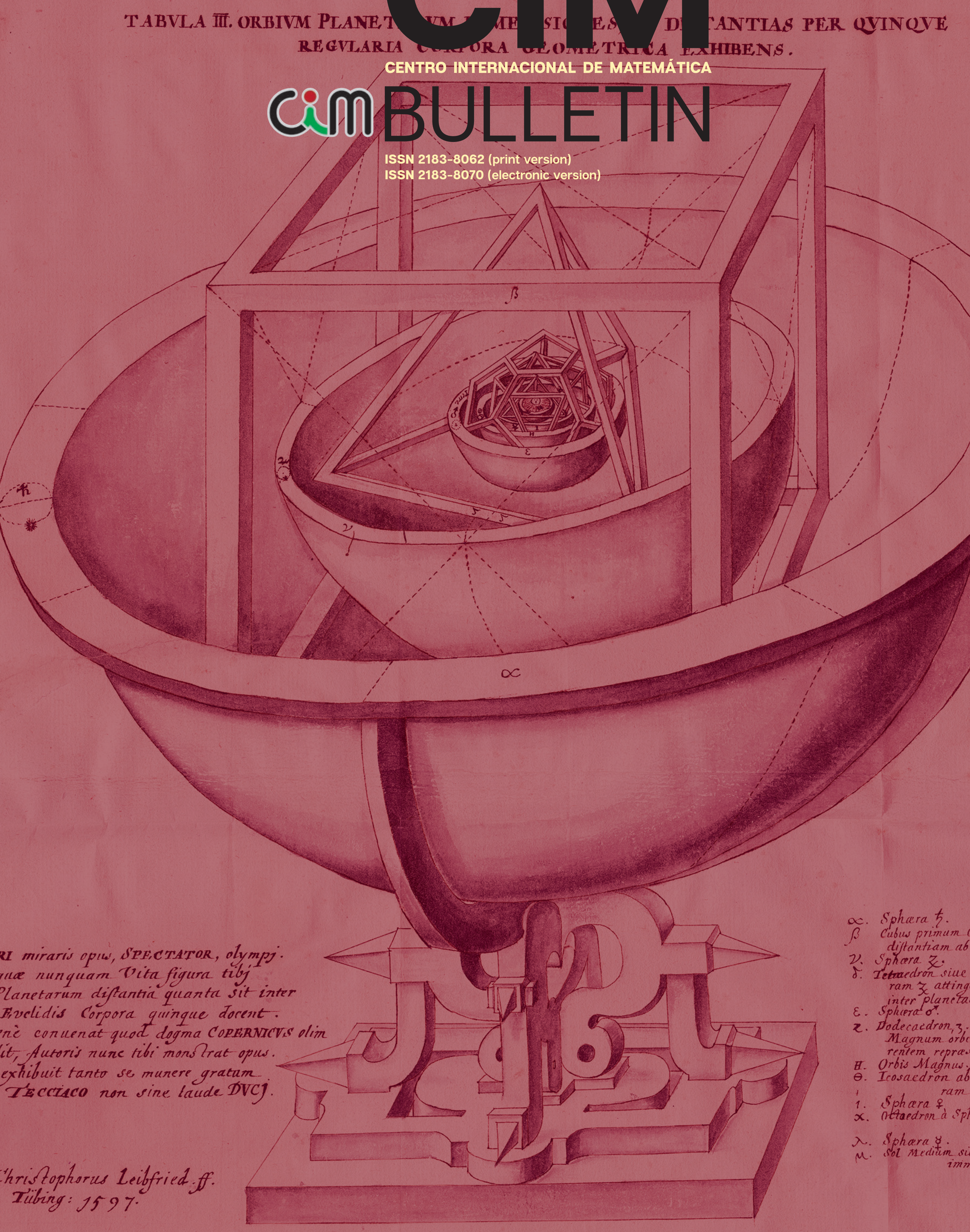
CIM

CENTRO INTERNACIONAL DE MATEMÁTICA

BULLETIN

ISSN 2183-8062 (print version)
ISSN 2183-8070 (electronic version)



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CIM

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Editorial

In March of 2024, a new direction board of CIM took office and the editorial board of the CIM bulletin was renewed. The editorial board is committed to continue the efforts of fulfilling the bulletin's main goals of promoting Mathematics and especially mathematical research.

In this issue, we include three articles on topical research subjects. One of them reviews the development of the theory of impulsive dynamical systems, which model real-world phenomena characterized by abrupt changes, and proposes future directions for its further advancement.

Another article reviews the main steps in the proof of Fermat's Last Theorem and explores Darmon's program for addressing the generalized Fermat equation $Ax^q + By^r = Cz^p$. Further, it highlights recent advances achieved by integrating the classical techniques with insights from Darmon's approach. The last article provides an elegant and concise exposition of the Lie theory needed to appreciate the eightfold way in particle physics.

Inserted in the cycle of historical articles, we feature an article dedicated to the Portuguese texts on the calculus in the 18th century. It explores the gradual introduction of the calculus in Portugal, emphasizing the contributions of Jacob de Castro Sarmento, the influence of Bézout's textbook and the work of José Anastácio da Cunha, the most original Portuguese mathematician of the 18th century.

We include an insightful article on the theme of Mathematics and Music, presenting a beautiful historical journey about the numerous interactions between these two fields, from Ancient Greece to the modern technological era.

We celebrate that Professor Jorge Buescu was awarded the *Grande Prémio Ciência Viva 2024*, that Professor José Francisco Rodrigues has been elected President of the Lisbon Academy of Sciences and that Professor Luís Nunes Vicente has been selected as a Fellow of the Society for Industrial and Applied Mathematics (SIAM).

As usual, we publish several summaries and reports regarding the activities partially supported by CIM during the last year.

We recall that the bulletin continues to welcome the submission of review, feature, outreach and research articles in Mathematics and its applications.

Maria Joana Torres

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Unraveling the Dynamics of Impulsive Semiflows: Ergodic and Topological Features

by **Jaqueline Siqueira***

Everything is logical. To understand is to justify.
— Eugène Ionesco

1 Introduction

The time evolution of various natural phenomena often exhibits sudden changes of state, occurring at certain points where the duration of these disturbances is either null or minimal in comparison to the overall duration of the phenomenon. These sudden changes are referred to as impulses and can be observed in fields such as physics, biology, economics, control theory, and information science, see [20, 24, 29, 32, 6, 26] and references therein.

Impulsive dynamical systems (IDS) are effective mathematical models of real world phenomena that display abrupt changes in their behavior. More precisely, an IDS is defined by three objects: a continuous semiflow on a phase space M; an impulsive region $D \subset M$, where the flow experiences sudden perturbations; and an impulsive map $I : D \to M$, which determines the change in the trajectory each time it hits the impulsive region D.

Under broad conditions an IDS generates a semiflow, called impulsive semiflow, that is not continuous in general. This inherent feature of the dynamics of an impulsive semiflow is a key challenge when trying to describe its topological and ergodic behavior.

In [21, 22], Kaul initiated the study of impulsive dynamical systems with impulses at variable times and since then several authors have contributed to the development of the theory. We mention the important contributions of Ciesielski [14, 15, 16], as well as those of Bonotto and his collaborators [9, 11, 12, 10].

However the study of impulsive semiflows from the perspective of smooth topological dynamics and ergodic theory has been initiated only recently. In this work we aim to outline the development of the theory to date and to propose potential directions for its further advancement.

Definition and first properties

Since the lack of continuity of an impulsive semiflow happens independently of the regularity of the IDS that generates it, we shall not consider the most general class of impulsive semiflows (see [10]). We thus start by assuming some regularity on the IDS.

Let M be a compact manifold endowed with the Riemannian metric d and let $\varphi : M \times \mathbb{R}_0^+ \to M$ be a semiflow: $\varphi_0(x) = x$ and $\varphi_{t+s}(x) = \varphi_t(\varphi_s(x))$ for all $x \in M$ and $t, s \in \mathbb{R}_0^+$. We assume that φ is generated by a C^1-vector field X, that D is a submanifold of codimension one transversal to X, and that $I : D \to M$ is a continuous map so that $I(D) \cap D \neq \emptyset$. The IDS (M, φ, D, I) generates an impulsive semiflow as follows.

The *first impulsive time map* $\tau_1 : M \to [0, +\infty]$ is the map that records the first visit of each φ-trajectory to D: $\tau_1(x) := \inf \{t > 0 : \varphi_t(x) \in D\}$ if $\varphi_t(x) \in D$ for some $t > 0$ and $\tau_1(x) := +\infty$, otherwise. Given $x \in M$, the *impulsive trajectory* γ_x and the subsequent *impulsive times* $\tau_2(x), \tau_3(x), \tau_4(x), \ldots$ are defined inductively. For $0 \leq t < \tau_1(x)$ we set $\gamma_x(t) = \varphi_t(x)$. Assuming that $\gamma_x(t)$ is defined for $t < \tau_n(x)$ for some $n \geq 2$, we set

$$\gamma_x(\tau_n(x)) = I(\varphi_{\tau_n(x)-\tau_{n-1}(x)}(\gamma_x(\tau_{n-1}(x)))).$$

Defining the $(n+1)^{\text{th}}$ impulsive time of x as

$$\tau_{n+1}(x) = \tau_n(x) + \tau_1(\gamma_x(\tau_n(x))),$$

for $\tau_n(x) < t < \tau_{n+1}(x)$, we set

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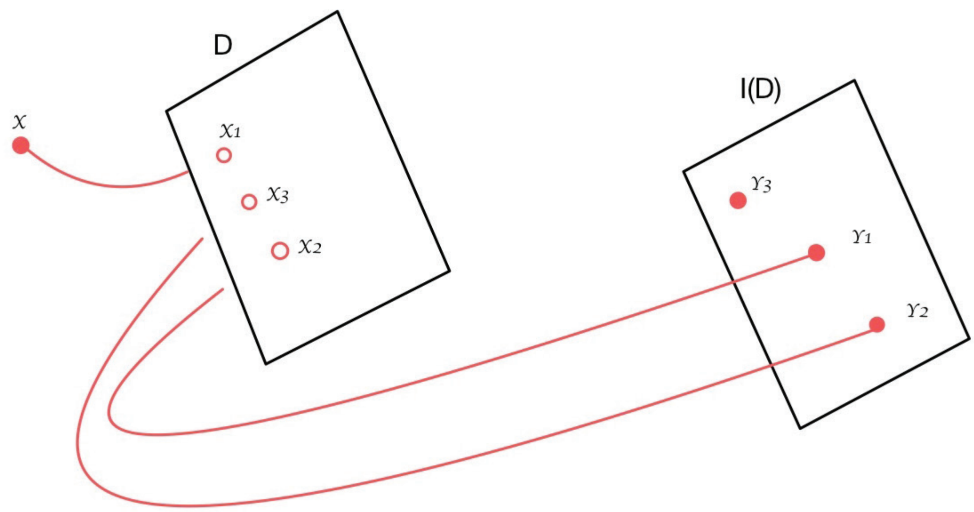

Figure 1.—Building the trajectory of a point x. Set $x_i = \varphi_{\tau_i(x)}(x)$, $y_i = I(x_i)$, $i = 1, 2, 3$.

$$\gamma_x(t) = \varphi_{t-\tau_n(x)}(\gamma_x(\tau_n(x))).$$

Since we assume $I(D) \cap D = \varnothing$ [3, Remark 1.1] we have that $T(x) := \sup\{\tau_n(x); n \geq 1\} = \infty$, thus impulsive trajectories are defined for all positive times.

According to [9, Proposition 2.1], the trajectories defined above indeed generate a semiflow which we call an *impulsive semiflow*:

$$\begin{array}{rccc} \psi : & \mathbb{R}_0^+ \times M & \longrightarrow & M \\ & (t, x) & \longmapsto & \gamma_x(t). \end{array}$$

In general terms, ψ-impulsive trajectories are built with segments of φ-trajectories in the following way. Given a point $x \in M$, consider its φ-trajectory until it hits D (x_1, in Figure 1). Then delete the intersection point x_1, restart the impulsive trajectory at the image of the deleted point under I ($y_1 = I(x_1)$). From there, follow its φ-trajectory until it hits D again, and repeat the process. Note that since $D \cap I(D) = \varnothing$, an impulsive trajectory of a point $x \in M$ intersects D only if $x \in D$. Moreover, no periodic trajectories intersect the impulsive region D.

2 Ergodic theory of impulsive semiflows

The field of ergodic theory has been developed with the goal of understanding the statistical behavior of a dynamical system via measures which remain invariant under its action. Describing the behavior of the orbits in a dynamical system can be challenging, particularly for systems with complex topological and geometrical structures. One powerful method for analyzing such systems is through invariant probability measures. For example, Birkhoff's Ergodic Theorem states that almost every initial condition within each ergodic component of an invariant measure shares the same statistical distribution in space.

We say that μ is an *invariant* probability measure under a semiflow φ if $\mu(\varphi_t^{-1}(A)) = \mu(A)$ for all Borel sets $A \subset M$ and for all $t > 0$. Denote by $\mathscr{M}(\varphi)$ the set of all invariant probability measures.

A point $x \in M$ is called *non-wandering* for φ if for every neighborhood $\mathscr{U}$ of x and for every $t > 0$, there exists $T \geq t$ so that $\varphi_T^{-1}(\mathscr{U}) \cap \mathscr{U} \neq \varnothing$. Denote by $\Omega(\varphi)$ the set of all non-wandering points for φ.

The compactness of M implies that $\Omega(\varphi) \neq \varnothing$ and is compact. Moreover, if φ is continuous, the non-wandering set is invariant: $\varphi_t(\Omega(\varphi)) \subset \Omega(\varphi)$ for all $t > 0$. However, this does not remain true in general for impulsive semiflows, which is quite a suprising phenomenon (see [3]).

For an invariant measure, the relevant points are the non-wandering ones, meaning that any invariant measure is supported on the non-wandering set. When we assume that φ is a continuous semiflow on a compact metric space there is always a φ-invariant measure [30, Theorem 2.1]. However, one can build examples of impulsive semiflows with no invariant measures (e.g. [3, Example 2.2]).

Set $\tau_D(x) = \tau_1(x)$ if $x \in \Omega(\psi)\backslash D$ and $\tau_D(x) = 0$ if $x \in \Omega(\psi) \cap D$. Let (M, φ, D, I) be an IDS so that φ is continuous on a compact metric space M, τ_D is continuous and $I(\Omega(\psi) \cap D) \subset \Omega(\psi)\backslash D$. Therefore, by [3, Theorem A], the impulsive semiflow ψ admits an invariant measure.

The existence of invariant measures can also be obtained by assuming some regularity and transversality conditions on the IDS [1, Theorem I] as follows.

A criterion for existence of invariant measures

Theorem 1.— Let $\varphi\colon M \times \mathbb{R}_0^+ \to M$ be a C^1-semiflow, D a compact submanifold of codimension one, transversal to the flow direction and $I\colon D \to M$ a continuous map. Then the impulsive semiflow ψ admits an invariant probability measure.

2.1 Topological pressure

The topological pressure of a semiflow with respect to a potential function is the rate of growth of trajectories of the semiflow where each point is weighted according to the potential. We recall the classical definition of topological pressure for continuous semiflows and generalize this concept to impulsive semiflows.

Topological pressure for continuous semiflows

Let $\varphi\colon M \times \mathbb{R}_0^+ \to M$ be a continuous semiflow. Given $\varepsilon > 0$ and $t \in \mathbb{R}^+$, a subset E of M is said to be $(\varphi, \varepsilon, t)$*-separated* if for any $x, y \in M$ with $x \neq y$ there is some $s \in [0, t]$ such that $d(\varphi_s(x), \varphi_s(y)) > \varepsilon$. Given $f : M \to \mathbb{R}$ a continuous potential, define

$$Z(\varphi, f, \varepsilon, t) = \sup_E \left\{ \sum_{x \in E} e^{\int_0^t f(\varphi_s(x))\, ds} \right\}, \text{ where}$$

the supremum is taken over $(\varphi, \varepsilon, t)$-separated sets.

We also define

$$P(\varphi, f, \varepsilon) = \limsup_{t \to +\infty} \frac{1}{t} \log Z(\varphi, f, \varepsilon, t).$$

The *topological pressure* of φ with respect to f is defined as

$$P(\varphi, f) = \lim_{\varepsilon \to 0} P(\varphi, f, \varepsilon).$$

The *topological entropy* of φ, $h(\varphi)$, is the topological pressure of φ with respect to the identically zero potential.

New definition for semiflows with discontinuities

Let $\psi : \mathbb{R}_0^+ \times M \to M$ be a semiflow possibly exhibiting discontinuities. Consider a function T assigning to each $x \in M$ a sequence $(T_n(x))_{n \in A(x)}$ of nonnegative numbers, where either $A(x) = \mathbb{N}$ or $A(x) = \{1, \dots, \ell\}$ for some $\ell \in \mathbb{N}$. We say that T is *admissible* if there exists $\eta > 0$ such that for all $x \in M$ and all $n \in \mathbb{N}$ with $n + 1 \in A(x)$ we have

1. $T_{n+1}(x) - T_n(x) \geq \eta$
2. $T_n(\psi_t(x)) = \begin{cases} T_n(x) - t, & \text{if } T_{n-1}(x) < t < T_n(x) \\ T_{n+1}(x), & \text{if } t = T_n(x). \end{cases}$

For each admissible function T, $x \in M$, $t > 0$ and $0 < \delta < \eta/2$, we define

$$J^T_{t,\delta}(x) = [0, t] \setminus \bigcup_{j \in A(x)}]T_j(x) - \delta, T_j(x) + \delta[.$$

Observe that $J^T_{t,\delta}(x) = [0, t]$, whenever $T_1(x) > t$.

