of a solution of (1) gives the profile in the phase plane of the trajectory of a travelling wave.* And the following simple facts may be proved.

1.A.—*If $f'(0)$ exists and equation $y'(u) = 2c\sqrt{y(u)} - 2f(u)$ has a solution $y(u)$ such that $y(0) = 0$ and $y(u) > 0$ in some interval $(0,\eta)$, then $c^2 \geq 4f'(0)$.*

1.B. [Lower solution criterion].—*If there exists a $C^1$ function $s : [0,1] \to \mathbb{R}$ such that $s(0) = 0$, $s(u) > 0$ if $u \in (0,1)$ and $\forall\, u \in [0,1]$,*

$$s'(u) \leq 2c\sqrt{s(u)} - 2f(u), \tag{7}$$

*then the first order problem $\psi'(u) = 2c\sqrt{\psi(u)} - 2f(u)$, $\psi(0) = \psi(1) = 0$ has a (unique) solution.*

1.C.—*The set of admissible speeds for $f$ is the set of numbers $c$ such that (1) has solutions. It is an unbounded closed interval $[c^*, +\infty)$ with $c^* > 0$.*

We sketch the proof of 1.B–C (for more general assertions and proofs see e.g. [11]). Let

$$M = \sup_{0<u<1} \frac{f(u)}{u}$$

(which exists by property $(H)$). If $c_0^2 \geq 4M$ and $c \geq c_0$, the equation $s'(u) = 2c_0\sqrt{s(u)} - 2Mu$ has a solution such that $s(0) = 0$ and $s(u) > 0$, for all $0 < u \leq 1$: take $s(u) = (Bu)^2$ with

$$B = \frac{c_0 \pm \sqrt{c_0^2 - 4M}}{2}.$$

Hence $s(u)$ is a lower solution of $y'(u) = 2c_0\sqrt{y(u)} - 2f(u)$, $y(0) = 0$. Therefore a positive solution $\bar{y}$ of this equation, with $\bar{y}(0) = 0$, exists. The (unique) solution $\tilde{u}$ of the same equation such that $\tilde{u}(1) = 0$ is the desired solution: its graph cannot meet either the graph of $\bar{y}$ (by uniqueness) or the $u$–axis (because of the sign of the slope) for $0 < u < 1$. Hence the set of admissible $c$ is a nonempty interval; by the equivalence between (1) and (5)–(6) any number $c_0$ such that $c_0 \geq 2\sqrt{M}$ is an admissible speed; it has a minimum element by an elementary compactness argument.

To each function $f$ satisfying our assumptions we thus associate a number $c^* > 0$ which is the minimum admissible speed of $f$. We write $c^* = c^*(f)$. We call it also the *critical speed* of $f$.

Remark 2.— Easy consequences are:

$$f \geq g \Rightarrow c^*(f) \geq c^*(g); \text{ and}$$

if $f'(0)$ exists,

$$2\sqrt{f'(0)} \le c^*(f) \le 2\sqrt{\sup_{0<u<1} \frac{f(u)}{u}}. \qquad (8)$$

In particular, if $f'(0)$ exists and $f(u) \le f'(0)u$ for every $u \in (0,1)$, then $c^* = 2\sqrt{f'(0)}$.

## 2 Asymptotic behaviour at infinity

Let us look at the behaviour of solutions of (1) at the endpoints of $[0,1]$: We assume that $f'(0)$ and $f'(1)$ exist. The objective is to compute the limits

$$\lim_{\xi\to-\pm\infty} \frac{u'(\xi)}{u(\xi)}.$$

2.A.—*Suppose that $f'(0)$ exists. If $\psi(u)$ solves $\psi'(u) = 2c\sqrt{\psi(u)} - 2f(u)$, $\psi(0) = 0$, with $\psi(u) > 0$ in some interval $(0,\eta)$, then the derivative $(\sqrt{\psi})'(0)$ exists and is a root of $x^2 - cx + f'(0) = 0$.*

Denote $\lambda^-(c) \le \lambda^+(c)$ the roots of $x^2 - cx + f'(0) = 0$ when they exist.

