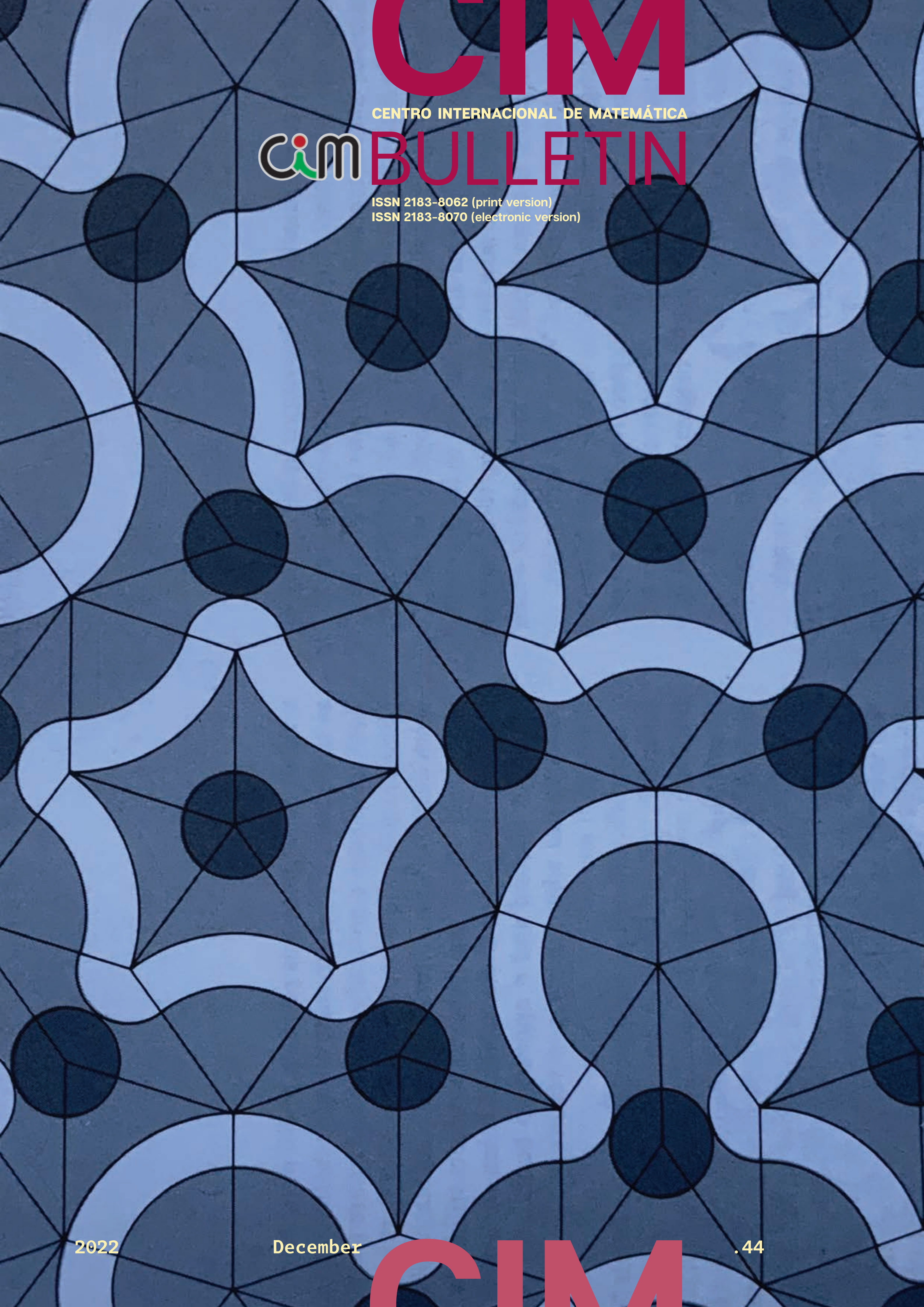

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Contents

Editorial

In this issue of the bulletin, we feature an interview honouring Luís Sanchez, who belonged to the Scientific Council of CIM and dedicated a considerable part of his life to the advancement and dissemination of Mathematics. As an indisputable testimony of this dedication, he authored an article, published in this issue, regarding the speed of travelling waves for Fisher-Kolmogorov-Petrovski-Piskounov equations in the framework of ordinary differential equations.

We include an article about some surprising properties of conditioned simple random walks in two dimensions, such as observing that if we performed a random walk of one meter steps on an imaginary galactic plane with at least the size of our galaxy, then, on average, we would revisit the origin around 30 times before leaving the Milky Way.

We present an article about generic behaviours of conservative dynamical systems, which includes a comprehensive and historical introduction to the theory of dynamical systems itself to motivate the study of generic properties.

We feature an interview with Sylvia Serfaty, who was the distinguished mathematician invited to deliver this year's Pedro Nunes' Lecture, which is one of the most emblematic joint initiatives of CIM, in collaboration with SPM.

As usual, we publish several summaries and reports regarding the activities partially supported by CIM during the last year.

We recall that the bulletin welcomes the submission of review, feature, outreach and research articles in Mathematics and its applications.

Ana Cristina Moreira Freitas

Faculdade de Economia and
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https://www.fep.up.pt/docentes/amoreira/

AN INTERVIEW WITH

Luís Sanchez

by **Maria do Rosário Grossinho***

Having lived in Moura, Alentejo, his first 18 years, Luís Fernando Sanchez Rodrigues entered the Faculty of Sciences of the University of Lisbon in 1966 to study Mathematics in the variant then oriented towards Pure Mathematics. During his pre-university education, which he completed at the Liceu de Beja, his mathematics teacher, of whom he speaks with high regard, gratitude and tenderness, was determinant in his decision to study Mathematics.

Since the beginning of his studies in Lisbon, at the Faculty of Sciences, his commitment and dedication to Mathematics became noticed. After graduation, Luís Sanchez joined the staff of the Department of Mathematics and became full professor in 1990. In his academic activity, where naturally teaching and research have had a prevalent role, he values in a special way his role as coordinator of CMAF (2004-2015) and CMAFcIO (2015-2017). After a life dedicated to the Academia, his jubilation took place in 2018, which nevertheless did not put an end to his connection to the University.

But let us hear in first-hand what Luís Sanchez can tell us about his life journey.

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How about starting at the end: what is your general impression of your academic career? Which points would you underline as satisfying experiences?

It is a good idea to start at the end, since at ages like mine what is recent fades from memory more quickly. I see my professional path as a commitment to pass on a taste for the subjects I have taught, and to carry out research, whenever possible on a collaborative basis, either with more experienced colleagues or with younger people. I believe that the circumstances of my time provided me with a smooth career. I would highlight (1) the guidance of some master and doctoral theses (the starting point was with you), (2) the years 2004-2017 when I coordinated CMAF (and then CMAFcIO) and (3) the period until 2007 when I was part of the editorial board of *Portugaliae Mathematica*, in the recovery phase of the journal after a problematic phase — perhaps these are my most relevant contributions for the mathematical and academic community. I would like to express my gratitude to all the colleagues and collaborators with whom I have interacted in the various tasks of the profession: I consider myself a lucky man, since they were people with intellectual and human qualities that made collaboration very pleasant. Let us say that I have tried to teach well but I have also learned a lot from others.

You belonged to the Scientific Council of CIM until 2002. Since then, quite a few years have passed. How do you see the changes in Mathematics in Portugal over all these years?

Many years have passed, indeed. If I recall here the great investment in Science in Portugal since the mid 1990s, by the consistency of policies and increased funding, I will be going along with the common place. It should be underlined that not only were the new opportunities welcomed enthusiastically, but autonomous proposals emerging from the base proved to be fruitful. The creation of the CIM was an example, whose output I could appreciate through the initiatives involving several research centres. But if I remember that time, or if I go back even further — for example, to the years when I started preparing my PhD — what is most fascinating is to contemplate the evolution of mathematical production until the present, in which we have teams that consistently produce work that competes at international level in diverse areas of Mathematics.

It is good to hear descriptions like the ones you give in an optimistic tone, but isn't there another side of the coin? Is everything always so uplifting?

No, in fact not everything can be seen from an optimistic perspective. If we want to look at the problems, one of the most obvious in the current situation is that many talented researchers have no secure career prospects ahead of them, either due to the ups and downs of ministerial policies, or due to constraints in options of the institutions that host them, or even both causes together. Even more worrying, concerning our country, is the fact that the business world does not show much interest in hiring PhDs. In another area, I will also mention, thinking of the moments in my professional experience which have generated the most stress, that the coordination of a centre has frequent accidents along the way: dealing with a large group of contradictory wills requires patience and the capacity to promote dialogue (which sometimes becomes exhausting). Also, the effort to maintain quality, especially in view of the complexity of the periodic evaluation process, can only result from intense work in which everyone's participation must be encouraged. And let me not forget the discomfort and pain caused by increasing bureaucracy that invades the university life.

Can we say that in some way you have stimulated a school in the area of ODEs ... ?

It can be said that, in fact, I had the opportunity to pass on the taste for non-linear analysis, with some emphasis on applications in the area of ODEs, to a set of students who became researchers with recognized work and who built the core of an important team at CMAF. One of the highlights of the team's visibility was our participation in a *Human capital and mobility* project led by Prof Jean Mawhin, 1994–97. But the legacy of the scientific contributions of the group members continued to make its way. With one, irreparable, loss out of time: Miguel Ramos.

Going further back: did you always know from your first studies that you wanted to pursue a career in Mathematics?

No. I remember that towards the end of secondary school, when the service teacher gave a test on imaginary numbers, I got a *no*. I was not able to grasp anything from his lessons. A little miracle occurred when a new teacher came to replace him: she taught like someone who shines a light on words. I started getting whole tests right and then decided I wanted to study more maths. The lady's name is Maria Teresa Caldeira de Sousa (she passed away in 2018). I thus appreciated, from my own experience, that the role of the teacher in successful teaching cannot be overemphasised.

You are obviously very grateful to your teacher. Are there any other names that have weighed on your path that you would like to mention?

It would be impossible to mention all those who influenced me. The truth is that when I studied at FCUL, although the situation was incomparable to the present one, I had excellent teachers: I'll mention Santos Guerreiro, who made me discover the taste for Analysis, and Sebastião e Silva, whose last course I attended at FCUL. In the continuation of my path at FCUL, the influence of Prof. João Paulo Dias was determinant. He directed my PhD thesis and remained an invaluable advisor in many occasions, and particularly in the directive board of CMAF for several years. I should also mention Prof. Alain Haraux, whose suggestions were decisive in the choice of some themes in the initial phase of my research work.

At a certain period you also had a role in the discussions

about syllabus and teaching of mathematics at pre-university level ... what do you remember of that contribution?

The pleasant memory in that regard is my participation in REANIMAT, a program sponsored by Fundação Calouste Gulbenkian. Together with other colleagues we tried to produce a model of what a textbook should be like, and how to implement it in practical classes, without missing the official syllabus guidelines (about which we had very critical views). We aimed at coherently structured texts, conveying mathematical insight and, why not say it, the esthetical flavour of mathematics. I believe that the materials we produced have been useful to many schoolteachers: there have been requests to access those texts until the last few months. Before that, I had participated in discussions about the syllabus when changes were introduced by 1996. That was, however, a less encouraging experience.

So, as not to end on a note of regret, can you tell me the most rewarding experience you remember from your teaching activity? And in your research activity?

I have always enjoyed teaching, so it would be difficult to point out a favourite subject. If I have to, however, I will say that the course that gave me the most pleasure to organize was the course on non-linear analysis that ran when the master's degree in Mathematics was launched at FCUL. But I also taught with great pleasure, for several years, the courses of multivariable Analysis for students of Mathematics and Physics. The reception used to be always very good, which put my mind at ease about the transmission of the message.

Speaking of research, I tackled several problems in my area of interest and I have very good memories of the interaction and collaboration with colleagues from various European centres, mainly Spain, Belgium and Italy. The teamwork and the conviviality implied in it have only left me with positive impressions. In retrospect, my appreciation of my production leads me to distinguish two series of works: the study of certain 4th order models, with emphasis on heteroclinics and positivity (2003-2007) and the study of travelling waves and critical velocities for FKPP models (2004–2015). I also like to recall that the work on heteroclinics in 4th order equations arose from conversations with Henrique Leitão, then a PhD student in Physics, in the favourable environment for collaboration between physicists and mathematicians that existed in the Interdisciplinary Complex of the University of Lisbon.

You still maintain activity in the Faculty of Sciences and

in the CMAFcIO. Is life difficult without the university?

In fact, I have been supporting some optional courses in Analysis at FCUL. This has been a gratifying task because the students who enrol in those courses are among the best in their class. I also keep reading papers and thinking about problems in math, but I maintain other centres of interest. Several people had warned me about the following fact and now I know that they are right: after retirement there is not enough time to do everything we would like to. The world is too interesting, for good and for bad reasons, to allow us spare time. On the other hand, life after retirement goes on in a natural way. It has difficult moments in store for us. For some years now, I have found out that plunging into a mathematical problem, big or small, has a soothing effect regarding troubles arising in the *real world*. Be it common worries in private life or discouragement in the presence of worrying trends in our societies, when common sense seems to have become a rare commodity, a little bit of a mathematical puzzle makes it easier to withstand the discomfort.

I know that you appreciate fiction novels and many kinds of music. Do you want to share some special items where in your opinion time is well spent?

There are so many that I am unable to make a choice. I can point to two or three names whose work has been a source of pleasure for me in recent years: Balzac, Stefan Zweig and Stephen Sondheim. But let me also mention the less static joys of walking and driving, and discovering the world that has been waiting to be enjoyed: Alentejo, Extremadura, Andalucia …

Epilogue.—The pleasant conversation went on. The excerpt registered above is a contribution to hear in the first person someone who gave, and still gives, a lot to the University, with discretion, and with the great value of dedication to Mathematics and to his students and collaborators.

As an epilogue to this interview, I would like to testify to the fruitful and friendly research environment that has always characterized the Luís Sanchez's research group, from which many people have benefited.

Having been his first PhD student, but certainly conveying the feelings of the other students and collaborators in whose path Luís Sanchez's work was important, I want to register the honour and privilege of having had him as a professor.

With all due respect, gratitude, and tenderness, from myself, as well as from your other *scientific descendants*, THANK YOU, Luís!

Short bio

Born in Lisbon, 1948. Graduated in Mathematics, University of Lisbon 1971; Ph. D., University of Lisbon, 1981 (thesis supervisor: João Paulo Dias). Professor of Mathematics at the Faculty of Sciences, University of Lisbon (1990), retired in November 2018. Member of the Editorial Board of Portugaliae Mathematica (1981–2007). Chairman of the Department of Mathematics of the University of Lisbon in 1990/91 and 1995/96. Member of the Scientific Committee of CIM (Centro Internacional de Matemática) up to 2002. Member of the Comisión Asesora Externa of IMAT (Instituto de Matemática, Univ. Santiago de Compostela), in 2018. Coordinator of Centro de Matemática e Aplicações Fundamentais (2004-2015) and Centro de Matemática, Aplicações Fundamentais e Investigação Operacional (2015-2017). Coordinator (with A. Machado) of the educational project REANIMAT sponsored by Fundação Calouste Gulbenkian — a three-year experiment in the teaching of Mathematics at high school level, 2001/2004, a contract involving FCG and FCUL.

Research interests

Boundary value problems for nonlinear ordinary differential equations; nonlinear functional analysis.

Ph. D. students

Maria do Rosário Grossinho (1988), To Fu Ma (1996), José Maria Gomes (2005), Ricardo Enguiça (2010). Also supervised master's theses of Miguel Ramos and Carlota Rebelo.

Authorship

Author or co-author of about 70 research papers. Reviewer of Mathematical Reviews and Zentralblatt für Mathematik.

Some surprising properties of a conditioned simple random walk in two dimensions

by **Serguei Popov***

Abstract.—We define the two-dimensional conditioned simple random walk as the Doob's h-transform of the simple random walk with respect to its potential kernel, and discuss some of its properties.
This is a very brief exposition of some of the topics presented in [12].

1 Introduction: simple random walks on integer lattices

This note is about the simple random walk[1] $(S_n, n \geq 0)$ on the integer lattice $\mathbb{Z}^d$ and we will pay special attention to the case $d = 2$. SRW is a discrete-time stochastic process defined as follows: if at a given time the walker is at $x \in \mathbb{Z}^d$, then at the next time moment it will be at one of x's $2d$ neighbours chosen uniformly at random, as shown on Figure 1. In other words, the probability that the walk follows a fixed length-n path of nearest-neighbour sites equals $(2d)^{-n}$. As a general fact, a random walk may be recurrent (i.e., almost surely it returns infinitely many times to its starting location) or transient (i.e., with positive probability it never returns to its starting location). An important result about SRWs on integer lattices is Pólya's classical theorem:

Theorem 1 ([10]).— Simple random walk in dimension d is recurrent for $d = 1, 2$ and transient for $d \geq 3$.

A well-known interpretation of this fact, attributed to Shizuo Kakutani, is: "a drunken man always returns home, but a drunken bird will eventually be lost". Still, despite recurrence, the drunken man's life is not so easy either: as we will see, it may take him *quite* some time to return home.

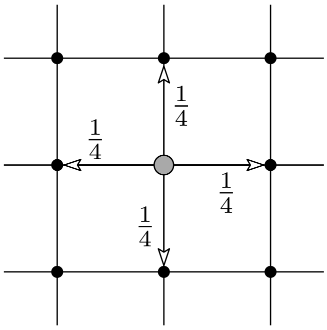

Figure 1.— Simple random walk in two dimensions.

Indeed, it is possible to obtain (see (10) and (6) below) that the probability that two-dimensional SRW gets more than distance n away from its starting position before revisiting it is approximately $(1.02937 + \frac{2}{\pi} \ln n)^{-1}$. While this probability does converge to zero as $n \to \infty$, it is important to notice how slow this convergence is. Here is a concrete example. Imagine a (two-dimensional) SRW taking place on the *galactic plane* of our galaxy, with the size of the walker's step being equal to 1m. What is the probability of reaching the galaxy's boundary before returning to the initial location? Since the walk is recurrent and the galaxy is enormous, one would expect this probability to be *extremely* small, correct? Now, let us do the calculations. The radius of the Milky Way galaxy is around 10^{21}m, and $(1.02937 + \frac{2}{\pi} \ln 10^{21})^{-1} \approx 0.031$, which is *surprisingly* large. Indeed, this means that the walker

[1] Also abbreviated as SRW.

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would revisit the origin only around 30 times on average, before leaving the galaxy; this is not something one would normally expect from a *recurrent* process.

Incidentally, these sorts of facts explain why it is difficult to verify conjectures about two-dimensional SRW using computer simulations. (For example, imagine that one needs to estimate via simulations how long we will wait until the walk returns to the origin, say, a hundred times.)

As we will see shortly, the recurrence of d-dimensional SRW is related to the divergence of the series $\sum_{n=1}^{\infty} n^{-d/2}$. Notice that this series diverges if and only if $d \leq 2$, and for $d = 2$ it is the harmonic series that diverges quite slowly. This might explain why the two-dimensional case is, in some sense, *really* critical. It is always interesting to study critical cases – they frequently exhibit behaviours not observable away from criticality.