Given $\varepsilon > 0$ and $t \in \mathbb{R}^+$, we say that $E \subset M$ is $(\psi, \delta, \varepsilon, t)$*-separated* if for any $x, y \in M$ with $x \neq y$ there is some $s \in J^T_{t,\delta}(x)$ so that $d(\psi_s(x), \psi_s(y)) > \varepsilon$. Given a continuous potential $f : M \to \mathbb{R}$, define

$$Z^T(\psi, f, \delta, \varepsilon, t) = \\ = \sup_E \left\{ \sum_{x \in E} \exp\left(\int_0^t f(\psi_s(x))\, ds \right) \right\},$$

where the supremum is taken over all finite separated sets. We also define

$$P^T(\psi, f, \delta, \varepsilon) = \limsup_{t \to +\infty} \frac{1}{t} \log Z^T_t(\psi, f, \delta, \varepsilon),$$

$$P^T(\psi, f, \delta) = \lim_{\varepsilon \to 0} P^T(\psi, f, \delta, \varepsilon).$$

Finally, define the T*-topological pressure* of ψ with respect to f as

$$P^T(\psi, f) = \lim_{\delta \to 0} P^T(\psi, f, \delta).$$

This concept was strongly inspired by the T−topological entropy introduced in [4]. Moreover, when f is identically null the T-topological pressure becomes the T-topological entropy.

Note that the sequence $\tau = \{\tau_n\}$ of impulsive times is admissible and we call $P^\tau(\varphi, f)$ the impulsive topological pressure.

The next result states that for continuous semiflows the classical and the new notion of topological pressure coincide [5, Theorem B].

Theorem 2.— Let φ be a continuous semiflow on a compact metric space M, T an admissible sequence and $f : X \to \mathbb{R}$ a continuous potential. Then $P^T(\varphi, f) = P(\varphi, f)$.

2.2 Variational Principle

The concept of topological entropy is, a priori, purely topological. Nevertheless it is intrinsically related to invariant measures. For instance, given a continuous flow, on a compact metric space, the classical variational principle [31] shows that the topological entropy is the supremum over all invariant measures of the measure theoretical entropy.

The *entropy* of a continuous map $g : M \to M$ with respect to a probability measure μ is given by:

$$h_\mu(g) = \int h_\mu(g, x)\, d\mu(x), \qquad \text{where}$$

$$h_\mu(g, x) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu\left(B_g(x, n, \varepsilon)\right), \text{ and}$$

$B_g(x, n, \varepsilon) = \{y \in M\colon\ d(g^j(x), g^j(y)) < \varepsilon, \forall j = 0, \cdots, n-1\}$.

The *entropy* of a semiflow ψ with respect to a probability measure μ is defined as $h_\mu(\psi) = h_\mu(\psi^1)$, where ψ^1 stands for the time-1 map.

A variational principle [1, Theorem II] holds for impulsive semiflows, (see[5] a topological pressure version).

Theorem 3.— Let $\varphi : M \times \mathbb{R}_0^+ \to M$ be a C^1-semiflow, D a compact submanifold of codimension one and $I : D \to M$ a 1-Lipschitz map. If D and $I(D)$ are transversal to the flow direction, then the impulsive semiflow ψ generated by (M, φ, D, I) satisfies

$$h^\tau_{top}(\psi) = \sup_{\mu \in \mathcal{M}(\psi)} \{h_\mu(\psi)\}.$$

2.3 Equilibrium states

In general a dynamical system admits more than one invariant measure, so it is important to choose a suitable one for analysis. One way to do this is by selecting measures that maximize the system's free energy, known as equilibrium states.

A ψ-invariant probability measure μ is said to be an *equilibrium state* for ψ with respect to a potential function $f : M \to \mathbb{R}$ if it satisfies:

$$h_\mu(\psi) + \int f\, d\mu = \sup_\eta \left\{ h_\eta(\psi) + \int f\, d\eta \right\}$$

where the supremum is taken over all ψ-invariant probability measures η.

Given an expansive continuous flow with the specification property and a Hölder continuous potential satisfying the Bowen property, there exists a unique equilibrium state, as shown in [17]. To extend this result to impulsive semiflows we first adapt the concepts involved.

Existence and uniqueness for impulsive semiflows

In simple terms, expansiveness means that the system has the property of pushing apart the trajectories of nearby points in its phase space over time.

Taking into account that a suitable concept of expansiveness for impulsive semiflows should ensure genuine separation of the trajectories rather than just the artificial ones generated by the impulse map, we introduce the following concept.

For a given $r > 0$, denote the set $B_r(D)$ by the r-neighborhood of D in M. The semiflow ψ is called *positively expansive* on M if for every $\delta > 0$ there exists $\epsilon > 0$ such that if $x, y \in M$ and a continuous map $s : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $s(0) = 0$ satisfy $d\left(\psi_t(x), \psi_{s(t)}(y)\right) < \epsilon$ for all $t \geq 0$ with $\psi_t(x), \psi_{s(t)}(y) \notin B_\epsilon(D)$, then either $y = \psi_t(x)$ or $x = \psi_t(y)$ for some $0 < t < \delta$.

The specification property refers to the ability of a dynamical system to approximate true trajectories with high precision, using only a finite number of segments from other orbits. The fact that the concept depends only on pieces of trajectories allows us to apply the classical concept to impulsive semiflows.

The semiflow ψ has the *specification property* on M if for all $\epsilon > 0$ there exists $L > 0$ such that for any sequence $x_0, \ldots, x_n$ of points in M and any sequence $0 \leq t_0 < \cdots < t_{n+1}$ such that $t_{i+1} - t_i \geq L$ for all $0 \leq i \leq n$, there are $y \in M$ and $r : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, which is constant on each interval $[t_i, t_{i+1}[$, whose values depend only on $x_0, \ldots, x_n$, that also satisfy

$$r([t_0, t_1[) = 0 \quad \text{and} \quad |r([t_{i+1}, t_{i+2}[) - r([t_i, t_{i+1}[)| < \epsilon,$$
$$\text{for which } d\left(\psi_{t+r(t)}(y), \psi_{t-t_i}(x_i)\right) < \epsilon,$$

for all $t \in [t_i, t_{i+1} - L[$ and $0 \leq i \leq n$.

In addition, the specification is called *periodic* if we can always choose y periodic with the minimum period in $[t_{n+1} - t_0 - n\epsilon, t_{n+1} - t_0 + n\epsilon]$.

Finally, we adapt the concept of Bowen potentials to include the action of the impulse on the potentials. Let $V^*(\psi)$ be the set of all continuous maps $f : M \to \mathbb{R}$ satisfying:

1. $f(x) = f(I(x))$ for all $x \in D$;
2. there are $K > 0$ and $\epsilon > 0$ such that for every $t > 0$ we have
$$\left| \int_0^t f(\psi_s(x))\, ds - \int_0^t f(\psi_s(y))\, ds \right| < K, \quad (1)$$
whenever $d(\psi_s(x), \psi_s(y)) < \epsilon$ for all $s \in [0, t]$ such that $\psi_s(x), \psi_s(y) \notin B_\epsilon(D)$.

Sufficient conditions to the existence and uniqueness of equlibrium states were first established in [5, Theorem A]. Here we state [1, Theorem III] that requires the following regularity and transversality of the IDS.

Theorem 4.— Let $\varphi : M \times \mathbb{R}_0^+ \to M$ be a C^1 semiflow, D a compact submanifold of codimension one and $I : D \to M$ a 1-Lipschitz map. If D and $I(D)$ are transversal to the flow direction, ψ is positively expansive and it has the periodic specification property in $\Omega(\psi)\backslash D$, then for any potential $f \in V^*(\psi)$ there is an equilibrium state. Moreover, if there is $k > 0$ such that $\#I^{-1}(\{y\}) \le k$ for every $y \in I(D)$ then the equilibrium state is unique.

3 Periodic Orbits of typical impulsive semiflows

Considering the space of impulsive semiflows parameterized by vector fields and impulse maps, a natural question is whether a typical impulsive semiflow admits periodic points. Also, recall the so called General Density Theorem (see [25]) that ensures the existence of a Baire residual subset of C^1 vector fields for which every element generates a C^1-flow φ such that the set, $Per(\varphi)$, of periodic orbits is dense in the non-wandering set $\Omega(\varphi)$.

We aim to provide a description of the non-wandering set for a generic class of Impulsive Semiflows. As the genericity depends on the topology with which the space is endowed, we present our results in the C^0 and C^1 topologies,via permanent and hyperbolic periodic points, respectively. All throughout let M be a compact Riemannian manifold of dimension $m \ge 3$. Given an IDS (M, φ, D, I), where φ is a flow generated by a vector field X, we denote its impulsive semiflow by $\psi_{X,I}$ to stress the dependence on the vector field and on the impulse map.

3.1 C^0-topology

We now consider impulsive semiflows for which the underlying flow is generated by Lipschitz continuous vector fields and impulses are chosen as homeomorphisms onto its image. We denote by $\mathfrak{X}^{0,1}(M)$ the space of Lipschitz vector fields endowed with the C^0-topology
$$\|X - Y\|_{C^0} := \max_{x \in M} \|X(x) - Y(x)\| < \varepsilon.$$

C^0-Baire generic impulses

Assume φ is a Lipschitz continuous flow generated by a vector field $X \in \mathfrak{X}^{0,1}(M)$ and D is a codimension one smooth submanifold of M, transversal to the flow direction, such that
$$\overline{D} \cap \mathrm{Sing}(\varphi) = \emptyset, \quad \text{(H)}$$
where $\overline{D}$ stands for the closure of D and $\mathrm{Sing}(\varphi)$ stands for the equilibrium points of φ. Let $\hat{D}$ be a codimension one submanifold transversal to X. Consider the space
$$\mathscr{I}_{D,\hat{D}} = \mathrm{Homeo}(D, \hat{D})$$
endowed with the C^0-distance. We have the following General Density Theorem [28, Theorem A].

Theorem 5.— There exists a C^0-Baire generic subset $\mathscr{R}_X \subset \mathscr{I}_{D,\hat{D}}$ of impulses such that
$$\overline{Per(\psi_{X,I})} \cap \mathring{D} = \Omega(\psi_{X,I}) \cap \mathring{D}$$
for every $I \in \mathscr{R}_X$, where $Per(\psi_{X,I})$ denotes the set of periodic orbits of $\psi_{X,I}$ and $\mathring{D}$ is the interior of D.

In general, one should not expect the density of periodic points in the all non-wandering set, however this is the case when the impulsive semiflow is generated by a minimal flow. Moreover, the following holds ([28, Corollary B]).

Corollary 6.— Let φ be a Lipschitz continuous flow generated by $X \in \mathfrak{X}^{0,1}(M)$ and $D, \hat{D} \subset M$ be smooth codimension one submanifolds transversal to the flow such that assumption (H) holds. The following hold:

1. if $I_0 \in \mathscr{I}_{D,\hat{D}}$ is such that $\Omega(\psi_{I_0}) \cap \partial D = \emptyset$ then there exist $\delta > 0$, an open neighborhood $\mathscr{V}$ of I_0 and a Baire generic subset $\mathscr{R} \subset \mathscr{V}$ so that, for every $I \in \mathscr{R}$ one can write the non-wandering set $\Omega(\psi_{X,I})$ as a (possibly non-disjoint) union $\Omega(\psi_{X,I}) = \overline{Per(\psi_{X,I})} \cup \Omega_2(\varphi, D)$, where $\Omega_2(\varphi, D) \subset \Omega(\psi_{X,I})$ is a φ-invariant set which does not intersect a δ-neighborhood of the cross-section D. Moreover, the set $\Omega(\psi_{X,I}) \setminus D$ is invariant under $\psi_{X,I}$.

2. if φ is minimal then there exists a Baire generic subset $\mathscr{R} \subset \mathscr{I}_{D,\hat{D}}$ so that, for every $I \in \mathscr{R}$, the set of periodic orbits is dense in $\Omega(\psi_{X,I})$. Moreover, the set $\Omega(\psi_{X,I}) \setminus D$ is a $\psi_{X,I}$-invariant subset of M.

3.2 C^1-topology

Let φ be the C^1-flow generated by a vector field X and let $D \subset M$ be a compact codimension one submanifold transversal to the flow direction.

We define the class of impulses $\mathscr{I}_D$ as the set of C^1-embeddings maps $I : D \to M$ so that $I(D) \pitchfork X$ and

$$\sup_{x \in I(D)} \left| \frac{d\tau_1}{dx}(x) \right| < +\infty.$$

Endow the space $\mathscr{I}_D$ with the distance $d_{C^1}(I_1, I_2)$ given by

$$\max \left\{ \sup_{x\in D} d\big(I_1(x), I_2(x)\big), \sup_{x\in D} \left\| DI_1(x) - DI_2(x) \right\| \right\},$$

where the expression on the right-hand side is well-defined after using parallel transport to identify the corresponding tangent spaces.

In this setting [27, Theorem A] establishes:

Theorem 7.— There exists a Baire residual subset $\mathscr{R}_X \subset \mathscr{I}_D$ of impulses such that the impulsive semiflow ψ_I determined by $I \in \mathscr{R}$ satisfies

$$\overline{Per_h(\psi_I)} \cap D = \Omega(\psi_I) \cap D$$

where $Per_h(\psi_I)$ denotes the set of hyperbolic periodic orbits of ψ_I.

We point out that the conclusion of Theorem 7 cannot be written using the landing region $I(D)$, as there exist C^1-open sets of impulses for which the equality $\overline{Per_h(\psi_I)} \cap I(D) = \Omega(\psi_I) \cap I(D)$ fails (see [27, Example 7.3]).

4 An invitation to the impulsive world

Despite their wide range of applications, IDS have only recently begun to be studied through the lens of ergodic theory. In this area, there remains vast potential for exploration. We conclude by presenting a few open questions, inviting the reader to delve into the dynamics of impulsive semiflows.

1. As mentioned in Section 2.3, in general a dynamical system admits more than one invariant measure, therefore it is necessary to choose a suitable one to analyze. While here we only focus on equilibrium states, criteria for the existence and finiteness of the number of absolutely continuous measures and/or physical measures are also not available. See [2] for the study of physical measures for a class of semiflows generated via impulsive perturbations of Lorenz flows.

2. In Theorem 6 we established the denseness of periodic points in the impulsive non-wandering set for a class of Baire generic impulses maps. The proof is based on the concept of uniform hyperbolicity and of perturbative results for discontinuous semiflows. A key tool in the proof is the following impulsive connecting lemma [27, Theorem 4.1].

Given $\delta > 0$, we say that a sequence $(x_k, t_k)_{k=0}^n$ in $M \times \mathbb{R}_+$ is a *δ-pseudo orbit* for the impulsive semiflow ψ_I if $d(\gamma_{x_k}(t_k), x_{k+1}) < \delta$, for every $k = 0 \dots n-1$. We say that y is a *chain iterate* of x (and write $x \dashv y$) if for any $\delta > 0$ there exists a δ-pseudo orbit $(x_k, t_k)_{k=0}^n$ such that $x_0 = x$ and $x_n = y$.

Theorem 8 (Impulsive connecting lemma).— Let φ be a C^1-flow generated by $X \in \mathfrak{X}^1(M)$ and let $D, \hat{D}$ be smooth submanifolds of codimension one transversal to X and $I \in \mathscr{I}_{D,\hat{D}}$ so that all ψ_I-periodic orbits whose orbit closure does not intersect ∂D are hyperbolic. If $x \dashv y$ then for any $\varepsilon > 0$ there exists an ε-C^1-perturbation J of I such that $y = \gamma_{J,x}(t)$, for some $t \geq 0$.

In the context of diffeomorphisms, connecting lemmas were obtained by Arnaud [7, Théorème 22] and Bonatti and Crovisier [8, Théorème 2.1], as refinements of the C^1-closing lemma by Pugh [25] and the C^1-connecting lemma of Hayashi [18].

A natural and interesting open question is to obtain a version of Theorem 6 when the vector field is perturbed. It seems that one should first establish a version of the last connecting lemma.

3. Considering the IDS (M, φ, I, D), even if φ is a flow, the IDS does not generate an impulsive *flow*,

but only a *semiflow*, when the impulse map I is not injective. It is well known that certain concepts do not present a direct adaptation from the setting of flows to the one of semiflows. For instance, the concept of expansiveness presented in this paper is stronger than the one for flows introduced by Bowen and Walters in [13] (see [19]). In [23] the concept of eventual expansiveness for continuous semiflows was introduced.

A continuous semiflow φ on a metric space M is *eventually expansive* if for every $\varepsilon > 0$ there is $\delta > 0$ so that if $x, y \in M$ and $s : [0, +\infty[\to [0, +\infty[$ is a continuous function with $s(0) = 0$ such that $d(\varphi(t, x), \varphi(t, y)) \leq \delta$ for all $t > 0$ then $\varphi(r, x) = \varphi(s, x)$ for some $r, s \geq 0$ with $|r - s| \leq \varepsilon$.

For an eventually expansive continuous semiflow its topological entropy is bounded from below by the growth rate of the periodic orbits [Theorem 1.5, [23]]. After adapting the concept of eventually expansiveness to impulsive semiflows, we would expect to be able to bound the τ-entropy introduced here by the growth rate of the periodic orbits.

Acknowledgements

The author was funded by the grant E-26/01‰2610/2019, Rio de Janeiro Research Foundation (FAPERJ), and by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES), Finance Code 001.

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The CIM Bulletin is published once a year. Material intended for publication shoud be sent to one of the editors. The Bulletin is available at: www.cim.pt. The CIM acknowledges the financial support of FCT- Fundação para a Ciência e Tecnologia.