2.B.—*Let c be an admissible speed of $u'' - cu' + f(u) = 0$.*

- *If $c = c^*$, $(\sqrt{\psi})'(0) = \lambda^+(c)$.*
- *If $c > c^*$, $(\sqrt{\psi})'(0) = \lambda^-(c)$.*

Let us point out the steps needed in the proof of 2.B.:

*Step 1.* Let $\eta > 0$, $0 < A < B$, $0 \le a < b$, $0 < c_1 < c_2 < 2A$ be constants such that

$a \le f(u)/u \le b, \quad 0 < u \le \eta;$

$A^2 - cA + b < 0 < B^2 - cB + a$, for every $c \in [c_1, c_2]$.

Then for $c \in [c_1, c_2]$ the initial value problem

$$\psi'(u) = 2c\sqrt{\psi(u)} - 2f(u), \quad \psi(0) = 0$$

has a unique solution such that $A^2u^2 \le \psi(u) \le B^2u^2$ for $0 \le u \le \eta$. Moreover the solution depends continuously on $c$.

(The proof uses the contraction operator

$$Tv(u) = 2c\int_0^u \sqrt{v(t)}dt - 2\int_0^u f(t)dt,$$

for $u \in [0,\eta]$, in the space $X$ of continuous functions $v$ such that $A^2u^2 \le v(u) \le B^2u^2$, for every $v \in [0,\eta]$.)

*Step 2.* Let $\bar{c} > 2\sqrt{f'(0)}$. Choose $A, B$ such that:

$$\frac{\bar{c}}{2} < A < \lambda^+(\bar{c}) < B.$$

Then there are numbers $0 \le a \le f'(0) < b$, $\eta > 0$ and an interval $[c_1, c_2]$ containing $\bar{c}$ such that all conditions of the precedent claim are satisfied.

*Step 3.* Given $c > 2\sqrt{f'(0)}$, there exists $\eta > 0$ such that $\psi'(u) = 2c\sqrt{\psi(u)} - 2f(u)$, $\psi(0) = 0$, has a unique solution $\psi$ in $[0,\eta]$ such that $(\sqrt{\psi})'(0) = \lambda^+(c)$.

*Step 4.* Let $c_0 > 2\sqrt{f'(0)}$, $c_0 \ge c^*(f)$ and $\psi$ be the solution of $\psi'(u) = 2c_0\sqrt{\psi(u)} - 2f(u)$, $\psi(0) = 0$, $\psi(1) = 0$, $\sqrt{\psi})'(0) = \lambda^-(c)$. Then $c_0 > c^*(f)$.

In terms of solutions of the second order equation this reads:

2.C *Let c be an admissible speed of $f$ and $u(t)$ a corresponding monotone heteroclinic solution.*

- *If $c = c^*$,*
$$\lim_{t\to-\infty} \frac{u'(t)}{u(t)} = \lambda^+(c).$$
- *If $c > c^*$,*
$$\lim_{t\to-\infty} \frac{u'(t)}{u(t)} = \lambda^-(c).$$

Asymptotic description of solutions near $+\infty$ can also be given. That discussion is simpler.

## 3 Finding some exact solutions

The form of (1) allows to obtain easily some exact heteroclinics.

Given a reaction term $f(u) = u^m - u^n$, $0 \le u \le 1$, where $1 \le m < n$, let us look for a solution of (1) of the form

$$y(u) = \lambda(u^\alpha - u^\beta)^2.$$

An easy computation shows:

- If $m = 1$ and $n = 2$, that is, for the simplest prototype, we obtain a solution with $\alpha = 1$, $\beta = 3/2$, $\lambda = 2/3$ and $c = 5/\sqrt{6}$. The profile obtained is
$$y(u) = \frac{2}{3}u^2(1-\sqrt{u})^2, \quad \text{with } c = \frac{5}{\sqrt{6}}$$
associated to a non-critical speed. From the equation $u' = \sqrt{y(u)}$ we obtain the expression of the heteroclinic.
$$u(t) = \frac{1}{((\sqrt{2}-1)e^{-\frac{t}{\sqrt{6}}} + 1)^2}$$
This solution was given by Ablowitz and Zeppetella [1].

