Mathematics and Music

by **José Francisco Rodrigues***

In four movements — Pythagorean Arithmiusic, Tone Algebra, Harmonisation of Analysis, Digital Musurgia — and through a few examples, we will present a brief introduction to the numerous interactions between mathematics and music throughout history, which can help us understand the modern interpretation of Leibniz's expression:

> *Musica est exercitium arithmeticae occultum nescientis se numerare animi.*[+]

1. Pythagorean Arithmusic

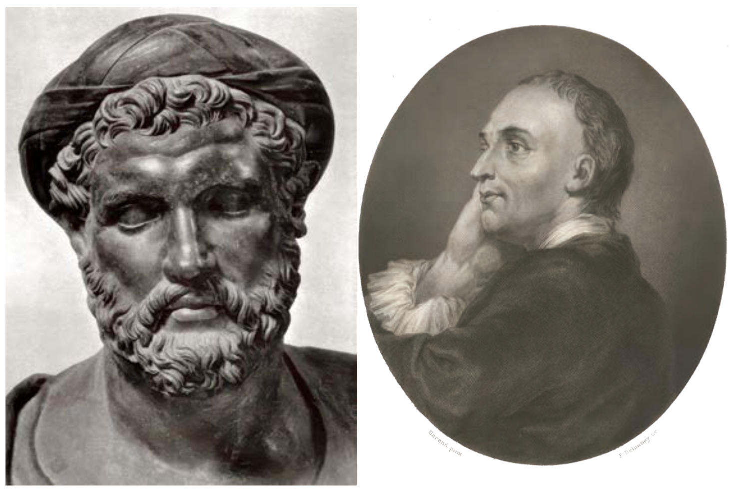
> **C'est par les nombres et non par le sens qu'il faut estimer la sublimité de la musique. Etudiez le monocorde.**[++]
>
> —Diderot, *Pythagoreanism*, Encyclopédie XII (1765)

Guido d'Arezzo (992–1050?) in the *Micrologus*, attributes to Pythagoras (6th century BCE) the fundamental discovery of the dependence of musical intervals on the quotients of the first integers numbers, writing:

> A certain Pythagoras, on one of his journeys, happened to pass a workshop where an anvil was being beaten with five hammers. Astonished by the pleasant harmony (*concordiam*) they produced, our philosopher approached them and, thinking at first that the quality of the sound and harmony (*modulationis)* lay in the different hands, exchanged the hammers. In this way, each hammer retained its own sound. After removing one that was dissonant, he weighed the others and, marvellously, by the grace of God, the first weighed *twelve*, the second *nine*, the third *eight*, the fourth *six* of I don't know what unit of weight.

For the Pythagorean School, the harmony of sounds was in direct correspondence with the arithmetic of proportions:

unison — ratio $1 : 1$	octave (*diapason*) $1 : 2$
fifth (*diapente*) $2 : 3$	fourth (*diatessaron*) $3 : 4$



Left: Bust of Pythagoras. Right: Denis Diderot (1713–1784)

These ratios can be obtained from those four numbers, corresponding respectively to a string length equal to 12 units (*unison*), halved to 6 (*octave*), 8 units (*fifth*) or 9 (*fourth*).

The Greek heritage, was transmited in particular by the Roman Boethius (6th century CE), "the great, astonishing and very sudden relationship (*concordiam*) that exists between music and the proportions of numbers (*numerum proportione*)".

[+] Music is a hidden arithmetic exercise of a mind unconscious that it is counting.
[++] It's by the numbers and not by the sense that one should evaluate the sublimity of music. Study the monochord.

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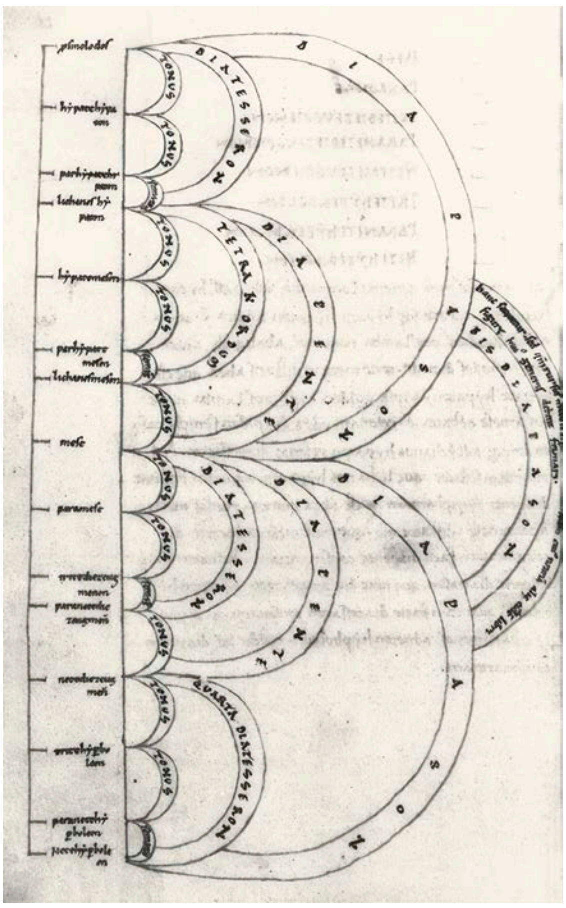


Top: Franchinus Gafurius (*Theorica musicae*, 1492)

Bottom: Boetius; c. 480–524, *De Institutione Musica*. Division of intervals (Paris, Bibl Nat, 12th cent.)

The arithmetic of proportion establishes: the product of 2/3 (fraction associated with the fifth) by 3/4 (fraction associated with the fourth) gives the fraction 1/2 associated with the octave; its division (subtraction of intervals) is associated with the fraction $8/9 = (2/3) \div (3/4)$ which represents a tone, i.e. the difference between a fifth and a fourth. Analogously, an octave is made up of two fourths and a tone $1/2 = 3/4 \times 3/4 \times 8/9$.

The *Sectio Canonis*, or the “Division of a monochord”, 300 BCE, by Euclid, has twenty propositions argued in the form of theorems, treatment of intervals as ratios between integers numbers and culminates with the division of the *Kanon*, For example, its 15th Proposition says “the fourth is less than two and a half tones and the fifth less than three and a half tones”, and others, like the 9 th (<= VIII.2), are consequences of the Book VIII of the *Elements*.