Mathematics and Music
25 years after the Diderot Mathematical Forum

by **José Francisco Rodrigues***

At the beginning of the third millennium of our era, Leibniz's famous expression, "Musica est exercitium arithmeticæ occultum nescientis se numerare animi" (Music is a hidden arithmetical exercise of a mind unconscious that it is counting) can be taken in a broad sense in a contemporary conception of art and science. In the European universities, since the middle ages, Music was part of the studies and Portugal was not an exception. The *Studium Generale* of Lisbon, instituted by King Dinis in 1290, marking the foundation of the Portuguese University, included the medieval version of the Liberal Arts of the Trivium (grammar, rhetoric and dialectic) and the Quadrivium (arithmetic, geometry, astronomy and music).

Nowadays the classical *Mathematical Physics* area does not appear explicitly in the *Mathematics Subject Classification*, but, of course, it currently includes a wide variety of areas and sub-areas of mathematics that are contained in that classification. In contrast, the somehow older subject *Mathematics and Music* has integrated that classification, under the reference 00A65, only since 2010. On the other hand, the *Journal of Mathematics and Music* has been publishing articles on computational and mathematical approaches to music composition, analysis and theory since 2007.

The European Mathematical Society (EMS) has promoted the *Fourth Diderot Mathematical Forum* simultaneously in Lisbon, Paris and Vienna, on 3–4 December

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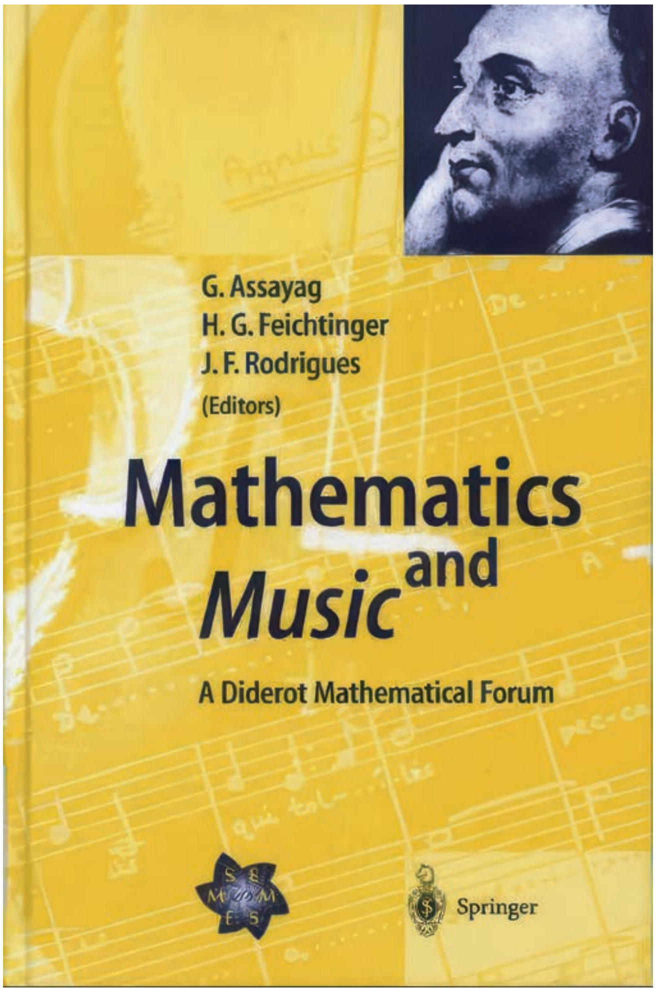
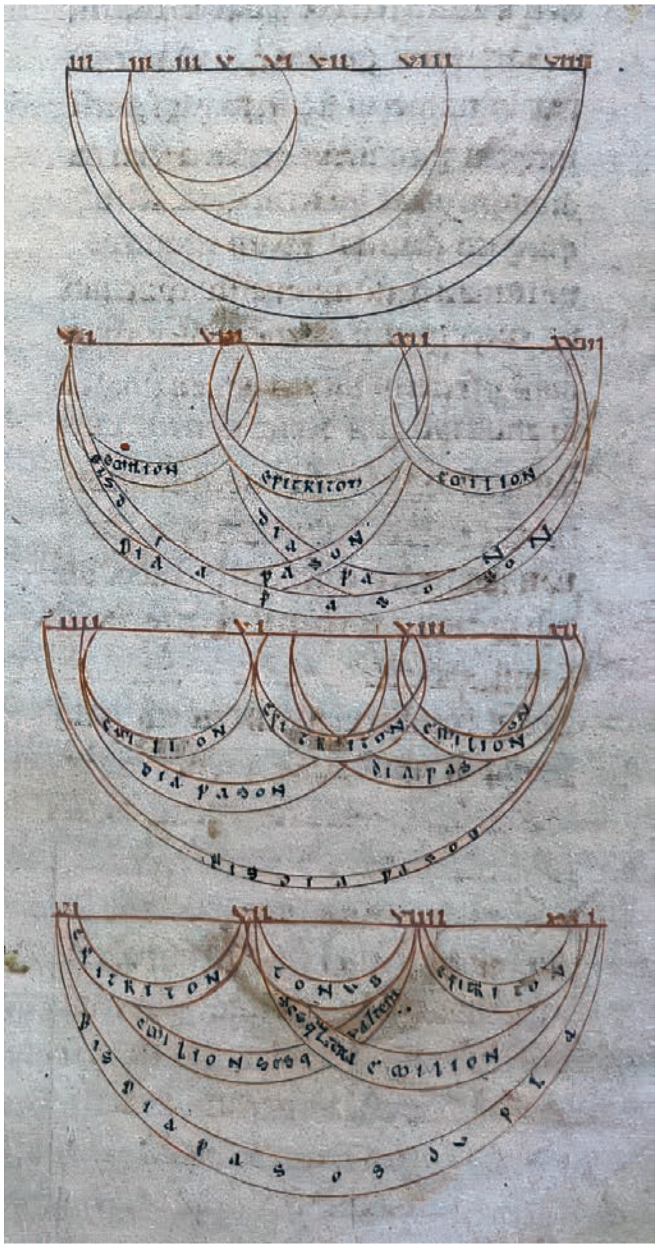




Book engraving from *Etymologiae*, by Isidore of Seville, belonging to the *Manuscrito de Santa Cruz* (12.° 17, fl.31 at *Biblioteca Pública Municipal do Porto*) with divisions of the monochord, in the medieval style of *De Institutione Musica* by Boethius (c. 475–524).

1999, which consisting of several conferences in parallel and, as a main feature of that Forum, a joint teleconference among the three cities, which took place under the theme "The relations between Mathematics and Music are natural or cultural relations?" The series of those *EMS Forums* took the name of the French philosopher Denis Diderot, who was the editor of the *Encyclopédie*, that he co-founded with the mathematician Jean D'Alembert (who also wrote on music), where he wrote "C'est par les nombres et non par les sens qu'il faut estimer la sublimité de la musique. Etudiez le monocorde." (It's by numbers and not by the senses that one should evaluate the sublimity of music. Study the monochord.), in the very classical Pythagorean tradition.

The contributions presented at that 1999 pioneering conference on *Relationships between Mathematics and Music* were published in 2002 in the influential book [MM] covered three complementary directions: "Historical Aspects", the topic addressed in Lisbon at the *Fundação Calouste Gulbenkian*; the "Mathematical Logic and Music Logic in the 20th century" at the IRCAM-*Centre Georges Pompidou*, in Paris, and the "Mathematical and computational methods in Music", at the *University of Vienna*.

The first of five texts on the historical topic of [MM], Proportion in Ancient and Medieval Music, by M. P. Ferreira, deals with the Pythagorean theory, from the Greek heritage and the Latin world to the late-medieval France and the decline of proportional thinking. The chapter by E. Knobloch on *The Sounding Algebra: Relations Between Combinatorics and Music from Mersenne to Euler* highlights the role of the Mersenne's *Harmonia Universalis* (1636) for the baroque music, when "to compose" was equivalent with "to combine", up to the Euler's contributions, with a reference to the Mozart's *Musicaliaches Würfelspiel* (Musical game of dice). The third article, by B. Scimemi, explains *The Use of Mechanical Devices and Numerical Algorithms in the 18th Century for the Equal Temperament of*

The Tonnetz explained by Andreas Matt to the author in Heidelberg the 6 July 2019.

the Musical Scale, from Zarlino and Tartini up to Strähle and Schröter. In the next article, J. Dhombres recalls Lagrange, *"Working Mathematician", on Music Considered as a Source for Science*, and the last one in this first part of that book presents some *Musical Patterns*, by W. Hodges and R. J. Wilson, with illustrations of mathematical devices, like canon, expansion, retrograde motion and inversion, used in music writing by composers.

The seven contributions in [MM] from the Paris meeting, going in general into an anti-Pythagorean direction, started with *Questions of Logic: Writing, Dialectics and Musical Strategies*, by F. Nicolas, followed with *The Formalization of Logic and the Issue of Meaning*, by M.-J. Durand-Richard, with *Musical Analysis Using Mathematical Proceedings in the* XX^{th} Century, by L. Fichet, with *Universal Prediction Applied to Stylistic Music Generation*, by S. Dubnov and G. Assayag, evolving into *Ethnomusicology, Ethnomathematics, The Logic Underlying Orally Transmitted Artistic Practices*, by the ethnomusicologist M. Chemillier, or into cognitive musicology with *Expressing Coherence of Musical Perception in Formal Logic*, by M. Leman. The last chapter on *The Topos Geometry of Musical Logic*, by G. Mazzola, searches for connections between the logic of musical composition and analysis with abstract algebraic geometry and logic structures.

In the last part of [MM], J.-C. Risset in *Computing Musical Sound* shows how mathematics is the pervasive tool of the computational craft of musical sound up to real-time musical performance, while E. Neuwirth gives an overview on The *Mathematics of Tuning Musical Instruments — a Simple Toolkit for Experiments*, the computer musicologist X. Serra describes *The Musical Communication Chain and its Modeling*, using contributions from music, electrical engineering, psychology and physics, and, completing the book, G. De Poli and D. Rocchesso review some of the most important *Computational Models for Musical Sound Sources* based on physical models and mathematical descriptions of sound sources, which are natural extensions of the classical cooperation and interaction between science and music.

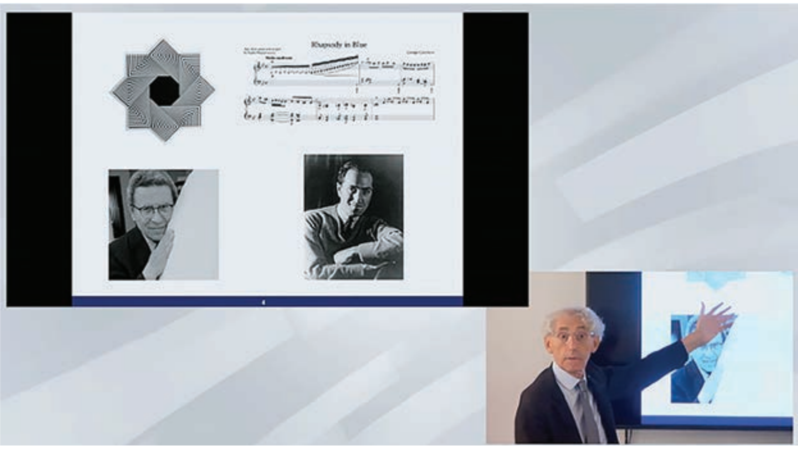
In the comprehensive and positive review of the book of that fourth Diderot Forum, S. Perrine in [P] acknowledges "a new alliance between music and mathematics" and states his conviction "that other Mathematics and Music initiatives need to be taken, and that there is no lack of topics to be covered".

In that year of 1999 that preceded the World Mathematical Year, WMY2000, and in fact to announce it, the Portuguese magazine of scientific culture COLÓQUIO/CIÊNCIAS published a series of articles dedicated to the interactions of Mathematics and Music. A first one [R] was a brief general introduction to the theme and to a series of seven conferences held in Lisbon, one per month, from January until July of 1998, at the *Fundação Calouste Gulbenkian* in Lisbon, organised with the collaboration of the *Centro de Matemática e Aplicações Fundamentais* of the University of Lisbon, which gave origin to a full issue of the COLÓQUIO/CIÊNCIAS magazine [CC]. Later, in 6–7 October 2006, the *Centro de Matemática da Universidade do Porto* organised a meeting on Música e Matemática, which included two concerts hold in *Casa da Música* of Porto and produced an interesting book [B] with thirteen contributions.

Actually, the article *A Matemática e a Música* [R] had the concept of an ambitious exhibition for the WMY2000 behind it, which proved impossible because it had too high a budget. The concept was based on the historical and conceptual connections and required, in addition to instruments and physical objects, several interactive exhibits to be developed in collaboration with an enthusiastic group of international experts.

Essentially the exhibition concept was composed in four movements connecting music with four mathematical areas: *Pythagorean Arithmusic*, based on the classical proportions and numbers related to harmonies (musical and celestial); *Algebra of Tones*, from the different temperaments to combinatorics and the musical symmetries; *Harmonisation of Analysis*, on the nature of the propagation of sound and the construction of instruments and *Digital Musurgia*, where, in the computer era, it is possible to produce music by calculating numbers. Indeed, if today we have mastered numerisation in the analysis and synthesis of musical sound, if we have begun to outline the mathematisation of certain musical structures and computers allow us to hear mathematical calculations and structures, i.e. paraphrasing Saccheri, we have *Pythagoras ab omni naevo vindicatus sive Conatus arithmeticus quo stabiliuntur prima ipsa universæ musicæ principia* (Pythagoras freed from all taint or the arithmetical attempt to establish the first principles of all music) and we can continue to agree with the Greek philosopher of the 4^{th} century BCE Aristoxenus of Tarentum and accept that the justification of music lies in the pleasure of hearing it and enjoying it.

In this century, the increase of events, articles and books on mathematics and music is showing that, indeed,



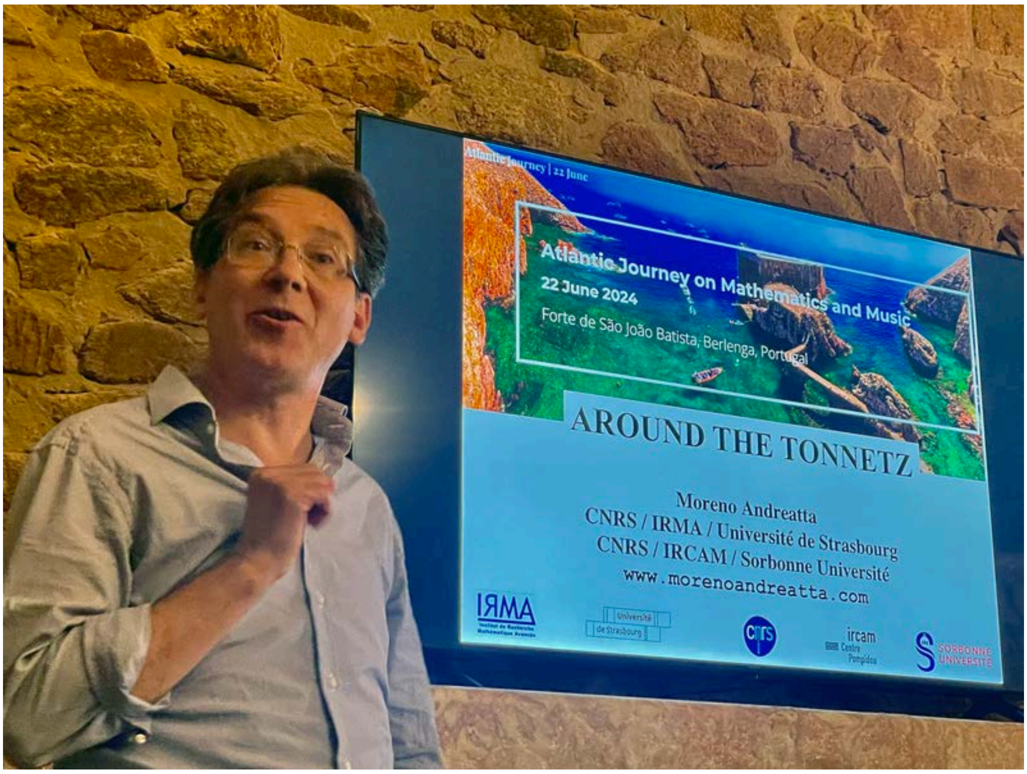
The mathematician Alfio Quarteroni showing how to play Morandini's pictures at Villa Toeplitz (RISM), Varèse, in 2021.

there is no lack of interesting topics to be covered, as it was observed before in several forums. In particular, the recent exhibition *LaLaLab–The Mathematics of Music* [LLL], organised by the *Imaginary*, which has the *Mathetisches Forschunginstitut Oberwolfach* as a shareholder, was shown in Heidelberg (May 2019–December 2020) and has about two dozen of interesting interactive exhibits, free and available online. This successful exhibition was based over music theory, current research in the connection of mathematics and music and also over art and entertainment, including artworks, talks and concerts.

Besides Music, as an artistic expression, Mathematics also interacts with other arts, like Architecture, Painting or Sculpture. An interesting recent paper [GQC], by P. Gervasio, A. Quarteroni and D. Cassani, using a mathematical method based in Fourier and Wavelet transform to extract similarities between image signals and audio signals, allowed, by minimizing a certain distance, not only to associate a given painting from a specific artist with music tracks, with the possibility of choosing the "optimal" one in a certain sense, but also "to generate the new music, is the most similar one to the painting chosen, in terms of intrinsic features". Those authors have also developed an original (and free) applet, which was applied to play some artworks of the Italian artist Marcelo Morandini.

