Unraveling the Dynamics of Impulsive Semiflows: Ergodic and Topological Features

by **Jaqueline Siqueira***

> Everything is logical. To understand is to justify.
> — Eugène Ionesco

1 Introduction

The time evolution of various natural phenomena often exhibits sudden changes of state, occurring at certain points where the duration of these disturbances is either null or minimal in comparison to the overall duration of the phenomenon. These sudden changes are referred to as impulses and can be observed in fields such as physics, biology, economics, control theory, and information science, see [20, 24, 29, 32, 6, 26] and references therein.

Impulsive dynamical systems (IDS) are effective mathematical models of real world phenomena that display abrupt changes in their behavior. More precisely, an IDS is defined by three objects: a continuous semiflow on a phase space M; an impulsive region $D \subset M$, where the flow experiences sudden perturbations; and an impulsive map $I : D \to M$, which determines the change in the trajectory each time it hits the impulsive region D.

Under broad conditions an IDS generates a semiflow, called impulsive semiflow, that is not continuous in general. This inherent feature of the dynamics of an impulsive semiflow is a key challenge when trying to describe its topological and ergodic behavior.

In [21, 22], Kaul initiated the study of impulsive dynamical systems with impulses at variable times and since then several authors have contributed to the development of the theory. We mention the important contributions of Ciesielski [14, 15, 16], as well as those of Bonotto and his collaborators [9, 11, 12, 10].

However the study of impulsive semiflows from the perspective of smooth topological dynamics and ergodic theory has been initiated only recently. In this work we aim to outline the development of the theory to date and to propose potential directions for its further advancement.

Definition and first properties

Since the lack of continuity of an impulsive semiflow happens independently of the regularity of the IDS that generates it, we shall not consider the most general class of impulsive semiflows (see [10]). We thus start by assuming some regularity on the IDS.

Let M be a compact manifold endowed with the Riemannian metric d and let $\varphi : M \times \mathbb{R}_0^+ \to M$ be a semiflow: $\varphi_0(x) = x$ and $\varphi_{t+s}(x) = \varphi_t(\varphi_s(x))$ for all $x \in M$ and $t, s \in \mathbb{R}_0^+$. We assume that φ is generated by a C^1-vector field X, that D is a submanifold of codimension one transversal to X, and that $I : D \to M$ is a continuous map so that $I(D) \cap D \neq \emptyset$. The IDS (M, φ, D, I) generates an impulsive semiflow as follows.

The *first impulsive time map* $\tau_1 : M \to [0, +\infty]$ is the map that records the first visit of each φ-trajectory to D: $\tau_1(x) := \inf\{t > 0 : \varphi_t(x) \in D\}$ if $\varphi_t(x) \in D$ for some $t > 0$ and $\tau_1(x) := +\infty$, otherwise. Given $x \in M$, the *impulsive trajectory* γ_x and the subsequent *impulsive times* $\tau_2(x), \tau_3(x), \tau_4(x), \ldots$ are defined inductively. For $0 \leq t < \tau_1(x)$ we set $\gamma_x(t) = \varphi_t(x)$. Assuming that $\gamma_x(t)$ is defined for $t < \tau_n(x)$ for some $n \geq 2$, we set

$$\gamma_x(\tau_n(x)) = I(\varphi_{\tau_n(x)-\tau_{n-1}(x)}(\gamma_x(\tau_{n-1}(x)))).$$

Defining the $(n+1)^{\text{th}}$ impulsive time of x as

$$\tau_{n+1}(x) = \tau_n(x) + \tau_1(\gamma_x(\tau_n(x))),$$

for $\tau_n(x) < t < \tau_{n+1}(x)$, we set

* Departamento de Matemática, Instituto de Matemática, Universidade Federal do Rio de Janeiro
Email: jaqueline@im.ufrj.br