- This example generalizes to: if $m = 1$ and $n > 1$, we find $\alpha = 1$, $\beta = (n+1)/2$, $\lambda = 2/(n+1)$ and $c = (n+3)/\sqrt{2(n+1)}$. The profile is

$$\psi(u) = \frac{2}{n+1}u^2\left(1 - u^{\frac{n-1}{2}}\right)^2$$

and the corresponding heteroclinic is

$$u(t) = \frac{1}{\left((2^{\frac{n-1}{2}} - 1)e^{-\frac{(n-1)t}{\sqrt{2(n+1)}}} + 1\right)^{\frac{2}{n-1}}}.$$

The critical speed is 2, for all $n$, and

$$c = c_n = \frac{n+3}{\sqrt{2(n+1)}} \to 2$$

as $n \to 1$.

- If $m = 2$ and $n = 3$, (Zeldovich's equation), the calculation shows that we can take $\alpha = 1$, $\beta = 2$, $\lambda = 1/2$ and $c = 1/\sqrt{2}$.

The profile thus obtained is

$$y(u) = \frac{1}{2}u^2(1-u)^2, \quad \text{with } c = \frac{1}{\sqrt{2}}.$$

In this case $f'(0) = 0$ and

$$\lim_{u\to 0}\frac{y(u)}{u^2} = \frac{1}{2}.$$

By 2.B. we conclude that $1/\sqrt{2}$ is the critical speed for Zeldovich. Solving $u' = y(u)$ we obtain the corresponding heteroclinic

$$u(t) = \frac{1}{1 + e^{-\frac{t}{\sqrt{2}}}}.$$

- More generally, if $m = (n+1)/2$ and $n > 1$ we obtain: $\alpha = 1$, $\beta = m$, $\lambda = 1/m$ and $c = 1/\sqrt{m}$.

The profile is

$$y(u) = \frac{1}{m}u^2\left(1 - u^{\frac{n-1}{2}}\right)^2, \quad \text{with } c = \frac{1}{\sqrt{m}}$$

and the speed is critical. (See [9].)

## 4 Sharp solutions

The more general equation with density dependent diffusion

$$u_t = (D(u)u_x)_x + g(u)$$

with $D > 0$ in $(0,1)$ has a corresponding ODE for travelling waves

$$(D(u)u')' - cu' + g(u) = 0. \qquad (d)$$

The substitution $\varphi(u) := D(u)u'$ leads to the first order problem

$$\psi'(u) = 2c\sqrt{\psi(u)} - 2f(u), \quad \psi(0) = \psi(1) = 0$$

where

$$\psi(u) := \varphi(u)^2, \quad f(u) = D(u)g(u)$$

and the heteroclinic is found via

$$u' = \frac{\sqrt{\psi(u)}}{D(u)}, \qquad u(0) = 1/2.$$

Then from the results in section 3 we find *sharp solutions*.

4.A.—*Assume that: $D \in C^1[0,1]$, $D > 0$ in $(0,1]$, $D(0) = 0$ and $D'(0) > 0$. Then, for a function $g$ of type A:*

(i) *There exists a stricly increasing solution of (d) with $u(-\infty) = 0$, $u(+\infty) = 1$, if and only if $c > c^*$;*

(ii) *If $c = c^*$, (d) has an increasing solution defined in $[0, \infty[$ with $u(0) = 0$, $u(\infty) = 1$ and $u'(0) = c^*/D'(0)$.*

*Those solutions are unique up to translation;*

(iii) *If $c < c^*$, the equation (d) has no increasing solution in any interval $(b, \infty)$ with*

$$\lim_{t\to b^-} u(t) = 0, \quad \lim_{t\to -\infty} u(t) = 1.$$

*See* [19, 17].

## 5 Further characterizations of the critical speed

We have been considering functions $f$ of *type A in* $[0,1]$. And, if there exists $\theta \in (0,1)$ such that $f \equiv 0$ in $[0,\theta]$ and $f$ is of type A in $[\theta, 1]$, then we say that $f$ *is of type B in* $[0,1]$.