Recently, the Institute of Mathematical sciences of the National University of Singapore organised a conference at the Yong Siew Toh Conservatory of Music, during the week 19-23 February 2024, celebrating the seminal event that took place in 1999 simultaneously in Lisbon, Paris and Vienna. The *Mathemusical Enconters in Singapore: a Diderot Legacy* consisted of about twenty two talks on different topics, such as, Mathematical and Computational approaches (Day 1), Machine Learning, Generativity, Interaction (Day 2), Computational Physiology/Medicine (Day 3), Education, Learning and Creativity (Day 4) and a Student Session during the morning of the fifth day. In addition, there were five concerts and recitals at the Conservatory Concert Hall and a Round Table, in hybrid format, on the Diderot Forum legacy and future perspec-

P. Codognet, J. F. Rodrigues, H. Chew and G. Assayag during the round table at the Mathmusical Encounters in Singapore, the 22 February 2024.



Moreno Andreatta introducing the interactive applet *Tonnetz* at the Atlantic Journey in Berlenga Island.

tives with an open online discussion with members of the Society for Mathematics and Computation in Music.

Immediately after the 9th International Congress on Mathematics and Computation in Music (MCM2024), that had taken place a the University of Coimbra, the 18–21 June 2024, the *Centro de Matemática, Aplicações Fundamentais e Investigação Operacional*, of the *Faculdade de Ciências da Universidade de Lisboa*, in collaboration with the *Centro Internacional de Matemática*, the National Agency for Scientific and Technological Culture – *Ciência Viva*, the *Academia das Ciências de Lisboa* and the *Associação Amigos da Berlenga*, organised an *Atlantic Journey on Mathematics and Music* at the Berlenga island the 22 June 2024. This small meeting was an extraordinary opportunity to get acquainted with some of the interactions of mathematics and music, not only through the six talks, covering from *Mathematics and Music in Historical Context*, by J. F. Rodrigues, *Music and Symmetries — from Bach to Jazz*, by C. Simões, *Symmetries and other mathematical beauties in music*, by E. Amiot, and *Conceptualising Tonality: Algebraic versus Statistical Approaches*, by T. Noll, and *Around theTonnetz*, by M. Andreatta, up to a talk and installation on *The MatheMusical Virtual Museum*, by G. Baroin, complemented with a Round Table on *How to make an Exhibition on Mathematics and Music?*, moderated by C. Florentino and R. Vargas, the Director of *Ciência Viva*, and the remote participation of D. Ramos, Chief Content Officer of the *Imaginary*. Possible future plans for 2026 were discussed in the Round Table.

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[CC] COLÓQUIO/CIÊNCIAS n. 24, Novembro 1999 (in Portuguese): I. Grattan-Guiness, *Mozart 18, Beethoven 32: Hidden shadows of integers in classical music*; E. Behrends, *Music: Structure and Chance*; G. Assayag, *Mathematics, the number and the computer*; G. Mazzola, *Listen to Mathematics?*; C. Simões, *The order of numbers in 20th century music*; B. Scimemi, *Musical counterpoint and geometric transformations*; J.P. Bourguignon, *Is it possible to hear the shape of a drum?*

[B] Encontro Música e Matemática — Actas, CMUP – Centro de Matemática do Porto, 2006.

[LLL] https://www.imaginary.org/exhibition/la-la-lab-the-mathematics-of-music

International Conference on Mathematical Analysis and Applications in Science and Engineering

20–22 June 2024, Porto, Portugal

by **Carla M. A. Pinto***

The International Conference on Mathematical Analysis and Applications in Science and Engineering (ICMASC'24) was held on June 20–22, 2024, at the Polytechnic of Porto's School of Engineering (ISEP) in Porto, Portugal. Dedicated to Prof. J.A. Tenreiro Machado, it united researchers from mathematics, science, and engineering to explore advances in differential equations, optimization, computational mathematics, and their applications in biology, finance, and robotics.

The program included plenary and invited talks, special sessions, and contributed papers. Open to all participants, the event promoted international collaboration, particularly benefiting the Portuguese scientific community. Reduced registration fees encouraged student participation, especially in applied mathematics.

ISEP, a leader in technology and sustainability, offers an ideal setting in Porto, a historic city known for its wine and vibrant culture.

Learn more on the ICMASC'24 website:
https://www2.isep.ipp.pt/icmasc/

* Politécnico do Porto — Instituto Superior de Engenharia

Portuguese texts on the calculus in the 18th century

by **João Caramalho Domingues***

Having appeared at the end of the seventeenth century, the calculus (differential and integral calculi for Leibniz, method of fluxions for Newton) was undoubtedly the most important branch of pure mathematics in the eighteenth century. This importance was recognized at the time:

"one of the most wonderful inventions in mathematics, which not only raised geometry almost to its highest peak, but also expanded the other disciplines to such an extent that one would have to write entire books if one wanted to specify the benefits of this calculus" [Zedler, 1731–1754, vol. 5, col. 190].

"Of all the discoveries that have ever been made in the sciences, there is none as important, nor as fruitful in applications, as that of infinitesimal analysis" [Bossut, 1784, lxxii].

This is also a common view among modern-day historians:

"Considered broadly, mathematical activity in the eighteenth century was characterized by a strong emphasis on analysis and mechanics. The great advances occurred in the development of calculus-related parts of mathematics and in the detailed elaboration of the program of inertial mechanics founded during the Scientific Revolution" [Fraser, 2003, 305].

"The Enlightenment in Mathematics is defined by the level achieved in the mastery of the new differential and integral calculus [...] The mark of the modernity of a work is the use made of the calculus, and undoubtedly a work whose content does not include the calculus can be said to be outdated" [Ausejo & Medrano, 2010, 26].

For all its centrality in eighteenth-century European mathematics, the adoption of the calculus in Portugal was slow [Domingues, 2021; to appear]. Before 1760, only a few isolated cases can be found of Portuguese individuals knowing about the calculus, and in each case one may wonder how profound was such knowledge. In the 1760s there were a couple of attempts at introducing the calculus into mathematical teaching, but they were not fruitful. The first successful case of the calculus being taught in Portugal occurred only as a result of the 1772 reform of the University of Coimbra, which created a Faculty of Mathematics (in section 2 we will look at the textbook used in that context).

1 Jacob de Castro Sarmento's explanation of fluxions (1737)

The very first text in Portuguese about the calculus was very short and non-technical.

It was written by Jacob de Castro Sarmento (born Henrique de Castro, 1691–1762), a "New Christian" physician who escaped to London in 1721 fleeing the

This work was partially financed by Portuguese Funds through FCT (Fundação para a Ciência e a Tecnologia) within the Project UID/00013: Centro de Matemática da Universidade do Minho (CMAT/UM).

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Inquisition and there converted to Judaism [Goldish, 1997]. In 1737 he published in London, but in Portuguese, a book on the Newtonian theory of tides which also includes an eulogy of Newton and a glossary of scientific words. Sarmento's treatment of tides does not involve any calculus, but the largest entry in the glossary is precisely "fluxions" (the Newtonian equivalent of differentials) [Sarmento, 1737, 129–131]. In two and a half pages, Sarmento expounds the Newtonian point of view of a line being generated by the motion of a point, a surface being generated by the motion of line, and a solid being generated by the motion of a surface; the velocity of each of these motions is the *fluxion* of the line, surface, or solid, while the line, surface, or solid is the *fluent* of that fluxion. The Direct Method of Fluxions is used to find the fluxion of any quantity, given the fluent (which is the quantity itself); the Inverse Method of Fluxions is used to find the fluent, given the fluxion. Finally, the direct method is useful in drawing tangents, solving problems of maxima and minima, etc.; while the inverse method is useful in calculating arclengths, areas and volumes. They have also plenty of use in physics and astronomy.

In short, this is a concise explanation for a layperson, not an introduction to the subject, by any means.

2 Bézout's textbook

2.1 The first translation (1774)

It was only as a consequence of the 1772 reform of the University of Coimbra that an introductory text on the calculus was published in Portuguese. This was a translation of a then recent text by the Frenchman Étienne Bézout (1730–1783).

Bézout, an examiner of the French navy schools, published between 1764 and 1769 a *Cours de Mathématiques* in six volumes (containing arithmetic, geometry and trigonometry, algebra, calculus, mechanics, and navigation) for the students of those schools. This course was hugely successful, and over the following decades either the full set or extracted parts were reprinted numerous times and translated into several languages.

For the new Faculty of Mathematics created in Coimbra in 1772, José Monteiro da Rocha (1734–1819), one of the main main people involved in the establishment of the Faculty, translated the first volume of Bézout's course, on arithmetic, and the section on plane trigonometry from the second volume — both to be used in the first year of mathematics. He also translated a textbook on mechanics by a different French author, to be used in the third year.

For the second year, which included algebra and the calculus, the parts from Bézout's course on these subjects were adopted. Their translations were published as volumes 1 and 2 of [Bézout, 1774]. It is not known who translated them into Portuguese, although by the 19th century it was said that the translator had been Fr. Joaquim de Santa Clara (1740–1818) — a Benedictine who graduated in theology but who also taught philosophy and mathematics in the early 1770s. Be as it may, there is a marked difference between this translation and those made by Monteiro da Rocha: while the latter adapted several passages and included additional material as he saw fit, [Bézout, 1774] is a very literal translation.

The volume on the calculus, [Bézout, 1774, II], presents a traditional Leibnizian version of the subject, with a strong geometrical tendency — particularly in the differential calculus.

Bézout casually accepts the existence of infinitely large and infinitely small quantities. His variable quantities increase (or decrease) by infinitely small degrees. Thus, the *differential* of a quantity is defined as the infinitely small difference between the values of that quantity in two consecutive moments. For instance, the differential of xy is $x\,dy + y\,dx$, because the difference between two consecutive states of xy is $(x+dx)(y+dy)-xy = x\,dy+y\,dx+dy\,dx$, and $dy\,dx$ must be omitted because it is infinitely small with regard to both $x\,dy$ and $y\,dx$. Accordingly, in order to calculate tangents, Bézout conceives a curve as a polygon with an infinite number of infinitely small sides. The tangent is the prolongation of one of these sides.

About two thirds of the differential calculus are taken up with geometrical applications — more precisely, applications to the study of curves: not only tangents (subtangents, subnormals) but also topics such as multiple points, points of inflexion, cusps, and radii of curvature. Yet another geometrical application is the determination of maxima and minima, which are treated as largest and smallest ordinates, so that the condition $dy/dx = 0$ comes from the tangent to a curve being parallel to the abscissas.

Another aspect of Bézout's calculus, consistent with this predominance of geometry, is the relative unimportance of the concept of function: the word "function" is first used 65 pages after "differential", in a section on multiple points of curves, as a mere abbreviation. Throughout the differential calculus, the object of study are *quantities*, fully geometrical or rep-

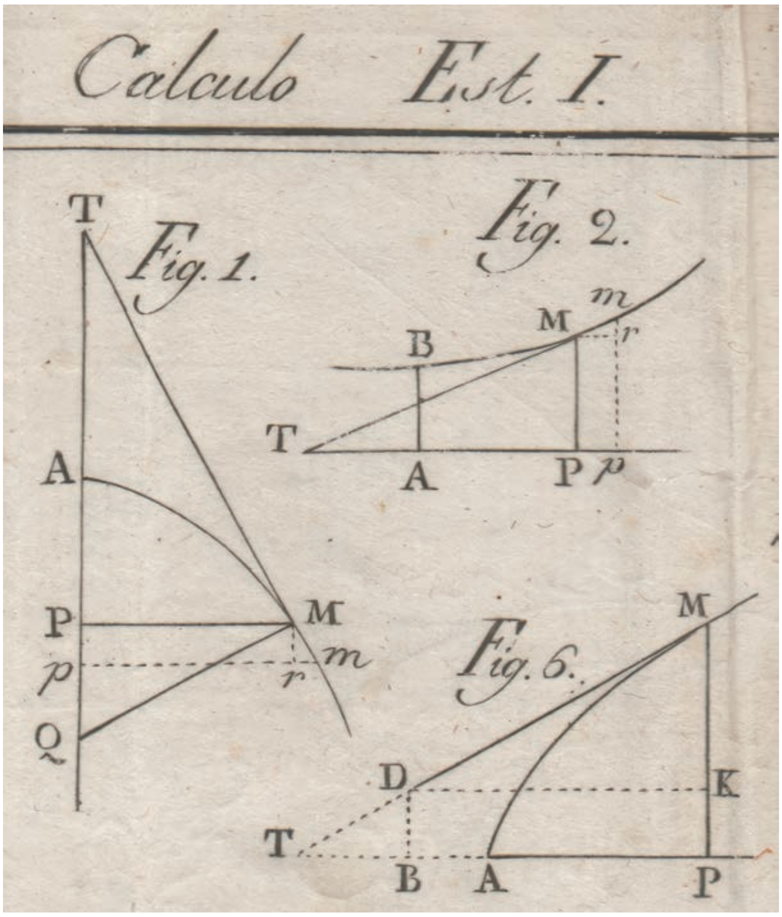

Figure 1.—Three figures from [Bézout, 1794] (similar to figures appearing in [Bézout, 1774]). In Fig. 1, line TM is tangent to the curve AM, and is obtained prolonging the infinitely small side Mm of this curve, regarded as a polygon; AP is the abscissa x, PM the ordinate y, $Pp = Mr = dx$, $mr = dy$; Mrm is an infinitely small triangle, similar to the finite triangle TPM, whence the subtangent PT is equal to $y\,dx/dy$.

resented geometrically.

This geometrical point of view was the norm at the time, in calculus textbooks all over Europe. The only major exception was Euler's treatises on the calculus (published between 1748 and 1770), where the primary object of study were *functions*. But these were treatises, not textbooks.

Still, it must be said that Bézout's integral calculus is much more analytical than his differential calculus: the integral is essentially defined as what we would call an antiderivative (which, again, was the most common approach at the time), so that the integral calculus deals naturally with expressions from the start; accordingly, the word "function" receives a definition at the beginning of the integral calculus[1]. There are some geometrical applications (areas, arc lengths and some volumes) but they occupy only about one fifth of the integral calculus.

2.2 The second translation (1794)

[Bézout, 1774] was too much of a literal translation, even including references to parts of Bézout's course that had not been translated nor adopted in Coimbra. In the 1790s, the same parts of Bézout's course were translated again, from scratch, by José Joaquim de Faria (1759–1828), who was at the time a substitute professor at the Faculty of Mathematics. The calculus volume appeared as [Bézout, 1794].

Unlike the previous one, this new translation is far from literal. There are minor adaptations to accommodate the text to the series of textbooks in use at the University, several calculations or arguments are shortened, and there are two small but relevant attempts to modernize the differential calculus: the word "function" is introduced somewhat earlier and is used more often; and a new (short) section is added, on Maclaurin and Taylor series.

At about the same time that this second translation came out, in France the calculus as a subject of teaching was changing dramatically. A new version of the calculus, inspired by Euler and Lagrange, and much more analytical than Bézout's, was being taught at the *École polytechnique*, founded in 1793. The changes introduced by José Joaquim de Faria were small steps in the same direction — certainly not caused by the new

[1] After having already been used, as was noticed above.

trends spreading from the *École polytechnique*; more likely, resulting from similar motivations, namely the growing gap between advanced, research-level works, which were typically very much analytical, and traditional introductory texts such as Bézout's. But Faria's changes were not enough to fundamentally change what was quickly becoming an outdated textbook.

3 José Anastácio da Cunha's fluxionary calculus

The earliest introduction to the calculus originally in Portuguese that is still extant[2] was written by the most original Portuguese mathematician of the 18th century, José Anastácio da Cunha (1744–1787), and it contains a remarkable definition of "fluxion" that has been described as the first rigorous analytic definition of the differential.

Cunha's introduction to the calculus was included in [Cunha, 1790], a short (little over 300 pages), very concise but very comprehensive, introduction to mathematics, from elementary geometry to some calculus of variations, organized in a logical way. This was a posthumous publication, and its text was never really finished, but as far as the calculus section is concerned, a manuscript text dated 1780 is known containing some of its main ideas.

Some of Cunha's personal opinions (such as his admiration for Newton, who preferred a geometrical style over algebra, his dislike of Euler, his distrust of conclusions drawn exclusively from analytical arguments) might lead us to expect from him a geometrically inclined version of the calculus. However, what we find is mostly analytical, albeit in an original way [Domingues, 2023]. It is certainly much more analytical than Bézout's calculus, and betrays more influence from Euler than Cunha would probably like to admit.

The book [Cunha, 1790] is divided into 21 "books" (so called following the Euclidean fashion; we would call them chapters). The calculus is introduced in "book" 15.

Book 15 opens with a crucial definition: "if an expression can assume more than one value, while another can assume only one, the latter will be called constant, and the former variable". This may seem trivial to a modern reader, but it was at the very least extremely unusual in the 18th century: a variable was almost always regarded as a quantity (not an expression) that *varied* (presumably over some sort of implicit time).

The second definition is built on the first one: "a variable always capable of assuming a value smaller than any proposed magnitude will be called infinitesimal". This means that, instead of infinitesimals being infinitely small quantities, as was then commonly the case, they are simply *expressions* that can assume arbitrarily small (but finite) values. In practice, Cunha's statements involving infinitesimals have the form "x infinitesimal makes $f(x)$ infinitesimal", which is equivalent to $\lim_{x\to 0} f(x) = 0$. Proposition 1 of book 15 states that if x is infinitesimal then $Ax + Bx^2 + Cx^3 + \&c.$ is also infinitesimal, and its proof is (as long as $Ax+Bx^2+Cx^3+\&c.$ is interpreted as a polynomial) an impeccable ε-δ argument.

The third definition is that of function: an expression A is a function of another expression B if the value of A depends on the value of B. This is not so remarkable, but it is worth noticing that "function" is defined at the outset (compare with what was said above about Bézout's text), which allows for the calculus to be about functions.

But the big highlight is the fourth definition, that of fluxion:

"Some magnitude having been chosen, homogeneous to an argument x, to be called fluxion of that argument, and denoted by dx; we will call fluxion of Γx, and will denote by $d\Gamma x$, the magnitude that would make $d\Gamma x/dx$ constant and $(\Gamma(x + dx) - \Gamma x)/dx - d\Gamma x/dx$ infinitesimal or zero, if dx were infinitesimal and all that does not depend on dx constant".