It is not our intention to present the proof of Theorem 1 in this note (one can find a modern proof of that result e.g. in [7]), but let us at least give a heuristic explanation of why it should be true. First, let us show that the number of visits to the origin is a.s. finite if and only if the *expected* number of visits to the origin is finite (note that this is something which is *not* true for general random variables). This is a useful fact, because, as it frequently happens, it is easier to control the expectation than the random variable itself.

Let $p_m(x, y) = \mathbb{P}_x[S_m = y]$ be the transition probability from x to y in m steps for the simple random walk in d dimensions. Let q_d be the probability that, starting at the origin, the walk eventually returns to the origin. If $q_d < 1$, then the total number of visits (counting the initial instance $S_0 = 0$ as a visit) is a geometric random variable with success probability $1 - q_d$, which has expectation $(1 - q_d)^{-1} < \infty$. If $q_d = 1$, then, clearly, the walk visits the origin infinitely many times a.s.. So, random walk is transient (i.e., $q_d < 1$) if and only if the *expected* number of visits to the origin is finite. This expected number equals[2]

$$\mathbb{E}_0 \sum_{k=0}^{\infty} \mathbf{1}\{S_k = 0\} = \sum_{k=0}^{\infty} \mathbb{E}_0 \mathbf{1}\{S_k = 0\} = \sum_{n=0}^{\infty} p_{2n}(0, 0)$$

(observe that the walk can be at the starting point only after an even number of steps). We thus obtain that the recurrence of the walk is *equivalent* to

$$\sum_{n=0}^{\infty} p_{2n}(0, 0) = \infty. \tag{1}$$

So, let us try to *understand* why Theorem 1 should hold. One can represent the d-dimensional simple random walk S as

$$S_n = X_1 + \cdots + X_n,$$

where $(X_k, k \geq 1)$ are i.i.d. random vectors, uniformly distributed on the set $\{\pm e_j, j = 1, \ldots, d\}$, where $e_1, \ldots, e_d$ is the canonical basis of $\mathbb{R}^d$. Since these random vectors are centered (expectation is equal to 0, component-wise), one can apply the (multivariate) Central Limit Theorem (CLT) to obtain that $S_n/\sqrt{n}$ converges in distribution to a (multivariate) centered Normal random vector with a diagonal covariance matrix. That is, it is reasonable to expect that S_n should be at distance of order $\sqrt{n}$ from the origin.

So, what about $p_{2n}(0, 0)$? If $x, y \in \mathbb{Z}^d$ are two *even* sites[3] at distance of order at most $\sqrt{n}$ from the origin, then our CLT intuition tell us that $p_{2n}(0, x)$ and $p_{2n}(0, y)$ should be *comparable*, i.e., their ratio should be bounded away from 0 and ∞. In fact, this statement can be made rigorous by using the *local* Central Limit Theorem (e.g., Theorem 2.1.1 of [7]). Now, if there are $O(n^{d/2})$ sites where $p_{2n}(0, \cdot)$ are comparable, then the value of these probabilities (including $p_{2n}(0, 0)$) should be of order $n^{-d/2}$. It remains only to observe that the series $\sum_{n=1}^{\infty} n^{-d/2}$ diverges only for $d = 1$ and 2 to convince oneself that Pólya's theorem indeed holds.

1.1 Potential kernel

Before starting the discussion on conditioned random walks, we need some technical preparations. Let us denote by

$$\tau_A = \min\{n \geq 0 : S_n \in A\}, \tag{2}$$

and

$$\tau_A^+ = \min\{n \geq 1 : S_n \in A\} \tag{3}$$

the entrance and the hitting times of a set A. Let $\partial A = \{x \in A : \exists y \in A^{\complement} \text{ such that } x \sim y\}$ be the boundary of $A \subset \mathbb{Z}^2$, and $\partial_e A = \partial A^{\complement}$ be its external boundary. Denote by $\mathsf{B}(x, r) = \{y : \|y - x\| \leq r\} \subset \mathbb{Z}^2$; $\mathsf{B}(r)$

[2] We can put the expectation inside the sum because of the Monotone Convergence Theorem.

[3] A site is called even if the sum of its coordinates is even; observe that the origin is even.

stands for $\mathsf{B}(0,r)$.

For transient random walks, a very important object is the Green's function, defined by $G(x,y) = \sum_{m=0}^{\infty} p_m(x,y)$, so that $G(x,y)$ is the expected number of visits to y starting from x. Its usefulness stems from the fact that $G(x, S_{n\wedge\tau_x})$ is a martingale, and martingales are really effective (see [9] for some interesting examples). However, for recurrent random walks that definition does not work since, as we know, the mean visit count equals infinity in that case. Fortunately, there is a way to amend that, essentially by considering the *difference* between mean visit counts starting from two different sites (of course, defining it properly). So, in two dimensions, let us define the *potential kernel* $a(\cdot)$ by

$$a(x) = \sum_{k=0}^{\infty} \big(\mathbb{P}_0[S_k = 0] - \mathbb{P}_x[S_k = 0]\big), \quad (4)$$

where $x \in \mathbb{Z}^2$. By definition, it holds that $a(0) = 0$, and one can show that the above series converges and that the resulting value is strictly positive for all $x \neq 0$ (here and in the sequel we refer to Section 4.4 of [7]). Also, the function a is harmonic outside the origin, i.e.,

$$a(x) = \frac{1}{4}\sum_{y\sim x} a(y) \quad \text{for all } x \neq 0. \quad (5)$$

It is possible to prove that, as $x \to \infty$,

$$a(x) = \frac{2}{\pi}\ln\|x\| + \gamma' + O(\|x\|^{-2}), \quad (6)$$

where $\gamma' = \pi^{-1}(2\gamma + \ln 8)$, with $\gamma = 0.57721\ldots$ the Euler-Mascheroni constant, cf. Theorem 4.4.4 of [7]. Another important observation is that $a(x) = 1$ if x is a neighbor of the origin.

Observe that the harmonicity of a outside the origin immediately implies that the following result holds:

Proposition 2.— The process $a(S_{k\wedge\tau_0})$ is a martingale.

Besides, note that, due to (6),

$$a(x+y) - a(x) = O\big(\tfrac{\|y\|}{\|x\|}\big) \quad (7)$$

for all $x, y \in \mathbb{Z}^2$ such that (say) $\|x\| > 2\|y\|$.

With some (slight) abuse of notation, we also consider the function

$$a(r) = \frac{2}{\pi}\ln r + \gamma'$$

of a *real* argument $r \geq 1$. Note that, in general, $a(x)$ need not be equal to $a(\|x\|)$, although they are of course quite close for large x. The advantage of using this notation is e.g. that, due to (6) and (7), we may write (for fixed x or at least x such that $2\|x\| \leq r$)

$$\sum_{y\in\partial\mathsf{B}(x,r)} \nu(y)a(y) = a(r) + O\big(\tfrac{\|x\|\vee 1}{r}\big) \quad (8)$$

as $r \to \infty$, for *any* probability measure ν on $\partial\mathsf{B}(x,r)$.

We need the following result for the probability of going a long distance before revisiting the origin:

Lemma 3.— Assume that $x \in \mathsf{B}(r)$ and $x \neq 0$. Then

$$\mathbb{P}_x\big[\tau_{\partial\mathsf{B}(r)} < \tau_0^+\big] = \frac{a(x)}{a(r) + O(r^{-1})}, \quad (9)$$

as $r \to \infty$.

Proof.— Indeed, use Proposition 2, and the optional stopping theorem to write (recall that $a(0) = 0$)

$$a(x) = \mathbb{P}_x\big[\tau_{\partial\mathsf{B}(r)} < \tau_0^+\big]\mathbb{E}_x\big(a(S_{\tau_{\partial\mathsf{B}(r)}}) \mid \tau_{\partial\mathsf{B}(r)} < \tau_0^+\big),$$

and then use (8). ∎

Note that Lemma 3 implies that (since, from the origin, on the next step the walk will go to a neighbor of the origin where the potential kernel equals 1)

$$\mathbb{P}_0[\tau_{\partial\mathsf{B}(r)} < \tau_0^+] = \frac{1}{a(r) + O(r^{-1})}. \quad (10)$$

With the technical facts established above, we are now ready to pass to the main subject of this note.

2 Random walk conditioned on never hitting the origin

2.1 Doob's h-transforms

Let us start with a one-dimensional example. Let $(S_n, n \geq 0)$ be the simple random walk in dimension 1. It is well known that for any $0 < x < R$

$$\mathbb{P}_x[\tau_R < \tau_0] = \frac{x}{R} \quad (11)$$

— this is the solution of Gambler's Ruin Problem for players of equal strength, and note also that it is straightforward to obtain it from the optional stopping theorem using the fact that S_n is a martingale. Now, how will the walk behave if we *condition* it to reach R before reaching the origin? Using (11), we write

$$\begin{aligned}
&\mathbb{P}_x[S_1 = x+1 \mid \tau_R < \tau_0] \\
&= \frac{\mathbb{P}_x[S_1 = x+1, \tau_R < \tau_0]}{\mathbb{P}_x[\tau_R < \tau_0]} \\
&= \frac{\mathbb{P}_x[S_1 = x+1]\mathbb{P}_{x+1}[\tau_R < \tau_0]}{\mathbb{P}_x[\tau_R < \tau_0]}
\end{aligned}$$

$$= \frac{\frac{1}{2} \times \frac{x+1}{R}}{\frac{x}{R}} = \frac{1}{2} \times \frac{x+1}{x},$$

which also implies that

$$\mathbb{P}_x[S_1 = x - 1 \mid \tau_R < \tau_0] = \frac{1}{2} \times \frac{x-1}{x}.$$

The above calculation does not yet formally show that the conditioned walk is a Markov process (we would have needed to condition on the whole history), but let us forget about that for now, and examine the new transition probabilities we just obtained,

$$\hat{p}(x, x-1) = \frac{1}{2} \times \frac{x-1}{x}$$

and

$$\hat{p}(x, x+1) = \frac{1}{2} \times \frac{x+1}{x}.$$

First, it is remarkable that they do not depend on R, which *suggests* that we can just send R to infinity and obtain "the random walk conditioned on never returning to the origin". Secondly, look at the arguments of $\hat{p}$'s and the second fraction in the right-hand sides: these new transition probabilities are related to the old ones (which are $p(x, y) = 1/2$ for $x \sim y$) in a special way:

$$\hat{p}(x, y) = p(x, y) \times \frac{h(y)}{h(x)} \tag{12}$$

with $h(x) = |x|$ (soon it will be clear why do we prefer to keep the function nonnegative). What is special about this function h is that it is harmonic outside the origin, so that $h(S_{n\wedge\tau_0})$ is a martingale. It is *precisely* this fact that permitted us to obtain (11) with the help of the optional stopping theorem.

Keeping the above discussion in mind, we pass to a more general setup. Consider a countable Markov chain on a state space Σ, and let $A \subset \Sigma$ be finite. Let $h : \Sigma \to \mathbb{R}_+$ be a nonnegative function which is zero on A and strictly positive and harmonic outside A, i.e., $h(x) = \sum_y p(x, y)h(y)$ for all $x \notin A$. We assume also that $h(x) \to \infty$ as $x \to \infty$; this implies that the Markov chain is recurrent (this follows e.g. from Theorem 2.4 of [12]).

Definition 4.— The new Markov chain with transition probabilities defined as in (12) is called *Doob's h-transform* of the original Markov chain with respect to h.

Observe that the harmonicity of h implies that $\hat{p}$'s are transition probabilities indeed:

$$\sum_y \hat{p}(x, y) = \frac{1}{h(x)} \sum_y p(x, y)h(y) = \frac{1}{h(x)} \times h(x) = 1.$$

To the best of the author's knowledge, this kind of object first appeared in [4], in the continuous-space-and-time context. Further information can be found e.g. in [1, 8, 14], and the book [5] provides a systematic treatment of the subject in full generality.

Note the following simple calculation: for any $x \notin A \cup \partial_e A$, we have (note that $h(y) \neq 0$ for all $y \sim x$ then)

$$\begin{aligned}
\mathbb{E}_x \frac{1}{h(\hat{X}_1)} &= \sum_{y\sim x} \hat{p}(x, y) \frac{1}{h(y)} \\
&= \sum_{y\sim x} p(x, y) \frac{h(y)}{h(x)} \frac{1}{h(y)} \\
&= \frac{1}{h(x)} \sum_{y\sim x} p(x, y) \\
&= \frac{1}{h(x)},
\end{aligned}$$

which implies the following:

Proposition 5.— The process $1/h(\hat{X}_{n\wedge\tau_{A\cup\partial_e A}})$ is a martingale and the Markov chain $\hat{X}$ is transient.

(The last statement follows from Theorem 2.5 of [12] since $1/h(x) \to 0$ as $x \to \infty$.)

Now let us try to get an idea about what the h-transformed chain really does. For technical reasons, let us make another assumption: there exists $c > 0$ such that $|h(x) - h(y)| \le c$ for all $x \sim y$ (for general Markov chains, $x \sim y$ means $p(x, y) + p(y, x) > 0$).

For $R > 0$, let us define

$$\Lambda_R = \{x \in \Sigma : h(x) \le R\};$$

under the previous assumptions, Λ_R is finite for any R. Note that the optional stopping theorem implies that, for $x_0 \in \Lambda_R \setminus A$

$$h(x_0) = \mathbb{P}_{x_0}[\tau_{\Lambda_R^\complement} < \tau_A]\mathbb{E}_{x_0}\big(h(X_{\tau_{\Lambda_R^\complement}}) \mid \tau_{\Lambda_R^\complement} < \tau_A\big),$$

(recall that $\mathbb{E}_{x_0}(h(X_{\tau_A}) \mid \tau_A < \tau_{\Lambda_R^\complement}) = 0$ because h vanishes on A) and, since the second factor in the preceding display is in $[R, R + c]$, we have

$$\mathbb{P}_{x_0}[\tau_{\Lambda_R^\complement} < \tau_A] = \frac{h(x_0)}{R}\big(1 + O(R^{-1})\big). \tag{13}$$

Then, we consider another countable Markov chain $\hat{X}$ on the state space $\Sigma \setminus A$ with transition probabilities $\hat{p}(\cdot, \cdot)$ defined as in (12) for $x \notin A$.

Now, consider a *path* $\wp = (x_0, \dots, x_{n-1}, x_n)$, where $x_0, \dots, x_{n-1} \in \Lambda_R \setminus A$ and $x_n \in \Sigma \setminus \Lambda_R$ (see Figure 2 (next page); here, *path* is simply a sequence of neighbouring sites; in particular, it need not be self-avoiding). The original weight of that path (i.e., the probability that the Markov chain X follows it starting from x_0) is

$$P_{\wp} = p(x_0,x_1)p(x_1,x_2)\dots p(x_{n-1},x_n),$$

and the weight of the path for the new Markov chain $\widehat{X}$ will be

$$\begin{aligned}\widehat{P}_{\wp} &= p(x_0,x_1)\frac{h(x_1)}{h(x_0)}\dots p(x_{n-1},x_n)\frac{h(x_n)}{h(x_{n-1})}\\ &= p(x_0,x_1)\dots p(x_{n-1},x_n)\frac{h(x_n)}{h(x_0)}\\ &= P_{\wp}\frac{h(x_n)}{h(x_0)}.\end{aligned} \tag{14}$$

Here comes the key observation: the last term in (2.1) actually equals $(1+O(R^{-1}))R/h(x_0)$, that is, it is *almost* inverse of the expression in the right-hand side of (13). So, we have

$$\widehat{P}_{\wp} = \frac{P_{\wp}}{\mathbb{P}_{x_0}[\tau_{\Lambda_R^{\complement}} < \tau_A]}\big(1+O(R^{-1})\big),$$

that is, the probability that the $\widehat{X}$ chain follows a path is almost the *conditional* probability that that the original chain X follows that path, under the condition that it goes out of Λ_R before reaching A (and the relative error goes to 0 as $R\to\infty$). Now, the (decreasing) sequence of events $\{\tau_{\Lambda_R^{\complement}} < \tau_A\}$ converges to $\{\tau_A=\infty\}$ as $R\to\infty$. Therefore, we can rightfully call $\widehat{X}$ the Markov chain *conditioned* on never reaching A, even though the probability of the latter event equals zero.

2.2 The conditioned SRW in two dimensions

Now, we will consider the two-dimensional SRW conditioned on never entering the origin, which is the Doob's h-transform of (unconditional) two-dimensional SRW with respect to its potential kernel a. It turns out that the conditioned walk $\widehat{S}$ is quite an interesting object on its own — some of its surprising properties are described later in this section.

By (5), the potential kernel a can play the role of the h (the one of the previous section), so let us define another random walk $(\widehat{S}_n, n\geq 0)$ on $\mathbb{Z}^2\setminus\{0\}$ in the following way: its transition probability matrix equals

$$\hat{p}(x,y) = \begin{cases}\dfrac{a(y)}{4a(x)}, & \text{if } x\sim y, x\neq 0,\\ 0, & \text{otherwise.}\end{cases} \tag{15}$$

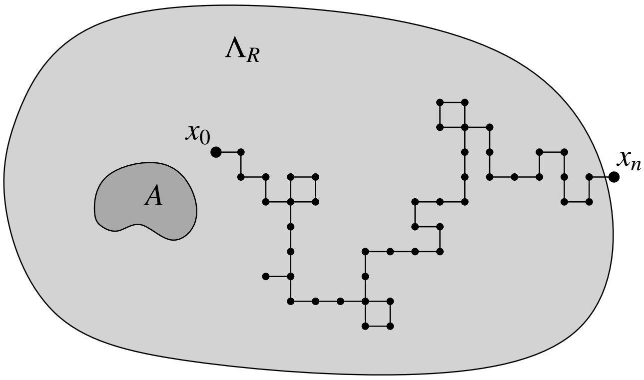

Figure 2.— Comparing the weights of the path..

The discussion of the previous section then means that the random walk $\widehat{S}$ is the Doob h-transform of the simple random walk, under the condition of not hitting the origin. Let $\hat{\tau}$ and $\hat{\tau}^+$ be the entrance and the hitting times for $\widehat{S}$; they are defined as in (1.1) and (1.1), only with $\widehat{S}$. We summarize the basic properties of the random walk $\widehat{S}$ in the following:

Proposition 6.— The following statements hold:

(i) The walk $\widehat{S}$ is reversible, with the reversible measure $\mu(x)=a^2(x)$.

(ii) In fact, it can be represented as a random walk on the two-dimensional lattice with the set of conductances $\big(a(x)a(y), x,y\in\mathbb{Z}^2, x\sim y\big)$.

(iii) The process $1/a(\widehat{S}_{n\wedge\hat{\tau}_{\mathcal{N}}})$ is a martingale.[4]

(iv) The walk $\widehat{S}$ is transient.