The ancient Greeks also divided the mathematical sciences into four parts:

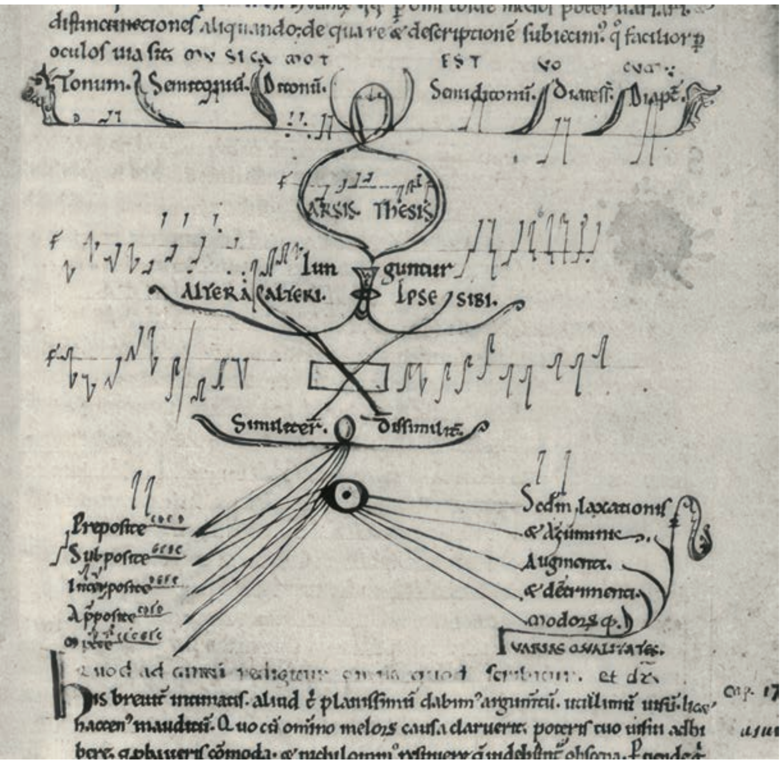
arithmetic (static discrete quantities)
music (discrete quantities in motion)
geometry (stationary magnitudes)
astronomy (dynamic magnitudes).

This classification constituted the *Quadrivium*, as part of the seven liberal arts of the medieval curriculum, which were complemented by the *Trivium* (grammar, dialectic and rhetoric).

Arithmetic, geometric and harmonic proportionality are present throughout medieval science and music, where the latter is defined as number associated with sound—*numerus relatus ad sonum*. For example, in the speculative treatise *Ars novae musicae* (1319), the Parisian mathematician and astronomer Jean de Muris wrote:

> Sound is generated by movement, since it belongs to the class of successive things. It therefore exists only as long as it is produced, ceasing to exist once it has been produced . . . All music, especially measurable music, is based on perfection, combining in itself number and sound.

Claudius Ptolemy (2nd century, CE), author of *Mathematike Syntaxis (Almagest)* and of the treatise *Harmonica*, in which he transmitted the myth of how the mathematical relationships underlying the structures of audible music constitute the forms of the essence and cause of harmonies both in the human soul and



Neumas, *Micrologus*, Guido d'Arezzo
Manuscript, 12th cent. (Biblioth. Nationale, Paris)

in the movements and configurations of the stars.

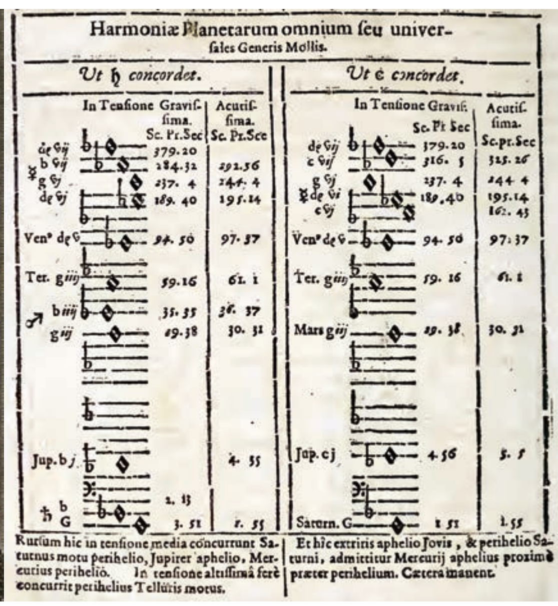
In the Boeotian terminology, this corresponded to musica instrumentalis (produced by the lyre, flute, etc.), musica humana (inaudible, produced in man by the interaction between body and soul), musica mundana (produced by the cosmos itself, also known as the music of the spheres).

The cube with 6 faces, 8 vertices and 12 edges, and therefore considered a harmonic solid, together with other more subtle parallelisms between arithmetic and geometry, led classical civilisation to the doctrine of the music of the spheres and, in Aristotle's expression, to consider that the whole sky is number and harmony.

For Joannes Kepler (1571–1630), the movement of the planets was still an immanent music of divine perfection, but this didn't prevent him to conclude the three laws of motion:

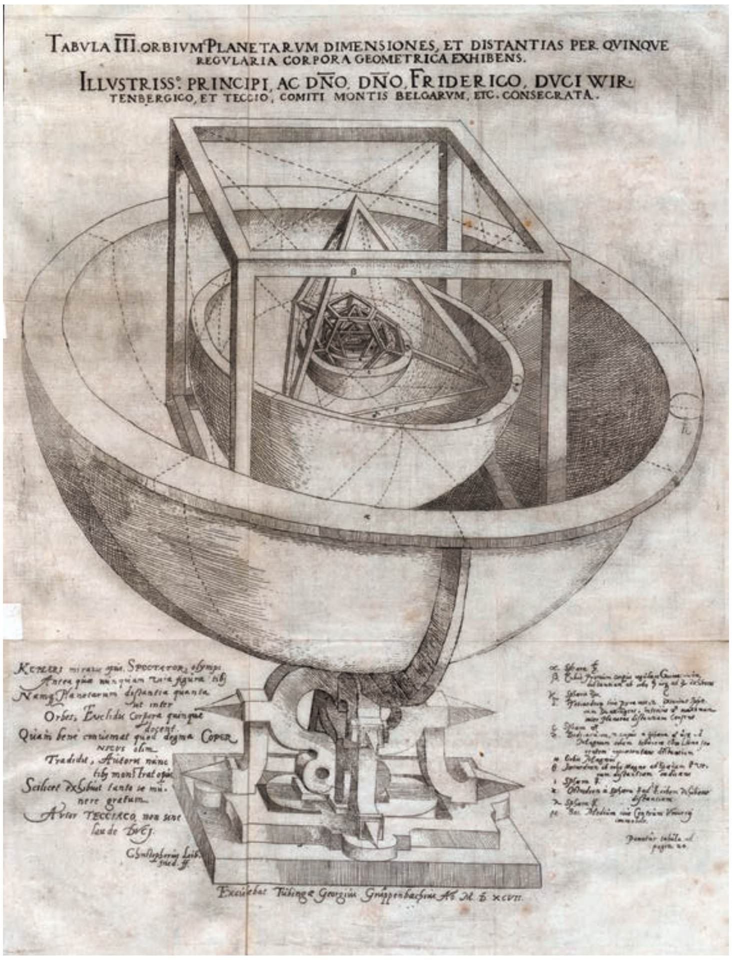
1. the planets revolve around the Sun in elliptical orbits;
2. with the Sun as a foci and their orbital areas are travelled in proportion to time;
3. the squares of the periods of revolution of each planet are proportional to the cubes of their average distances from the Sun.