Notice that Cunha seems to combine the two main traditions in the calculus: the word "fluxion" is Newtonian, while the notation dx, $d\Gamma x$ is Leibnizian. However, this definition does not belong in either tradition. Youschkevitch [1973] said of it that "it was Cunha who, for the first time, formulated a rigorous analytical definition of the differential, taken up again and used later by the mathematicians of the nineteenth century". Mawhin [1990] was more specific, saying that it "corresponds to the modern definition of differential": $d\Gamma x$ is a linear function of dx (since $d\Gamma x/dx$ is constant) such that

$$\lim_{dx\to 0} \frac{\Gamma(x + dx) - \Gamma x - d\Gamma x}{dx} = 0.$$

[2] José Monteiro da Rocha is known to have written an introduction to the calculus in the 1760s, that was never published. The manuscript was at the Academy of Sciences of Lisbon in 1825, but its present whereabouts is unknown.

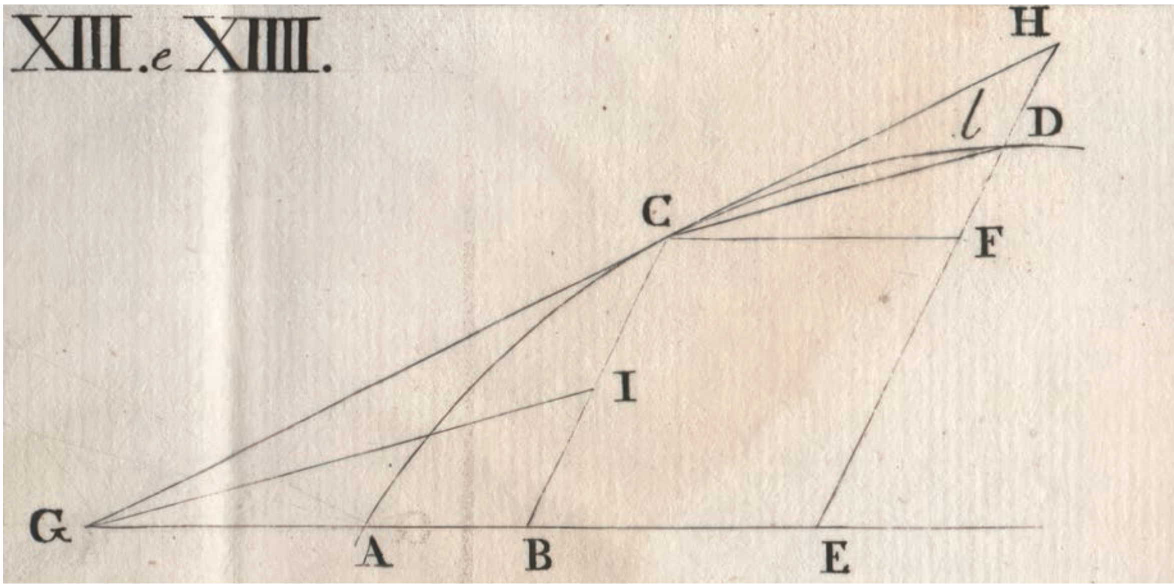

Figure 2.—A diagram for propositions 13 and 14 from book 15 of [Cunha, 1790]. *AB* is an abscissa and *BC* an ordinate (oblique coordinates!) of a curve *AD*. In prop. 13, Cunha proves that if *BE* is the fluxion of the abscissa *AB*, then parallelogram *BF* (that is, the parallelogram with diagonal *BF*) is the fluxion of the area *ACB*; for that, he assumes that the ordinate function is monotonic and trusts the diagram to convince the reader that area *CDF* is contained in the parallelogram with diagonal *CD* (not drawn): BF/BE is the height of parallelogram *BF*, and calling $(BF/BE)+\pi$ the distance from *D* to the axis of abscissas, it will be $CDF/BE < \pi$; this means that *AB* constant and *BE* infinitesimal would make BF/BE and $(ADE-ACB)/BE-BF/BE(=CDF/BE<\pi)$ infinitesimal, fulfilling the conditions in the definition of fluxion.

Such correspondence is not complete: for instance, $d\Gamma x$ is not explicitly stated to be a function of dx; and the existence of $d\Gamma x$ is not questioned — like all his contemporaries, Cunha assumed all functions to be differentiable. However, Cunha is definitely closer to a modern definition of differential than the usual definitions in the 18th century.

The propositions in book 15 can be roughly divided into two groups. Up to proposition 12, we find, in a very concise way, the fundamentals of what we call differential calculus: for instance, $d(x^n) = n x^{n-1} dx$ (prop. 2), $dx = x\,dlx$, l standing for hyperbolic (that is, natural) logarithm (prop. 8), the Taylor series of a function Γx (prop. 11), and the equality of mixed higher-order derivatives (prop. 12). All of this is proven (not always up to modern standards of proof, naturally) in an analytical way. For example, prop. 8 is obtained from the power series of the exponential:

$$\begin{aligned} dx &= d\Big(1 + lx + \frac{1}{2}(lx)^2 + \frac{1}{6}(lx)^3 + \\ &\qquad + \frac{1}{24}(lx)^4 + \frac{1}{120}(lx)^5 + \&\text{c.}\Big) = \\ &= dlx + \frac{2}{2}(lx)dlx + \frac{3}{6}(lx)^2 dlx + \frac{4}{24}(lx)^3 dlx + \\ &\qquad + \frac{5}{120}(lx)^4 dlx + \&\text{c.} = \\ &= \Big(1 + lx + \frac{1}{2}(lx^2) + \frac{1}{6}(lx)^3 + \frac{1}{24}(lx)^4 + \\ &\qquad + \&\text{c.}\Big) dlx = \\ &= x\,dlx. \end{aligned}$$

In the rest of the book appear the simplest geometrical applications: the fluxions of the area under a curve, of the arc length of a curve, and of the volume of a simple solid.

Most of the following books are dedicated to applications or particular topics in the calculus. Book 16 is instead dedicated to trigonometry, but with a peculiar organization (for an introductory text) that makes the calculus central: one of the earliest propositions gives the fluxion of the sine, from there the power series for the sine and cosine are derived, and it is from these that comes the formula for the sine of the sum of two arcs.

In book 17, we find topics of elementary differential geometry of curves: multiple points, asymptotes, radius of curvature. In other words, geometrical applications similar to those that are so important in [Bézout, 1774].

Book 18 gives several techniques of integration (such as partial fraction decomposition), L'Hôpital's rule (proven using Taylor series expansions of the numerator and of the denominator), and the Bernoulli series of a function Γx.

Book 19 addresses very quickly (in only 6 pages) several aspects of differential equations: "exact fluxions", homogeneous equations, integrating factors, higher-order linear equations [Baroni, 2001].

Book 20 gives an introduction to the calculus of finite differences.

Book 21 is a miscellany, probably compiled from several short manuscripts left by Cunha on diverse topics, by whoever arranged for the final publication of [Cunha, 1790]. Some of these topics are not related to the calculus, while others are. The latter include a couple of improper integrals, the condition $d\Gamma x = 0$ for a maximum of Γx (which had not been given before), and a very short introduction to the calculus of variations.

Summing up, as an introduction to the calculus, the relevant sections in [Cunha, 1790] are very ambitious in scope, but often too brief; it was, generally, an up to date text at the time (more so than Bézout's); and, of course, its definition of fluxion (along with its handling of infinitesimals) was very innovative, even in an European context.

4 Early attempts at research

The Academy of Sciences of Lisbon was founded in the final days of 1779. This was the first institution in Portugal with the goal of promoting scientific research — including mathematical research.

In the 1790s two volumes of memoirs were published containing mathematics. In total, four of those memoirs can be classified under "calculus": three in the first volume, and one in the second.

In the first volume (published in 1797 but with articles written in the 1780s), two pieces concern an approximation method for integrals by Alexis Fontaine (1704–1771). The Academy had proposed for 1785 a prize for a proof of Fontaine's method and a study of its (rate of) convergence, which was won by Manuel Joaquim Coelho da Maia (1750–1817), one of the first batch of doctors in mathematics from Coimbra. The winning entry was the subject of harsh criticism by José Anastácio da Cunha, which prompted Monteiro da Rocha to write some additional comments, in defence of the Academy's honour. Coelho da Maia's solution [Maia, 1797] is indeed mediocre, but [Monteiro da Rocha, 1797] contains valuable additions about the rate of convergence of the method [Figueiredo, 2011, ch. 9].

Also in the first volume, there is a memoir by Francisco Garção Stockler (1759–1829) on the "true principles of the Method of Fluxions" — like Anastácio da Cunha, Stockler admired Newton and d'Alembert, and his purpose was to expand on ideas that those two mathematicians had supposedly only sketched. But he was neither very original nor very clear. Briefly, Stockler

1. defined "fluent" as a variable quantity, in the traditional 18th-century sense, explicitly admitting that a fluent increases or decreases in intervals of time — a modern reader might interpret Stockler's fluents as functions of a time variable;
2. then considered "hypothetical fluxions", which were ratios between increments or decrements of fluents and the corresponding time intervals, and "proper fluxions", which had an unclear definition (the increments or decrements that the fluents' "tendency" to increase or decrease could produce in a unit of time) but could be calculated as limits of hypothetical fluxions;
3. and finally used power series expansions to calculate those limits.

Stockler also published with the Academy a small booklet on limits [Stockler, 1794], far less ambitious but quite interesting. Inspired by the Swiss Simon l'Huilier (1750–1840), Stockler assumed that there are two cases when a variable has a limit: it can be an *increasing limit* or a *decreasing limit* (that is, they assumed that only monotonic variables could had limits). But, unlike l'Huilier, Stockler devoted a great deal of attention to *variables that decrease without limit*; in modern terms, these are variables with limit zero. The first section of [Stockler, 1794] uses elementary but careful ε-δ arguments to develop an extensive arithmetic of such variables. This is then used in the second section, on variables with (non-zero) limits, by means of a Fundamental Principle: if $Z = A \mp z$, where Z is a variable, A a constant, and z a variable that decreases without limit, then A is the limit of Z. This makes Stockler's proofs much less tiresome than those of l'Huilier, who had mostly written in algebraic language Greek-style exhaustion arguments. The third and fourth section deal with trigonometric, logarithmic, and exponential functions.[3] In spite of Stockler's careful treatment of more elementary limits, his handling of infinite series is not up to modern standards; for instance, he uses them to "prove" that the limit of any function of a variable equals the function of the limit of the variable. However, in a time when almost all limit arguments were vague at best, [Stockler, 1794] is worthy of note.

The second volume of memoirs from the Academy of Sciences (published in 1799) includes another article by Stockler on the calculus [Stockler, 1799]. This is a quite long (100 pages) attempt at simplifying and systematizing conditions for exact differentials (Stockler calls them "exact fluxions"). It is explicitly inspired by (early) works of Condorcet, who had tried to create a general theory of integration [Gilain, 1988], himself inspired by works of Fontaine and Euler.

5 Final remarks

The effective introduction of the calculus in Portugal occurred relatively late. However, in the final three decades of the 18th century a number of texts were published in Portuguese about the calculus, which was by then well established as part of mathematical curricula.

Also, the calculus had a prominent place in the first attempts at organized mathematical research in Portugal — partly reflecting the place it had in contemporary European mathematical research.

Most of the 19th century would be a period of stagnation in Portuguese mathematics, but that was not foreseeable around 1800.

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[3] For more details on the contents of [Stockler, 1794], see [Saraiva, 2001].

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Mathematics and Music

by **José Francisco Rodrigues***

In four movements — Pythagorean Arithmiusic, Tone Algebra, Harmonisation of Analysis, Digital Musurgia — and through a few examples, we will present a brief introduction to the numerous interactions between mathematics and music throughout history, which can help us understand the modern interpretation of Leibniz's expression:

Musica est exercitium arithmeticae occultum nescientis se numerare animi.[+]

1. Pythagorean Arithmusic

C'est par les nombres et non par le sens qu'il faut estimer la sublimité de la musique. Etudiez le monocorde.[++]

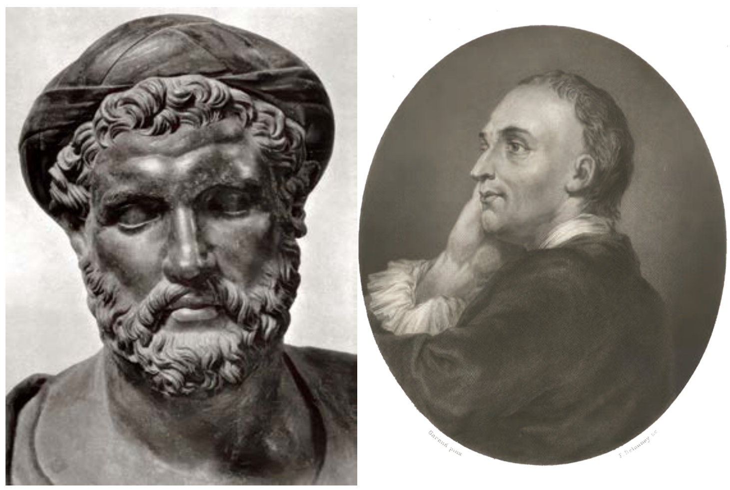
—Diderot, *Pythagoreanism*, Encyclopédie XII (1765)

Guido d'Arezzo (992–1050?) in the *Micrologus*, attributes to Pythagoras (6th century BCE) the fundamental discovery of the dependence of musical intervals on the quotients of the first integers numbers, writing:

> A certain Pythagoras, on one of his journeys, happened to pass a workshop where an anvil was being beaten with five hammers. Astonished by the pleasant harmony (*concordiam*) they produced, our philosopher approached them and, thinking at first that the quality of the sound and harmony (*modulationis)* lay in the different hands, exchanged the hammers. In this way, each hammer retained its own sound. After removing one that was dissonant, he weighed the others and, marvellously, by the grace of God, the first weighed *twelve*, the second *nine*, the third *eight*, the fourth *six* of I don't know what unit of weight.

For the Pythagorean School, the harmony of sounds was in direct correspondence with the arithmetic of proportions:

unison — ratio 1 : 1	octave (*diapason*) 1 : 2
fifth (*diapente*) 2 : 3	fourth (*diatessaron*) 3 : 4



Left: Bust of Pythagoras. Right: Denis Diderot (1713–1784)

These ratios can be obtained from those four numbers, corresponding respectively to a string length equal to 12 units (*unison*), halved to 6 (*octave*), 8 units (*fifth*) or 9 (*fourth*).

The Greek heritage, was transmited in particular by the Roman Boethius (6th century CE), "the great, astonishing and very sudden relationship (*concordiam*) that exists between music and the proportions of numbers (*numerum proportione*)".

[+] Music is a hidden arithmetic exercise of a mind unconscious that it is counting.
[++] It's by the numbers and not by the sense that one should evaluate the sublimity of music. Study the monochord.

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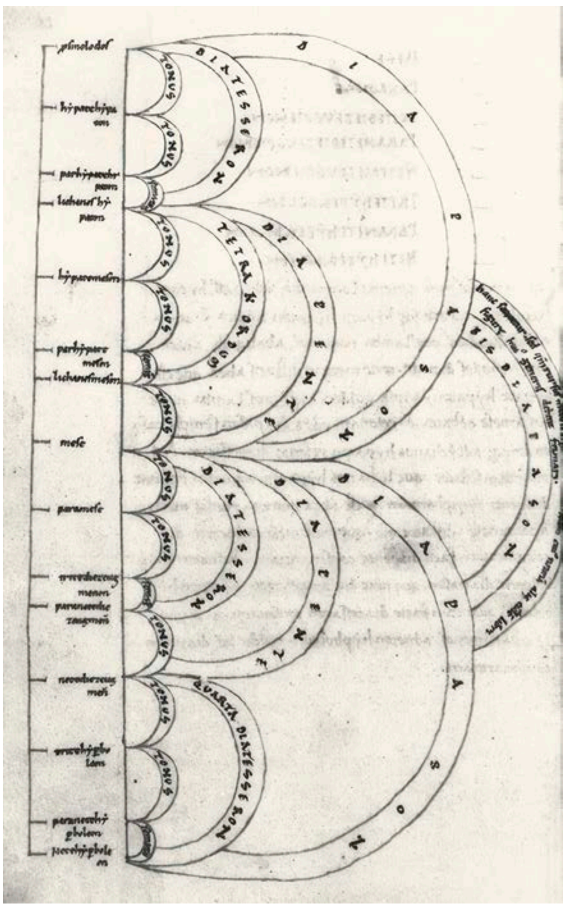


Top: Franchinus Gafurius (*Theorica musicae*, 1492)

Bottom: Boetius; c. 480–524, *De Institutione Musica*. Division of intervals (Paris, Bibl Nat, 12^{th} cent.)

The arithmetic of proportion establishes: the product of 2/3 (fraction associated with the fifth) by 3/4 (fraction associated with the fourth) gives the fraction 1/2 associated with the octave; its division (subtraction of intervals) is associated with the fraction $8/9 = (2/3) \div (3/4)$ which represents a tone, i.e. the difference between a fifth and a fourth. Analogously, an octave is made up of two fourths and a tone $1/2 = 3/4 \times 3/4 \times 8/9$.

The *Sectio Canonis*, or the "Division of a monochord", 300 BCE, by Euclid, has twenty propositions argued in the form of theorems, treatment of intervals as ratios between integers numbers and culminates with the division of the *Kanon*, For example, its 15^{th} Proposition says "the fourth is less than two and a half tones and the fifth less than three and a half tones", and others, like the 9 th (<= VIII.2), are consequences of the Book VIII of the *Elements*.