Proof.— Indeed, for (i) and (ii) note that

$$a^2(x)\hat{p}(x,y) = \frac{a(x)a(y)}{4} = a^2(y)\hat{p}(y,x)$$

for all adjacent $x,y\in\mathbb{Z}^2\setminus\{0\}$, and, since a is harmonic outside the origin,

$$\frac{a(x)a(y)}{\sum_{z\sim x}a(x)a(z)} = \frac{a(y)}{4\sum_{z\sim x}\frac{1}{4}a(z)} = \frac{a(y)}{4a(x)} = \hat{p}(x,y).$$

Items (iii) and (iv) are Proposition 5. ∎

The Green's function of the conditioned walk (which is transient) is defined in the usual way: for $x,y\in\mathbb{Z}^2\setminus\{0\}$

$$\widehat{G}(x,y) = \mathbb{E}_x\sum_{k=0}^{\infty}\mathbf{1}\{\widehat{S}_k = y\}. \tag{16}$$

[4] $\mathcal{N}$ is the set of the four neighbours of the origin.

One can calculate this function in terms of the potential kernel a (this is Theorem 1.1 of [11]): for all $x, y \in \mathbb{Z}^2 \setminus \{0\}$ it holds that

$$\widehat{G}(x,y) = \frac{a(y)}{a(x)}\big(a(x) + a(y) - a(x-y)\big). \tag{17}$$

Now, we are able to obtain exact expressions (in terms of Green's function) for one-site escape probabilities, and probabilities of (not) hitting a given site. Indeed, since, under $\mathbb{P}_x$, the number of visits (counting the one at time 0) to x is geometric with success probability $\mathbb{P}_x[\hat{\tau}_x = \infty]$, we have

$$\mathbb{P}_x[\hat{\tau}_x^+ < \infty] = 1 - \frac{1}{\widehat{G}(x,x)} = 1 - \frac{1}{2a(x)} \tag{18}$$

for $x \neq 0$. Also, since

$$\widehat{G}(x,y) = \mathbb{P}_x[\hat{\tau}_y^+ < \infty]\widehat{G}(y,y) \quad \text{for } x \neq y,\ x,y \neq 0$$

(one needs to go to y first, to start counting visits there), we have

$$\mathbb{P}_x[\hat{\tau}_y < \infty] = \frac{\widehat{G}(x,y)}{\widehat{G}(y,y)} = \frac{a(x) + a(y) - a(x-y)}{2a(x)}. \tag{19}$$

Let us also observe that (19) (together with (6)) implies the following surprising[5] fact: for any $x \neq 0$,

$$\lim_{y\to\infty} \mathbb{P}_x[\hat{\tau}_y < \infty] = \frac{1}{2}. \tag{20}$$

It is interesting to note that this fact permits us to obtain a criterion for recurrence of a set with respect to the conditioned walk. We say that *a set* is recurrent with respect to a (transient) Markov chain, if it is visited infinitely many times almost surely; a set is called *transient*, if it is visited only finitely many times almost surely (note that, trivially, for a transient Markov chain every finite set is transient). Recall that, for SRW in dimensions $d \geq 3$, the characterization of recurrent/transient sets is provided by *Wiener's criterion* (see e.g. Corollary 6.5.9 of [7]) formulated in terms of capacities of intersections of the set with exponentially growing annuli. Although this result does provide a complete classification, it may be difficult to apply it in practice, because it is not always trivial to calculate (even to *estimate*) capacities. Now, it turns out that for the conditioned two-dimensional walk $\widehat{S}$ the characterization of recurrent and transient sets is particularly simple:

Theorem 7 ([6]).— A set $A \subset \mathbb{Z}^2$ is recurrent with respect to $\widehat{S}$ if and only if A is infinite.

Proof.— We only need to prove that every infinite subset of $\mathbb{Z}^d$ is recurrent for $\widehat{S}$. As mentioned before, this is basically a consequence of (20). Indeed, let $\widehat{S}_0 = x_0$; since A is infinite, by (20) one can find $y_0 \in A$ and R_0 such that $\{x_0, y_0\} \subset \mathsf{B}(R_0)$ and

$$\mathbb{P}_{x_0}\big[\hat{\tau}_{y_0} < \hat{\tau}_{\partial\mathsf{B}(R_0)}\big] \geq \frac{1}{3}.$$

Then, for any $x_1 \in \partial\mathsf{B}(R_0)$, we can find $y_1 \in A$ and $R_1 > R_0$ such that $y_1 \in \mathsf{B}(R_1) \setminus \mathsf{B}(R_0)$ and

$$\mathbb{P}_{x_1}\big[\hat{\tau}_{y_1} < \hat{\tau}_{\partial\mathsf{B}(R_1)}\big] \geq \frac{1}{3}.$$

Continuing in this way, we can construct a sequence $R_0 < R_1 < R_2 < \dots$ (depending on the set A) such that, for each $k \geq 0$, the walk $\widehat{S}$ hits A on its way from $\partial\mathsf{B}(R_k)$ to $\partial\mathsf{B}(R_{k+1})$ with probability at least $\frac{1}{3}$, regardless of the past. This clearly implies that A is a recurrent set. ∎

Next, we state an even more surprising result, which attests the "fractal" behaviour of $\widehat{S}$'s trajectories. For a set $T \subset \mathbb{Z}_+$ (thought of as a set of time moments) let

$$\widehat{S}_T = \bigcup_{m\in T} \{\widehat{S}_m\}$$

be the *range* of the walk $\widehat{S}$ with respect to that set, i.e., it is made of sites that are visited by $\widehat{S}$ over T. For simplicity, we assume in the following that the walk $\widehat{S}$ starts at a fixed neighbour x_0 of the origin, and we write $\mathbb{P}$ for $\mathbb{P}_{x_0}$. For a nonempty and finite set $A \subset \mathbb{Z}^2$, let us consider the random variable

$$\mathcal{R}(A) = \frac{\big|A \cap \widehat{S}_{[0,\infty)}\big|}{|A|};$$

that is, $\mathcal{R}(A)$ is the proportion of visited sites of A by the walk $\widehat{S}$ (and, therefore, $1 - \mathcal{R}(A)$ is the proportion of unvisited sites of A). A natural question is: how should $\mathcal{R}(A)$ behave for "large" sets? By (20), in average approximately half of A should be covered, i.e., $\mathbb{E}\mathcal{R}(A)$ should be close to 1/2. Surprisingly, for a "typical" large set (e.g., a disk, a rectangle, a segment) the random variable $\mathcal{R}(A)$ does not concentrate, and instead the following holds: the proportion of visited sites is a *random variable* which is close in distribution to Uniform$[0,1]$. The paper [6] contains the corresponding results in greater generality, but here we content ourselves in stating the result for a particular case of a large disk which does not "touch" the origin:

[5] We know that the conditioned walk is transient and there is "a lot of space" in $\mathbb{Z}^2$, so one would rather expect that the probability to eventually hit a very distant site would go to zero.

Theorem 8.— Let $D \subset \mathbb{R}^2$ be a closed disk such that $0 \notin D$, and denote $D_n = nD \cap \mathbb{Z}^2$. Then, for all $s \in [0,1]$, we have, with positive constant c_1 depending only on D,

$$\left|\mathbb{P}[\mathscr{R}(D_n) \leq s] - s\right| \leq c_1 \left(\frac{\ln \ln n}{\ln n}\right)^{1/3}. \tag{21}$$

The last result we mention here is a quantitative assessment of how fast the transience of $\widehat{S}$ happens. Let us define the *future minima process*

$$M_n := \min_{m \geq n} |\widehat{S}_m|;$$

so far, we only know that $M_n \to \infty$ a.s. by transience. It is possible to obtain some finer asymptotic properties of M_n:

Theorem 9 ([13]).— For every $0 < \delta < \frac{1}{2}$ we have, almost surely,

$$M_n \leq n^\delta \text{ i.o.} \quad \text{but} \quad M_n \geq \frac{\sqrt{n}}{\ln^\delta n} \text{ i.o.}$$

The above result means that the transience of the conditioned SRW is "very irregular": sometimes it goes to infinity in the usual "diffusive" way, but sometimes slows down quite dramatically.

As a concluding remark, we also mention that in [2] even finer results were obtained for the continuous analogue of the conditioned SRW (which is the Brownian motion conditioned on never hitting the unit disk — one is then able to use the fact that it is radially symmetric, something that does not hold in the discrete setting). We are now working on extending the results of [2] to the discrete case.

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French–German–Portuguese Conference on Optimization

May 03–06, 2022, Porto, Portugal

by Maria do Rosário Pinho and
Fernando A.C.C. Fontes**

The international conference on **OPTIMIZATION 2022** – FRENCH-GERMAN-PORTUGUESE (FGP2022) was organized by a team of researchers from various Portuguese universities and research centers, in close collaboration with SYSTEC, *Institute for Systems and Robotics*, *Faculty of Engineering of the University of Porto*, and *Faculty of Economics of the University of Porto*.

The CIM — *International Center of Mathematics*, INESC-TEC, ARISE, CMUC, CMA-UNL, NOVA-LINCS and FCT supported the organization of this event.

The FGP2022 conference joined 168 participants from 12 countries, and it was an opportunity to discuss recent scientific research results and developments in the field.

The scientific program consisted of 11 plenary talks with leading invited and renowned keynote speakers, 15 minisymposia gathering a group of talks in a specialized area, and 23 contributed talks.

Thanks to the support of the sponsors, various Ph.D. students were able to participate in the conference.

Further information on the event can be found on www.fe.up.pt/fgp22

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REPORT

New trends in Lyapunov exponents

February 7th–11th 2022, Lisbon, Portugal

by **Pedro Miguel Duarte*** and **João Lopes Dias****

Between the 7th and 11th of February it took place at ISEG (*Lisbon School of Economics & Management*) the international workshop *New Trends in Lyapunov Exponents* (see its webpage at https://toni.iseg.ulisboa.pt/lxds/workshop/), also broadcasted via Zoom. The main topic of this event was the theory of Lyapunov exponents in their various manifestations in Dynamical Systems along with their applications to other areas of mathematics. The workshop was organized under the scope of the homonymous project (PTDC/MAT-PUR/29126/2017) funded by FCT. The poster and a couple of meeting photos go attached with this report.

The event had around 30 in-person daily participants (mainly but not exclusively speakers and organizers) as well as around 10-15 daily online participants from different countries. The list of speakers below includes renowned experts in the field from 10 different nationalities and 8 different countries. All speakers and non local organizers stayed at Lisbon São Bento Hotel with their accommodation expenses covered by FCT through the referenced project. The lunches took place at ISEG's rooftop and their expenses were partly covered by CMAT and FCT. The coffee- breaks and a working room for the participants were kindly offered by ISEG. Finally we are grateful for the support of CIM in sponsoring the workshop's proceedings which will be published in CIM Springer Series.

The event was a great scientific success, promoting new connections and interactions between speakers, organizers and the other participants. For the organizers it was extremely rewarding to see the joy of most participants gathering for their first in-person international event after two years of restrictions caused by the pandemic. Lisbon's good weather (São Pedro) also helped with the lunches, coffee-breaks and the conference dinner, which all took place in open air spaces, thus facilitating safe social interactions between participants.

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List of speakers

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Jamerson Bezerra
Copernicus University in Toruń
Kristian Bjerklöv
KTH
Alex Blumenthal
Georgia Tech
David Burguet
Sorbonne Université
Ao Cai
PUC-Rio
Sylvain Crovisier
Université Paris-Saclay
Anton Gorodetski
UC Irvine
Anders Karlsson
Université de Genève
Victor Kleptsyn
Institut de Recherche Mathématique de Rennes
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Reza Mohammadpour
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Sixth Workshop New Trends in Quaternions and Octonions

by Patrícia Damas Beites*

The *Sixth Workshop New Trends in Quaternions and Octonions* (NTQO 2022) took place in November 25–26, 2022, at the *University of Beira Interior* (Covilhã, Portugal). The event was organized by the *Centre of Mathematics and Applications of the University of Beira Interior* (CMA-UBI), jointly with the *Center for Research & Development in Mathematics and Applications* (CIDMA-UA) and the *Centre of Mathematics of the University of Minho* (CMAT-UMinho). There were several short communications, and four invited lectures by **Helmuth Malonek** (University of Aveiro, Portugal), **Nichol Furey** (Humboldt University of Berlin, Germany), **Rui Pacheco** (University of Beira Interior, Portugal) and **Vladislav Kravchenko** (CINVESTAV, Mexico) (see https://w3.math.uminho.pt/NTQO/).

As in the previous editions, held at the *University of Trás-os-Montes e Alto Douro* (2021), *Polytechnic Institute of Guarda* (2020), *University of Coimbra* (2019), *University of Minho* (2016) and *University of Aveiro* (2015), NTQO 2022 aimed to present recent advances in the research on quaternions and octonions, and their applications. Furthermore, this workshop was an opportunity for the 62 participants, from 13 countries (Brasil, Germany, Israel, Italy, Lithuania, Mexico, Pakistan, Portugal, Russia, Spain, Turkey, USA, Uzbekistan), to discuss recent developments related to quaternions and octonions. Next edition – the seventh – will take place in 2023, at the *Polytechnic Institute of Leiria*.

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On behalf of the Oganizing Committee

Generic behaviours of conservative dynamical systems

by **Maria Joana Torres***

Science treats only the general; there is no science of the individual.
—Aristotle[1]

Dynamical systems is the study of the long-term behaviour of evolving systems. The foundations were set by the innovative work of Henri Poincaré (1854–1912), *Les Méthodes Nouvelles de la Mécanique Céleste* (1892–1899), with fundamental questions concerning the stability of the solar system. Attempts to answer those questions revealed the incapacity of solving *exactly* the mathematical questions arising from physical systems. It appeared that understanding *typical* systems, or systems *in general*, was mathematically more fruitful.

The aim of the modern theory of dynamical systems is thus to describe the behaviour of *typical* trajectories, for *typical* evolution laws. Furthermore, when dealing with real-world systems, neither the initial data nor the evolution law are known exactly. For these reasons, one is most interested in properties that are *stable*, i.e., that are persistent under small perturbations of the evolution law.

1 What is a generic behaviour?

Ce qui limite le vrai, ce n'est pas le faux, c'est l'insignifiant.
—René Thom[2]

One of the oldest aspirations of humanity is to understand the motion of celestial bodies–the sun, moons, planets and visible stars of the solar system. The first complete mathematical formulation of the classical n-Body Problem was presented by Isaac Newton (1643–1727) in his masterpiece *Philosophiae Naturalis Principia Mathematica*, first published in 1687. Informally, the n-Body Problem can be stated as (see [15]): *Given only the present positions and velocities of a group of n celestial bodies, predict their motions for all future time and deduce them for all past time.* The Two-Body Problem is known as the *Kepler problem*, in honor of Johannes Kepler (1571–1630), who provided inspiration for Newton's gravitational model, with his laws on planetary motions deduced from the astronomical observational data of Tycho Brahe (1546–1601). Unfortunately, the Kepler problem revealed to be the only easy case among the n-Body Problem.

Most of the great mathematicians of the eigh-

[1] Quoted in *Itinerary for a Science of the Detail*, René Thom.

[2] In *Prédire n'est pas expliquer.*

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teenth and nineteenth century tackled the equations of the Three-Body Problem but were unable to make much progress. Until the final decade of the nineteenth century, the goal was to obtain exact results, to integrate equations and obtain complete solutions. But physical phenomena are in general nonlinear and some are even chaotic. At the end of the nineteenth century, a new *qualitative* era was opened by Poincaré, who introduced geometric, topological and probabilistic methods in order to understand qualitatively the complex behaviour of most of the solutions of the Three-Body Problem (see [13]). Poincaré's work on this problem provided a glimpse of chaotic behaviour in a dynamical system, a feature entirely understood by the mathematical community only three quarters of a century later, after George D. Birkhoff (1884–1944) and then Stephen Smale (1930–) show its importance.

At the end of the fifties, the development of dynamical systems as a theory was greatly influenced by the program of classifying singularities of differential mappings. One of the main goals of this program was to classify functions from $\mathbb{R}^n$ to $\mathbb{R}^p$ by the kind of singularities exhibited. But since such a characterization for *all* such functions is impossible, mathematicians settle for a classification of *almost all* functions. The approach was then to derive the type of singularities a certain *prototype* function can have, and then prove that any other function could be approximated by the prototype. This property of prototypes was expressed by René Thom (1923–2002) in terms of a *generic property*. Thom picked up the term *generic* from Italian algebraic geometers, who had already defined a generic property as *a property that is satisfied for all points of a space except for the points of a thin submanifold of that particular space* (see [12]). As did Thom in [34, p. 357], consider the space $L^m_{n,p}$ of functions of class C^m from $\mathbb{R}^n$ to $\mathbb{R}^p$ equipped with the C^m topology, which is a Baire space. Thom defined a *generic property* (P) in the space $L^m_{n,p}$ as a property that holds for all functions belonging to this space, except for a *rare* subset of that space. A generic property was then known as a property that is satisfied by the elements that form an *open and dense* subspace of the domain, which is the complement of a closed subspace without interior points (see [12]).

In the context of dynamical systems theory, the term generic (borrowed from Thom) was used for the first time by Mauricio Peixoto (1921–2019) in [25], motivated by his interest on structural stability.

In 1967, Smale presented in [31] the advances of the theory of dynamical systems so far. At that moment, the mathematical definitions needed to be adapted to a new classification program. Smale proposes that the term *generic* should be associated with a behaviour that holds for a *residual* subset of systems. Given a topological space M, a subset of the space is called *residual* or *topologically generic* if it contains a countable intersection of open and dense sets. Note that if M is a Baire space—for example, if M is completely metrizable—a countable intersection of dense open subsets is still dense. A property (P) is generic if it is verified on a residual subset. We observe that a countable intersection of residual sets is a residual set.

2 Two main theories in dynamical systems

In the 1960s two main theories in dynamical systems were developed: the *hyperbolic* theory for general systems and the *KAM* (for Kolmogorov–Arnold–Moser) theory for a distinguished class of conservative systems.

In the study of physical systems which evolve in time as solutions of certain differential equations one is led naturally to the consideration of measure preserving—*conservative*—systems. These conservative (or incompressible) flows are associated to *divergence-free vector fields*, they preserve a volume form on the ambient manifold and thus come equipped with a natural invariant measure μ which we call Lebesgue measure.

The hyperbolic theory was initiated by Smale, in the west, and Dmitri Anosov (1936–2014), Yakov Sinai (1935–) and Vladimir Arnold (1937–2010), in the former Soviet Union. It was part of a revolution in our vision of determinism, providing a mathematical foundation for the fact that deterministic systems often present chaotic behaviour in a robust way. Let M be a compact smooth Riemannian manifold and $f : M \to M$ be a diffeomorphism. Recall that, an *invariant* (i.e. $f(\Lambda) = \Lambda$) compact set $\Lambda \subset M$ is a *uniformly hyperbolic set* for f if the tangent bundle over Λ admits a continuous decomposition $T_\Lambda M = E^u \oplus E^s$, invariant under the derivative, and for which there are constants $C > 0$ and $\lambda \in (0,1)$ so that $\|Df^{-n}(x)|_{E^u_x}\| \leq C\lambda^n$ and $\|Df^n(x)|_{E^s_x}\| \leq C\lambda^n$, for every $x \in \Lambda$ and $n \geq 1$. The diffeomorphism f is called *Anosov (or globally hyperbolic)* if the whole manifold M is a uniformly hyperbolic set. The definition of uniform hyperbolicity for a smooth flow $f^t : M \to M, t \in \mathbb{R}$, is analogous except that (unless

Λ consists of equilibria) the previous decomposition becomes $T_\Lambda M = E^u \oplus E^0 \oplus E^s$, where E^0 is a line bundle tangent to the flow lines.