Following his third law, in 1619 Kepler wrote: musical modes or tones are reproduced in a certain way at the extremities of planetary movements. Considering the seven consonant intervals of the octave of his time, he established the following harmonies of



Left
Johannes Kepler (1571–1630)

Right
Harmonices Mundi, 1619



Kepler's *Mysterium cosmographicum* (1596), with the embeding of the cube (Saturn-Jupiter), tetrahedron (Jupiter-Mars), dodecahedron (Mars-Earth), icosahedron (Earth-Venus) and octahedron (Venus-Mercury) in the sphere.

the six known planets:

Saturn 4 : 5 (a major tertia)
Jupiter 5 : 6 (a minor tertia)
Mars 2 : 3 (a fifth)
Earth 5 : 16 (a half-tone)
Venus 24 : 25 (a sharp)
Mercury 5 : 12 (an octave and a minor tertia);

by calculating the aphelion/perihelion ratios for each of them: Saturn travels an arc of 106 or 135 seconds per day when it is at its furthest point (aphelion) or closest (perihelion) to the Sun, respectively, obtaining the ratio 106/135 ~ 4/5.

Kepler's metaphysics goes so far as to states that the Earth sings the notes MI, FA, MI, so that from them it can be conjectured that misery (MIseria) and hunger (FAmes) prevail in our midst.

2. Tone Algebra

Pythagorean scales are based on the elementary "rational" intervals (octave, fifth and fourth) and their alternating successions, i.e., starting from a sound from a sound $f_0 = f$ and the sound $f_1 = 3f/2$ located a fifth higher on the scale, the sound $f_2 = 3f_1/4 = 9f/8$ will be one fourth below f_1, the sound f_3 a fifth above f_2 and so on. This gives the *cycle of fifths* as

$$f_n = \left(\frac{3}{2}\right)^n \left(\frac{1}{2}\right)^p f$$

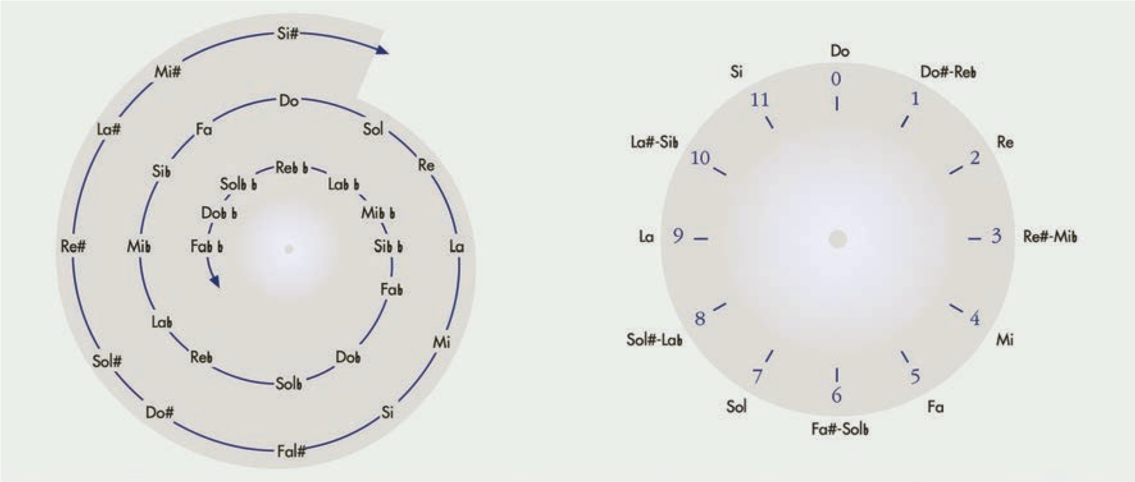
which isn't a real cycle, because if it were, there would have to be two integers n and p such that $3^n = 2^{n+p}$; but an odd number is different from an even number, so it is impossible!

In classical solfege "12 fifths correspond to 7 octaves", mathematically it would be $3^{12} = 2^{19}$, which is false. We have that

$$\frac{3^{12}}{2^{19}} = \frac{531441}{524288} \approx 1.$$

This only translates into a certain tolerance of the ear to that tuning and this difference is the Pythagorean coma.

A theoretical formulation of equal temperament can already be found in the work *De musica* by F. Salinas, published in Salamanca in 1577, which states that *the octave must be divided into twelve equally proportional parts, which will be the equal semitones.*



Left: A spiral of fifths. Right: A 12-tone chromatic clock

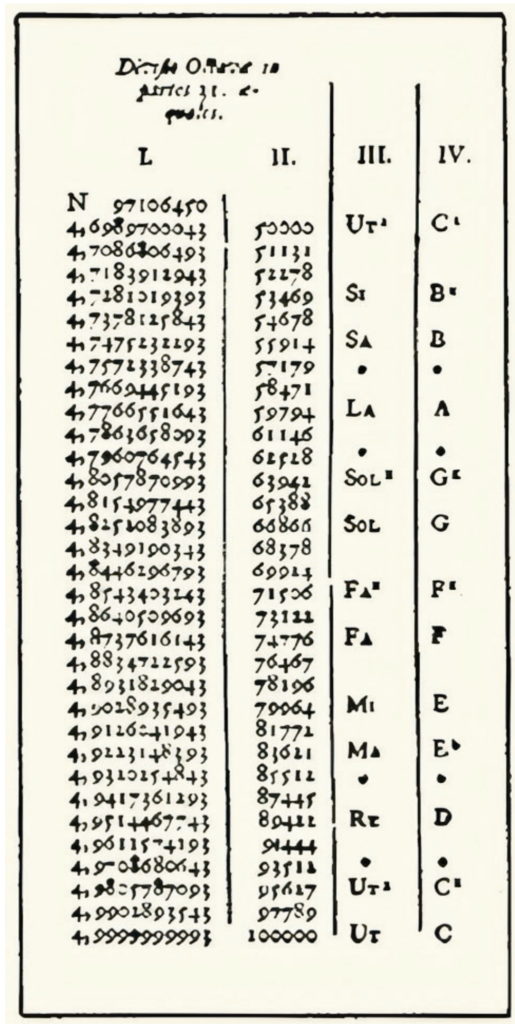
If τ is the interval between two consecutive tones, it is equal to the irrational number

$$\tau = \sqrt[12]{2} = 1,059463094\ldots$$

and represents the ratio of the respective geometric progression. So the frequencies associated with the seven notes of the usual scale are given by Dó $= f$, Ré $= \sqrt[6]{2}f$, Mi $= \sqrt[3]{2}f$, Fá $= \sqrt[12]{2^5}f$, Sol $= \sqrt[12]{2^7}f$, Lá $= \sqrt[4]{2^3}f$, Si $= \sqrt[12]{2^{11}}f$ e Dó $= 2f$.

Methods for mumerical approximations of equal temperament can be found in Zarlino in the 16th cent. and in M. Mersenne's *Harmonie Universelle* (1636–7) or in A. Kircher's *Musurgia Universalis* (1650).

The theorising of equal temperament in the 17th century will use logarithms. C. Huygens (1629–1695) in *Novus Cyclus Harmonicus* (1691) theorised the division of the octave into 31 equal intervals and was one of the first to introduce the calculation of logarithms into music.