The ancient Greeks also divided the mathematical sciences into four parts:

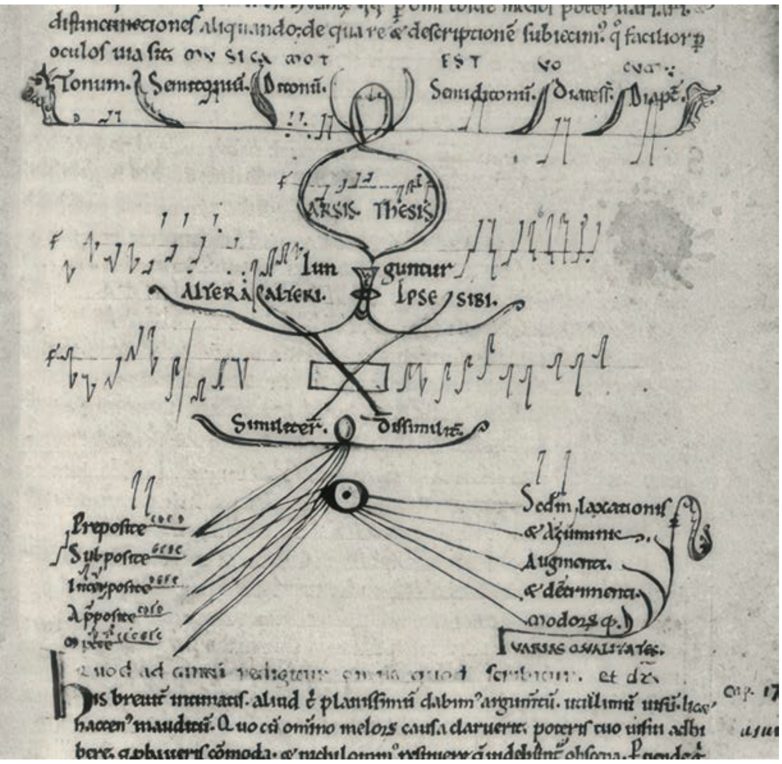
arithmetic (static discrete quantities)
music (discrete quantities in motion)
geometry (stationary magnitudes)
astronomy (dynamic magnitudes).

This classification constituted the *Quadrivium*, as part of the seven liberal arts of the medieval curriculum, which were complemented by the *Trivium* (grammar, dialectic and rhetoric).

Arithmetic, geometric and harmonic proportionality are present throughout medieval science and music, where the latter is defined as number associated with sound—*numerus relatus ad sonum*. For example, in the speculative treatise *Ars novae musicae* (1319), the Parisian mathematician and astronomer Jean de Muris wrote:

> Sound is generated by movement, since it belongs to the class of successive things. It therefore exists only as long as it is produced, ceasing to exist once it has been produced . . . All music, especially measurable music, is based on perfection, combining in itself number and sound.

Claudius Ptolemy (2^{nd} century, CE), author of *Mathematike Syntaxis (Almagest)* and of the treatise *Harmonica*, in which he transmitted the myth of how the mathematical relationships underlying the structures of audible music constitute the forms of the essence and cause of harmonies both in the human soul and



Neumas, *Micrologus*, Guido d'Arezzo
Manuscript, 12th cent. (Biblioth. Nationale, Paris)

in the movements and configurations of the stars.

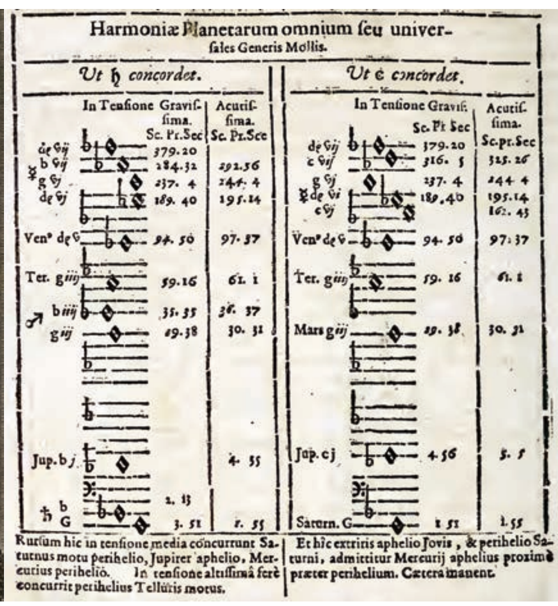
In the Boeotian terminology, this corresponded to musica instrumentalis (produced by the lyre, flute, etc.), musica humana (inaudible, produced in man by the interaction between body and soul), musica mundana (produced by the cosmos itself, also known as the music of the spheres).

The cube with 6 faces, 8 vertices and 12 edges, and therefore considered a harmonic solid, together with other more subtle parallelisms between arithmetic and geometry, led classical civilisation to the doctrine of the music of the spheres and, in Aristotle's expression, to consider that the whole sky is number and harmony.

For Joannes Kepler (1571–1630), the movement of the planets was still an immanent music of divine perfection, but this didn't prevent him to conclude the three laws of motion:

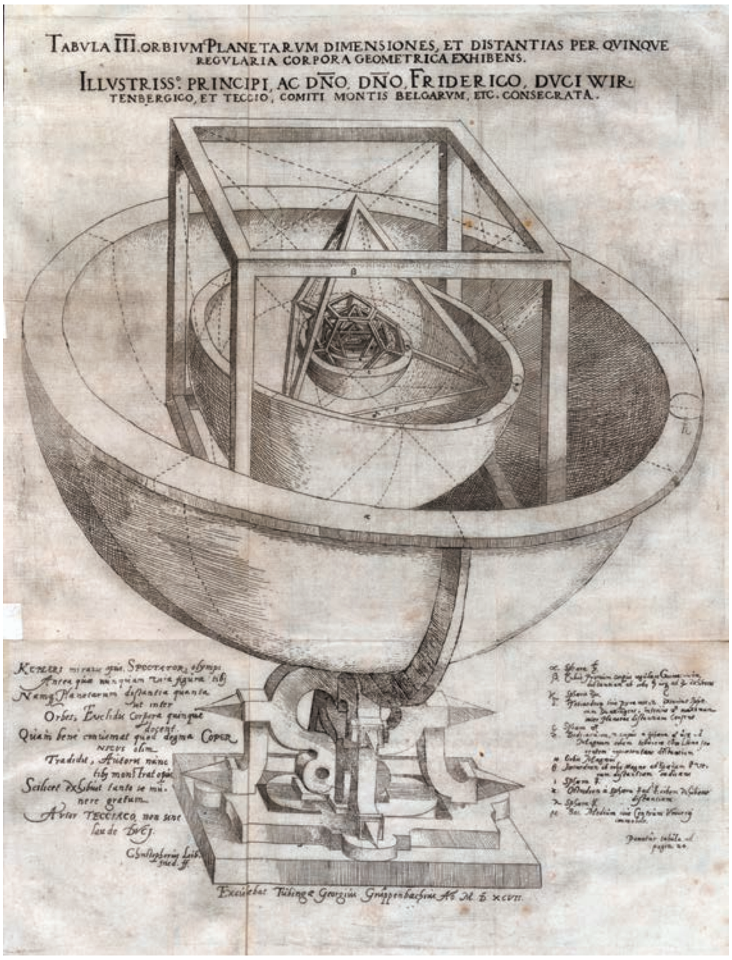
1. the planets revolve around the Sun in elliptical orbits;
2. with the Sun as a foci and their orbital areas are travelled in proportion to time;
3. the squares of the periods of revolution of each planet are proportional to the cubes of their average distances from the Sun.

Following his third law, in 1619 Kepler wrote: musical modes or tones are reproduced in a certain way at the extremities of planetary movements. Considering the seven consonant intervals of the octave of his time, he established the following harmonies of



Left
Johannes Kepler (1571–1630)

Right
Harmonices Mundi, 1619



Kepler's *Mysterium cosmographicum* (1596), with the embeding of the cube (Saturn-Jupiter), tetrahedron (Jupiter-Mars), dodecahedron (Mars-Earth), icosahedron (Earth-Venus) and octahedron (Venus-Mercury) in the sphere.

the six known planets:

Saturn 4 : 5 (a major tertia)
Jupiter 5 : 6 (a minor tertia)
Mars 2 : 3 (a fifth)
Earth 5 : 16 (a half-tone)
Venus 24 : 25 (a sharp)
Mercury 5 : 12 (an octave and a minor tertia);

by calculating the aphelion/perihelion ratios for each of them: Saturn travels an arc of 106 or 135 seconds per day when it is at its furthest point (aphelion) or closest (perihelion) to the Sun, respectively, obtaining the ratio 106/135 ~ 4/5.

Kepler's metaphysics goes so far as to states that the Earth sings the notes MI, FA, MI, so that from them it can be conjectured that misery (MIseria) and hunger (FAmes) prevail in our midst.

2. Tone Algebra

Pythagorean scales are based on the elementary "rational" intervals (octave, fifth and fourth) and their alternating successions, i.e., starting from a sound from a sound $f_0 = f$ and the sound $f_1 = 3f/2$ located a fifth higher on the scale, the sound $f_2 = 3f_1/4 = 9f/8$ will be one fourth below f_1, the sound f_3 a fifth above f_2 and so on. This gives the *cycle of fifths* as

$$f_n = \left(\frac{3}{2}\right)^n \left(\frac{1}{2}\right)^p f$$

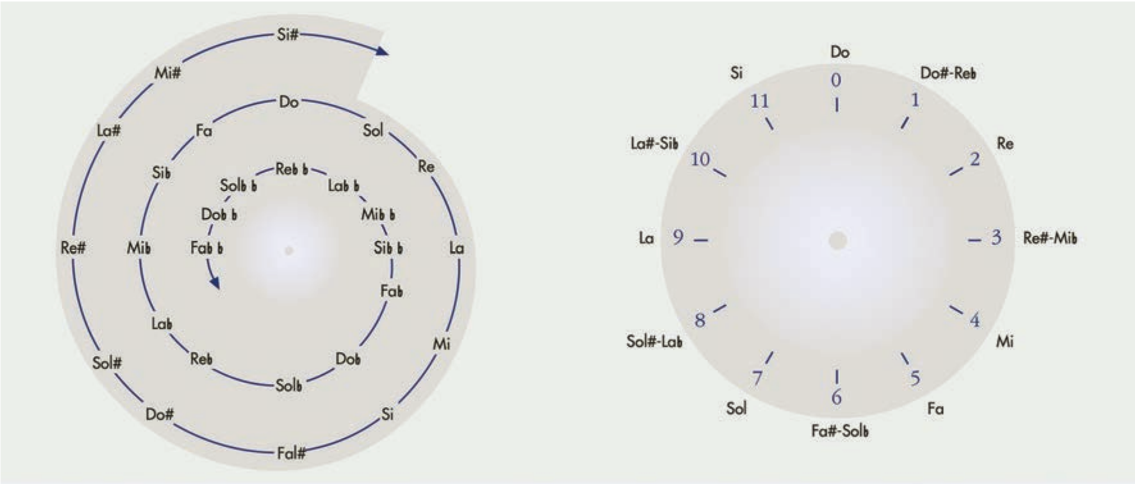
which isn't a real cycle, because if it were, there would have to be two integers n and p such that $3^n = 2^{n+p}$; but an odd number is different from an even number, so it is impossible!

In classical solfege "12 fifths correspond to 7 octaves", mathematically it would be $3^{12} = 2^{19}$, which is false. We have that

$$\frac{3^{12}}{2^{19}} = \frac{531441}{524288} \approx 1.$$

This only translates into a certain tolerance of the ear to that tuning and this difference is the Pythagorean coma.

A theoretical formulation of equal temperament can already be found in the work *De musica* by F. Salinas, published in Salamanca in 1577, which states that *the octave must be divided into twelve equally proportional parts, which will be the equal semitones.*



Left: A spiral of fifths. Right: A 12-tone chromatic clock

If τ is the interval between two consecutive tones, it is equal to the irrational number

$$\tau = \sqrt[12]{2} = 1,059463094\ldots$$

and represents the ratio of the respective geometric progression. So the frequencies associated with the seven notes of the usual scale are given by $\text{Dó} = f$, $\text{Ré} = \sqrt[6]{2}f$, $\text{Mi} = \sqrt[3]{2}f$, $\text{Fá} = \sqrt[12]{2^5}f$, $\text{Sol} = \sqrt[12]{2^7}f$, $\text{Lá} = \sqrt[4]{2^3}f$, $\text{Si} = \sqrt[12]{2^{11}}f$ e $\text{Dó} = 2f$.

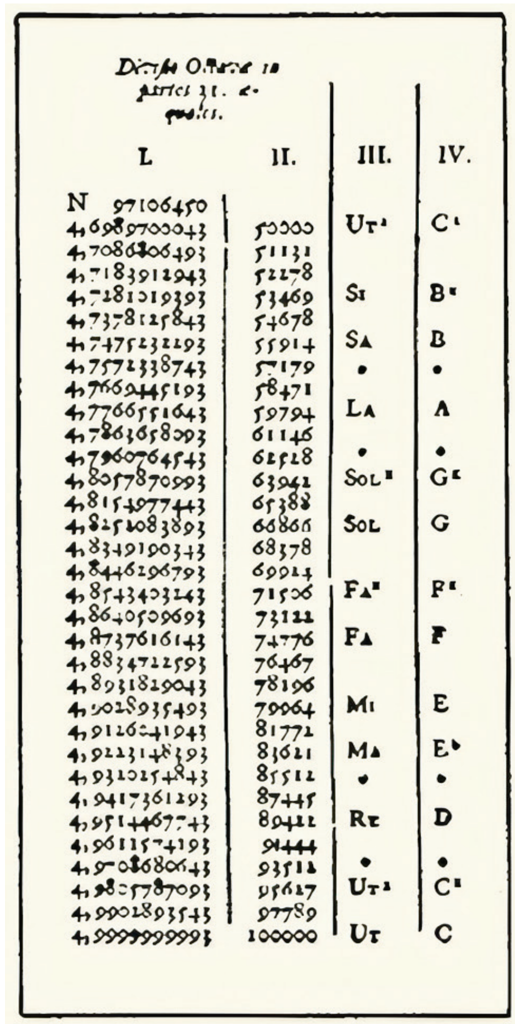
Methods for mumerical approximations of equal temperament can be found in Zarlino in the 16th cent. and in M. Mersenne's *Harmonie Universelle* (1636–7) or in A. Kircher's *Musurgia Universalis* (1650).

The theorising of equal temperament in the 17th century will use logarithms. C. Huygens (1629–1695) in *Novus Cyclus Harmonicus* (1691) theorised the division of the octave into 31 equal intervals and was one of the first to introduce the calculation of logarithms into music.

Referring to Salinas and Mersenne as authors who had already considered this division to be of no great consequence, Huygens remarked that if their predecessors had been mistaken because "they hadn't known how to divide the octave into 31 equal parts (. . .) for this the intelligence of Logarithms was necessary."

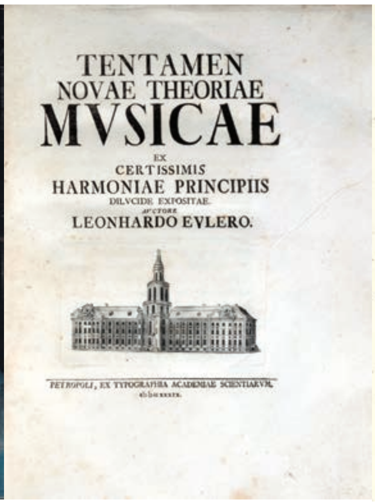
In Euler (1707–1783) we find one of the most ingenious algebraic theories of the division of the octave and the degree of consonance of musical intervals.

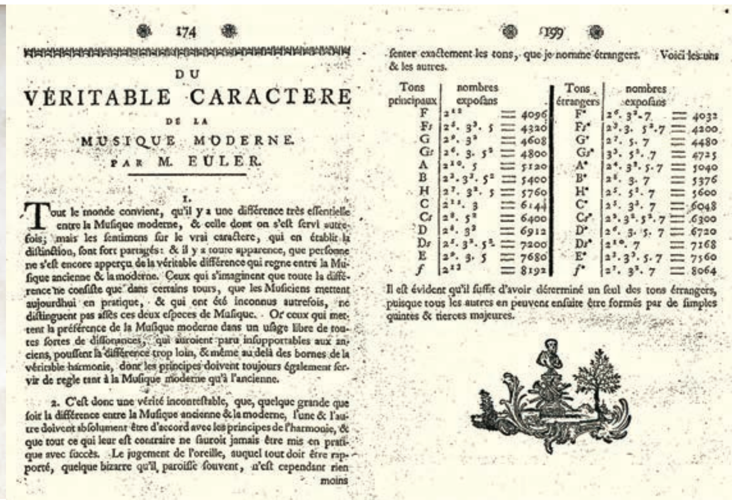
In the Essay on a new theory of music (*Tentamen novae theoriae musicae*, 1739), Euler develops an argument in which proportions generate musical pleasure, via order and perfection — music is the science of combining sounds in a pleasing harmony — so that, for this mathematician, a musical object is a simple arithmetical object.


Divisio Octavae in partes 31. aequales.