3 Generic dynamical behaviours

The properties of generic dynamical systems depend mostly on the dimension of the manifold and of the C^r-topology considered, $r \geq 0$. We refer the reader to [11], for a recent overview on the subject.

3.1 Generic dynamics in low regularity

3.1.1 *Metric and topological transitivity*

Before Poincaré's work, the founders of statistical mechanics, James Maxwell (1831–1879) and Ludwig Boltzmann (1844–1906) tried to provide a rigorous formulation of the kinetic theory of gases and statistical mechanics. A key ingredient was Boltzmann's Fundamental Principle, which asserts that the time and space averages of an observable (a function on the phase space) can be set equal (see [22, 19]). The ergodic theorems of Birkhoff and John von Neumann (1903–1957) ("time averages exist a.e.") set the foundation for the current definition of *ergodicity*: *any invariant set has zero measure or full measure.* If this holds, then time averages coincide with space averages at least for typical points—Boltzmann's Fundamental Principle.

An important question in the 1930s was then: *Is ergodicity with respect to volume a typical property?* The question was first addressed by John Oxtoby (1910–1991) and Stanisław Ulam (1909–1984) in [24], who proved that a generic volume-preserving homeomorphism of a compact manifold is ergodic. A natural question, still open in general, is whether such a result extends to the space of volume-preserving C^1 diffeomorphisms. If one considers the other extreme of regularity, C^∞ diffeomorphisms, ergodicity is not a typical property at all: KAM theory assures that there are open sets of volume-preserving C^∞ diffeomorphisms that are not ergodic (see [35, 1]).

In their 1912 article, Paul (1880–1933) and Tatiana (1876–1964) Ehrenfest discussed questions related with the ergodic hypothesis and then proposed the alternative *quasi-ergodic hypothesis* (see [22]): *some orbit of the system will pass arbitrarily close to every point of the phase space*, i.e., the system is *(topologically) transitive*. It is the topological counterpart of an ergodic system.

Concerning the continuous-time counterpart of the Oxtoby-Ulam theorem, there is a lack in the literature. Motivated by this, the main goal in [10] was to study the abundance of transitivity-like properties of C^0 conservative flows generated by Lipschitz vector fields and to establish a weaker (topological) counterpart of the Oxtoby-Ulam theorem: C^0-generic flows are strongly transitive: *the shortest hitting time from a ball to any other ball of the same radius is uniformly bounded above by a constant depending only on the radius.* For Lipschitz divergence-free vector fields without singularities it was proved in [10, Theorem A] that:

Theorem 1.— C^0-generic non-singular Lipschitz divergence-free vector fields generate conservative flows that are strongly transitive.

It was also proved that C^0-generic Lipschitz divergence-free vector fields generate transitive flows (see [10, Theorem B]).

3.1.2 *Perturbation of orbits: Closing lemma*

The problem of closing a nonperiodic trajectory is a well-known problem in the theory of dynamical systems, whose origin remounts to Poincaré (see [26, vol 1, p.82]). We want to *close*, in the sense that we transform into a periodic orbit, a given orbit with some return property (for example, non-trivial recurrence or non-wandering) by performing a small perturbation on the original system.

Poincaré believed that such a closing could be done in quite general situations. However, until now, there are positive answers only if the perturbations are with respect to coarse topologies like e.g., C^0, Hölder, Sobolev-$(1, p)$ and C^1. The C^0 closing lemma can readily be proved, except perhaps for the geodesic flows. But the C^1 closing lemma reveals some fundamental difficulties. The C^1 closing lemma for non-conservative systems was first established by Pugh [27] in the late 1960s and for conservative systems was prove by Pugh and Robinson [28] in the early 1980s. A non-conservative version of the closing lemma for the Sobolev-$(1, p)$ topology was recently presented in [17]. In [2], it was given a simpler and different proof of the Sobolev closing lemma which also works in the conservative case.

More precisely, let U be an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary and let $1 \leq p, q \leq \infty$. A measurable map $f = (f_1, \dots, f_n) : U \to \mathbb{R}^n$ is in the Sobolev class $W^{1,p}(U, \mathbb{R}^n)$ if, for all $i = 1, \dots, n$,

f_i and all its distributional partial derivatives $\partial f_i/\partial x_j$ are in $L^p(U)$. We are interested only in Sobolev maps that are continuous up to the boundary, i.e., we consider the space $W^{1,p}(U,\mathbb{R}^n)\cap C^0(\overline{U},\mathbb{R}^n)$. Finally we define $\mathbb{W}^{1,p}(U)$ as the set of all homeomorphisms $f : U \to U$ such that $f \in W^{1,p}(U,\mathbb{R}^n)\cap C^0(\overline{U},\mathbb{R}^n)$. We also define $\mathbb{W}^{1,p,q}(U)$ as the set of all elements in $\mathbb{W}^{1,p}(U)$ whose inverse is in $\mathbb{W}^{1,q}(U)$. In $\mathbb{W}^{1,p}(U)$ and $\mathbb{W}^{1,p,q}(U)$ we consider the natural metrics defined by $d_{\infty,p}(f,g) = \|f-g\|_\infty + \|D(f-g)\|_p$ and $(f,g) \mapsto d_{\infty,p}(f,g)+d_{\infty,q}(f^{-1},g^{-1})$, respectively. We consider also the subspaces $\mathbb{W}^{1,p}_\mu(U)$ and $\mathbb{W}^{1,p,q}_\mu(U)$ of volume preserving elements. The spaces $\mathbb{W}^{1,p}(U)$, $\mathbb{W}^{1,p}_\mu(U)$, $\mathbb{W}^{1,p,q}(U)$ and $\mathbb{W}^{1,p,q}_\mu(U)$ satisfy the Baire property.

Recall that a point $x \in U$ is said to be a *non-wandering point* for f if for any neighbourhood V of x there exists $n \in \mathbb{N}$ such that $f^n(V)$ intersects V. The set of non-wandering points $\Omega(f)$ contains the set $\mathrm{Per}(f)$ of periodic points. It was proved in [2, Theorem A] the Sobolev-$(1,p)$ closing lemma:

Theorem 2.— Let $n \geq 2$. Consider X to be any of the spaces $\mathbb{W}^{1,p}(U)$, $\mathbb{W}^{1,p}_\mu(U)$, $\mathbb{W}^{1,p,q}(U)$, $\mathbb{W}^{1,p,q}_\mu(U)$, where $p,q \in [1,\infty[$. Let $f \in X$ and $z \in \Omega(f)$. Then, for all $\varepsilon > 0$ there exists $y_\varepsilon \in U$ and $h_\varepsilon \in X$ such that $\lim_{\varepsilon\to 0} |y_\varepsilon - z| = 0$, $\lim_{\varepsilon\to 0} \|h_\varepsilon - f\|_X = 0$ and $y_\varepsilon \in \mathrm{Per}(h_\varepsilon)$.

3.1.3 *Density of periodic orbits*

Another fundamental result in the theory of dynamical systems is the *general density theorem*. It asserts that *generically the closure of the set of periodic orbits is the set where the dynamics is truly relevant*: in the non-conservative case this set is the non-wandering set and in the conservative case this set is the whole manifold. The general density theorem has been proved in many different settings and there is a vast literature on the subject (see [2] and references therein).

The general density theorem is a direct consequence of the combination of the closing lemma and the stability of the closed orbits. As a consequence, the general density theorem turns out to be easier in the C^1 case when compared to the C^0 one, because in the differentiable case the stability of periodic points can be expressed through hyperbolicity but in the topological case is more subtle. Within Sobolev homeomorphisms, in order to use hyperbolicity we had to request for differentiability at least for a map arbitrarily close from the Sobolev perspective. But this bypass through a differentiable map is very difficult to obtain. In fact, regularization of Sobolev-$(1,p)$ homeomorphisms is available only for planar domains. In [2, Theorem B], it was proved the planar general density theorem for Sobolev-$(1,p)$ maps:

Theorem 3.— Let $U \subset \mathbb{R}^2$. There exists a $\mathbb{W}^{1,p}$-residual subset $\mathscr{R} \subset \mathbb{W}^{1,p}(U)$ such that if $f \in \mathscr{R}$, then $\overline{Per(f)} = \Omega(f)$.

3.2 Smooth Generic Dynamics

3.2.1 *Palis-Smale stability conjecture*

Structural stability is one of the most fundamental topics in dynamical systems and contains some of the hardest conjectures in the area. The concept of *structural stability* was introduced in the mid 1930s by Andronov and Pontryagin. Roughly speaking it means that under small perturbations the whole orbit structure remains the same: there exists a homeomorphism of the ambient manifold mapping orbits of the initial system into orbits of the modified one. The aim of Smale's program in the early 1960s was to prove the genericity of structurally stable systems. Although Smale's program was proved to have serious flaws one decade later, it played a major role in the development of the theory of smooth dynamical systems. It led to the construction of hyperbolic theory, studying uniform hyperbolicity, and characterizing structural stability as being essentially equivalent to uniform hyperbolicity. Indeed, one of the high points in the development of smooth dynamics is the proof by Robbin, Robinson, Mañé and Hayashi [21, 29, 30, 20] that structural stability indeed characterizes hyperbolic dynamical systems. For C^1-diffeomorphisms this was achieved in the 1980s, for flows in the 1990s. The C^r structural stability conjecture for $r \geq 2$ remains wide open. In the conservative setting we highlight the seminal paper of Newhouse [23] where it was proved that *a symplectic diffeomorphism of a compact manifold is structurally stable if and only if it is Anosov*. In the continuous-time setting, similar results were obtained for conservative flows in [7] and for Hamiltonian flows in [6], but in lower dimension (three and four, respectively). These results were generalized in [18] and [8] for arbitrary dimension, respectively. Let us describe the Hamiltonian framework.

A *Hamiltonian system* can be seen as the apotheosis of mathematical models of classical mechanics. The mathematician William R. Hamilton (1805–1865) developed a formalism for the equations of the dynamics, which played a major role in the development of the theory of classical dynamical systems. The *Hamiltonian formalism* was originally formulated combining the formulation of mechanics of Joseph-Louis Lagrange (1736–1813) (itself deduced from Newton's laws) with variational methods (see [16]). In the modern language, Hamiltonian systems are a part of *symplectic geometry*.

Let (M,ω) be a symplectic manifold, where M is a $2n$-dimensional ($n \geq 2$), closed,[3] connected and smooth Riemannian manifold, endowed with a symplectic structure, i.e. a closed and nondegenerate 2-form ω. A *Hamiltonian* is a real-valued C^r function on M, $2 \leq r \leq \infty$. The associated *Hamiltonian vector field* X_H is defined by $\omega(X_H(p),u) = dH_p(u)$, for all $u \in T_pM$; this vector field generates the Hamiltonian flow X^t_H. From now on, we shall be restricted to the C^2-topology and thus we set $r = 2$. Observe that H is C^2 if and only if the associated Hamiltonian vector field X_H is C^1. A scalar $e \in H(M) \subset \mathbb{R}$ is called an *energy* of H. An *energy hypersurface* $\mathscr{E}_{H,e}$ is a connected component of $H^{-1}(\{e\})$ and it is *regular* if it does not contain singularities. Observe that a regular energy hypersurface is a X^t_H-invariant, compact and $(2n-1)$-dimensional manifold. A *Hamiltonian system* is a triple $(H,e,\mathscr{E}_{H,e})$, where H is a Hamiltonian, e is an energy and $\mathscr{E}_{H,e}$ is a regular connected component of $H^{-1}(\{e\})$. Fixing a small neighbourhood $\mathscr{W}$ of a regular $\mathscr{E}_{H,e}$, there exist a small neighbourhood $\mathscr{U}$ of H and $\epsilon > 0$ such that, for all $\tilde{H} \in \mathscr{U}$ and $\tilde{e} \in (e-\epsilon, e+\epsilon)$, $\tilde{H}^{-1}(\{\tilde{e}\}) \cap \mathscr{W} = \mathscr{E}_{\tilde{H},\tilde{e}}$. We call $\mathscr{E}_{\tilde{H},\tilde{e}}$ the *analytic continuation* of $\mathscr{E}_{H,e}$. In the space of Hamiltonian systems we consider the topology generated by a fundamental systems of neighbourhoods. Given a Hamiltonian system $(H,e,\mathscr{E}_{H,e})$ we say that $\mathscr{V}(\mathscr{U},\epsilon)$ is a *neighbourhood* of $(H,e,\mathscr{E}_{H,e})$ if there exist a small neighbourhood $\mathscr{U}$ of H and $\epsilon > 0$ such that for all $\tilde{H} \in \mathscr{U}$ and $\tilde{e} \in (e-\epsilon, e+\epsilon)$ one has that the analytic continuation $\mathscr{E}_{\tilde{H},\tilde{e}}$ of $\mathscr{E}_{H,e}$ is well-defined. A Hamiltonian system $(H,e,\mathscr{E}_{H,e})$ is said to be *Anosov* if $\mathscr{E}_{H,e}$ is uniformly hyperbolic for the Hamiltonian flow X^t_H associated to H. We say that the Hamiltonian system $(H,e,\mathscr{E}_{H,e})$ is *structurally stable* if there exists a neighbourhood $\mathscr{V}$ of $(H,e,\mathscr{E}_{H,e})$ such that, for any $(\tilde{H},\tilde{e},\mathscr{E}_{\tilde{H},\tilde{e}}) \in \mathscr{V}$, there exists a homeomorphism $h_{\tilde{H},\tilde{e}}$ between $\mathscr{E}_{H,e}$ and $\mathscr{E}_{\tilde{H},\tilde{e}}$, preserving orbits and their orientations. Moreover, $h_{\tilde{H},\tilde{e}}$ is continuous on the parameters $\tilde{H}$ and $\tilde{e}$, and converges to id when $\tilde{H}$ C^2-converges to H and $\tilde{e}$ converges to e. The stability conjecture for Hamiltonians was established in [8, Theorem 2]:

Theorem 4.— If $(H,e,\mathscr{E}_{H,e})$ is a structurally stable Hamiltonian system, then $(H,e,\mathscr{E}_{H,e})$ is Anosov.

3.2.2 *Shades of hyperbolicity*

The characterization of structurally stable systems using topological and geometrical dynamical properties has been one of the main objects of interest in the global qualitative theory of dynamical systems in the last 40 years. Here, we shall focus on *tracing orbit properties*, namely, the *shadowing* and *specification* properties (see [9] where the topological stability and expansiveness properties were also considered).

A dynamical system has the *shadowing property* if for any *almost orbit* (obtained, for example, by a numerical method with good accuracy) there is a close *true orbit*. In this case, an approximate pattern of trajectories given by numerical modeling reflects the exact structure of trajectories. A dynamical system has the *specification property* if one can shadow distinct n pieces of orbits, which are sufficiently time-spaced, by a single orbit. We say that the specification property is *weak* if $n = 2$.

Let (M,d) be a compact metric space and let $(X^t)_{t\in\mathbb{R}}$ be a continuous flow on M. Fix real numbers $\delta, T > 0$. We say that a pair of sequences $((x_i),(t_i))_{i\in\mathbb{Z}}$ $(x_i \in M, t_i \in \mathbb{R}, t_i \geq T)$ is a (δ,T)-*pseudo-orbit* if $d(X^{t_i}(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. For the sequence $(t_i)_{i\in\mathbb{Z}}$ we write $\sigma(n) = t_0 + t_1 + \ldots + t_{n-1}$ if $n > 0$, $\sigma(n) = -(t_n + \ldots + t_{-2} + t_{-1})$ if $n < 0$, and $\sigma(0) = 0$. Let $x_0 \star t$ denote a point on a (δ,T)-chain t units time from x_0. More precisely, for $t \in \mathbb{R}$, $x_0 \star t = X^{t-\sigma(i)}(x_i)$ if $\sigma(i) \leq t < \sigma(i+1)$. In continuous-time setting the shadowing property should reflect the speed at which different points travel in their trajectories. For that reason we need to consider orbits up to reparametrization. By Rep we denote the set of all increasing homeomorphisms $\alpha : \mathbb{R} \to \mathbb{R}$, such that $\alpha(0) = 0$, called *(time) reparameterizations*. Fixing $\varepsilon > 0$, we define the set

$$\mathrm{Rep}(\varepsilon) = \left\{\alpha \in \mathrm{Rep} : \left|\frac{\alpha(t)-\alpha(s)}{t-s} - 1\right| < \varepsilon, s,t \in \mathbb{R}\right\},$$

of the reparameterizations ε-close to the identity. The flow $(X^t)_t$ satisfies the *shadowing property* if for any $\varepsilon > 0$ there exist $\delta, T > 0$ such that for any

[3] That is, compact and without boundary.

(δ, T)-pseudo-orbit $((x_i),(t_i))_{i\in\mathbb{Z}}$ there exist a point $z \in M$ and a reparametrization $\alpha \in \mathrm{Rep}(\varepsilon)$ such that $d(X^{\alpha(t)}(z), x_0 \star t) < \varepsilon$, for every $t \in \mathbb{R}$.

The flow $(X^t)_{t\in\mathbb{R}}$ has the *specification property* if for any $\varepsilon > 0$ there exists a $T = T(\varepsilon) > 0$ such that: given any finite collection τ of intervals $I_i = [a_i, b_i]$ ($i = 1 \dots m$) of the real line satisfying $a_{i+1} - b_i \geq T(\varepsilon)$ for every i and every map $P : \bigcup_{I_i\in\tau} I_i \to M$ such that $X^{t_2}(P(t_1)) = X^{t_1}(P(t_2))$ for any $t_1, t_2 \in I_i$ there exists $z \in M$ so that $d(X^t(z), P(t)) < \varepsilon$ for all $t \in \bigcup_i I_i$.

It is well-known that Anosov systems, and thus, structurally stable Hamiltonian systems, satisfy the shadowing property. Moreover, mixing Anosov systems satisfy the specification property. In the context of Hamiltonian systems, it was proved in [9, Theorem 1] that if one requires the stability of shadowing (or weak specification) property under perturbation, then the Hamiltonian system is Anosov. Thus, we call these properties *shades of hyperbolicity*. We say that a property *stably holds* for some system if it holds for any system in some neighbourhood of that system.

Theorem 5.— Let $(H, e, \mathscr{E}_{H,e})$ be a Hamiltonian system. If any of the following statements hold:

(1) $(H, e, \mathscr{E}_{H,e})$ is stably shadowable;

(2) $(H, e, \mathscr{E}_{H,e})$ has the stable weak specification property,

then the Hamiltonian system $(H, e, \mathscr{E}_{H,e})$ is Anosov.