Divisio Octavæ in partes 31. æquales.

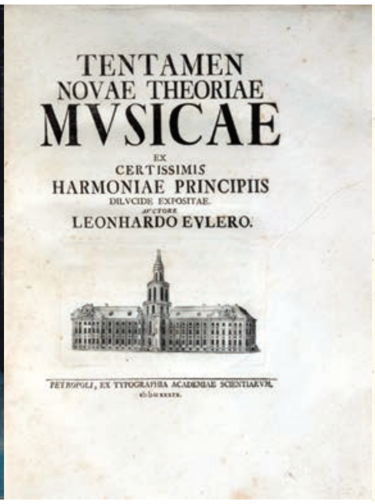
I.	II.	III.	IV.
N 97106450			
4,6989700043	50000	Ut	C
4,7086806493	51131		
4,7183912943	52278		
4,7281019393	53469	Si	B
4,7378125843	54678		
4,7475232293	55914	Sa	B
4,7572338743	57179	•	•
4,7669445193	58471		
4,7766551643	59794	La	A
4,7863658093	61146		
4,7960764543	62528	•	•
4,8057870993	63942	Sol	G
4,8154977443	65388		
4,8252083893	66866	Sol	G
4,8349190343	68378		
4,8446296793	69924		
4,8543403243	71506	Fa	F
4,8640509693	73122		
4,8737616143	74776	Fa	F
4,8834722593	76467		
4,8931829043	78196		
4,9028935493	79964	Mi	E
4,9126041943	81772		
4,9223148393	83621	Ma	E
4,9320254843	85512	•	•
4,9417361293	87445		
4,9514467743	89422	Re	D
4,9611574193	91444		
4,9708680643	93512	•	•
4,9805787093	95627	Ut	C
4,9902893543	97789		
4,9999999993	100000	Ut	C

Calculations of 1691 of the division of the octave into 31 tones by Huygens using logarithms

Referring to Salinas and Mersenne as authors who had already considered this division to be of no great consequence, Huygens remarked that if their predecessors had been mistaken because "they hadn't known how to divide the octave into 31 equal parts (. . .) for this the intelligence of Logarithms was necessary."

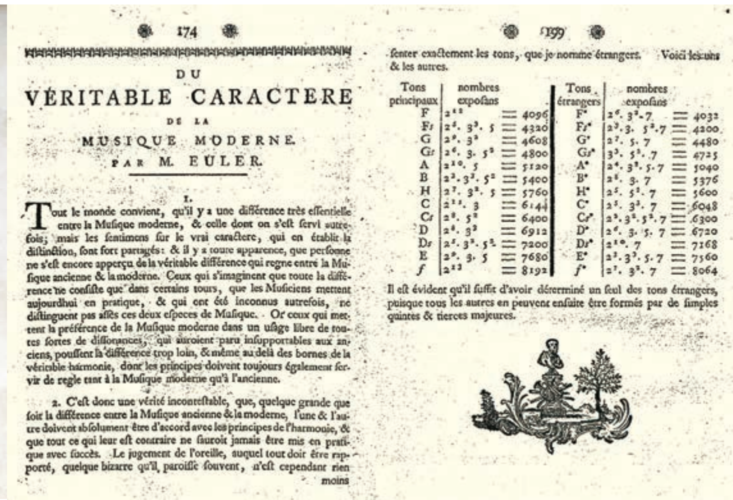
In Euler (1707–1783) we find one of the most ingenious algebraic theories of the division of the octave and the degree of consonance of musical intervals.

In the Essay on a new theory of music (*Tentamen novae theoriae musicae*, 1739), Euler develops an argument in which proportions generate musical pleasure, via order and perfection — music is the science of combining sounds in a pleasing harmony — so that, for this mathematician, a musical object is a simple arithmetical object.



TENTAMEN
NOVAE THEORIAE
MVSICAE
EX
CERTISSIMIS
HARMONIAE PRINCIPIIS
DILVCIDE EXPOSITAE.
AVCTORE
LEONHARDO EVLERO.

PETROPOLI, EX TYPOGRAPHIA ACADEMIAE SCIENTIARVM.



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DU
VÉRITABLE CARACTERE
DE LA
MUSIQUE MODERNE.
PAR M. EULER.

1.

Tout le monde convient, qu'il y a une différence très essentielle entre la Musique moderne, & celle dont on s'est servi autrefois; mais les sentimens sur le vrai caractere, qui en établit la distinction, sont fort partagés: & il y a toute apparence, que personne ne s'est encore apperçu de la véritable différence qui regne entre la Musique ancienne & la moderne. Ceux qui s'imaginent que toute la différence ne consiste que dans certains tours, que les Musiciens mettent aujourdhui en pratique, & qui ont été inconnus autrefois, ne distinguent pas assés ces deux especes de Musique. Or ceux qui mettent la préférence de la Musique moderne dans un usage libre de toutes sortes de dissonances, qui auroient paru insupportables aux anciens, poussent la différence trop loin, & même au delà des bornes de la véritable harmonie, dont les principes doivent toujours également servir de regle tant à la Musique moderne qu'à l'ancienne.

2. C'est donc une vérité incontestable, que, quelque grande que soit la différence entre la Musique ancienne & la moderne, l'une & l'autre doivent absolument être d'accord avec les principes de l'harmonie, & que tout ce qui leur est contraire ne sauroit jamais être mis en pratique avec succès. Le jugement de l'oreille, auquel tout doit être rapporté, quelque bizarre qu'il paroisse souvent, n'est cependant rien moins

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senter exactement les tons, que je nomme étrangers. Voici les uns & les autres.

Tons principaux	nombres exposans		Tons étrangers	nombres exposans	
F	2^{12}	= 4096	F*	$2^6 \cdot 3^2 \cdot 7$	= 4032
Fs	$2^5 \cdot 3^3 \cdot 5$	= 4320	Fs*	$2^3 \cdot 3 \cdot 5^2 \cdot 7$	= 4200
G	$2^9 \cdot 3^2$	= 4608	G*	$2^7 \cdot 5 \cdot 7$	= 4480
Gs	$2^6 \cdot 3 \cdot 5^2$	= 4800	Gs*	$3^3 \cdot 5^2 \cdot 7$	= 4725
A	$2^{10} \cdot 5$	= 5120	A*	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	= 5040
B	$2^3 \cdot 3^3 \cdot 5^2$	= 5400	B*	$2^8 \cdot 3 \cdot 7$	= 5376
H	$2^7 \cdot 3^2 \cdot 5$	= 5760	H*	$2^5 \cdot 5^2 \cdot 7$	= 5600
C	$2^{11} \cdot 3$	= 6144	C*	$2^5 \cdot 3^3 \cdot 7$	= 6048
Cs	$2^8 \cdot 5^2$	= 6400	Cs*	$2^2 \cdot 3^2 \cdot 5^2 \cdot 7$	= 6300
D	$2^8 \cdot 3^3$	= 6912	D*	$2^6 \cdot 3 \cdot 5 \cdot 7$	= 6720
Ds	$2^5 \cdot 3^2 \cdot 5^2$	= 7200	Ds*	$2^{10} \cdot 7$	= 7168
E	$2^9 \cdot 3 \cdot 5$	= 7680	E*	$2^3 \cdot 3^3 \cdot 5 \cdot 7$	= 7560
f	2^{13}	= 8192	f*	$2^7 \cdot 3^2 \cdot 7$	= 8064

Il est évident qu'il suffit d'avoir déterminé un seul des tons étrangers, puisque tous les autres en peuvent ensuite être formés par de simples quintes & tierces majeures.