In such cases there exists a unique "admissible speed" (see e.g. [7]):

5.A.—*Let $f$ be of type B in $[0,1]$. Then there exists a number $c^* > 0$ so that problem (1st order with zero boundary conditions) admits a positive solution in $(0,1)$ if and only if $c = c^*$.*

Now the symbol $c^*(f)$ is meaningful if $f$ is of type A or B, and the construction of $c^*$ preserves monotonicity with respect to $f$.

5.B.—*Let $f$ be of type A in $[0,1]$. Let $f_1 \le f_2 \le \cdots$ be a sequence of functions of type B in $[0,1]$ such that*

$\lim_{n\to\infty} f_n(x) = f(x)$ *for every* $x \in [0,1]$. *Then* $c^*(f_n)\uparrow c^*(f)$.

Variational characterizations of the critical speed are also possible. In addition to the one given by Benguria and Depassier [6], the following was given (in a slightly different form) in [2].

5.C.—*Let* $F(u) = \int_0^u f(s)\,ds$ *(where* $f$ *is defined outside* $[0,1]$ *with value* 0*) and set*

$$X = \left\{ v \in C(\mathbb{R}_+) \;\middle|\; v(0) = 0 \text{ and } \int_0^\infty v'^2 < \infty \right\}.$$

*Then*

$$c^* = \sqrt{\frac{1}{\lambda}}, \quad \lambda = \inf_{v\in X,\, \int_0^\infty \frac{F(v(s))}{s^2}\,ds = 1} \int_0^\infty \frac{v'(s)^2}{2}\,ds$$

*and the* inf *is attained if* $c^* > 2\sqrt{f'(0)}$.

*Outline of proof in the simple case where* $f'(0) = 0$: Consider a minimizing sequence $v_n$, that is

$$\lim_{n\to\infty} \int_0^\infty \frac{v_n'(s)^2}{2}\,ds = \lambda, \qquad \int_0^\infty \frac{F(v_n(s))}{s^2}\,ds = 1.$$

By Hardy's inequality there is $C \in \mathbb{R}$ so that

$$\int_0^\infty \frac{(v_n(s))^2}{s^2}\,ds < C.$$

It is clear that one may suppose $0 \le v_n \le 1$. Since $f'(0) = 0$, given $\varepsilon > 0$, exists $\delta > 0$ such that $F(z) \le \varepsilon z^2$ if $0 \le z \le \delta$. Also, there exists $\eta > 0$ such that $0 \le t \le \eta \Rightarrow v_n(t) \le C_1\sqrt{\eta} = \delta$ for every $n$. Hence, for large $n$,

$$\int_0^\eta \frac{F(v_n(s))}{s^2}\,ds \le \varepsilon C.$$

The *tail*

$$\int_A^\infty \frac{F(v_n(s))}{s^2}\,ds$$

for large $A$ is clearly uniformly small. From $v_n \to v$ uniformly in compact intervals, $v_n' \to v'$ weakly in $L^2(0,\infty)$ we obtain by a standard argument that $v$ is a minimizer. Moreover, exploiting the homogenity of the constrained problem (induced by changes of variable $s = kt$, $k > 0$), we see that $v$ minimizes the map

$$w \mapsto \int_0^\infty \frac{(w'(s))^2}{2}\,ds - \lambda \int_0^\infty \frac{F(w(s))}{s^2}\,ds.$$

The minimizer solves

$$v''(s) + \lambda\frac{f(v(s))}{s^2} = 0, \quad s > 0.$$

The change of variable $s = e^t$ yields

$$u''(t) - u'(t) + \lambda f(u(t)) = 0, \quad -\infty < t < +\infty$$

and considering the behaviour (and integrability property) of the solution as it approaches 0, we conclude that $\lambda = 1/c^*(f)^2$.