A natural question is whether these results are extensible to the subclass of Hamiltonians formed by the geodesic flows. Let (M, g) be a Riemannian manifold, where M is a closed, connected, Riemannian manifold of dimension ≥ 2 and $g \in \mathscr{R}^r$. Here $\mathscr{R}^r$ stands for the set of C^r Riemannian metrics, $2 \leq r \leq \infty$. Given a tangent vector $v \in T_xM$ at a point $x \in M$, denote by $\gamma_{x,v} : \mathbb{R} \to M$ the geodesic such that $\gamma_{x,v}(0) = x$ and $\dot\gamma_{x,v}(0) = v$. The geodesic flow of g is the flow on TM defined by $\varphi_g^t(x, v) = (\gamma_{x,v}(t), \dot\gamma_{x,v}(t))$. Since geodesics travel with constant speed, the unit tangent bundle $S_gM = \{(x, v) \in TM : g_x(v, v) = 1\}$ is preserved by φ_g^t. It is widely known that the geodesic flow is a Hamiltonian flow given by $(x, v) \mapsto \frac{1}{2} g_x(v, v)$ on TM for a symplectic form depending on g. The perturbation tools for geodesic flows are very delicate as opposed to the general Hamiltonian case. We can only perturb the metric, hence the perturbation is never a local issue in phase space. This is the main reason why it is not known a closing lemma and a general density theorem for geodesic flows in the C^1 topology, i.e. C^2 in the metric. As a consequence, the hyperbolic structure of the closure of the periodic orbits cannot be extrapolated to the whole energy level and we are not able to assure global hyperbolicity. It was proved in [4] that:

Theorem 6.— There is a set $\mathscr{G}_1 \subset \mathscr{R}^2$ where $\mathscr{G}_1$ is C^2-open in $\mathscr{R}^2$ and $\mathscr{G}_1 \cap \mathbb{R}^\infty$ is C^∞-dense in $\mathscr{R}^\infty$ such that if $g \in \mathscr{G}_1$ and the geodesic flow satisfies any of the properties:

(1) is stably shadowable;

(2) has the stable weak specification property,

then $\overline{\mathrm{Per}(g)}$ is a uniformly hyperbolic set.

Notice that we were only able to show the result for a residual set of metrics, in contrast to the Hamiltonian case and also in contrast to the 2-dimensional case (for geodesic flows) previously considered in [5].

3.2.3 *Topological entropy*

The complexity of a dynamical system can be measured by the *topological entropy*. The topological entropy is a nonnegative real number that, roughly speaking, measures the rate of exponential growth of the number of distinguishable orbits with finite but arbitrary precision as time advances.

Let (M, d) be a compact metric space and $f : M \to M$ be a continuous map. For each $n \geq 1$, let $d_n(x, y) = \max\{d(f^i(x), f^i(y)) : 0 \leq i < n - 1\}$. A subset F of the phase space M is said to be (n, ϵ)-*spanning* if M is covered by the union of the dynamical balls $\{y : d_n(x, y) < \epsilon\}$ centered at the points $x \in F$. Denote by $N(n, \epsilon)$ the minimal cardinality of a (n, ϵ)-spanning set. Roughly, this gives the number of orbit segments that one can distinguish up to some precision. The *topological entropy* is then the exponential growth rate of this number as the precision increases,

$$h_{top}(f) = \lim_{\epsilon\to 0}\left(\limsup_{n\to\infty} \frac{1}{n} \log N(n, \epsilon)\right).$$

If a manifold has negative sectional curvature, its geodesic flow is Anosov and hence it has positive topological entropy. On manifolds with non negative curvature it is not so clear that one can perturb the metric to obtain positive topological entropy. Recently, Contreras proved in [14] that positive topological entropy is a generic property among geodesic flows on any closed manifold with dimension ≥ 2.

Very recently, it was obtained in [3] a similar result for billiards in generic convex bodies.

Tell me these things, Olympian Muses, tell
From the beginning, which came first to be?
Chaos was first of all
—Hesiod, Theogeny, II, 114-116[4]

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AN INTERVIEW WITH

Sylvia Serfaty

by João P. Nunes*

Photo credits: **Olivier Boulanger**

Sylvia Serfaty is currently Silver Professor of Mathematics at the Courant Institute of New York University. Since obtaining her PhD from Université Paris-Sud in 1999, she has embarked on a stellar mathematical career in the fields of analysis, partial differential equations and mathematical physics. She has received many distinctions which include, among others, the European Mathematical Society Prize (2004) and the Henri Poincaré Prize (2012). Since 2019, she is a member of the American Academy of Arts and Sciences.

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In the beginning of your career, why did you choose the field of PDE and Ginzburg-Landau models ? Did you hesitate between this and any other topic ?

It really happened by chance. I didn't know what to specialize in, I liked analysis but I also thought of differential geometry and dynamical systems. I decided to follow many different graduate courses, and along the way I ended up liking Fabrice Bethuel's course, which is how I did my PhD with him, and he proposed the topic of Ginzburg-Landau vortices.

Can you describe, in a few words, what are the main long-term goals of your research ?

After a long streak on Ginzburg-Landau, I am now mostly focusing on statistical mechanics models of particles with Coulomb interaction, similar to the vortex interactions but in general dimension, and also their dynamics. In the long term, I would like to understand better what happens in these systems: the phase transitions (such as the Kosterlitz-Thouless phase transition and the possibly solid/liquid phase transition in two dimensional one-component plasmas). But maybe it will be too hard and I will think of something else! A lot of research is not planned, but happens as you go along.

In recent work, you relate the Cohn-Kumar conjecture on energy minimizing configurations of points in dimensions 2, 8 and 24 to conjectures on systems with Coulomb interactions. What makes these dimensions special ?

The conjecture is only made for these dimensions. What is special about them is the existence of lattices which are such that the norms of all vectors in the lattice are the square root of an even number (I am simplifying a bit here). There could be other dimensions where the existence of such a lattice occurs, but for sure, it is not true in all dimensions. For instance in dimension 3, there is no such lattice, and the Cohn-Kumar conjecture (that there is a universally minimizing lattice) is wrong, as the optimal lattice depends on the precise nature of the monotone interaction.

Among the very many results that you have produced over the years, is there one that you consider to be mathematically the most beautiful ?

I would say there are two competitors: one is this work with Étienne Sandier you alluded to, where we bridge between the Ginzburg-Landau model of superconductivity and the Cohn-Kumar conjecture in dimension 2 — essentially we prove why the Abrikosov (triangular) lattice happens. It comes as the culmination of a long program with technical buildup and is striking both from the mathematical and physical point of view.

The other is the introduction of the modulated energy method for deriving the mean-field limit for Ginzburg-Landau dynamics with many vortices, and which ends up working for more general discrete dynamics. I think I particularly like it because I thought about it on and off for 17 years before finding the right approach, which in the end is quite simple to phrase and elegant, in my view.

When you moved from France to the US, back in 2001, what were the main differences that you found from the academic point of view ?

Everything is similar and everything is different at the same time. I think I was shocked that in the US you can get a whole undergraduate education in math without having seen a proof-based course except for the last two courses in the last year. At the same time, the situation of the faculty is much more comfortable than in France and even more valued in society. It seemed a really strange use of the intellectual power of the faculty to make them teach these undergraduate courses. I felt like I was being used as a high school teacher, academically and also emotionally. In office hours, US students told me about their life problems and expected me to hold their hand in their studies in a way that students in France never would. And the contrast with graduate courses was huge, much bigger than in Europe.

How often in your work have fully-blown theorems and results start by some particular, simple calculation or observation that you can identify ?

This has definitely happened several times, in fact most theorems start this way, with a little calculation. I remember particularly the time early on where I found an identity which provided an *entropy* and thus a lower bound (in a sort of calibration way) for the model of micromagnetics I was studying with Tristan Rivière. I remember I was very excited, it felt like I had stumbled upon it practically by accident. The modulated energy method and the Gamma-convergence of gradient flows were also in that category although the computation came more from a conceptual reasoning.

What kind of teaching do you do at Courant ? Both graduate and undergraduate ? How does teaching interact with research ?

Yes, I do both graduate and undergraduate. The graduate teaching very directly interacts with research, as most of my PhD students follow my courses, and also the students of other colleagues. The more advanced *special topics* courses are places of discussion with the students, I sometimes ask them to present papers that are recent research, and often research questions or progress happens in relation. In fact the interaction with the PhD students is, I would say, my favorite part of the job. I am quite proud of my students!

I have read that you play the piano. How often do you play, what role does music play in your life ?

It depends on the periods, often these days I don't play enough because I am too busy. I try to do it at least a couple of times a week. When I finally do it is always a moment to relax and let my thoughts wander. It is for me a way to connect to music, and an activity that is totally *free* in the sense that it is not serving to achieve any goal or duty, and we don't have many of these.

Do you play in the periods when you are more intensely involved in a hard point of a research problem? Does it help ?

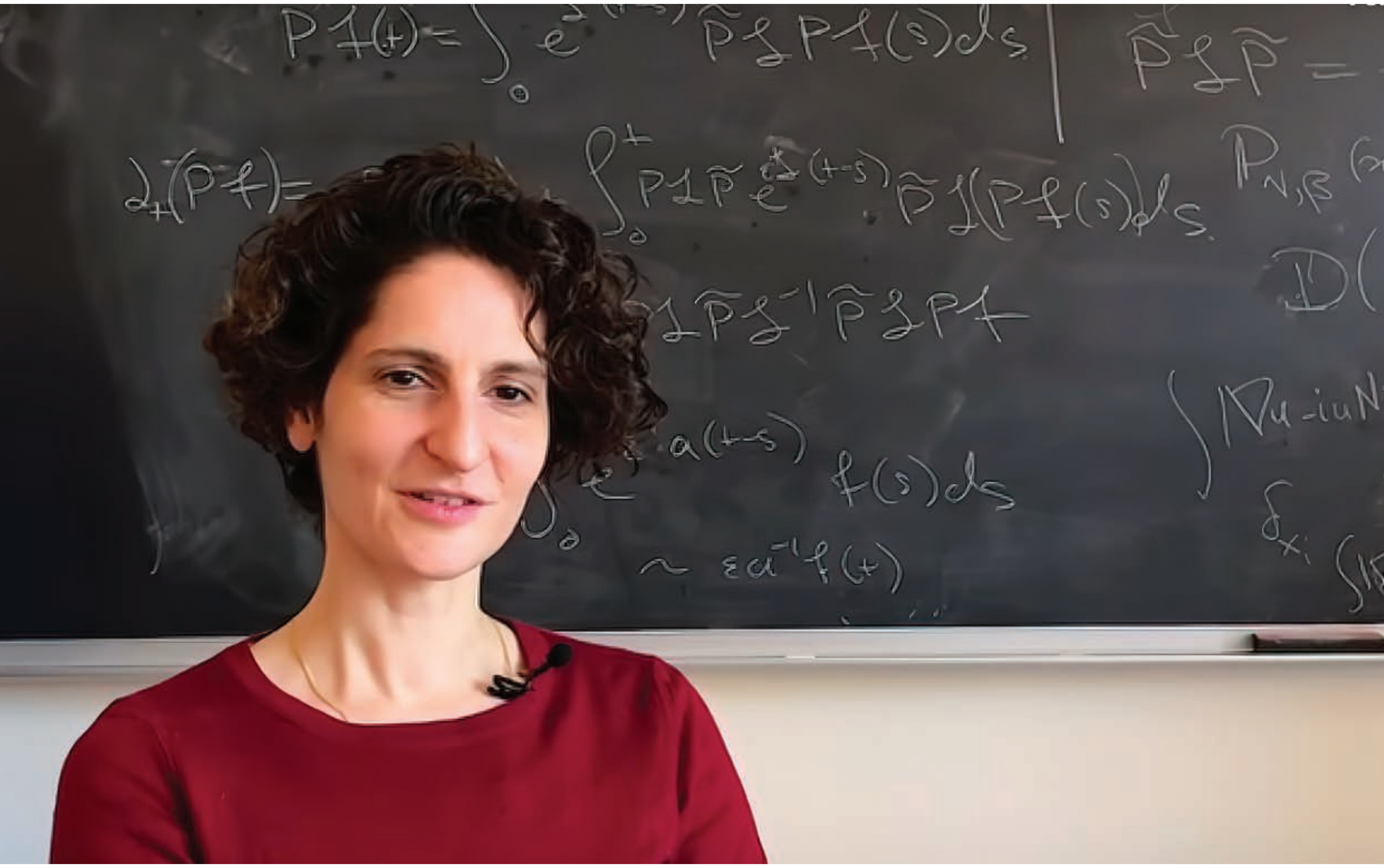


Photo credits: **Stephan Falke**

I haven't observed any correlation but I will pay more attention now!

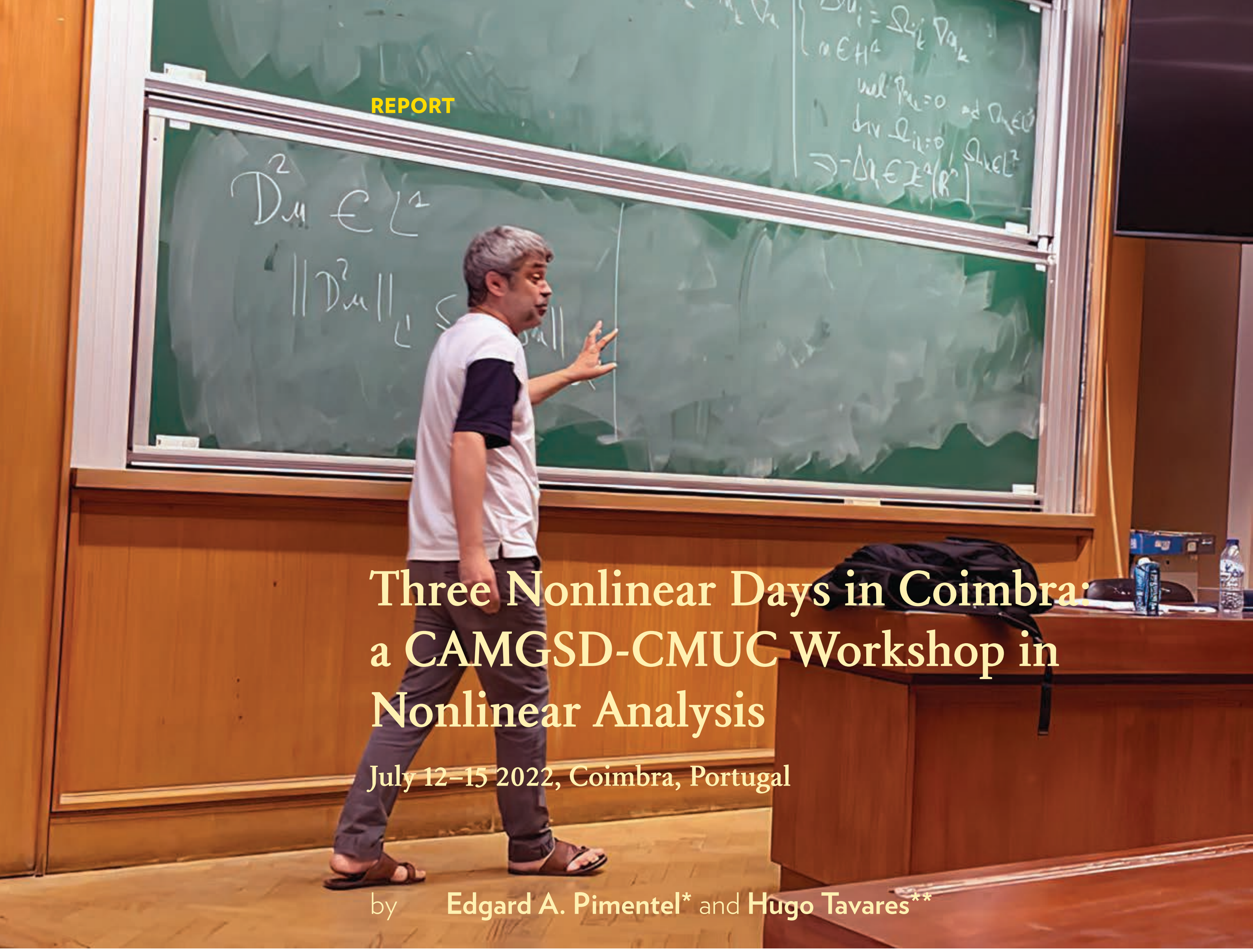
You have mentioned that your interest in Mathematics arose in high school. Can you imagine yourself with another occupation? What would have come in second after Mathematics? Would you be also a researcher? In what field?

Good question. I never thought of anything else seriously and never had a real plan B! I don't think I would be a good researcher in other fields, I am sure I would be terrible at experiments, or at things geared in data or concrete life, honestly. I think if I wasn't a mathematician I would like to do rather something else creative or artistic, sometimes now with age I start to think of writing ... But it is not clear I have enough talent for other things!

If you had to choose a little piece of elementary Mathematics that lies the closest to your heart, what would it be ? (Sorry, I know this is an unfair question.)

I remember that when I was a student I had an esthetic shock when first learning about Z/pZ and how one can answer arithmetic questions mod p. I also liked any kind of functional inequality. For instance Cauchy-Schwarz or Minkowski, or the elementary proof by Fourier of the isoperimetric inequality in 2D ...

Thank you so much, Sylvia.



REPORT

Three Nonlinear Days in Coimbra: a CAMGSD-CMUC Workshop in Nonlinear Analysis

July 12–15 2022, Coimbra, Portugal

by Edgard A. Pimentel* and Hugo Tavares**

Between the 13 and 15 of July 2022, the event entitled *Three Nonlinear Days in Coimbra: a CAMGSD-CMUC Workshop in nonlinear Analysis* took place at the University of Coimbra. Jointly organised by the *Centre for Mathematics at the University of Coimbra* (CMUC) and the *Centre for Mathematical Analysis, Geometry and Dynamical Systems* (CAMGSD-IST), the workshop gathered international specialists working in the fields of nonlinear analysis, variational problems and methods, regularity theory for elliptic equations, free boundary problems, and other related topics.

The structure of the event comprised two mini-courses and nine plenary talks. The list of plenary speakers included **Isabeau Birindelli** (Rome), **Denis Bonheure** (Brussels), **Jean-Baptiste Casteras** (Lisbon), **Simão Correia** (Lisbon), **Cristina De Filippis** (Parma), **Manuel del Pino** (Bath), **Jean Dolbeault** (Paris), **Fabiana Leoni** (Rome), and **Gianmaria Verzini** (Milan).

The material presented in those talks provided the audience with an in-depth and in-breadth account of recent results in the field. Showcasing important advances and newly developed techniques, this part of the scientific program equipped the participants with a general perspective on trendy, further directions of research yet to be pursued.