Euler introduced a mesure of the degree of consonance (agrément) of an interval through an algebraic formula in which p_i are prime numbers and m_i, integer exponents:

$$\alpha(I) = \sum_{i=1}^{n} \left(m_i p_i - m_i\right) + 1.$$

Euler also wrote other essays, such as *Du véritable caractere de la musique moderne* (On the true character of modern music), in Mémoires de l'Académie des Sciences de Berlin (1764), 1766.

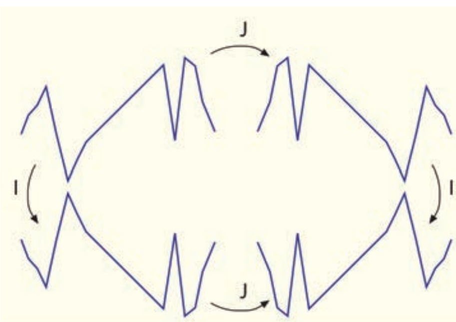
But the algebra of tones is not limited to the problems associated with temperament, but also appears in the structure of sounds and in musical composition itself.

Musical notes can be grouped into equivalence classes and hence called by the same name, i.e. two notes are said to be equivalent if they are separated by an exact number of octaves, i.e. if they have frequencies p and q, the interval between them is of the form $p/q = 2k$, with $k = 0, \pm 1, \pm 2, \ldots$ and will be denoted by $p \sim q$.

In the 12-note tempered system, the interval is characterised by the number of semitones and the notes can be associated with the set of integers

$$\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$$

which is a group for addition (mod 12).



O - identidade

I - inversão (simetria horizontal)

J - retrogradação (simetria vertical)

K = I + J = J + I (composição de I com J)

+	O	I	J	K
O	O	I	J	K
I	I	O	K	J
J	J	K	O	I
K	K	J	I	O

Geometric and algebraic representations of a group

Inversion (horizontal symmetry), Petrushka by Igor Stravinsky

Another typical example is given in J.S. Bach's *Musical Offering* of 1747, which presents three types of transformations: translations (upward transpositions, as in the canon ascendenteque *Modulationem ascendat gloria Regis*), horizontal symmetries (melodic inversions, as in the canon *Per Motum Contrarium*) and vertical symmetries (retrogrades, as in the canon a 2 which plays the same theme starting on the last note and moving backwards to the first). Also known as *palindromes* or *crab canons*: $y = -x$.

3. Harmonisation of Analysis

Marin Mersenne (1588–1648) is credited with establishing the basic laws of modern string acoustics. *Harmonie universelle* (1636), establishes the experimental laws on the proportionality of the period of vibration of the string, in relation to its length, to the inverse of the square root of its tension and to the square root of its thickness or cross-sectional area.

Galileo Galilei, in *Discorsi e dimostrazioni matematiche . . .* (1638) refers to the question of vibrating strings and consonance as follows:

> . . . the first and immediate reason on which the ratios of musical intervals depend is neither the length of the strings nor their thickness, but the proportion existing between the frequencies of the vibrations, and therefore of the waves which, propagating in the air, reach the eardrum of the ear causing it to vibrate at the same intervals of time.

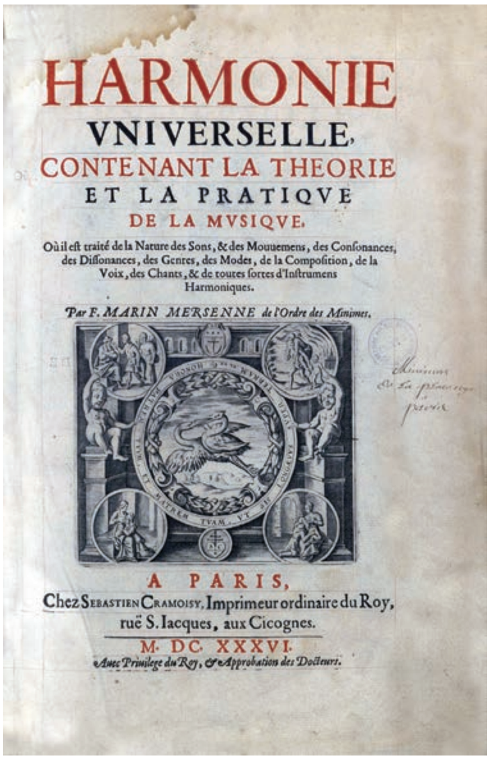
The mathematical analysis of the sound starts with the modeling of the vibrating string, namely with the computation of its fundamental period by B. Taylor in 1713, with the first ODE analysis by Jean Bernoulli in 1727 and the famous controversy between D'Alembert and Euler on the admissible initial conditions on the wave equation.

It is above all with the introduction of the equation

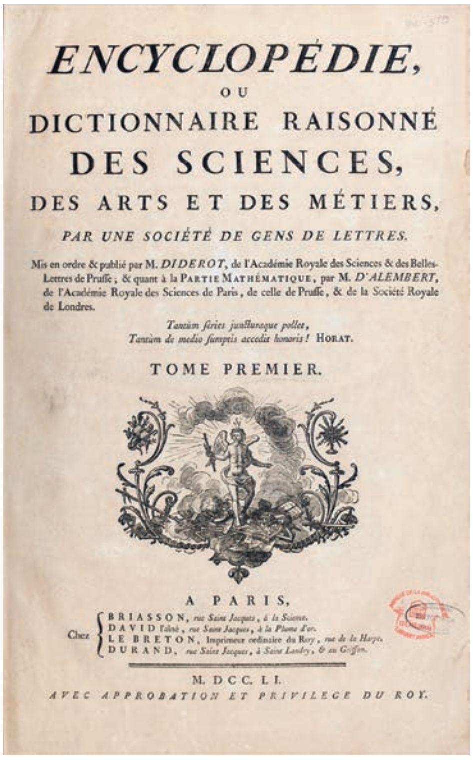
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

in D'Alembert's 1747 memoire published by the Berlin Academy, *Recherche sur la courbe que forme une corde tendue mise en vibration*, and with the subsequent works of Euler, Daniel Bernoulli and Lagrange, that the mathematical theory of the "musical string" acquires the appropriate model for small vibrations, which will be decisive in the study of oscillations in continuous media, in particular the propagation of sound in air.