## 6 The case of nonlinear diffusion: basic properties

Consider the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u)\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right] + g(u), \qquad (9)$$

where $p > 1$, $g$ is of type A, $u = 0$ and $u = 1$ being equilibrium solutions. We look for travelling wave solutions $u(t,x) = U(x - ct)$ for some $c > 0$ where $U$ is monotone and connects the equilibria. To facilitate the checking of details to the interested reader we now assume, as in [11], that $U$ is decreasing and $U(-\infty) = 1$, $U(+\infty) = 0$ (of course, a sign change allows to reduce to the case where $U$ is increasing as previously). The problem is therefore

$$(D(u)|u'|^{p-2}u')' + cu' + g(u) = 0 \qquad (10)$$

with limit conditions

$$u(-\infty) = 1,\; u(+\infty) = 0. \qquad (11)$$

Looking for solutions with $u' < 0$ in their whole domain, we set

$$-v := D(u)|u'|^{p-2}u',$$

for such solutions, then $v$ may be seen as a function of $u$. If we define

$$y(u) = v(u)^q$$

the function $y$ will solve (2) with $(1/p)+(1/q) = 1$ and $y(0) = 0 = y(1)$, provided that we set

$$f(u) = D(u)^{q-1}g(u).$$

Then, as in the case $p = 2$, we obtain results on the admissible speeds and critical speed.

6.A.—*Assume that* $f$ *is a function of type A in* $[0,1]$ *satisfying*

$$\sup_{u\in(0,1)} \frac{f(u)}{u^{q-1}} = \mu < +\infty,$$

*or the stronger property*

$$\lim_{u\to 0^+} \frac{f(u)}{u^{q-1}} = \lambda < +\infty.$$

*Then there exists a constant* $c^* > 0$ *(depending on* $f$ *and* $p$*) such that*

$$y'(u) = q(c\, y_+(u)^{\frac{1}{p}} - f(u)), \quad y(0) = y(1) = 0$$

*for* $0 \le u \le 1$, *admits a unique positive solution if and only if* $c \ge c^*$. *Moreover we have the estimate* $(\lambda q)^{1/q} p^{1/p} \le c^* \le q^{1/q} p^{1/p} \mu^{1/q}$.

*If in addition* $\mu = \lambda$, *then* $c^* = q^{1/q} p^{1/p} \lambda^{1/q}$.

As in the case $p = 2$, the behaviour of solutions at the origin is related to the corresponding value of $c$:

6.B.—*If* $c > c^*$, *then* $y$ *satisfies*

$$\lim_{u\to 0} \frac{y(u)}{u^q} = \omega_c^-(\lambda).$$

*And, if* $c = c^*$, *then* $y$ *satisfies*

$$\lim_{u\to 0} \frac{y(u)}{u^q} = \omega_c^+(\lambda).$$

Here $\omega_c^-(\lambda) \le \omega_c^+(\lambda)$ stand for the positive roots of the function $x \mapsto x - cx^{1/p} + \lambda$.

Going back to the 2nd order problem, we can state:

6.C.—

(A) *Let* $1 < p \le 2$. *If* $g$ *is a function of type A and* $D \in C^1[0,1]$ *with* $D > 0$ *in* $[0,1]$, *and*

$$\sup_{u\in(0,1)} \frac{g(u)}{u^{q-1}} < +\infty, \qquad \sup_{u\in(0,1)} \frac{g(u)}{(1-u)^{p-1}} < +\infty,$$

*then there exists* $c^*$ *such that*

$$\begin{aligned} &(D(u)|u'|^{p-2}u')' + cu' + g(u) = 0,\\ &u(-\infty) = 1,\\ &u(+\infty) = 0 \end{aligned}$$

*has a decreasing solution* $u(t)$ *taking values in* $(0,1)$ *if and only if* $c \ge c^*$. *That solution is unique up to translation.*

*Here the number* $c^*$ *is associated to* $f = D^{q-1} g$ *according to the theory for the first order equation.*

(B) *If, further,* $g^*(0) \equiv \lim_{u\to 0^+} g(u)/u^{q-1}$ *exists, then*

$$\lim_{t\to+\infty} \frac{u'(t)}{u(t)^{q-1}} = \begin{cases} -\frac{\omega_c^-(D(0)^{q-1}g^*(0))^{1/p}}{D(0)^{q-1}}, & c > c^*\\ -\frac{\omega_{c^*}^+(D(0)^{q-1}g^*(0))^{1/p}}{D(0)^{q-1}}, & c = c^*. \end{cases}$$

Sharp solutions can also be found in this case, cf. [11].

Also, a variational definition of $c^*$ is possible in the nonlinear diffusion setting. Consider the critical speed $c^*$ for

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x}\right] + f(u).$$

where $p > 1$ and $f$ is type A. Then we have (see [14]).