I.	II.	III.	IV.
N 97106450			
4,6989700043	50000	Ut'	C'
4,7086806493	51131		
4,7183912943	52278		
4,7281019393	53469	Si	B'
4,7378125843	54678		
4,7475232293	55914	Sa	B
4,7572338743	57179	•	•
4,7669445193	58471		
4,7766551643	59794	La	A
4,7863658093	61146		
4,7960764543	62528	•	•
4,8057870993	63942	Sol'	G'
4,8154977443	65388		
4,8252083893	66866	Sol	G
4,8349190343	68378		
4,8446296793	69924		
4,8543403243	71506	Fa'	F'
4,8640509693	73122		
4,8737616143	74776	Fa	F
4,8834722593	76467		
4,8931829043	78196		
4,9028935493	79964	Mi	E
4,9126041943	81772		
4,9223148393	83621	Ma	E'
4,9320254843	85512	•	•
4,9417361293	87445		
4,9514467743	89422	Re	D
4,9611574193	91444		
4,9708680643	93512	•	•
4,9805787093	95627	Ut''	C''
4,9902893543	97789		
4,9999999993	100000	Ut	C

Calculations of 1691 of the division of the octave into 31 tones by Huygens using logarithms





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DU

VÉRITABLE CARACTERE

DE LA

MUSIQUE MODERNE.

PAR M. EULER.

I.

Tout le monde convient, qu'il y a une différence très essentielle entre la Musique moderne, & celle dont on s'est servi autrefois; mais les sentimens sur le vrai caractere, qui en établit la distinction, sont fort partagés: & il y a toute apparence, que personne ne s'est encore apperçu de la véritable différence qui regne entre la Musique ancienne & la moderne. Ceux qui s'imaginent que toute la différence ne consiste que dans certains tours, que les Musiciens mettent aujourdhui en pratique, & qui ont été inconnus autrefois, ne distinguent pas assés ces deux especes de Musique. Or ceux qui mettent la préférence de la Musique moderne dans un usage libre de toutes sortes de dissonances, qui auroient paru insupportables aux anciens, poussent la différence trop loin, & même au delà des bornes de la véritable harmonie, dont les principes doivent toujours également servir de regle tant à la Musique moderne qu'à l'ancienne.

2. C'est donc une vérité incontestable, que, quelque grande que soit la différence entre la Musique ancienne & la moderne, l'une & l'autre doivent absolument être d'accord avec les principes de l'harmonie, & que tout ce qui leur est contraire ne sauroit jamais être mis en pratique avec succès. Le jugement de l'oreille, auquel tout doit être rapporté, quelque bizarre qu'il paroisse souvent, n'est cependant rien moins

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senter exactement les tons, que je nomme étrangers. Voici les uns & les autres.

Tons principaux	nombres exposans		Tons étrangers	nombres exposans	
F	2^{12}	= 4096	F*	$2^6.3^2.7$	= 4032
Fs	$2^5.3^3.5$	= 4320	Fs*	$2^3.3.5^2.7$	= 4200
G	$2^9.3^2$	= 4608	G*	$2^7.5.7$	= 4480
Gs	$2^6.3.5^2$	= 4800	Gs*	$3^3.5^2.7$	= 4725
A	$2^{10}.5$	= 5120	A*	$2^4.3^2.5.7$	= 5040
B	$2^3.3^3.5^2$	= 5400	B*	$2^8.3.7$	= 5376
H	$2^7.3^2.5$	= 5760	H*	$2^5.5^2.7$	= 5600
C	$2^{11}.3$	= 6144	C*	$2^5.3^3.7$	= 6048
Cs	$2^8.5^2$	= 6400	Cs*	$2^2.3^2.5^2.7$	= 6300
D	$2^8.3^3$	= 6912	D*	$2^6.3.5.7$	= 6720
Ds	$2^5.3^2.5^2$	= 7200	Ds*	$2^{10}.7$	= 7168
E	$2^9.3.5$	= 7680	E*	$2^3.3^3.5.7$	= 7560
f	2^{13}	= 8192	f*	$2^7.3^2.7$	= 8064

Il est évident qu'il suffit d'avoir déterminé un seul des tons étrangers, puisque tous les autres en peuvent ensuite être formés par de simples quintes & tierces majeures.

Euler introduced a mesure of the degree of consonance (agrément) of an interval through an algebraic formula in which p_i are prime numbers and m_i, integer exponents:

$$\alpha(I) = \sum_{i=1}^{n} \left(m_i p_i - m_i\right) + 1.$$

Euler also wrote other essays, such as *Du véritable caractere de la musique moderne* (On the true character of modern music), in Mémoires de l'Académie des Sciences de Berlin (1764), 1766.

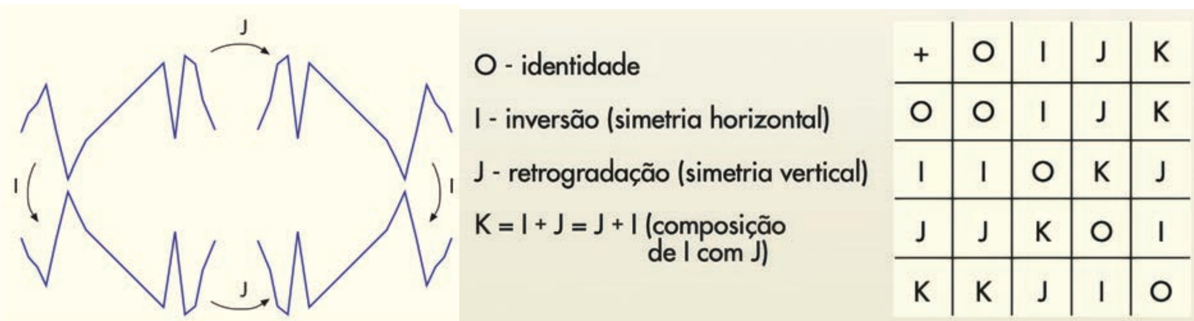
But the algebra of tones is not limited to the problems associated with temperament, but also appears in the structure of sounds and in musical composition itself.

Musical notes can be grouped into equivalence classes and hence called by the same name, i.e. two notes are said to be equivalent if they are separated by an exact number of octaves, i.e. if they have frequencies p and q, the interval between them is of the form $p/q = 2k$, with $k = 0, \pm1, \pm2, \dots$ and will be denoted by $p \sim q$.

In the 12-note tempered system, the interval is characterised by the number of semitones and the notes can be associated with the set of integers

$$\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$$

which is a group for addition (mod 12).



Geometric and algebraic representations of a group

Inversion (horizontal symmetry), Petrushka by Igor Stravinsky

Another typical example is given in J.S. Bach's *Musical Offering* of 1747, which presents three types of transformations: translations (upward transpositions, as in the canon ascendenteque *Modulationem ascendat gloria Regis*), horizontal symmetries (melodic inversions, as in the canon *Per Motum Contrarium*) and vertical symmetries (retrogrades, as in the canon a 2 which plays the same theme starting on the last note and moving backwards to the first). Also known as *palindromes* or *crab canons*: $y = -x$.

3. Harmonisation of Analysis

Marin Mersenne (1588–1648) is credited with establishing the basic laws of modern string acoustics. *Harmonie universelle* (1636), establishes the experimental laws on the proportionality of the period of vibration of the string, in relation to its length, to the inverse of the square root of its tension and to the square root of its thickness or cross-sectional area.

Galileo Galilei, in *Discorsi e dimostrazioni matematiche . . .* (1638) refers to the question of vibrating strings and consonance as follows:

> . . . the first and immediate reason on which the ratios of musical intervals depend is neither the length of the strings nor their thickness, but the proportion existing between the frequencies of the vibrations, and therefore of the waves which, propagating in the air, reach the eardrum of the ear causing it to vibrate at the same intervals of time.

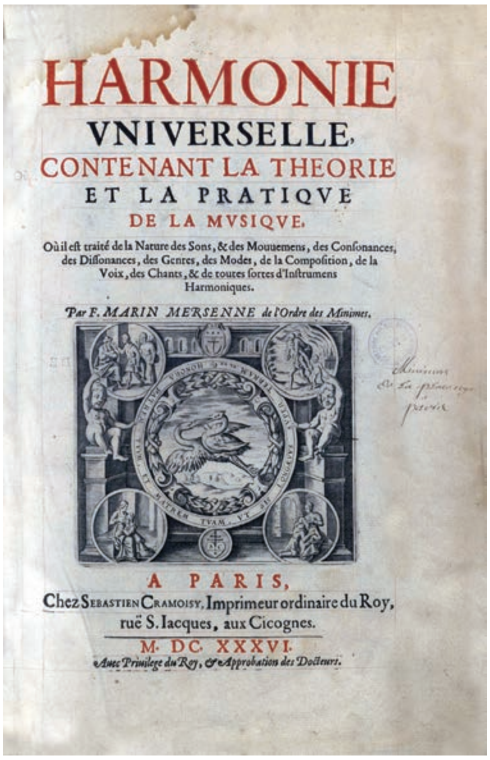
The mathematical analysis of the sound starts with the modeling of the vibrating string, namely with the computation of its fundamental period by B. Taylor in 1713, with the first ODE analysis by Jean Bernoulli in 1727 and the famous controversy between D'Alembert and Euler on the admissible initial conditions on the wave equation.

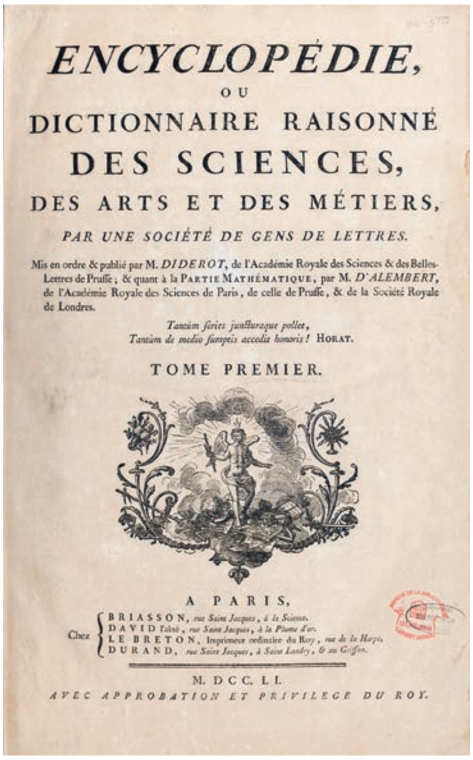
It is above all with the introduction of the equation

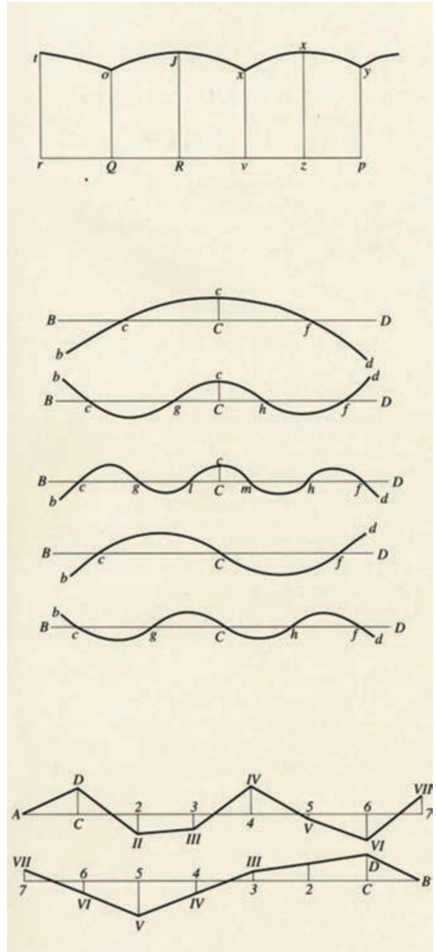
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

in D'Alembert's 1747 memoire published by the Berlin Academy, *Recherche sur la courbe que forme une corde tendue mise en vibration*, and with the subsequent works of Euler, Daniel Bernoulli and Lagrange, that the mathematical theory of the "musical string" acquires the appropriate model for small vibrations, which will be decisive in the study of oscillations in continuous media, in particular the propagation of sound in air.

During the course of the famous "vibrating string controversy", a scientific dispute involving the leading mathematicians of the 1700s, Daniel Bernoulli, in





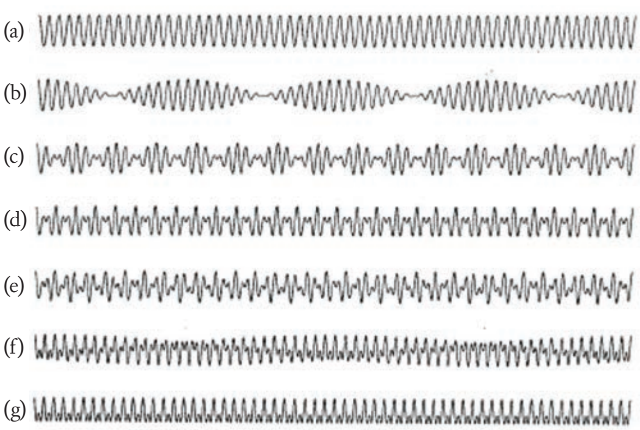


a 1753 letter, established the principle of the superposition of small harmonic oscillations as a physical law and not so much as a mathematical result, concluding that

> every sounding body potentially contains an infinity of sounds and a corresponding infinity of ways of producing their respective vibrations.

In a memoir by the Turinese mathematician Lagrange (1736-1813), we find a formula for the solution of the wave equation which, in the 19th century, after the work of Fourier, will allow us to demonstrate D. Bernoulli's principle of superposition of waves. Lagrange not only sought to analyse the propagation of sound, he also tried to provide a scientific explanation for Tartini's theory of the combination of tones, set out in his Treatise on Music of 1754.

The musical string is just the first mathematical example of sound analysis. Both the sound produced by



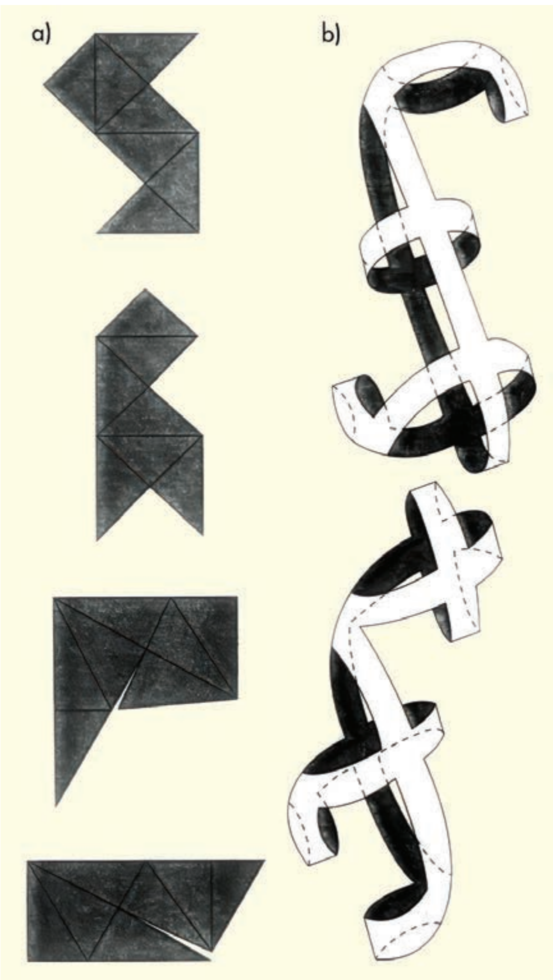
Soit $\frac{X}{a}$ la raison générale des indices des Y et des V au nombre m, X dénotera la partie de l'axe qui leur est correspondante dans le premier état du système; donc, si l'on emploie le signe intégral $\int$ pour exprimer la somme de toutes ces suites, on aura

$$y=\frac{2}{a}\int dx\,Y\left(\sin\frac{\varpi X}{2a}\sin\frac{\varpi x}{2a}\cos\frac{\varpi H t}{2T}+\sin\frac{2\varpi X}{2a}\sin\frac{2\varpi x}{2a}\cos\frac{2\varpi H t}{2T}+\sin\frac{3\varpi X}{2a}\sin\frac{3\varpi x}{2a}\cos\frac{3\varpi H t}{2T}+\dots\right)$$
$$+\frac{4T}{\varpi H a}\int dx\,V\left(\sin\frac{\varpi X}{2a}\sin\frac{\varpi x}{2a}\sin\frac{\varpi H t}{2T}+\frac{1}{2}\sin\frac{2\varpi X}{2a}\sin\frac{2\varpi x}{2a}\sin\frac{2\varpi H t}{2T}+\frac{1}{3}\sin\frac{3\varpi X}{2a}\sin\frac{3\varpi x}{2a}\sin\frac{3\varpi H t}{2T}+\dots\right),$$

Left: Ratios of frequencies of two pure tones (a) 1:1 (b) 15:16 (c) 4:5 (d) 2:3 (e) 20:31 (f) 30:59 (g) 1:2.
Right: An excerpt of *Recherches sur la nature de la propagation du son* (1579) by Lagrange.

most musical instruments and the human ear itself require mathematical models that take into account the various dimensions of physical space and geometry.

In the mathematical analysis of the sound a famous question arouse: *is it possible to hear the shape of a drum*? This question, which has a precise and profound meaning in maths, consists of knowing whether from the same family of eigenvalues, i.e., numbers $\lambda = \lambda_n$, $n = 1, 2, \ldots$, that satisfy the equation $\Delta u + \lambda u = 0$ in two domains Ω_1 and Ω_2 it is possible to say that these regions are congruent in the sense of Euclidean geometry. Of all the drums with the same area, the round one has the deepest sound.