During the course of the famous "vibrating string controversy", a scientific dispute involving the leading mathematicians of the 1700s, Daniel Bernoulli, in



HARMONIE
VNIVERSELLE,
CONTENANT LA THEORIE
ET LA PRATIQVE
DE LA MVSIQVE,
Où il est traité de la Nature des Sons, & des Mouuemens, des Consonances, des Dissonances, des Genres, des Modes, de la Composition, de la Voix, des Chants, & de toutes sortes d'Instrumens Harmoniques.
Par F. MARIN MERSENNE de l'Ordre des Minimes.
A PARIS,
Chez SEBASTIEN CRAMOISY, Imprimeur ordinaire du Roy, ruë S. Iacques, aux Cicognes.
M. DC. XXXVI.
Auec Priuilege du Roy, & Approbation des Docteurs.



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OU
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DES ARTS ET DES MÉTIERS,
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Tantùm series juncturaque pollet,
Tantùm de medio sumptis accedit honoris! HORAT.
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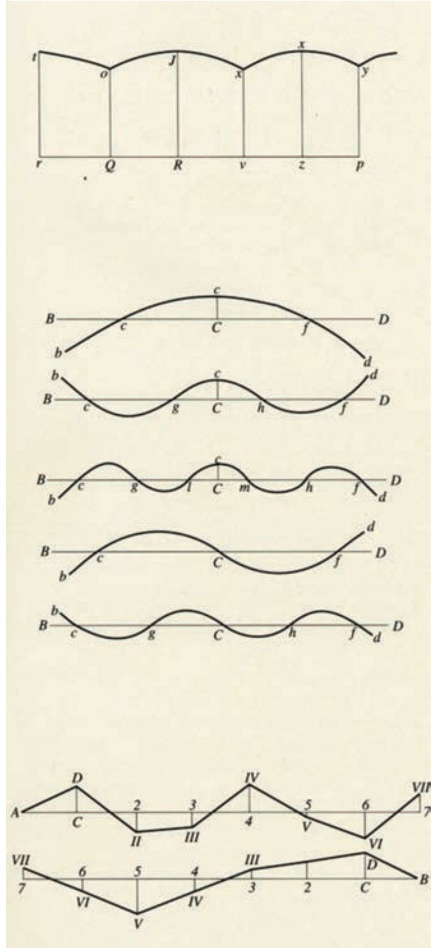
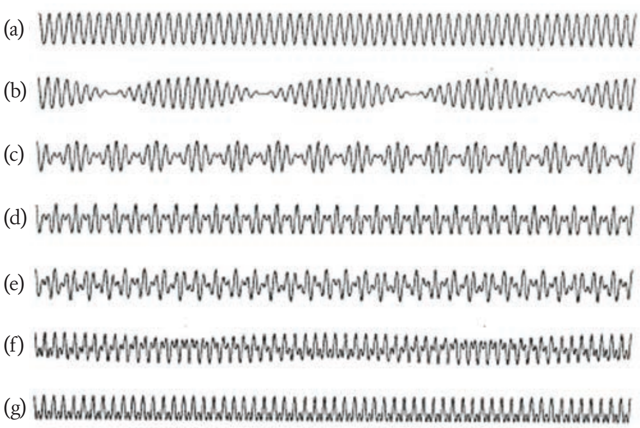


a 1753 letter, established the principle of the superposition of small harmonic oscillations as a physical law and not so much as a mathematical result, concluding that

> every sounding body potentially contains an infinity of sounds and a corresponding infinity of ways of producing their respective vibrations.

In a memoir by the Turinese mathematician Lagrange (1736-1813), we find a formula for the solution of the wave equation which, in the 19th century, after the work of Fourier, will allow us to demonstrate D. Bernoulli's principle of superposition of waves. Lagrange not only sought to analyse the propagation of sound, he also tried to provide a scientific explanation for Tartini's theory of the combination of tones, set out in his Treatise on Music of 1754.

The musical string is just the first mathematical example of sound analysis. Both the sound produced by



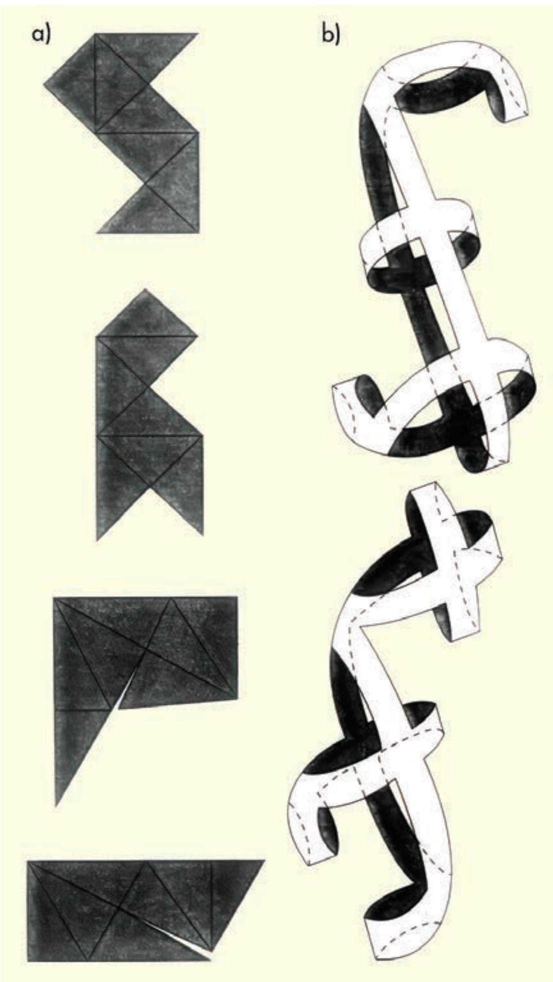
Soit $\frac{X}{a}$ la raison générale des indices des Y et des V au nombre m, X dénotera la partie de l'axe qui leur est correspondante dans le premier état du système; donc, si l'on emploie le signe intégral $\int$ pour exprimer la somme de toutes ces suites, on aura

$$y = \frac{2}{a}\int dx\,Y\left(\sin\frac{\varpi X}{2a}\sin\frac{\varpi x}{2a}\cos\frac{\varpi H t}{2T} + \sin\frac{2\varpi X}{2a}\sin\frac{2\varpi x}{2a}\cos\frac{2\varpi H t}{2T} + \sin\frac{3\varpi X}{2a}\sin\frac{3\varpi x}{2a}\cos\frac{3\varpi H t}{2T} + \ldots\right)$$
$$+ \frac{4T}{\varpi H a}\int dx\,V\left(\sin\frac{\varpi X}{2a}\sin\frac{\varpi x}{2a}\sin\frac{\varpi H t}{2T} + \frac{1}{2}\sin\frac{2\varpi X}{2a}\sin\frac{2\varpi x}{2a}\sin\frac{2\varpi H t}{2T} + \frac{1}{3}\sin\frac{3\varpi X}{2a}\sin\frac{3\varpi x}{2a}\sin\frac{3\varpi H t}{2T} + \ldots\right),$$

Left: Ratios of frequencies of two pure tones (a) 1:1 (b) 15:16 (c) 4:5 (d) 2:3 (e) 20:31 (f) 30:59 (g) 1:2.
Right: An excerpt of *Recherches sur la nature de la propagation du son* (1579) by Lagrange.

most musical instruments and the human ear itself require mathematical models that take into account the various dimensions of physical space and geometry.