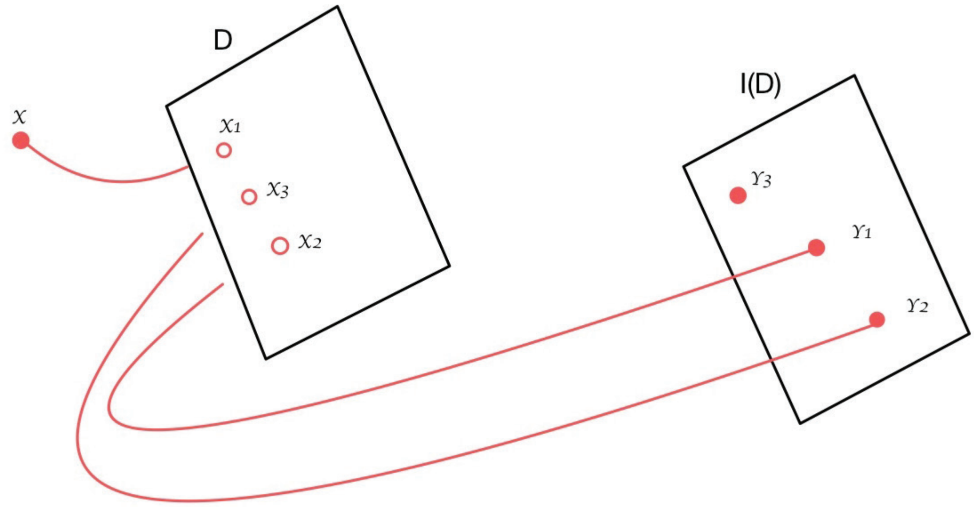

Figure 1.—Building the trajectory of a point x. Set $x_i = \varphi_{\tau_i(x)}(x)$, $y_i = I(x_i)$, $i = 1, 2, 3$.

$$\gamma_x(t) = \varphi_{t-\tau_n(x)}(\gamma_x(\tau_n(x))).$$

Since we assume $I(D) \cap D = \varnothing$ [3, Remark 1.1] we have that $T(x) := \sup\{\tau_n(x); n \geq 1\} = \infty$, thus impulsive trajectories are defined for all positive times.

According to [9, Proposition 2.1], the trajectories defined above indeed generate a semiflow which we call an *impulsive semiflow*:

$$\begin{array}{rccc} \psi : & \mathbb{R}_0^+ \times M & \longrightarrow & M \\ & (t, x) & \longmapsto & \gamma_x(t). \end{array}$$

In general terms, ψ-impulsive trajectories are built with segments of φ-trajectories in the following way. Given a point $x \in M$, consider its φ-trajectory until it hits D (x_1, in Figure 1). Then delete the intersection point x_1, restart the impulsive trajectory at the image of the deleted point under I ($y_1 = I(x_1)$). From there, follow its φ-trajectory until it hits D again, and repeat the process. Note that since $D \cap I(D) = \varnothing$, an impulsive trajectory of a point $x \in M$ intersects D only if $x \in D$. Moreover, no periodic trajectories intersect the impulsive region D.

2 Ergodic theory of impulsive semiflows

The field of ergodic theory has been developed with the goal of understanding the statistical behavior of a dynamical system via measures which remain invariant under its action. Describing the behavior of the orbits in a dynamical system can be challenging, particularly for systems with complex topological and geometrical structures. One powerful method for analyzing such systems is through invariant probability measures. For example, Birkhoff's Ergodic Theorem states that almost every initial condition within each ergodic component of an invariant measure shares the same statistical distribution in space.

We say that μ is an *invariant* probability measure under a semiflow φ if $\mu(\varphi_t^{-1}(A)) = \mu(A)$ for all Borel sets $A \subset M$ and for all $t > 0$. Denote by $\mathscr{M}(\varphi)$ the set of all invariant probability measures.

A point $x \in M$ is called *non-wandering* for φ if for every neighborhood $\mathscr{U}$ of x and for every $t > 0$, there exists $T \geq t$ so that $\varphi_T^{-1}(\mathscr{U}) \cap \mathscr{U} \neq \varnothing$. Denote by $\Omega(\varphi)$ the set of all non-wandering points for φ.

The compactness of M implies that $\Omega(\varphi) \neq \varnothing$ and is compact. Moreover, if φ is continuous, the non-wandering set is invariant: $\varphi_t(\Omega(\varphi)) \subset \Omega(\varphi)$ for all $t > 0$. However, this does not remain true in general for impulsive semiflows, which is quite a suprising phenomenon (see [3]).

For an invariant measure, the relevant points are the non-wandering ones, meaning that any invariant measure is supported on the non-wandering set. When we assume that φ is a continuous semiflow on a compact metric space there is always a φ-invariant measure [30, Theorem 2.1]. However, one can build examples of impulsive semiflows with no invariant measures (e.g. [3, Example 2.2]).