As concerns the mini-courses, there were lectures by **Jose Carrillo** (Oxford), on the topic of *nonlocal aggregation-diffusion equations*, and by **Yannick Sire** (Johns Hopkins U.), on *harmonic maps*. We highlight that both lines of research are still very incipient within the Portuguese mathematical community. As a consequence, those mini-courses presented an important opportunity

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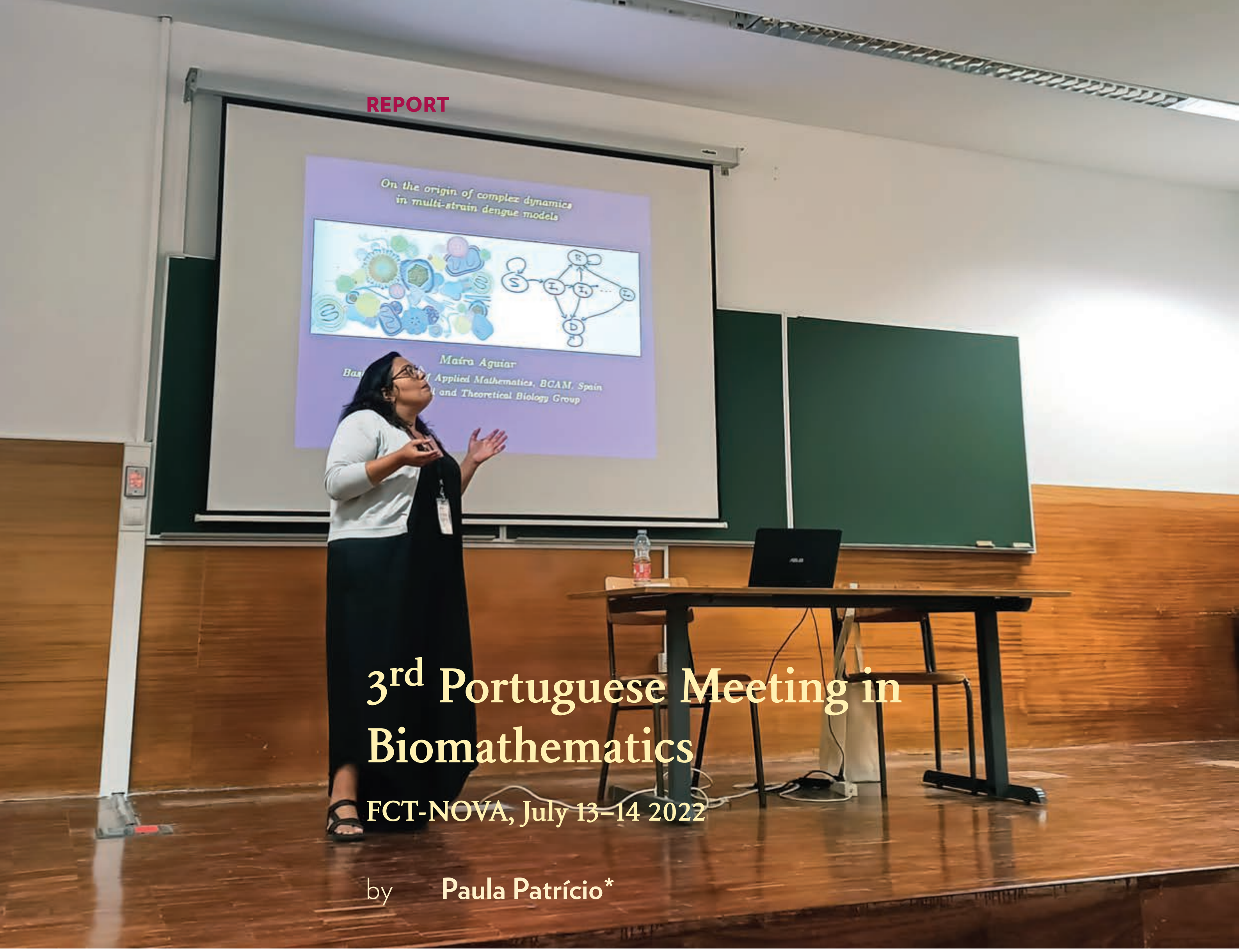
for advanced students and young researchers to get acquainted with beautiful, and mathematically very challenging classes of problems.

A series of contributed talks completed the program. In addition to being a space of scientific interaction, this part of the workshop also succeeded in giving young researchers the opportunity to talk about their work. It reinforces the systematic concern of the organisation with creating training opportunities for new graduates in the context of the local community. The list of contributed speakers included **Makson Santos** (Lisbon), **Delia Schiera** (Lisbon), **Rafayel Teymurazyan** (Coimbra) and **David Jesus** (Coimbra).

The event also offered an opportunity for long-term collaborators to catch up with their research agendas. We also expect new collaborations to arise from the several discussions taking place in the workshop environment. In particular, we highlight the networking opportunities, created mostly by young researchers and advanced students.

REPORT



3rd Portuguese Meeting in Biomathematics

FCT-NOVA, July 13–14 2022

by **Paula Patrício***

The Portuguese Meeting on Biomathematics is a biannual event, which aims to bring together national researchers interested in Biomathematics and, at the same time, to promote interaction between mathematicians working on models arising from the field of biology, with other researchers working in areas of biology and who use mathematics as an important tool in their investigation.

After two attempts postponed by the current COVID-19 pandemic, the *3rd Portuguese Meeting in Biomathematics* (III EPB), took place on the 13th and 14th of July of 2022, at *NOVA School of Science and Technology* (https://eventos.fct.unl.pt/3epb). III EPB was a joint organization of the *NOVA MATH* (*Center for Mathematics and Applications*) of the *NOVA School of Science and Technology*, *NOVA University of Lisbon* and the *Center for Research and Development in Mathematics and Applications of the University of Aveiro* (CIDMA), integrated in its thematic lines Mathematical Biology and Biomathematics.

The III EPB had 6 invited speakers, 17 contributed talks and 5 posters, with more than 50 participants, some of them international. The plenary talks were on exciting and diversified topics and were given by the renowned Portuguese and international scientists in this area of research. Additionally, there was a short course *Stochastic processes theory and applications*, on the 12 and 15 Jully, by **Nico Stollenwerk**, **Maíra Aguiar**, *Basque Center for Applied Mathematics* (BCAM), *Ikerbasque*, *Basque Foundation for Science*, Bilbao, Spain. The course had 10 participants, most of them master and PhD students.

The III EPB has been sponsored by the Portuguese research centres *CMA* and *CIDMA* and the *Centro Internacional de Matemática*, to whom we are grateful.

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Invited speakers

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INRIA Paris Research Centre
Ana Jacinta Soares
CMAT, Universidade do Minho
Carlos Ramos
CIMA, Universidade de Évora
Cristiana J. Silva
CIDMA, Universidade de Aveiro
Jorge Orestes Cerdeira
CMA, Universidade Nova de Lisboa
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IFCS 2022

Classification and Data Science in the Digital Age

by **Paula Brito***

Last July took place in Porto the 17th conference of the *International Federation of Classification Societies* (IFCS) —IFCS 2022, *Classification and Data Science in the Digital Age*. The conference was a joint organization of the *Portuguese Association for Classification and Data Analysis* (CLAD) and the *Faculty of Economics of the University of Porto*, and was supported by CIM — see https://ifcs2022.fep.up.pt/ .

IFCS,[1] of which CLAD is a member, is a federation of national, regional or language-based classification societies, and organises an international conference every two years. IFCS 2022 was its first edition in Portugal.

The conference gathered 330 participants from 32 countries in Europe, Asia, Africa, Australia and the Americas, including more than 70 students. Prestigious Keynote Speakers delivered lectures on highly topical subjects:

Charles Bouveyron (Université Côte d'Azur, France): *Statistical Learning with Dynamic Interaction Data for Public Health*

Dianne Cook (Monash University, Australia): *A Showcase of New Methods for High Dimensional Data Viewing with Linear Projections and Sections*

Genevera Allen (Rice University, USA): *Fast Minipatch Ensemble Strategies for Learning and Inference*

João Gama (University of Porto, Portugal): *Trends in Data Stream Mining*

Plenary lectures were also given by **Edwin Diday** (Université Paris Dauphine-PSL, France), who was awarded the *IFCS Medal for Outstanding Research Achievements in Classification*, and **Rebecca Nugent** (Carnegie Mellon University, USA), President-Elect of IFCS.

The scientific programme further included semi-plenary invited sessions on current topics in Data Science: Categorical Data Analysis and Visualization, Clustering and Classification of Time Series, Covid Data Analysis, Dimension Reduction, and Explainable Machine Learning. There were 67 thematic sessions and two Poster sessions.

The Social Programme provided moments of relaxation and informal networking. It counted with an Ice-Breaker kindly offered by Comissão de Viticultura da Região dos Vinhos Verdes (CVRVV), a Cocktail at Praia da Luz, a boat trip on Douro River and finally the conference dinner at Taylor's, preceded by a visit to the Port Wine cellars. We believe that all will have fond memories of IFCS 2022 in Porto !

[1] https://ifcs.boku.ac.at/site/doku.php

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Travelling waves and their speeds for FKPP equations — an overview in the framework of ODEs

by **Luís Sanchez***

Abstract.—We review the basics on admissible speeds of travelling waves to FKPP (Fisher-Kolmogorov-Petrovski-Piskounov) equations, starting from the classical setting and pointing out some changes needed to deal with nonlinear diffusion. Most of the material is based on elementary theory of ordinary differential equations (ODEs).

Consider the following curious (and simple) problem concerning a class of first order ODEs: given a function $f : [0,1] \to \mathbb{R}$ of **type A**, i.e. continuous, $f(0) = f(1) = 0$ and $f(s) > 0\ \forall s \in]0,1[$, *find the values of the parameter $c > 0$ so that the problem*

$$y' = 2(c\sqrt{y} - f(u)), \quad y(0) = y(1) = 0 \tag{1}$$

has a solution $y(u) \geq 0$, $0 \leq u \leq 1$.

More generally, we shall look also at the problem where the equation is instead

$$y' = q(c\, {y_+}^{1/p} - f(u)), \tag{2}$$

(p, q being positive conjugate, i.e. $1/p + 1/q = 1$.)

This problem turns out to be useful in the fields of Applied Mathematics where the equations known as *FKPP (Fisher-Kolmogorov-Petrovski-Piskounov) equations* provide relevant models. Unsurprisingly, the elementary problem is interesting in itself, since it carries a lot of important information and features from its motivating source.

In 1937 R. Fisher [12] proposed a partial differential equation model for the propagation of an advantageous gene in a one dimensional spatial setting. In the same year, Kolmogorov, Petrovsky and Piskunov [15] obtained significant properties of the PDE model. The simplest protopype is given by the PDE

$$u_t = u_{xx} + u(1-u). \tag{3}$$

u denoting the frequence of the gene, taking values from 0 to 1.

Other problems in the applied sciences lead to (FKPP) equations

$$u_t = u_{xx} + f(u) \tag{4}$$

where f is a function of type A. An example is the Zeldovich equation, $u_t = u_{xx} + u^2(1-u)$, from the theory of combustion, where u means temperature and $u^2(1-u)$ represents the generated heat.

Equation (4) has two equilibria $u = 0$ e $u = 1$. Meaningful solutions $u(x,t)$ take values between 0 and 1. Let us look for **travelling waves** with **speed** c: $u(x,t) = u(\xi)$, $\xi = x+ct$, whose profile u is increasing and connects the equilibria: $u(-\infty) = 0$, $u(\infty) = 1$. Substitution in $u_t = u_{xx} + f(u)$ leads to

$$u''(\xi) - cu'(\xi) + f(u(\xi)) = 0, \quad \xi \in \mathbb{R}. \tag{5}$$

Therefore we look for increasing solutions of (5) such that

$$u(-\infty) = 0, \quad u(+\infty) = 1. \tag{6}$$

The study of the FKPP equations has generated a rich literature. The reader is referred to the (somewhat arbitrary) short selection [12, 15, 3, 13, 16, 5] and to their references for an account of the mathematical development of the subject.

Travelling wave solutions $u(x+ct)$ and their speeds are of great interest, because under certain conditions they shape the behaviour of solutions $u(x,t)$ as $t \to +\infty$. As we shall recall in a moment, the admissible speeds form a half-line $c \geq c^*$, where the minimum $c^* > 0$ (called **critical speed**) has a special

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role. For example, as we show below, $c^* = 2$ for the simple model where $f(u) = u(1-u)^\alpha$, $\alpha \geq 1$; and (see [15]) if $u(x,t)$ denotes the solution such that $u(x,0)$ is the Heaviside function, then $u(x + a(t), t) \to u(x)$ for every x, as $t \to +\infty$, where $a(t) = 2t + o(t)$ at ∞ and $u(x)$ is the profile of the travelling wave with speed 2.

This article reviews, in an elementary setting and in as a self-contained manner as possible, the basics about some classic and more recent results on travelling waves for FKPP. In the final sections we sketch the corresponding results for the analogous model with nonlinear diffusion.

In what follows, we consider equation (5) where f is continuous, of type A in $[0,1]$, and in addition

$$(H) \qquad \exists k > 0 \,:\, f(u) \leq \min(ku,\ k(1-u)),$$

for every $u \in [0,1]$.

A number $c \in \mathbb{R}$ is called an *admissible speed* to (5), or with respect to f, if there exists an increasing solution of (5)-(6), that is, a monotone heteroclinic connecting the equilibria 0 and 1.

Remark 1.— When f is differentiable, linearization about the equilibrium at the origin yields immediately that *increasing* travelling waves can exist only if $c \geq 2\sqrt{f'(0)}$.

1 Reduction to a first order problem.

Let $u = u(\xi)$ be an increasing solution of

$$u''(\xi) - cu'(\xi) + f(u(\xi)) = 0$$

in $\mathbb{R}$. Then $u'(\xi) > 0, \xi \in \mathbb{R}$, there exists $\xi(u)$, the inverse function of $u = u(\xi)$, and u' may be given as a function of u: $\phi(u) = u'(\xi(u))$. Therefore $\phi \ :]0,1[\to \mathbb{R}^+$ is C^1, and may be extended to $[0,1]$ with $\phi(0) = \phi(1) = 0$. ϕ is a solution of $\phi(u)\phi'(u) - c\phi(u) + f(u) = 0$. Hence, setting $\psi(u) := \phi(u)^2$, ψ solves the problem (1) (the boundary conditions come from the fact that $u'(\pm\infty) = 0$).

Conversely, if ψ solves (1) and we consider the Cauchy problem $u' = \sqrt{\psi(u)}$, $\quad u(0) = 1/2$, it can be shown that its solution is defined in the whole real line $(-\infty, \infty)$ (using assumption (H)). That solution $u(t)$ satisfies (5–6) and $u'(t) > 0$, for every t.

In summary, $u = u(\xi)$ is an increasing solution connecting the two equilibria if and only if $\phi(u)^2$ solves (1). Hence, *the square root of a solution of (1) gives the profile in the phase plane of the trajectory of a travelling wave.* And the following simple facts may be proved.

1.A.—*If $f'(0)$ exists and equation $y'(u) = 2c\sqrt{y(u)} - 2f(u)$ has a solution $y(u)$ such that $y(0) = 0$ and $y(u) > 0$ in some interval $(0,\eta)$, then $c^2 \geq 4f'(0)$.*

1.B. [Lower solution criterion].—*If there exists a C^1 function $s : [0,1] \to \mathbb{R}$ such that $s(0) = 0$, $s(u) > 0$ if $u \in (0,1)$ and $\forall\, u \in [0,1]$,*

$$s'(u) \leq 2c\sqrt{s(u)} - 2f(u), \qquad (7)$$

then the first order problem $\psi'(u) = 2c\sqrt{\psi(u)} - 2f(u)$, $\quad \psi(0) = \psi(1) = 0$ has a (unique) solution.

1.C.—*The set of admissible speeds for f is the set of numbers c such that (1) has solutions. It is an unbounded closed interval $[c^*, +\infty)$ with $c^* > 0$.*

We sketch the proof of 1.B–C (for more general assertions and proofs see e.g. [11]). Let

$$M = \sup_{0<u<1} \frac{f(u)}{u}$$

(which exists by property (H)). If $c_0^2 \geq 4M$ and $c \geq c_0$, the equation $s'(u) = 2c_0\sqrt{s(u)} - 2Mu$ has a solution such that $s(0) = 0$ and $s(u) > 0$, for all $0 < u \leq 1$: take $s(u) = (Bu)^2$ with

$$B = \frac{c_0 \pm \sqrt{c_0^2 - 4M}}{2}.$$

Hence $s(u)$ is a lower solution of $y'(u) = 2c_0\sqrt{y(u)} - 2f(u)$, $y(0) = 0$. Therefore a positive solution $\bar{y}$ of this equation, with $\bar{y}(0) = 0$, exists. The (unique) solution $\tilde{u}$ of the same equation such that $\tilde{u}(1) = 0$ is the desired solution: its graph cannot meet either the graph of $\bar{y}$ (by uniqueness) or the u–axis (because of the sign of the slope) for $0 < u < 1$. Hence the set of admissible c is a nonempty interval; by the equivalence between (1) and (5)–(6) any number c_0 such that $c_0 \geq 2\sqrt{M}$ is an admissible speed; it has a minimum element by an elementary compactness argument.

To each function f satisfying our assumptions we thus associate a number $c^* > 0$ which is the minimum admissible speed of f. We write $c^* = c^*(f)$. We call it also the *critical speed* of f.

Remark 2.— Easy consequences are:

$$f \geq g \Rightarrow c^*(f) \geq c^*(g); \text{ and}$$

if $f'(0)$ exists,

$$2\sqrt{f'(0)} \leq c^*(f) \leq 2\sqrt{\sup_{0<u<1} \frac{f(u)}{u}}. \qquad (8)$$

In particular, if $f'(0)$ exists and $f(u) \leq f'(0)u$ for every $u \in (0,1)$, then $c^* = 2\sqrt{f'(0)}$.

2 Asymptotic behaviour at infinity

Let us look at the behaviour of solutions of (1) at the endpoints of $[0,1]$: We assume that $f'(0)$ and $f'(1)$ exist. The objective is to compute the limits

$$\lim_{\xi\to-\pm\infty} \frac{u'(\xi)}{u(\xi)}.$$

2.A.—*Suppose that $f'(0)$ exists. If $\psi(u)$ solves $\psi'(u) = 2c\sqrt{\psi(u)} - 2f(u)$, $\quad \psi(0) = 0$, with $\psi(u) > 0$ in some interval $(0,\eta)$, then the derivative $(\sqrt{\psi})'(0)$ exists and is a root of $x^2 - cx + f'(0) = 0$.*

Denote $\lambda^-(c) \leq \lambda^+(c)$ the roots of $x^2 - cx + f'(0) = 0$ when they exist.

2.B.—*Let c be an admissible speed of $u'' - cu' + f(u) = 0$.*

- *If $c = c^*$, $(\sqrt{\psi})'(0) = \lambda^+(c)$.*
- *If $c > c^*$, $(\sqrt{\psi})'(0) = \lambda^-(c)$.*

Let us point out the steps needed in the proof of 2.B.:

Step 1. Let $\eta > 0$, $0 < A < B$, $0 \leq a < b$, $0 < c_1 < c_2 < 2A$ be constants such that

$a \leq f(u)/u \leq b, \quad 0 < u \leq \eta;$

$A^2 - cA + b < 0 < B^2 - cB + a$, for every $c \in [c_1, c_2]$.