6.D.—*Let* $F$ *be the primitive of* $f$ *with* $F(0) = 0$; $\mathcal{F} = \{v \mid v \text{ is defined in } [0,\infty[,\ v(0) = 0\}$ *so that we can define*

$$\gamma = \inf_{v\in\mathcal{F}\setminus 0} \frac{\frac{1}{q}\int_0^{+\infty} |v'(s)|^q\, ds}{\int_0^{+\infty} \frac{F(v(s))}{s^q}\, ds}.$$

*Then the number* $c^*$ *is given by*

$$\gamma = \frac{q}{pc^{*q}}.$$

*Moreover* $\gamma$ *is attained if* $\mu p^q \gamma < 1$ *where*

$$\mu := \lim_{u\to 0^+} \frac{f(u)}{u^{q-1}}.$$

## 7 Examples: computation of $c^*$

As a first example consider again the ODE for the $p$-Laplacian led diffusion

$$(|u'|^{p-2}u')' + cu' + f(u) = 0$$

where $f(u) = u^q(1-u)^{q-1}$ (that is, the analogue of Zeldovich's equation).

We compute an exact solution, for the corresponding 1st order equation, of the form

$$y = \alpha u^q (1-u)^q,$$

with $\alpha = 1/2$ and $c = 2^{-1/q}$.

Since $\lim_{u\to 0} \frac{f(u)}{u^{q-1}} = 0$ and $\lim_{u\to 0} \frac{y(u)}{u^q} = \frac{1}{2}$, we conclude from previous information on asymptotics that in fact $c^* = 2^{-1/q}$.

Next, we give a second example in the presence of advection: consider

$$\frac{\partial u}{\partial t} + ku\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x}\right] + f(u)$$

where $k > 0$.

The 2nd order ODE for travelling waves now is

$$(|u'|^{p-2}u')' + (c - ku)u' + f(u) = 0$$

and it may be studied by reduction to the first order equation (the boundary conditions are as before)

$$y' = q((c - ku)\, y^{1/p} - f(u)).$$

Similar arguments enable us to see that there exists a critical speed $c^*$ and the corresponding solution may be identified by its behaviour near the origin. Let

$$f(u) = u^{q-1}(1-u)^{q-1}$$

(see [18] for $p = 2$). The 1st order equation has a solution of the form $y = \alpha u^q (1-u)^q$: substitution into the equation shows that in fact this is the case, provided that

$$\alpha = \left(\frac{k}{2}\right)^q \quad \text{and} \quad c = \frac{k}{2} + \left(\frac{2}{k}\right)^{q-1}.$$

Minimizing $c$ we obtain

$$k_0 = 2(q-1)^{1/q} \quad \text{and} \quad c_0 = q^{1/q} p^{1/p}.$$

Computing $y(u)/u^q$, $f(u)/u^{q-1}$ at $u = 0$ and comparing with $w_+$, that turns out to be $w_+ = (k/2)^q$, we conclude that if $k \geq k_0$ then the critical speed is $c^* = k/2 + (2/k)^{q-1}$.

A different argument, based on the monotonicity of $c$ with respect to $k$ and the known result corresponding to $k = 0$ leads to (see [8])

$$c^* = q^{1/q} p^{1/p} \text{ for } 0 \leq k \leq k_0.$$

## 8 Final note

Recently, Audrito and Vázquez [4] have considered the model with doubly nonlinear diffusion

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\left|\frac{\partial u^m}{\partial x}\right|^{p-2}\frac{\partial u^m}{\partial x}\right] + g(u)$$

and its $N$-dimensional analogue. Among other results, they found that there is a *critical speed* $c^* > 0$ so that:

*If $m > 0$, $p > 1$ and $m(p-1) > 1$ then there are travelling waves for speeds $c \geq c^*$, the profiles of the waves being positive everywhere if $c > c^*$ and* finite *if $c = c^*$.* Here finite means that the profile vanishes in a half-line.

The critical speed $c^*$ has a threshold role with respect to propagation of disturbances with bounded support, as in the case of linear diffusion.

Further new interesting results are also found in Drábek and Takáč [10] who consider degeneracy and singular diffusion coefficients.

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