In the mathematical analysis of the sound a famous question arouse: *is it possible to hear the shape of a drum*? This question, which has a precise and profound meaning in maths, consists of knowing whether from the same family of eigenvalues, i.e., numbers $\lambda = \lambda_n$, $n = 1, 2, \ldots$, that satisfy the equation $\Delta u + \lambda u = 0$ in two domains Ω_1 and Ω_2 it is possible to say that these regions are congruent in the sense of Euclidean geometry. Of all the drums with the same area, the round one has the deepest sound.



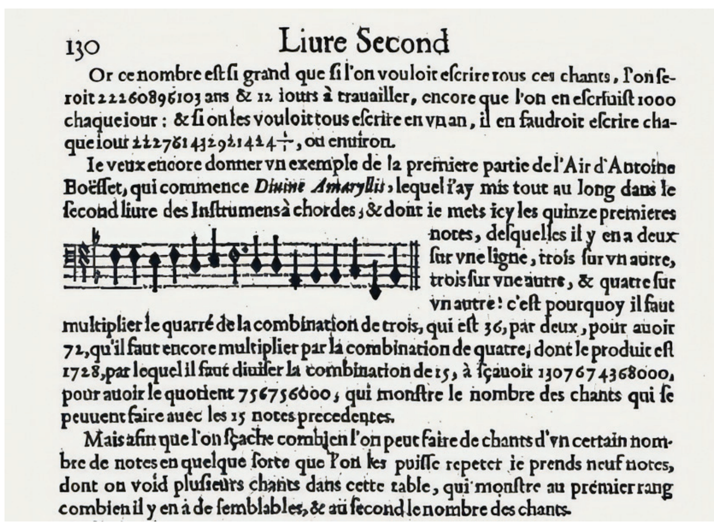
a) Isospectral (reproducing the same sound) drum shapes (flat polygons) with different shapes (C. Gordon and D. Webb, 1991).
b) Isospectral spatial shapes of bells (Riemannian surfaces) by P. Buser (1986).

4. Digital Musurgia

As early as the 17th century, an obscure German mathematician, K. Schott, following the ideas of Mersenne and his teacher Kircher, author of a *Musurgia Universalis* (1650), argued in his *Organum mathematicum* (1668) that to compose harmonic chants it was enough to master the new art of music-arithmetic, which consisted of combining the *bacilli musurgici* (the musical keys) and using the *abaci melothetici* and the *tabulae musarithmeticae*.

These ideas were based on the new combinatorial art of Mersenne, in *Harmonie Universelle* (1636), for whom composing was reduced to combining, he had distinguished permutations without repetition of a given number of n notes (ordinary combinations which he calculated up to $n = 64$) $S_n = n!$, from permutations with repetition of n notes (p are different), which he used to calculate the table of chants that can be made from 9 notes, he also calculated arrangements without repetition (p different notes among n given) and also combinations without repetition.

In the evolution of mathematics from the 17th to the 18th century, particularly for G.W. Leibniz, the mathematical sciences acquired a broader role as a science about the representations of all possible relationships and dependencies of the simplest elements, seeking a universal language and an algebra of reasoning, perfect-



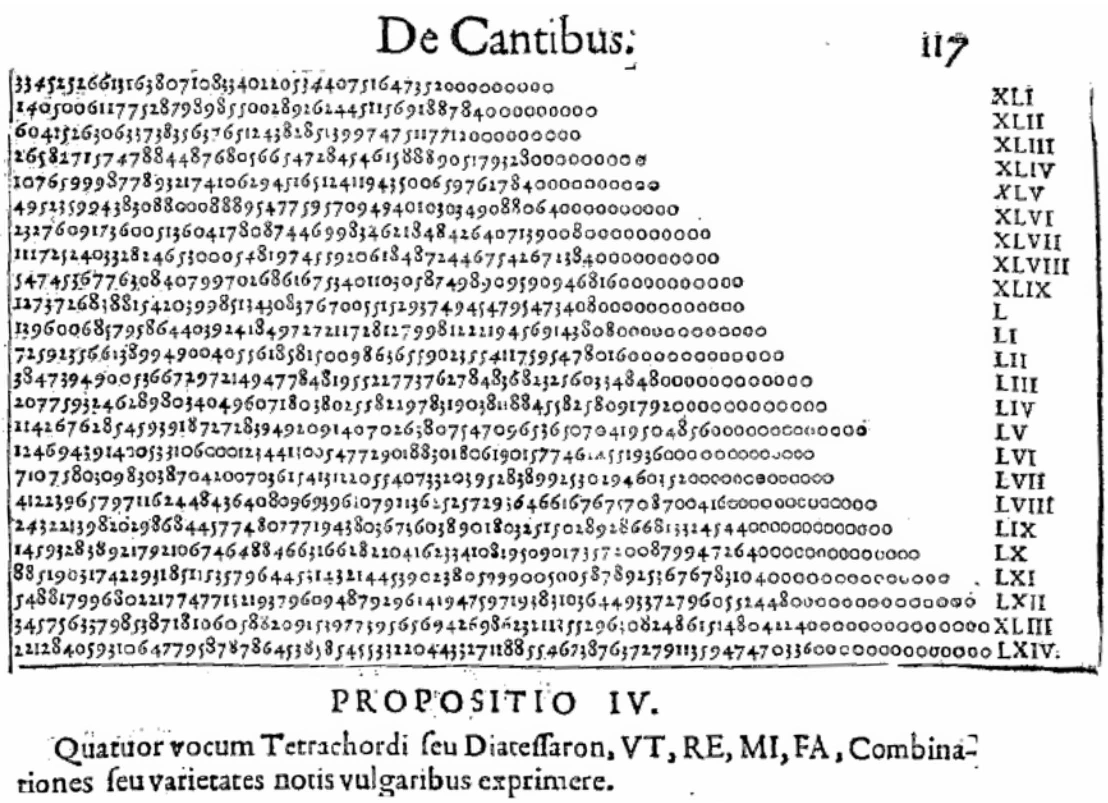
130 Liure Second

Or ce nombre est si grand que si l'on vouloit escrire tous ces chants, l'on seroit 22260896103 ans & 12 iours à trauailler, encore que l'on en escriuist 1000 chaque iour: & si on les vouloit tous escrire en vn an, il en faudroit escrire chaque iour 2227814329214 2/4, ou enuiron.

Ie veux encore donner vn exemple de la premiere partie de l'Air d'Antoine Boësset, qui commence *Diuine Amaryllis*, lequel i'ay mis tout au long dans le second liure des Instrumens à chordes, & dont ie mets icy les quinze premieres notes, desquelles il y en a deux sur vne ligne, trois sur vn autre, trois sur vne autre, & quatre sur vn autre: c'est pourquoy il faut multiplier le quarré de la combination de trois, qui est 36, par deux, pour auoir 72, qu'il faut encore multiplier par la combination de quatre, dont le produit est 1728, par lequel il faut diuiser la combination de 15, à sçauoir 1307674368000, pour auoir le quotient 756756000, qui monstre le nombre des chants qui se peuuent faire auec les 15 notes precedentes.