Then for $c \in [c_1, c_2]$ the initial value problem

$$\psi'(u) = 2c\sqrt{\psi(u)} - 2f(u), \quad \psi(0) = 0$$

has a unique solution such that $A^2u^2 \leq \psi(u) \leq B^2u^2$ for $0 \leq u \leq \eta$. Moreover the solution depends continuously on c.

(The proof uses the contraction operator

$$Tv(u) = 2c\int_0^u \sqrt{v(t)}dt - 2\int_0^u f(t)dt,$$

for $u \in [0,\eta]$, in the space X of continuous functions v such that $A^2u^2 \leq v(u) \leq B^2u^2$, for every $v \in [0,\eta]$.)

Step 2. Let $\bar{c} > 2\sqrt{f'(0)}$. Choose A, B such that:

$$\frac{\bar{c}}{2} < A < \lambda^+(\bar{c}) < B.$$

Then there are numbers $0 \leq a \leq f'(0) < b$, $\eta > 0$ and an interval $[c_1, c_2]$ containing $\bar{c}$ such that all conditions of the precedent claim are satisfied.

Step 3. Given $c > 2\sqrt{f'(0)}$, there exists $\eta > 0$ such that $\psi'(u) = 2c\sqrt{\psi(u)} - 2f(u)$, $\psi(0) = 0$, has a unique solution ψ in $[0,\eta]$ such that $(\sqrt{\psi})'(0) = \lambda^+(c)$.

Step 4. Let $c_0 > 2\sqrt{f'(0)}$, $c_0 \geq c^*(f)$ and ψ be the solution of $\psi'(u) = 2c_0\sqrt{\psi(u)} - 2f(u)$, $\psi(0) = 0$, $\psi(1) = 0$, $\sqrt{\psi})'(0) = \lambda^-(c)$. Then $c_0 > c^*(f)$.

In terms of solutions of the second order equation this reads:

2.C *Let c be an admissible speed of f and $u(t)$ a corresponding monotone heteroclinic solution.*

- *If $c = c^*$,*
$$\lim_{t\to-\infty} \frac{u'(t)}{u(t)} = \lambda^+(c).$$
- *If $c > c^*$,*
$$\lim_{t\to-\infty} \frac{u'(t)}{u(t)} = \lambda^-(c).$$

Asymptotic description of solutions near $+\infty$ can also be given. That discussion is simpler.

3 Finding some exact solutions

The form of (1) allows to obtain easily some exact heteroclinics.

Given a reaction term $f(u) = u^m - u^n$, $0 \leq u \leq 1$, where $1 \leq m < n$, let us look for a solution of (1) of the form

$$y(u) = \lambda(u^\alpha - u^\beta)^2.$$

An easy computation shows:

- If $m = 1$ and $n = 2$, that is, for the simplest prototype, we obtain a solution with $\alpha = 1$, $\beta = 3/2$, $\lambda = 2/3$ and $c = 5/\sqrt{6}$. The profile obtained is
$$y(u) = \frac{2}{3}u^2(1-\sqrt{u})^2, \quad \text{with} \quad c = \frac{5}{\sqrt{6}}$$
associated to a non-critical speed. From the equation $u' = \sqrt{y(u)}$ we obtain the expression of the heteroclinic.
$$u(t) = \frac{1}{((\sqrt{2}-1)e^{-\frac{t}{\sqrt{6}}} + 1)^2}$$
This solution was given by Ablowitz and Zeppetella [1].

- This example generalizes to: if $m = 1$ and $n > 1$, we find $\alpha = 1$, $\beta = (n+1)/2$, $\lambda = 2/(n+1)$ and $c = (n+3)/\sqrt{2(n+1)}$. The profile is
$$\psi(u) = \frac{2}{n+1}u^2\left(1 - u^{\frac{n-1}{2}}\right)^2$$
and the corresponding heteroclinic is
$$u(t) = \frac{1}{\left((2^{\frac{n-1}{2}} - 1)e^{-\frac{(n-1)t}{\sqrt{2(n+1)}}} + 1\right)^{\frac{2}{n-1}}}.$$
The critical speed is 2, for all n, and
$$c = c_n = \frac{n+3}{\sqrt{2(n+1)}} \to 2$$
as $n \to 1$.

- If $m = 2$ and $n = 3$, (Zeldovich's equation), the calculation shows that we can take $\alpha = 1$, $\beta = 2$, $\lambda = 1/2$ and $c = 1/\sqrt{2}$.

  The profile thus obtained is
$$y(u) = \frac{1}{2}u^2(1-u)^2, \quad \text{with} \;\; c = \frac{1}{\sqrt{2}}.$$
  In this case $f'(0) = 0$ and
$$\lim_{u\to 0}\frac{y(u)}{u^2} = \frac{1}{2}.$$
  By 2.B. we conclude that $1/\sqrt{2}$ is the critical speed for Zeldovich. Solving $u' = y(u)$ we obtain the corresponding heteroclinic
$$u(t) = \frac{1}{1 + e^{-\frac{t}{\sqrt{2}}}}.$$

- More generally, if $m = (n+1)/2$ and $n > 1$ we obtain: $\alpha = 1$, $\beta = m$, $\lambda = 1/m$ and $c = 1/\sqrt{m}$.

  The profile is
$$y(u) = \frac{1}{m}u^2\left(1 - u^{\frac{n-1}{2}}\right)^2, \quad \text{with} \;\; c = \frac{1}{\sqrt{m}}$$
  and the speed is critical. (See [9].)

4 Sharp solutions

The more general equation with density dependent diffusion
$$u_t = (D(u)u_x)_x + g(u)$$
with $D > 0$ in $(0,1)$ has a corresponding ODE for travelling waves
$$(D(u)u')' - cu' + g(u) = 0. \qquad (d)$$

The substitution $\varphi(u) := D(u)u'$ leads to the first order problem
$$\psi'(u) = 2c\sqrt{\psi(u)} - 2f(u), \qquad \psi(0) = \psi(1) = 0$$
where
$$\psi(u) := \varphi(u)^2, \quad f(u) = D(u)g(u)$$
and the heteroclinic is found via
$$u' = \frac{\sqrt{\psi(u)}}{D(u)}, \qquad u(0) = 1/2.$$
Then from the results in section 3 we find *sharp solutions.*

4.A.—*Assume that:* $D \in C^1[0,1]$, $D > 0$ *in* $(0,1]$, $D(0) = 0$ *and* $D'(0) > 0$. *Then, for a function* g *of type A:*

(i) *There exists a stricly increasing solution of (d) with* $u(-\infty) = 0$, $u(+\infty) = 1$, *if and only if* $c > c^*$*;*

(ii) *If* $c = c^*$, *(d) has an increasing solution defined in* $[0, \infty[$ *with* $u(0) = 0$, $u(\infty) = 1$ *and* $u'(0) = c^*/D'(0)$.

Those solutions are unique up to translation;

(iii) *If* $c < c^*$, *the equation (d) has no increasing solution in any interval* (b, ∞) *with*
$$\lim_{t\to b^-} u(t) = 0, \qquad \lim_{t\to -\infty} u(t) = 1.$$
See [19, 17].

5 Further characterizations of the critical speed

We have been considering functions f of *type A in* $[0,1]$. And, if there exists $\theta \in (0,1)$ such that $f \equiv 0$ in $[0,\theta]$ and f is of type A in $[\theta, 1]$, then we say that f *is of type B in* $[0,1]$.

In such cases there exists a unique "admissible speed" (see e.g. [7]):

5.A.—*Let* f *be of type B in* $[0,1]$. *Then there exists a number* $c^* > 0$ *so that problem (1st order with zero boundary conditions) admits a positive solution in* $(0,1)$ *if and only if* $c = c^*$.

Now the symbol $c^*(f)$ is meaningful if f is of type A or B, and the construction of c^* preserves monotonicity with respect to f.

5.B.—*Let* f *be of type A in* $[0,1]$. *Let* $f_1 \leq f_2 \leq \cdots$ *be a sequence of functions of type B in* $[0,1]$ *such that*

$\lim_{n\to\infty} f_n(x) = f(x)$ *for every* $x \in [0,1]$. *Then* $c^*(f_n) \uparrow c^*(f)$.

Variational characterizations of the critical speed are also possible. In addition to the one given by Benguria and Depassier [6], the following was given (in a slightly different form) in [2].

5.C.—*Let* $F(u) = \int_0^u f(s)\,ds$ *(where* f *is defined outside* $[0,1]$ *with value* 0*) and set*

$$X = \left\{ v \in C(\mathbb{R}_+) \;\middle|\; v(0) = 0 \text{ and } \int_0^\infty v'^2 < \infty \right\}.$$

Then

$$c^* = \sqrt{\frac{1}{\lambda}}, \quad \lambda = \inf_{v\in X,\, \int_0^\infty \frac{F(v(s))}{s^2}\,ds = 1} \int_0^\infty \frac{v'(s)^2}{2}\,ds$$

and the inf *is attained if* $c^* > 2\sqrt{f'(0)}$.

Outline of proof in the simple case where $f'(0) = 0$: Consider a minimizing sequence v_n, that is

$$\lim_{n\to\infty} \int_0^\infty \frac{v_n'(s)^2}{2}\,ds = \lambda, \qquad \int_0^\infty \frac{F(v_n(s))}{s^2}\,ds = 1.$$

By Hardy's inequality there is $C \in \mathbb{R}$ so that

$$\int_0^\infty \frac{(v_n(s))^2}{s^2}\,ds < C.$$

It is clear that one may suppose $0 \le v_n \le 1$. Since $f'(0) = 0$, given $\varepsilon > 0$, exists $\delta > 0$ such that $F(z) \le \varepsilon z^2$ if $0 \le z \le \delta$. Also, there exists $\eta > 0$ such that $0 \le t \le \eta \Rightarrow v_n(t) \le C_1\sqrt{\eta} = \delta$ for every n. Hence, for large n,

$$\int_0^\eta \frac{F(v_n(s))}{s^2}\,ds \le \varepsilon C.$$

The *tail*

$$\int_A^\infty \frac{F(v_n(s))}{s^2}\,ds$$

for large A is clearly uniformly small. From $v_n \to v$ uniformly in compact intervals, $v_n' \to v'$ weakly in $L^2(0,\infty)$ we obtain by a standard argument that v is a minimizer. Moreover, exploiting the homogenity of the constrained problem (induced by changes of variable $s = kt$, $k > 0$), we see that v minimizes the map

$$w \mapsto \int_0^\infty \frac{(w'(s))^2}{2}\,ds - \lambda \int_0^\infty \frac{F(w(s))}{s^2}\,ds.$$

The minimizer solves

$$v''(s) + \lambda \frac{f(v(s))}{s^2} = 0, \quad s > 0.$$

The change of variable $s = e^t$ yields

$$u''(t) - u'(t) + \lambda f(u(t)) = 0, \quad -\infty < t < +\infty$$

and considering the behaviour (and integrability property) of the solution as it approaches 0, we conclude that $\lambda = 1/c^*(f)^2$.

6 The case of nonlinear diffusion: basic properties

Consider the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(u) \left| \frac{\partial u}{\partial x} \right|^{p-2} \frac{\partial u}{\partial x} \right] + g(u), \tag{9}$$

where $p > 1$, g is of type A, $u = 0$ and $u = 1$ being equilibrium solutions. We look for travelling wave solutions $u(t,x) = U(x - ct)$ for some $c > 0$ where U is monotone and connects the equilibria. To facilitate the checking of details to the interested reader we now assume, as in [11], that U is decreasing and $U(-\infty) = 1$, $U(+\infty) = 0$ (of course, a sign change allows to reduce to the case where U is increasing as previously). The problem is therefore

$$(D(u)|u'|^{p-2}u')' + cu' + g(u) = 0 \tag{10}$$

with limit conditions

$$u(-\infty) = 1, \;\; u(+\infty) = 0. \tag{11}$$

Looking for solutions with $u' < 0$ in their whole domain, we set

$$-v := D(u)|u'|^{p-2}u',$$

for such solutions, then v may be seen as a function of u. If we define

$$y(u) = v(u)^q$$

the function y will solve (2) with $(1/p)+(1/q) = 1$ and $y(0) = 0 = y(1)$, provided that we set

$$f(u) = D(u)^{q-1} g(u).$$

Then, as in the case $p = 2$, we obtain results on the admissible speeds and critical speed.

6.A.—*Assume that* f *is a function of type A in* $[0,1]$ *satisfying*

$$\sup_{u\in(0,1)} \frac{f(u)}{u^{q-1}} = \mu < +\infty,$$

or the stronger property

$$\lim_{u\to 0^+} \frac{f(u)}{u^{q-1}} = \lambda < +\infty.$$

Then there exists a constant $c^* > 0$ *(depending on* f *and* p*) such that*

$$y'(u) = q(c\, y_+(u)^{\frac{1}{p}} - f(u)), \quad y(0) = y(1) = 0$$

for $0 \le u \le 1$, admits a unique positive solution if and only if $c \ge c^*$. Moreover we have the estimate $(\lambda q)^{1/q} p^{1/p} \le c^* \le q^{1/q} p^{1/p} \mu^{1/q}$.

If in addition $\mu = \lambda$, then $c^* = q^{1/q} p^{1/p} \lambda^{1/q}$.

As in the case $p = 2$, the behaviour of solutions at the origin is related to the corresponding value of c:

6.B.—*If $c > c^*$, then y satisfies*

$$\lim_{u\to 0} \frac{y(u)}{u^q} = \omega_c^-(\lambda).$$

And, if $c = c^*$, then y satisfies

$$\lim_{u\to 0} \frac{y(u)}{u^q} = \omega_c^+(\lambda).$$

Here $\omega_c^-(\lambda) \le \omega_c^+(\lambda)$ stand for the positive roots of the function $x \mapsto x - cx^{1/p} + \lambda$.

Going back to the 2nd order problem, we can state:

6.C.—

(A) *Let $1 < p \le 2$. If g is a function of type A and $D \in C^1[0,1]$ with $D > 0$ in $[0,1]$, and*

$$\sup_{u\in(0,1)} \frac{g(u)}{u^{q-1}} < +\infty, \qquad \sup_{u\in(0,1)} \frac{g(u)}{(1-u)^{p-1}} < +\infty,$$

then there exists c^* such that

$$\begin{aligned} &(D(u)|u'|^{p-2}u')' + cu' + g(u) = 0,\\ &u(-\infty) = 1,\\ &u(+\infty) = 0 \end{aligned}$$

has a decreasing solution $u(t)$ taking values in $(0,1)$ if and only if $c \ge c^*$. That solution is unique up to translation.

Here the number c^* is associated to $f = D^{q-1}g$ according to the theory for the first order equation.

(B) *If, further, $g^*(0) \equiv \lim_{u\to 0^+} g(u)/u^{q-1}$ exists, then*

$$\lim_{t\to+\infty} \frac{u'(t)}{u(t)^{q-1}} = \begin{cases} -\frac{\omega_c^-(D(0)^{q-1}g^*(0))^{1/p}}{D(0)^{q-1}}, & c > c^*\\ -\frac{\omega_{c^*}^+(D(0)^{q-1}g^*(0))^{1/p}}{D(0)^{q-1}}, & c = c^*. \end{cases}$$

Sharp solutions can also be found in this case, cf. [11].

Also, a variational definition of c^* is possible in the nonlinear diffusion setting. Consider the critical speed c^* for

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x}\right] + f(u).$$

where $p > 1$ and f is type A. Then we have (see [14]).

6.D.—*Let F be the primitive of f with $F(0) = 0$; $\mathcal{F} = \{v \,|\, v$ is defined in $[0,\infty[$, $v(0) = 0\}$ so that we can define*

$$\gamma = \inf_{v\in\mathcal{F}\setminus 0} \frac{\frac{1}{q}\int_0^{+\infty} |v'(s)|^q\, ds}{\int_0^{+\infty} \frac{F(v(s))}{s^q}\, ds}.$$

Then the number c^* is given by

$$\gamma = \frac{q}{pc^{*q}}.$$

Moreover γ is attained if $\mu p^q \gamma < 1$ where

$$\mu := \lim_{u\to 0^+} \frac{f(u)}{u^{q-1}}.$$

7 Examples: computation of c^*

As a first example consider again the ODE for the p-Laplacian led diffusion

$$(|u'|^{p-2}u')' + cu' + f(u) = 0$$

where $f(u) = u^q(1-u)^{q-1}$ (that is, the analogue of Zeldovich's equation).

We compute an exact solution, for the corresponding 1st order equation, of the form

$$y = \alpha u^q (1-u)^q,$$

with $\alpha = 1/2$ and $c = 2^{-1/q}$.

Since $\lim_{u\to 0} \frac{f(u)}{u^{q-1}} = 0$ and $\lim_{u\to 0} \frac{y(u)}{u^q} = \frac{1}{2}$, we conclude from previous information on asymptotics that in fact $c^* = 2^{-1/q}$.

Next, we give a second example in the presence of advection: consider

$$\frac{\partial u}{\partial t} + ku\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[\left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x}\right] + f(u)$$

where $k > 0$.

The 2nd order ODE for travelling waves now is

$$(|u'|^{p-2}u')' + (c - ku)u' + f(u) = 0$$

and it may be studied by reduction to the first order equation (the boundary conditions are as before)

$$y' = q((c - ku)\, y^{1/p} - f(u)).$$

Similar arguments enable us to see that there exists a critical speed c^* and the corresponding solution may be identified by its behaviour near the origin. Let

$$f(u) = u^{q-1}(1-u)^{q-1}$$

(see [18] for $p = 2$). The 1st order equation has a solution of the form $y = \alpha u^q (1-u)^q$: substitution into the equation shows that in fact this is the case, provided that

$$\alpha = \left(\frac{k}{2}\right)^q \quad \text{and} \quad c = \frac{k}{2} + \left(\frac{2}{k}\right)^{q-1}.$$

Minimizing c we obtain

$$k_0 = 2(q-1)^{1/q} \quad \text{and} \quad c_0 = q^{1/q}p^{1/p}.$$

Computing $y(u)/u^q$, $f(u)/u^{q-1}$ at $u = 0$ and comparing with w_+, that turns out to be $w_+ = (k/2)^q$, we conclude that if $k \geq k_0$ then the critical speed is $c^* = k/2 + (2/k)^{q-1}$.

A different argument, based on the monotonicity of c with respect to k and the known result corresponding to $k = 0$ leads to (see [8])

$$c^* = q^{1/q}p^{1/p} \ \text{ for } \ 0 \leq k \leq k_0.$$

8 Final note

Recently, Audrito and Vázquez [4] have considered the model with doubly nonlinear diffusion

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\left|\frac{\partial u^m}{\partial x}\right|^{p-2}\frac{\partial u^m}{\partial x}\right] + g(u)$$

and its N-dimensional analogue. Among other results, they found that there is a *critical speed* $c^* > 0$ so that:

If $m > 0$, $p > 1$ and $m(p-1) > 1$ then there are travelling waves for speeds $c \geq c^*$, the profiles of the waves being positive everywhere if $c > c^*$ and finite *if $c = c^*$.* Here finite means that the profile vanishes in a half-line.