Set $\tau_D(x) = \tau_1(x)$ if $x \in \Omega(\psi)\backslash D$ and $\tau_D(x) = 0$ if $x \in \Omega(\psi) \cap D$. Let (M, φ, D, I) be an IDS so that φ is continuous on a compact metric space M, τ_D is continuous and $I(\Omega(\psi) \cap D) \subset \Omega(\psi)\backslash D$. Therefore, by [3, Theorem A], the impulsive semiflow ψ admits an invariant measure.

The existence of invariant measures can also be obtained by assuming some regularity and transversality conditions on the IDS [1, Theorem I] as follows.

A criterion for existence of invariant measures

Theorem 1.— Let $\varphi : M \times \mathbb{R}_0^+ \to M$ be a C^1-semiflow, D a compact submanifold of codimension one, transversal to the flow direction and $I : D \to M$ a continuous map. Then the impulsive semiflow ψ admits an invariant probability measure.

2.1 Topological pressure

The topological pressure of a semiflow with respect to a potential function is the rate of growth of trajectories of the semiflow where each point is weighted according to the potential. We recall the classical definition of topological pressure for continuous semiflows and generalize this concept to impulsive semiflows.

Topological pressure for continuous semiflows

Let $\varphi : M \times \mathbb{R}_0^+ \to M$ be a continuous semiflow. Given $\varepsilon > 0$ and $t \in \mathbb{R}^+$, a subset E of M is said to be $(\varphi, \varepsilon, t)$-*separated* if for any $x, y \in M$ with $x \neq y$ there is some $s \in [0, t]$ such that $d(\varphi_s(x), \varphi_s(y)) > \varepsilon$. Given $f : M \to \mathbb{R}$ a continuous potential, define

$$Z(\varphi, f, \varepsilon, t) = \sup_E \left\{ \sum_{x \in E} e^{\int_0^t f(\varphi_s(x))\, ds} \right\}, \text{ where}$$

the supremum is taken over $(\varphi, \varepsilon, t)$-separated sets.

We also define

$$P(\varphi, f, \varepsilon) = \limsup_{t \to +\infty} \frac{1}{t} \log Z(\varphi, f, \varepsilon, t).$$

The *topological pressure* of φ with respect to f is defined as

$$P(\varphi, f) = \lim_{\varepsilon \to 0} P(\varphi, f, \varepsilon).$$

The *topological entropy* of φ, $h(\varphi)$, is the topological pressure of φ with respect to the identically zero potential.

New definition for semiflows with discontinuities

Let $\psi : \mathbb{R}_0^+ \times M \to M$ be a semiflow possibly exhibiting discontinuities. Consider a function T assigning to each $x \in M$ a sequence $(T_n(x))_{n \in A(x)}$ of nonnegative numbers, where either $A(x) = \mathbb{N}$ or $A(x) = \{1, \ldots, \ell\}$ for some $\ell \in \mathbb{N}$. We say that T is *admissible* if there exists $\eta > 0$ such that for all $x \in M$ and all $n \in \mathbb{N}$ with $n + 1 \in A(x)$ we have

1. $T_{n+1}(x) - T_n(x) \geq \eta$
2. $T_n(\psi_t(x)) = \begin{cases} T_n(x) - t, & \text{if } T_{n-1}(x) < t < T_n(x) \\ T_{n+1}(x), & \text{if } t = T_n(x). \end{cases}$

For each admissible function T, $x \in M$, $t > 0$ and $0 < \delta < \eta/2$, we define

$$J_{t,\delta}^T(x) = [0, t] \setminus \bigcup_{j \in A(x)}]T_j(x) - \delta, T_j(x) + \delta[.$$

Observe that $J_{t,\delta}^T(x) = [0, t]$, whenever $T_1(x) > t$.