Mais afin que l'on sçache combien l'on peut faire de chants d'vn certain nombre de notes en quelque sorte que l'on les puisse repeter ie prends neuf notes, dont on void plusieurs chants dans cette table, qui monstre au premier rang combien il y en a de semblables, & au second le nombre des chants.

M. Mersenne, *Harmonie Universelle* (1636–37)



De Cantibus. 117

33452526613163807108334012053440751647352000000000	XLI
1405006117752879898550028926144511569188784000000000	XLII
60415263063373835637651243828513997475117712000000000	XLIII
2658271574788448768056654728454615888905179328000000000	XLIV
107659998778932174106294516512411943500659761784000000000	XLV
4952359924383088000888954775957094940103034908806400000000000	XLVI
232760917360051360417808744699834621848426407139008000000000	XLVII
11172524033282465300054819745920618487244675426713840000000000	XLVIII
547455677630840799702686167534011030587498909590946816000000000 0	XLIX
27373268388154203998511430837670055152937494547954734080000000000000	L
1396006857958644039241849727211728127998112219456914380800000000000000	LI
72592356613899490040556185815009865655902355411759547801600000000000000	LII
3847394900536672972149477848195522773762784836823256033484800000000000000	LIII
207759324628980340496071803802558219783190381188455825809179200000000000000	LIV
11426762854593918727283949209140702638075470965365070419504856000000000000000	LV
1246943914205331060001234411005477290188301806190157746145519360000000000000000	LVI
71075803098303870420070361541311205540733203952838992530194601520000000000000000	LVII
4122396579711624484364080969396107911362525729364661676757087004160000000000000000	LVIII
243221398202986844577480777194380356736038901803251502892866813321454400000000000000000	LIX
1459328389217921067464884663166282204162334108195090173572008799472640000000000000000	LX
88519031742293185115357964453143214453902380599900500587892536767831040000000000000000	LXI
548817996802217747715219379609487929614194759719383103644933727960552448000000000000000000	LXII
34575633579853871810605882091519775956569426986232111355296308248615148041240000000000000000000	XLIII
2212840593106477958787864538385455331204433271188554673876372791135947470336000000000000000000	LXIV.

PROPOSITIO IV.

Quatuor vocum Tetrachordi ſeu Diateſſaron, VT, RE, MI, FA, Combinationes ſeu varietates notis vulgaribus exprimere.

M. Mersenne, *Harmonie Universelle* (1636-37)

ing calculation and creating new algorithms to which it became necessary to give a symbolism appropriate to the essence of the concepts and operations.

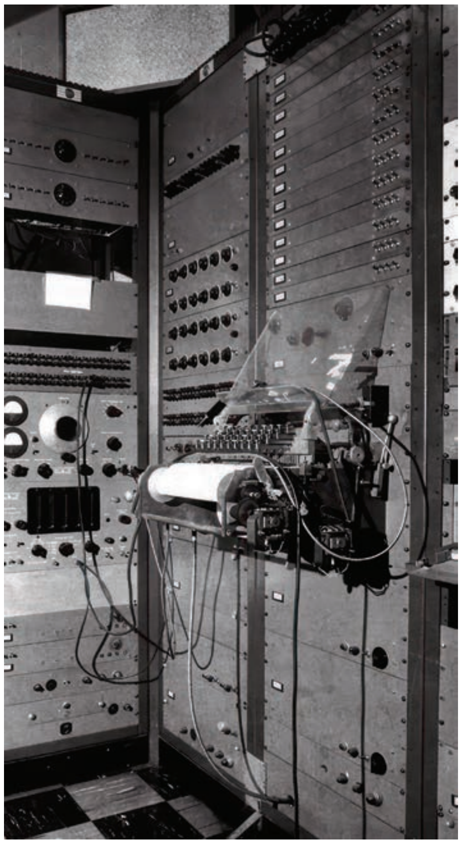
In his dissertation on the art of combinatorics (1666), the young Leibniz already intended to reorganise logic, but it was after the creation of the Calculus that he referred to binary notation in a 1701 letter to J. Bernoulli: Many years ago an original idea occurred to me about a type of arithmetic where everything is expressed with 0 and 1.

However, this new type of binary arithmetic was only realised in modern computers, where each bit represents an electrical state: on (current) is associated with the number 1; off (no current) is associated with 0; and sequences of electrical impulses, such as 01000001 which represents the number 65 in the binary system, and which can also be assigned to the capital letter A using another code.

Forerunners of modern calculators, the machines of the 17th century had a limited impact, in particular those of W. Schickard (1592–1635) and B. Pascal (1623–1662), capable of adding and subtracting mechanically, or that of Leibniz in 1671, which could also multiply and divide.

Left:
G. W. Leibniz (1646–1716)

Right;
Binary system designed by Leibniz, which reads “one created everything out of nothing” at the top and “one is necessary” at the bottom.



Left:
Ada Lovelace (1815-1852)

Right:
RCA Mark II
Electronic Music
Synthesizer,
H. Olson e H. Belar (1957)

However, only C. Babbage's (1791-1871) mechanical machines, namely the Difference Engine (1821) and the Analytical Engine (1834), are considered to be the forerunners of electronic computers, even though they were never built.

In a passage on the conception of that machine, Ada Lovelace specifically states that its operative mechanism could act on things other than numbers, objects such that their fundamental reciprocal relationships could be expressed by the abstract science of operations and, as a concrete example within the framework of the operative notation and mechanisms of the Analytical Engine, explicitly supposes that the fundamental relationships of sounds determined in the science of harmony and musical composition could be expressed and adaptable to its action; the machine could compose scientific and elaborate musical pieces, with any degree of complexity or extension.

However, a sufficiently powerful mechanism capable of incorporating the science of operations only appeared with the modern computer in the second half of the 20th century.

The first experiments in computer-assisted musical composition appeared from the start L. Hiller in 1956 in the USA, followed by P. Barbaud and I. Xenakis in France and others. At Bell Laboratories, in 1957, M. Mathews and his collaborators made the first numerical record and the first computer synthesis of sounds and, in 1965, J.C. Risset computer-simulated the first sounds of musical instruments.

In 1973, the first numerical synthesiser was built, Synclavier, which was then commercialised, and about ten years later theten years later, the public had access to digital recording CD's (compact discs).

Since 1983, the MIDI (Musical Instrumental Digital Interface) standard has allowed computers to record and edit music.

If today we have the mastery of numerisation in the analysis and synthesis of musical sound, if we have begun to outline the mathematisation of certain musical structures and computers allow us to hear mathematical calculations and structures, i.e. to paraphrase Saccheri we have *Pythagoras ab omni naevo vindicatus sive Conatus arithmeticus quo stabiliuntur prima ipsa universa musica principia (Pythagoras freed from all taint or the arithmetical attempt to establish the first principles of all music*). we can continue to agree with Aristoxenus and accept that the justification of music lies in the pleasure of its hearing and its enjoyment.