The critical speed c^* has a threshold role with respect to propagation of disturbances with bounded support, as in the case of linear diffusion.

Further new interesting results are also found in Drábek and Takáč [10] who consider degeneracy and singular diffusion coefficients.

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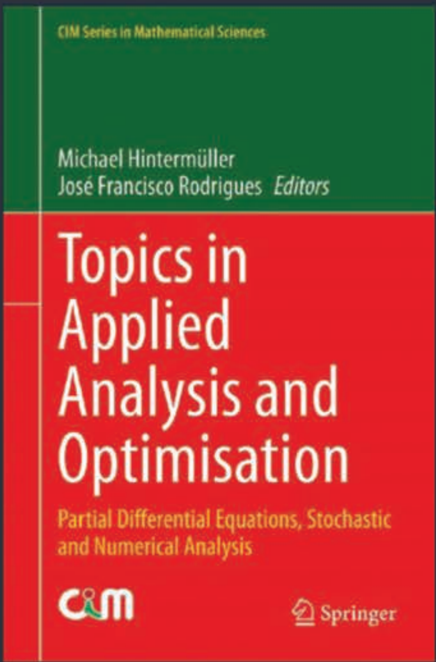

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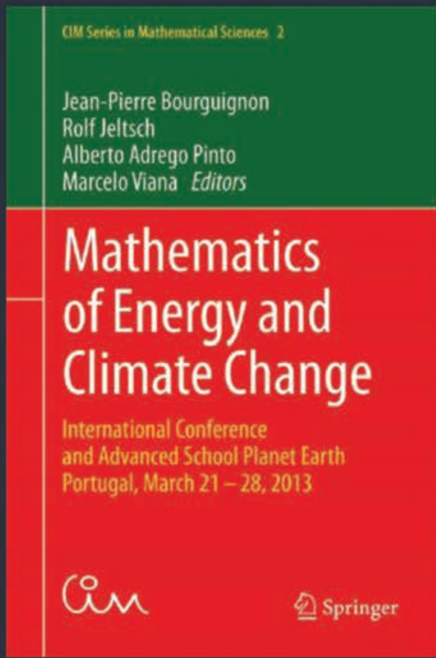

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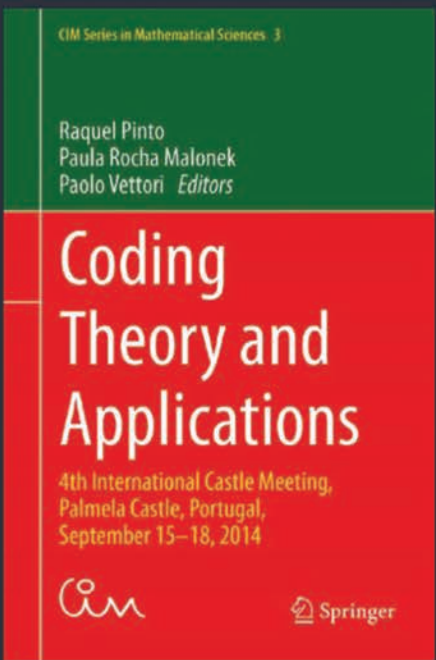

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REPORT

2nd Women in Mathematics Meeting

by **Ana Cristina Casimiro***, **Ana Cristina Ferreira****
Ana Jacinta Soares*** and **Lucile Vandembroucq******

The conference *2nd Women in Mathematics Meeting* was held at the *School of Sciences of the University of Minho*, Braga, from 7th through 9th September 2022.

The event received financial support from the following institutions: *Centro de Matemática da Universidade do Minho* (CMAT), *Centro de Matemática e Aplicações da Universidade Nova de Lisboa* (NOVAMath), *Centro Internacional de Matemática* (CIM), *Foundation Compositio Mathematica*, *Fundação para a Ciência e a Tecnologia* (FCT).

This was the second edition of the *Women in Mathematics Meeting* in Portugal (WM22) which took place at the *School of Sciences of the University of Minho*, Braga, from the 7th to 9th of September 2022.

One of the main goals of this conference was to cover, as broadly as possible, the diversity of interests of Portu-

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guese female mathematicians. It brought together more than 50 participants, coming from various countries, and including several experts in their fields as well as graduate students and early-stage post-docs.
In Portugal, as in many countries, the top positions of the higher education institutions, and also the invited speakers for national and international conferences happen to include very few women. Also, the number of female undergraduate students proceeding to a PhD in Mathematics is very limited and needs to grow. Besides the presentation of scientific results, the WM22 meeting was also intended to discuss these aspects. The conference was, therefore, an opportunity to share different experiences, and, building on previous editions, helped to develop a more supportive community that can inspire future women mathematicians.

In the WM22 meeting, beyond the Plenary Talks, there were Contributed Talks and Posters, and a Special Session followed by a Panel Discussion on the gender gap situation in Portugal.

Further details can be found at:
https://w3.math.uminho.pt/wm22/

Plenary talks

Ana Luísa Custódio
Nova University of Lisbon, Portugal

Enide Andrade
University of Aveiro, Portugal

Ilka Agricola
University of Marburg, Germany

Leonor Godinho
University of Lisbon, Portugal

Maria Joana Torres
University of Minho, Portugal

Marina Ferreira
University of Helsinki, Finland

Special session

Alexandra Moura (chairwoman)
University of Lisbon, Portugal

Fernanda Estrada
University of Minho, Portugal

Isabel Labouriau
University of Porto, Portugal

Paula de Oliveira
University of Coimbra, Portugal

Patrícia Gonçalves
University of Lisbon, Portugal

Contributed talks

Evelin Krulikovski
Nova University of Lisbon, Portugal

Inês Serôdio Costa
University of Aveiro, Portugal

Liliana Garrido da Silva
University of Porto, Portugal

Marisa Toste
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Panel discussion (round table)

Eurica Henriques (moderator)
University of Trás-os-Montes e Alto Douro, Portugal

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Digital Libraries and Electronic Access to Mathematical Information

by Peter Gothen* and José Francisco Rodrigues**

* Centro de Matemática UP and Dep. de Matemática, Faculdade de Ciências, Universidade do Porto

** CMAFcIO and Dep. de Matemática, Faculdade de Ciências, Universidade de Lisboa

On the 21st of January of 2022, the Workshop *Digital Libraries and Electronic Access to Mathematical Information* was held virtually. It was promoted by CIM, the *International Center for Mathematics*, and organized by **Isabel Narra Figueiredo** (CIM), **Peter Gothen** (CNM, the *National Commission for Mathematics*), **José Francisco Rodrigues** (SPM, the *Portuguese Mathematical Society*). It had about 30 participants covering most of the main national research institutions in mathematics.

As is well known, we are currently witnessing a rapid development in the forms of access to information and scientific publishing. The purpose of the workshop was to take stock of the situation in the area of Mathematics and discuss future prospects, including topics such as digital and physical access to national archives, coordination at national level, the inclusion of content on b-on, and FCT's adoption of the O*pen Access initiative Plan S*. The invited speakers were

José Borbinha
Instituto Superior Técnico, Universidade de Lisboa

José Francisco Rodrigues
Faculdade de Ciências, Universidade de Lisboa

Peter Gothen
Faculdade de Ciências, Universidade do Porto

Joana Novais
Manager of the national digital library *b-on*
Fundação para a Ciência e a Tecnologia

The speakers provided an informative overview on the history of digitizing and digital access to mathematical information in Portugal, the current situation with respect to subscription of digital resources by individual institutions, and future perspectives for the *b-on* and the implementation of Plan S.

This was followed by a lively round table discussion moderated by Isabel Narra Figueiredo (Universidade de Coimbra). Among the various topics discussed were the perspectives for inclusion in the *b-on* of additional publishers such as those of scientific societies, including the *European Mathematical Society*, which is publishing the P*ortugaliae Mathematica*, and the *American Mathematical Society* with its *MathSciNet*.

Further information on the event can be found at
https://www.cim.pt/agenda/event/221

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REPORT

Portugal-Italy Conference on Nonlinear Equations and Applications

University of Évora, July 4–6 2022, Évora, Portugal

by Feliz Minhós*

The *Portugal-Italy Conference on Nonlinear Differential Equations and Applications* (PICNDEA22), https://www.picndea22.uevora.pt/ , was held on July 4-6, 2022, at the University of Évora, Colégio Espírito Santo, Évora, Portugal, sponsored by CIMA (*Centro de Investigação em Matemática e Aplicações*) and CMAFcIO (*Centro de Matemática, Aplicações Fundamentais e Investigação Operacional*).

The main scientific topics of the conference were *Ordinary and Partial Differential Equations*, with particular regard to non-linear problems originating in applications, and their treatment with the methods of Numerical Analysis.

The fundamental main purpose was to bring together Italian and Portuguese researchers in the above fields to create new and amplify previous collaborations, and to follow and discuss new topics in the area.

During these days, 65 researchers and Ph.D. students present and attempt two plenary lectures and three parallel sessions, where recent and classical results were presented and discussed.

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José Sebastião e Silva
From Mértola to Lisboa,
via Évora and Roma

REPORT

IV International Workshop on Non-Associative Algebras

by **Ivan Kaygorodov***, **Isabel Cunha**** and **Patrícia Beites*****

In the last week of October 2022, the *IV International Workshop on Non-Associative Algebras* in Covilhã was held at the *University of Beira Interior* (Covilhã, Portugal). The event was promoted and organized by CMA-UBI, in collaboration with CMUC, CMUP, and CIM. The series of workshops on non-associative algebras aims to bring together researchers from around the world, working in non-associative algebras, and to share the latest results and challenges in the mentioned field. The referred series started in 2018 at the *University of Cádiz* (Spain); after the first edition, the following ones were held at the *University of Porto* (Portugal) in 2019 and at the *University*

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of Málaga (Spain) in 2020. With a target audience of researchers, technical staff, undergraduate and graduate students (PhD students or Postdoctoral fellows), with interest in non-associative algebras and related topics, the most recent edition counted with 30 participants, engaged in lively discussions with the speakers representing 13 countries, namely:

Adela Latorre (Spain), **Alexandre Quesney** (Spain), **Andrey Krutov** (Czech Republic), **Arne Van Antwerpen** (Belgium), **Bernardo Leite da Cunha** (Spain), **Diogo Diniz** (Brazil), **Jorge Garcés** (Spain), **Ilya Gorshkov** (Russia), **Jakob Palmkvist** (Sweden), **María Victoria Velasco** (Spain), **Marina Tvalavadze** (Canada), **Matthew Westaway** (UK), **Mina Monadjem** (Austria), **Mykola Khrypchenko** (Portugal), **Paola Stefanelli** (Italy), **Pasha Zusmanovich** (Czech Republic), **Pilar Páez-Guillán** (Austria), **Quentin Ehret** (France), **Rosa Navarro** (Spain), **Salvatore Siciliano** (Italy), **Samuel Lopes** (Portugal), **Sofiane Bouarroudj** (UAE), **Stephane Launois** (UK), **Vladimir Tkatjev** (Sweden).

Next editions, the fifth and the sixth of this series of international workshops on non-associative algebras will be respectively organized by the *University of Espírito Santo* (Vitória, Brazil), be held in December 2022, and by the *Polytechnic University of Madrid* (Spain), to be held in June 2023.

Encontro Nacional da SPM 2022

by Mário Bessa*

By the beginning of the XIV Century, grounded on heresy accusations, the Templars were extinguished, only to be brought to life some years later in Tomar, the so called *last city of the templars*. Exactly, seven centuries after that, the face to face national meetings of the *Portuguese Mathematical Society* (SPM) were resumed, after the cancellation of the 2020 event courtesy of COVID, but brought to life in 2021 as the online event whose impact and quality could be witnessed by the 700+ participants. The Direction of the SPM in the term 20-22 had the goal of promoting the National Meetings of SPM as a major forum to gather the top experts on research, education, outreach and history, and for that many world class scientists from Portugal as well as foreigners were invited to deliver plenary talks or parallel sessions.

From July 18 to 20, 2022 Tomar, with the high patronage of the President of the Republic of Portugal and a panel of speakers of enormous quality that included 8 ERC (European Research Council grant) recipients, was the stage for a meeting organized around SPM's main pillars: research, teaching, history and outreach. It was three days of intense work, learning and discussion of ideas in the pleasant Tomar's summer; a city that knows how to welcome and a local organization by the *Polytechnic Institute of Tomar* for whom the highest compliments will always be scarce. There were 250 participants spread all over 132 lectures in parallel sessions with a panel of speakers that covered most of the areas and aspects of mathematics. To organize this event it was necessary an impressive mobilisation of the Portuguese mathemati-

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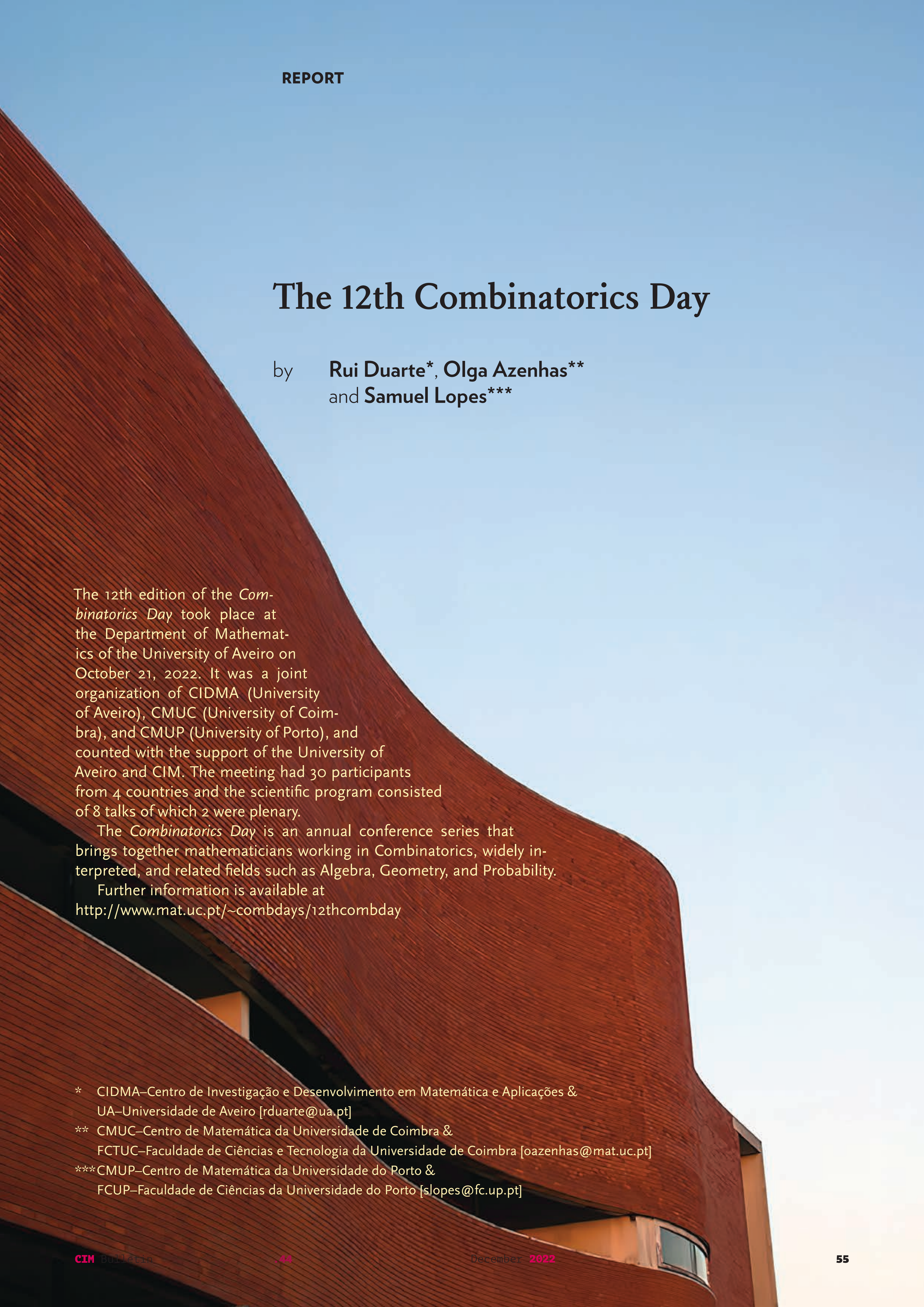
cal community, from all corners of Portugal. There was a magnificent talk open to the entire city and to all audiences by Rogério Martins, the main character of the internationally awarded program *Isto é Matemática* on SIC notícias. At the end, the enormous value of the Portuguese mathematician Afonso Bandeira of the Swiss ETH Zurich was justly acknowledged with the *António Aniceto Monteiro Prize*, a prize that was awarded for the first time in this event and had an international jury of unquestionable reputation, and the sponsorship of SPM-UNILABS. A final word of thanks to all the entities that sponsored the meeting, including many research centers and the CIM, and to the countless mathematicians and staff that worked very hard to make the event possible.



The 12th Combinatorics Day

by Rui Duarte*, Olga Azenhas**
and Samuel Lopes***

The 12th edition of the *Combinatorics Day* took place at the Department of Mathematics of the University of Aveiro on October 21, 2022. It was a joint organization of CIDMA (University of Aveiro), CMUC (University of Coimbra), and CMUP (University of Porto), and counted with the support of the University of Aveiro and CIM. The meeting had 30 participants from 4 countries and the scientific program consisted of 8 talks of which 2 were plenary.

The *Combinatorics Day* is an annual conference series that brings together mathematicians working in Combinatorics, widely interpreted, and related fields such as Algebra, Geometry, and Probability.

Further information is available at
http://www.mat.uc.pt/~combdays/12thcombday

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PEDRO NUNES LECTURES

SYLVIA SERFATY

SYSTEMS OF POINTS WITH COULOMB INTERACTIONS

NOVEMBER, 16
2022

Photo credits: **Stephan Falke**

Sylvia Serfaty is Silver Professor of Mathematics at the Courant Institute of Mathematical Sciences in New York University. She studied at Ecole Normale Supérieure, Paris (MSc Mathematics (1995) and Doctoral Studies), at Université Paris-Sud, Orsay (PhD in mathematics (1999), under the supervision of Fabrice Bethuel) and Université Pierre et Marie Curie Paris 6 ("habilitation" to direct research (2002)).

Her research interests revolve around analysis, Partial Differential Equations and mathematical physics. She has focused in particular on the Ginzburg-Landau model of superconductivity, and recently on the statistical mechanics of Coulomb-type systems.

Among many other distinctions, she was one of the laureates of the 2004 European Mathematical Society Prize and of the 2012 Henri Poincaré prize of the IAMP, the recipient of the 2013 Grand prix Mergier-Bourdeix de l'Académie des Sciences de Paris and was plenary speaker at the 2012 European Congress of Mathematics and at the 2018 International Congress of Mathematicians in Rio de Janeiro. In 2019 she was elected to the American Academy of Arts and Sciences.

VIDEO RECORDING

http://www.mat.uc.pt/~clcmat/videos/PNL_2022.mp4

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