Given $\varepsilon > 0$ and $t \in \mathbb{R}^+$, we say that $E \subset M$ is $(\psi, \delta, \varepsilon, t)$-*separated* if for any $x, y \in M$ with $x \neq y$ there is some $s \in J_{t,\delta}^T(x)$ so that $d(\psi_s(x), \psi_s(y)) > \varepsilon$. Given a continuous potential $f : M \to \mathbb{R}$, define

$$Z^T(\psi, f, \delta, \varepsilon, t) = \\ = \sup_E \left\{ \sum_{x \in E} \exp\left(\int_0^t f(\psi_s(x))\, ds \right) \right\},$$

where the supremum is taken over all finite separated sets. We also define

$$P^T(\psi, f, \delta, \varepsilon) = \limsup_{t \to +\infty} \frac{1}{t} \log Z_t^T(\psi, f, \delta, \varepsilon),$$

$$P^T(\psi, f, \delta) = \lim_{\varepsilon \to 0} P^T(\psi, f, \delta, \varepsilon).$$

Finally, define the T-*topological pressure* of ψ with respect to f as

$$P^T(\psi, f) = \lim_{\delta \to 0} P^T(\psi, f, \delta).$$

This concept was strongly inspired by the $T-$topological entropy introduced in [4]. Moreover, when f is identically null the T-topological pressure becomes the T-topological entropy.

Note that the sequence $\tau = \{\tau_n\}$ of impulsive times is admissible and we call $P^\tau(\varphi, f)$ the impulsive topological pressure.

The next result states that for continuous semiflows the classical and the new notion of topological pressure coincide [5, Theorem B].

Theorem 2.— Let φ be a continuous semiflow on a compact metric space M, T an admissible sequence and $f : X \to \mathbb{R}$ a continuous potential. Then $P^T(\varphi, f) = P(\varphi, f)$.

2.2 Variational Principle

The concept of topological entropy is, a priori, purely topological. Nevertheless it is intrinsically related to invariant measures. For instance, given a continuous flow, on a compact metric space, the classical variational principle [31] shows that the topological entropy is the supremum over all invariant measures of the measure theoretical entropy.

The *entropy* of a continuous map $g : M \to M$ with respect to a probability measure μ is given by:

$$h_\mu(g) = \int h_\mu(g, x)\, d\mu(x), \qquad \text{where}$$

$$h_\mu(g, x) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu\left(B_g(x, n, \varepsilon)\right), \text{ and}$$

$B_g(x, n, \varepsilon)=\{y\in M\colon\ d(g^j(x), g^j(y))<\varepsilon, \forall j=0,\cdots,n-1\}$.

The *entropy* of a semiflow ψ with respect to a probability measure μ is defined as $h_\mu(\psi) = h_\mu(\psi^1)$, where ψ^1 stands for the time-1 map.

A variational principle [1, Theorem II] holds for impulsive semiflows, (see[5] a topological pressure version).

Theorem 3.— Let $\varphi : M \times \mathbb{R}_0^+ \to M$ be a C^1-semiflow, D a compact submanifold of codimension one and $I : D \to M$ a 1-Lipschitz map. If D and $I(D)$ are transversal to the flow direction, then the impulsive semiflow ψ generated by (M, φ, D, I) satisfies

$$h^\tau_{top}(\psi) = \sup_{\mu\in\mathscr{M}(\psi)} \{h_\mu(\psi)\}.$$

2.3 Equilibrium states

In general a dynamical system admits more than one invariant measure, so it is important to choose a suitable one for analysis. One way to do this is by selecting measures that maximize the system's free energy, known as equilibrium states.

A ψ-invariant probability measure μ is said to be an *equilibrium state* for ψ with respect to a potential function $f : M \to \mathbb{R}$ if it satisfies:

$$h_\mu(\psi) + \int f\, d\mu = \sup_\eta \left\{ h_\eta(\psi) + \int f\, d\eta \right\}$$

where the supremum is taken over all ψ-invariant probability measures η.

Given an expansive continuous flow with the specification property and a Hölder continuous potential satisfying the Bowen property, there exists a unique equilibrium state, as shown in [17]. To extend this result to impulsive semiflows we first adapt the concepts involved.

Existence and uniqueness for impulsive semiflows

In simple terms, expansiveness means that the system has the property of pushing apart the trajectories of nearby points in its phase space over time.

Taking into account that a suitable concept of expansiveness for impulsive semiflows should ensure genuine separation of the trajectories rather than just the artificial ones generated by the impulse map, we introduce the following concept.

For a given $r > 0$, denote the set $B_r(D)$ by the r-neighborhood of D in M. The semiflow ψ is called *positively expansive* on M if for every $\delta > 0$ there exists $\epsilon > 0$ such that if $x, y \in M$ and a continuous map $s : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $s(0) = 0$ satisfy $d(\psi_t(x), \psi_{s(t)}(y)) < \epsilon$ for all $t \geq 0$ with $\psi_t(x), \psi_{s(t)}(y) \notin B_\epsilon(D)$, then either $y = \psi_t(x)$ or $x = \psi_t(y)$ for some $0 < t < \delta$.

The specification property refers to the ability of a dynamical system to approximate true trajectories with high precision, using only a finite number of segments from other orbits. The fact that the concept depends only on pieces of trajectories allows us to apply the classical concept to impulsive semiflows.

The semiflow ψ has the *specification property* on M if for all $\epsilon > 0$ there exists $L > 0$ such that for any sequence $x_0, \ldots, x_n$ of points in M and any sequence $0 \leq t_0 < \cdots < t_{n+1}$ such that $t_{i+1} - t_i \geq L$ for all $0 \leq i \leq n$, there are $y \in M$ and $r : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, which is constant on each interval $[t_i, t_{i+1}[$, whose values depend only on $x_0, \ldots, x_n$, that also satisfy

$$r([t_0, t_1[) = 0 \quad \text{and} \quad |r([t_{i+1}, t_{i+2}[)-r([t_i, t_{i+1}[)| < \epsilon,$$

$$\text{for which } d\left(\psi_{t+r(t)}(y), \psi_{t-t_i}(x_i)\right) < \epsilon,$$

for all $t \in [t_i, t_{i+1} - L[$ and $0 \leq i \leq n$.

In addition, the specification is called *periodic* if we can always choose y periodic with the minimum period in $[t_{n+1} - t_0 - n\epsilon, t_{n+1} - t_0 + n\epsilon]$.

Finally, we adapt the concept of Bowen potentials to include the action of the impulse on the potentials. Let $V^*(\psi)$ be the set of all continuous maps $f : M \to \mathbb{R}$ satisfying:

1. $f(x) = f(I(x))$ for all $x \in D$;
2. there are $K > 0$ and $\epsilon > 0$ such that for every $t > 0$ we have
$$\left| \int_0^t f(\psi_s(x))\, ds - \int_0^t f(\psi_s(y))\, ds \right| < K, \quad (1)$$
whenever $d(\psi_s(x), \psi_s(y)) < \epsilon$ for all $s \in [0, t]$ such that $\psi_s(x), \psi_s(y) \notin B_\epsilon(D)$.

Sufficient conditions to the existence and uniqueness of equlibrium states were first established in [5, Theorem A]. Here we state [1, Theorem III] that requires the following regularity and transversality of the IDS.

Theorem 4.— Let $\varphi : M \times \mathbb{R}_0^+ \to M$ be a C^1 semiflow, D a compact submanifold of codimension one and $I : D \to M$ a 1-Lipschitz map. If D and $I(D)$ are transversal to the flow direction, ψ is positively expansive and it has the periodic specification property in $\Omega(\psi)\backslash D$, then for any potential $f \in V^*(\psi)$ there is an equilibrium state. Moreover, if there is $k > 0$ such that $\#I^{-1}(\{y\}) \leq k$ for every $y \in I(D)$ then the equilibrium state is unique.

3 Periodic Orbits of typical impulsive semiflows

Considering the space of impulsive semiflows parameterized by vector fields and impulse maps, a natural question is whether a typical impulsive semiflow admits periodic points. Also, recall the so called General Density Theorem (see [25]) that ensures the existence of a Baire residual subset of C^1 vector fields for which every element generates a C^1-flow φ such that the set, $Per(\varphi)$, of periodic orbits is dense in the non-wandering set $\Omega(\varphi)$.

We aim to provide a description of the non-wandering set for a generic class of Impulsive Semiflows. As the genericity depends on the topology with which the space is endowed, we present our results in the C^0 and C^1 topologies,via permanent and hyperbolic periodic points, respectively. All throughout let M be a compact Riemannian manifold of dimension $m \geq 3$. Given an IDS (M, φ, D, I), where φ is a flow generated by a vector field X, we denote its impulsive semiflow by $\psi_{X,I}$ to stress the dependence on the vector field and on the impulse map.

3.1 C^0-topology

We now consider impulsive semiflows for which the underlying flow is generated by Lipschitz continuous vector fields and impulses are chosen as homeomorphisms onto its image. We denote by $\mathfrak{X}^{0,1}(M)$ the space of Lipschitz vector fields endowed with the C^0-topology
$$\|X - Y\|_{C^0} := \max_{x \in M} \|X(x) - Y(x)\| < \varepsilon.$$

C^0-Baire generic impulses

Assume φ is a Lipschitz continuous flow generated by a vector field $X \in \mathfrak{X}^{0,1}(M)$ and D is a codimension one smooth submanifold of M, transversal to the flow direction, such that
$$\overline{D} \cap \text{Sing}(\varphi) = \emptyset, \quad \text{(H)}$$
where $\overline{D}$ stands for the closure of D and $\text{Sing}(\varphi)$ stands for the equilibrium points of φ. Let $\hat{D}$ be a codimension one submanifold transversal to X. Consider the space
$$\mathscr{I}_{D,\hat{D}} = \text{Homeo}(D, \hat{D})$$
endowed with the C^0-distance. We have the following General Density Theorem [28, Theorem A].

Theorem 5.— There exists a C^0-Baire generic subset $\mathscr{R}_X \subset \mathscr{I}_{D,\hat{D}}$ of impulses such that
$$\overline{Per(\psi_{X,I})} \cap \mathring{\hat{D}} = \Omega(\psi_{X,I}) \cap \mathring{\hat{D}}$$
for every $I \in \mathscr{R}_X$, where $Per(\psi_{X,I})$ denotes the set of periodic orbits of $\psi_{X,I}$ and $\mathring{\hat{D}}$ is the interior of D.

In general, one should not expect the density of periodic points in the all non-wandering set, however this is the case when the impulsive semiflow is generated by a minimal flow. Moreover, the following holds ([28, Corollary B]).

Corollary 6.— Let φ be a Lipschitz continuous flow generated by $X \in \mathfrak{X}^{0,1}(M)$ and $D, \hat{D} \subset M$ be smooth codimension one submanifolds transversal to the flow such that assumption (H) holds. The following hold:

1. if $I_0 \in \mathcal{I}_{D,\hat{D}}$ is such that $\Omega(\psi_{I_0}) \cap \partial D = \emptyset$ then there exist $\delta > 0$, an open neighborhood $\mathcal{V}$ of I_0 and a Baire generic subset $\mathcal{R} \subset \mathcal{V}$ so that, for every $I \in \mathcal{R}$ one can write the non-wandering set $\Omega(\psi_{X,I})$ as a (possibly non-disjoint) union $\Omega(\psi_{X,I}) = \overline{Per(\psi_{X,I})} \cup \Omega_2(\varphi, D)$, where $\Omega_2(\varphi, D) \subset \Omega(\psi_{X,I})$ is a φ-invariant set which does not intersect a δ-neighborhood of the cross-section D. Moreover, the set $\Omega(\psi_{X,I}) \setminus D$ is invariant under $\psi_{X,I}$.

2. if φ is minimal then there exists a Baire generic subset $\mathcal{R} \subset \mathcal{I}_{D,\hat{D}}$ so that, for every $I \in \mathcal{R}$, the set of periodic orbits is dense in $\Omega(\psi_{X,I})$. Moreover, the set $\Omega(\psi_{X,I}) \setminus D$ is a $\psi_{X,I}$-invariant subset of M.

3.2 C^1-topology

Let φ be the C^1-flow generated by a vector field X and let $D \subset M$ be a compact codimension one submanifold transversal to the flow direction.

We define the class of impulses $\mathcal{I}_D$ as the set of C^1-embeddings maps $I : D \to M$ so that $I(D) \pitchfork X$ and

$$\sup_{x \in I(D)} \left| \frac{d\tau_1}{dx}(x) \right| < +\infty.$$

Endow the space $\mathcal{I}_D$ with the distance $d_{C^1}(I_1, I_2)$ given by

$$\max \left\{ \sup_{x \in D} d\big(I_1(x), I_2(x)\big), \sup_{x \in D} \left\| DI_1(x) - DI_2(x) \right\| \right\},$$

where the expression on the right-hand side is well-defined after using parallel transport to identify the corresponding tangent spaces.

In this setting [27, Theorem A] establishes:

Theorem 7.— There exists a Baire residual subset $\mathcal{R}_X \subset \mathcal{I}_D$ of impulses such that the impulsive semiflow ψ_I determined by $I \in \mathcal{R}$ satisfies

$$\overline{Per_h(\psi_I)} \cap D = \Omega(\psi_I) \cap D$$

where $Per_h(\psi_I)$ denotes the set of hyperbolic periodic orbits of ψ_I.

We point out that the conclusion of Theorem 7 cannot be written using the landing region $I(D)$, as there exist C^1-open sets of impulses for which the equality $\overline{Per_h(\psi_I)} \cap I(D) = \Omega(\psi_I) \cap I(D)$ fails (see [27, Example 7.3]).

4 An invitation to the impulsive world

Despite their wide range of applications, IDS have only recently begun to be studied through the lens of ergodic theory. In this area, there remains vast potential for exploration. We conclude by presenting a few open questions, inviting the reader to delve into the dynamics of impulsive semiflows.

1. As mentioned in Section 2.3, in general a dynamical system admits more than one invariant measure, therefore it is necessary to choose a suitable one to analyze. While here we only focus on equilibrium states, criteria for the existence and finiteness of the number of absolutely continuous measures and/or physical measures are also not available. See [2] for the study of physical measures for a class of semiflows generated via impulsive perturbations of Lorenz flows.

2. In Theorem 6 we established the denseness of periodic points in the impulsive non-wandering set for a class of Baire generic impulses maps. The proof is based on the concept of uniform hyperbolicity and of perturbative results for discontinuous semiflows. A key tool in the proof is the following impulsive connecting lemma [27, Theorem 4.1].

Given $\delta > 0$, we say that a sequence $(x_k, t_k)_{k=0}^n$ in $M \times \mathbb{R}_+$ is a *δ-pseudo orbit* for the impulsive semiflow ψ_I if $d(\gamma_{x_k}(t_k), x_{k+1}) < \delta$, for every $k = 0 \dots n-1$. We say that y is a *chain iterate* of x (and write $x \dashv y$) if for any $\delta > 0$ there exists a δ-pseudo orbit $(x_k, t_k)_{k=0}^n$ such that $x_0 = x$ and $x_n = y$.

Theorem 8 (Impulsive connecting lemma).— Let φ be a C^1-flow generated by $X \in \mathfrak{X}^1(M)$ and let $D, \hat{D}$ be smooth submanifolds of codimension one transversal to X and $I \in \mathcal{I}_{D,\hat{D}}$ so that all ψ_I-periodic orbits whose orbit closure does not intersect ∂D are hyperbolic. If $x \dashv y$ then for any $\varepsilon > 0$ there exists an ε-C^1-perturbation J of I such that $y = \gamma_{J,x}(t)$, for some $t \geq 0$.

In the context of diffeomorphisms, connecting lemmas were obtained by Arnaud [7, Théorème 22] and Bonatti and Crovisier [8, Théorème 2.1], as refinements of the C^1-closing lemma by Pugh [25] and the C^1-connecting lemma of Hayashi [18].

A natural and interesting open question is to obtain a version of Theorem 6 when the vector field is perturbed. It seems that one should first establish a version of the last connecting lemma.

3. Considering the IDS (M, φ, I, D), even if φ is a flow, the IDS does not generate an impulsive *flow*,

but only a *semiflow*, when the impulse map I is not injective. It is well known that certain concepts do not present a direct adaptation from the setting of flows to the one of semiflows. For instance, the concept of expansiveness presented in this paper is stronger than the one for flows introduced by Bowen and Walters in [13] (see [19]). In [23] the concept of eventual expansiveness for continuous semiflows was introduced.

A continuous semiflow φ on a metric space M is *eventually expansive* if for every $\varepsilon > 0$ there is $\delta > 0$ so that if $x, y \in M$ and $s : [0, +\infty[\to [0, +\infty[$ is a continuous function with $s(0) = 0$ such that $d(\varphi(t, x), \varphi(t, y)) \leq \delta$ for all $t > 0$ then $\varphi(r, x) = \varphi(s, x)$ for some $r, s \geq 0$ with $|r - s| \leq \varepsilon$.

For an eventually expansive continuous semiflow its topological entropy is bounded from below by the growth rate of the periodic orbits [Theorem 1.5, [23]]. After adapting the concept of eventually expansiveness to impulsive semiflows, we would expect to be able to bound the τ-entropy introduced here by the growth rate of the periodic orbits.

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Editor-in-Chief

Maria Joana Torres
[jtorres@math.uminho.pt]

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*Responsible for the production and graphic design of the bulletin

Address

Centro de Matemática
Escola de Ciências da Universidade do Minho
Campus de Gualtar
4710-057 Braga
Portugal

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