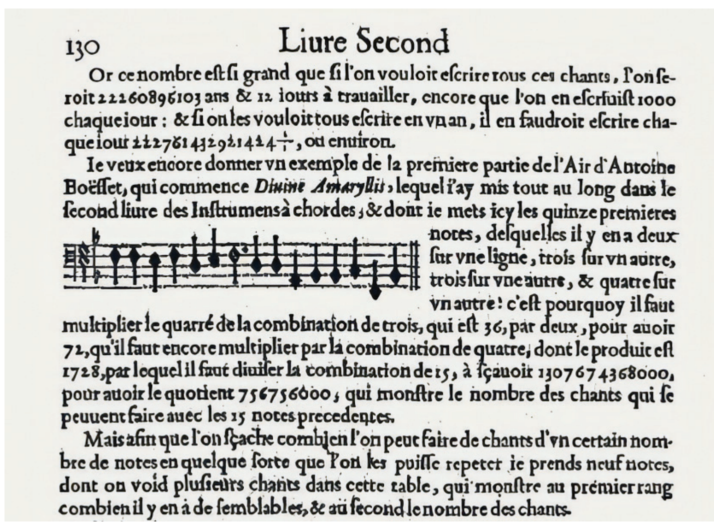
a) Isospectral (reproducing the same sound) drum shapes (flat polygons) with different shapes (C. Gordon and D. Webb, 1991).
b) Isospectral spatial shapes of bells (Riemannian surfaces) by P. Buser (1986).

4. Digital Musurgia

As early as the 17th century, an obscure German mathematician, K. Schott, following the ideas of Mersenne and his teacher Kircher, author of a *Musurgia Universalis* (1650), argued in his *Organum mathematicum* (1668) that to compose harmonic chants it was enough to master the new art of music-arithmetic, which consisted of combining the *bacilli musurgici* (the musical keys) and using the *abaci melothetici* and the *tabulae musarithmeticae*.

These ideas were based on the new combinatorial art of Mersenne, in *Harmonie Universelle* (1636), for whom composing was reduced to combining, he had distinguished permutations without repetition of a given number of n notes (ordinary combinations which he calculated up to $n = 64$) $S_n = n!$, from permutations with repetition of n notes (p are different), which he used to calculate the table of chants that can be made from 9 notes, he also calculated arrangements without repetition (p different notes among n given) and also combinations without repetition.

In the evolution of mathematics from the 17th to the 18th century, particularly for G.W. Leibniz, the mathematical sciences acquired a broader role as a science about the representations of all possible relationships and dependencies of the simplest elements, seeking a universal language and an algebra of reasoning, perfect-



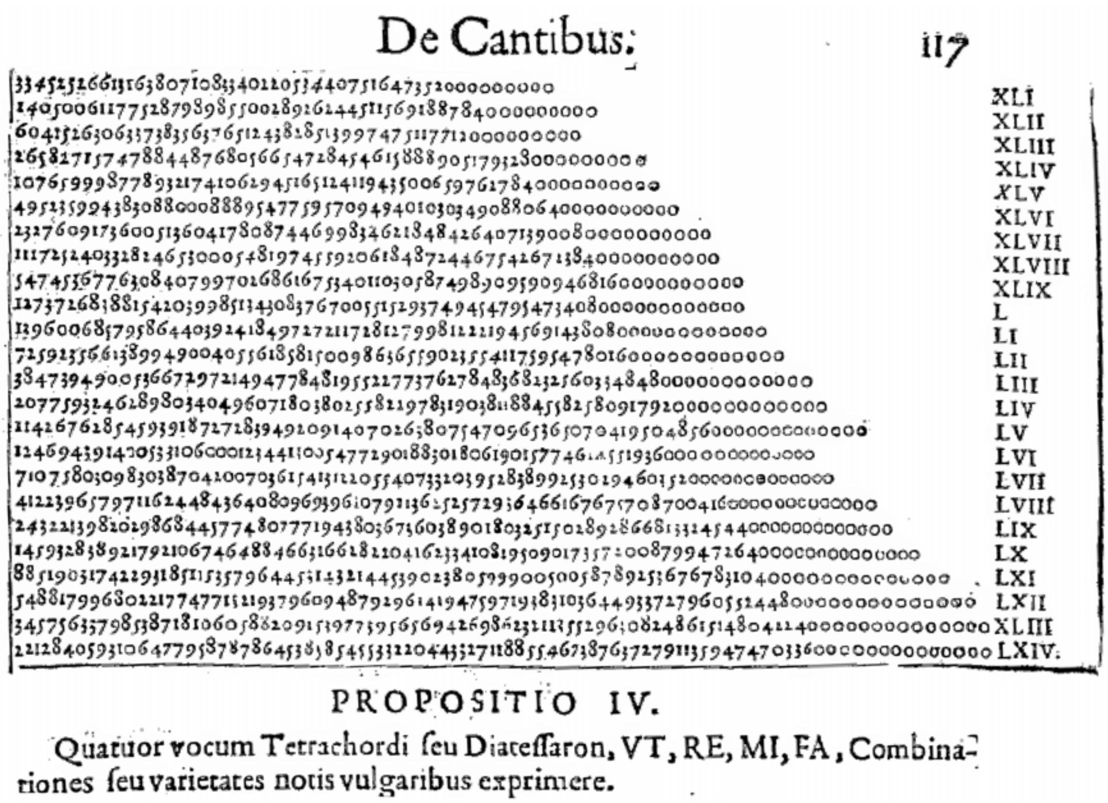
130 Liure Second

Or ce nombre est si grand que si l'on vouloit escrire tous ces chants, l'on seroit 22260896103 ans & 12 iours à trauailler, encore que l'on en escriuist 1000 chaque iour : & si on les vouloit tous escrire en vn an, il en faudroit escrire chaque iour 2227814329214 1/4, ou enuiron.

Ie veux encore donner vn exemple de la premiere partie de l'Air d'Antoine Boësset, qui commence *Diuine Amaryllis*, lequel i'ay mis tout au long dans le second liure des Instrumens à chordes, & dont ie mets icy les quinze premieres notes, desquelles il y en a deux sur vne ligne, trois sur vn autre, trois sur vne autre, & quatre sur vn autre : c'est pourquoy il faut multiplier le quarré de la combination de trois, qui est 36, par deux, pour auoir 72, qu'il faut encore multiplier par la combination de quatre, dont le produit est 1728, par lequel il faut diuiser la combination de 15, à sçauoir 1307674368000, pour auoir le quotient 756756000, qui monstre le nombre des chants qui se peuuent faire auec les 15 notes precedentes.

Mais afin que l'on sçache combien l'on peut faire de chants d'vn certain nombre de notes en quelque sorte que l'on les puisse repeter ie prends neuf notes, dont on void plusieurs chants dans cette table, qui monstre au premier rang combien il y en à de semblables, & au second le nombre des chants.

M. Mersenne, *Harmonie Universelle* (1636–37)


De Cantibus: 117

PROPOSITIO IV.

Quatuor vocum Tetrachordi seu Diatessaron, VT, RE, MI, FA, Combinationes seu varietates notis vulgaribus exprimere.

M. Mersenne, *Harmonie Universelle* (1636-37)

ing calculation and creating new algorithms to which it became necessary to give a symbolism appropriate to the essence of the concepts and operations.

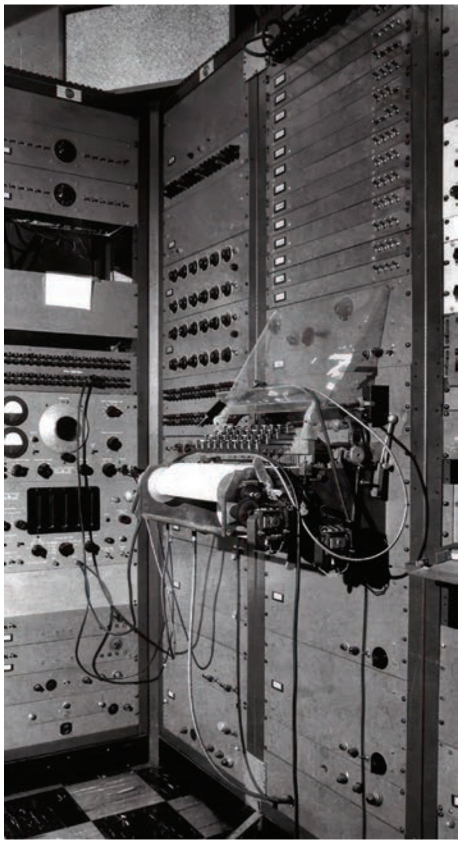
In his dissertation on the art of combinatorics (1666), the young Leibniz already intended to reorganise logic, but it was after the creation of the Calculus that he referred to binary notation in a 1701 letter to J. Bernoulli: Many years ago an original idea occurred to me about a type of arithmetic where everything is expressed with 0 and 1.

However, this new type of binary arithmetic was only realised in modern computers, where each bit represents an electrical state: on (current) is associated with the number 1; off (no current) is associated with 0; and sequences of electrical impulses, such as 01000001 which represents the number 65 in the binary system, and which can also be assigned to the capital letter A using another code.

Forerunners of modern calculators, the machines of the 17th century had a limited impact, in particular those of W. Schickard (1592–1635) and B. Pascal (1623–1662), capable of adding and subtracting mechanically, or that of Leibniz in 1671, which could also multiply and divide.

Left:
G. W. Leibniz (1646–1716)

Right;
Binary system designed by Leibniz, which reads “one created everything out of nothing” at the top and “one is necessary” at the bottom.



Left:
Ada Lovelace (1815-1852)

Right:
RCA Mark II
Electronic Music
Synthesizer,
H. Olson e H. Belar (1957)

However, only C. Babbage's (1791-1871) mechanical machines, namely the Difference Engine (1821) and the Analytical Engine (1834), are considered to be the forerunners of electronic computers, even though they were never built.

In a passage on the conception of that machine, Ada Lovelace specifically states that its operative mechanism could act on things other than numbers, objects such that their fundamental reciprocal relationships could be expressed by the abstract science of operations and, as a concrete example within the framework of the operative notation and mechanisms of the Analytical Engine, explicitly supposes that the fundamental relationships of sounds determined in the science of harmony and musical composition could be expressed and adaptable to its action; the machine could compose scientific and elaborate musical pieces, with any degree of complexity or extension.

However, a sufficiently powerful mechanism capable of incorporating the science of operations only appeared with the modern computer in the second half of the 20th century.

The first experiments in computer-assisted musical composition appeared from the start L. Hiller in 1956 in the USA, followed by P. Barbaud and I. Xenakis in France and others. At Bell Laboratories, in 1957, M. Mathews and his collaborators made the first numerical record and the first computer synthesis of sounds and, in 1965, J.C. Risset computer-simulated the first sounds of musical instruments.

In 1973, the first numerical synthesiser was built, Synclavier, which was then commercialised, and about ten years later theten years later, the public had access to digital recording CD's (compact discs).

Since 1983, the MIDI (Musical Instrumental Digital Interface) standard has allowed computers to record and edit music.

If today we have the mastery of numerisation in the analysis and synthesis of musical sound, if we have begun to outline the mathematisation of certain musical structures and computers allow us to hear mathematical calculations and structures, i.e. to paraphrase Saccheri we have *Pythagoras ab omni naevo vindicatus sive Conatus arithmeticus quo stabiliuntur prima ipsa universa musica principia (Pythagoras freed from all taint or the arithmetical attempt to establish the first principles of all music*). we can continue to agree with Aristoxenus and accept that the justification of music lies in the pleasure of its hearing and its enjoyment.

Professor José Francisco Rodrigues has been elected **President of the Lisbon Academy of Sciences**

José Francisco Rodrigues, former President of CIM, has been elected President of the Lisbon Academy of Sciences (ACL) for the 2025-2026 term. The election took place at the Academy's plenary session on December 19, 2024.

José Francisco Rodrigues, a Full Professor at the Faculty of Sciences of the University of Lisbon and a researcher at its Centre for Mathematics (CMAFcIO), has extensive experience in mathematical research and a keen interest in the history and dissemination of science.

In his acceptance speech, José Francisco emphasised the ACL's role as a public utility institution and the need to promote scientific research, cultural enrichment, and collaboration with educational institutions. He also mentioned the upcoming 250th anniversary of the ACL in 2029 and the importance of celebrating this milestone with initiatives that promote the development of Science and Portuguese Culture.

The new President also highlighted the importance of interdisciplinarity in research, citing the first joint session of the Classes of Sciences and Letters in 2025, dedicated to Artificial Intelligence in Scientific Discovery and its Impact on Science, as an example.

Concluding his speech, Professor Rodrigues invoked Mário Soares, on the centenary of his birth, sharing his vision that the Academy should aspire to be "the most expressive forum of the best Portuguese intelligence," as it was at its foundation and in other high points of its history.

With this election, the Lisbon Academy of Sciences reaffirms its commitment to society, strengthening its position as one of the oldest and most important scientific institutions in the country.

LxDS Spring School 2024

by Telmo Peixe*,**

The LxDS Spring School 2024, organized by the LxDS-Lisbon Dynamical Systems Group in collaboration with CEMAPRE and CMAF-CIO, took place from May 27 to 29, 2024. The event was held at the Faculdade de Ciências, Universidade de Lisboa (FCUL). This spring school focused on various topics of dynamical systems, providing an opportunity for participants to enhance their knowledge through courses delivered by internationally recognized experts.

The school featured three comprehensive courses on dynamical systems, presented by distinguished scholars:

Arnold Diffusion through Geometric Methods
Professor Tere M-Seara
Universitat Politècnica de Catalunya

This course delved into the geometric methods used to study Arnold diffusion, a phenomenon in Hamiltonian systems where trajectories exhibit slow drift over long periods.

* On behalf of the organizing committee.
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Nonautonomous Dynamical Systems: Theory and Applications
Professor Peter Ashwin
University of Exeter

Professor Ashwin's course covered the theoretical foundations and practical applications of nonautonomous dynamical systems, which involve time-dependent changes in the system's parameters.

Topological and Ergodic Properties of Hyperbolic Flows
Professor Paulo Varandas
Universidade Federal da Bahia and CMUP

This course explored the topological and ergodic characteristics of hyperbolic flows, providing insights into their behavior and properties.

The event attracted around 20 participants, including speakers, organizers, PhD students from the Universidade do Porto and Universidade de Aveiro, master's students in mathematics from FCUL, and researchers with an interest in dynamical systems.

In addition to the courses, the school featured a session for oral presentations, where participants had the chance to present their latest research. Notable presentations included:

Qualitative Analysis of Prey Predator Model
Muhammad Ajaz
CMUP

Ajaz presented the complex dynamics of two-dimensional discrete-time predator-prey models, focusing on a modified Leslie–Gower model with prey harvesting.

A Dynamical Journey Around Double Standard Maps
Ana Rodrigues
Universidade de Évora

Rodrigues presented the results obtained so far for the investigation of the family of double standard maps

$$f_{a,b}(x) = 2x + a + \frac{b}{\pi}\sin(2\pi x) \pmod 1.$$

from topological results to ergodic theory.

Thanks to the financial support from CIM (Centro Internacional de Matemática), the school was able to cover the travel, lodging, and meal expenses for the participating PhD students. This support was crucial in facilitating their attendance and participation in the event.

The LxDS Spring School 2024 was a successful event that provided valuable learning and networking opportunities for all attendees. The collaboration between LxDS-Lisbon Dynamical Systems Group, CEMAPRE, and CMAF-CIO, along with the support from CIM, ensured a productive and enriching experience for everyone involved. The courses and presentations highlighted some of the latest advancements in dynamical systems, contributing to the growth and development of this field.

More information about the event can be found at https://sites.google.com/view/lxds-ss-2024/

The organizing committee:

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Universidade de Lisboa, ISEG, CEMAPRE

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The Generalized Fermat Equation

by **Nicolas Billerey*** and **Nuno Freitas****

Abstract.—In this note we will review the main steps in the proof of Fermat's Last Theorem and discuss Darmon's program to tackle the generalized Fermat equation $Ax^q + By^r = Cz^p$. Finally, we discuss how combining the classical approach with some ideas of Darmon led to recent results for equations of the form $x^r + y^r = Cz^p$.

1 Introduction

After Wiles' proof [27] of Fermat's Last Theorem (FLT) attention shifted towards the so-called generalized Fermat equation (GFE)

$$Ax^r + By^q = Cz^p \quad \text{with} \quad \mathcal{X} := \frac{1}{r} + \frac{1}{q} + \frac{1}{p} < 1, \quad (1.1)$$

where A, B, C are fixed non-zero coprime integers and $r, q, p \geq 2$ are integers. The triple (r, q, p) is called the *signature* of the GFE. A solution $(a, b, c) \in \mathbb{Z}^3$ to (1.1) is called *primitive* if $\gcd(a, b, c) = 1$ and *non-trivial* if $abc \neq 0$.

The condition $\mathcal{X} < 1$ is required to guarantee finiteness of solutions. More precisely, Darmon and Granville [13] proved that if we fix both the coefficients A, B, C and the exponents r, q, p satisfying $\mathcal{X} < 1$ then there are only finitely many primitive solutions to (1.1). But more is conjectured (see [4]): it is expected that the number of primitive solutions remains finite if we fix the coefficients but allow the three exponents to vary while still verifying $\mathcal{X} < 1$. On the other hand, if $\mathcal{X} > 1$ then the set of solutions is either empty or infinite by a result of Beukers [3] and, for $\mathcal{X} = 1$, the problem reduces to the determination of rational points on genus-1 curves. A very natural question is whether the strategy that proved FLT, which is now known as *the modular method*, can be used to establish more cases of the aforementioned conjecture.

As we shall see below, to apply the modular method to other instances of (1.1) one needs to start with the construction of a Frey curve. However, there are only a few choices of the exponents r, q, p in (1.1) for which Frey curves are known (see [10, p.14] for a complete list of rational Frey curves). To circumvent this issue, Darmon described in [11] a remarkable program to study (1.1) where he replaces Frey curves by higher dimensional abelian varieties. However, applying the rest of his program is very challenging because several of the main steps rely on open conjectures.

The objective of this expository note is to briefly discuss some recent results regarding the subfamily of (1.1) of the shape $x^r + y^r = Cz^p$ obtained by combining the classical approach with Frey curves and some of the ideas in the Darmon's program. For a brief introduction to Diophantine equations including a quick discussion of the modular method we refer the reader to [22].

2 Elliptic curves

For this section, the main reference is [24].

Let K be a field. An *elliptic curve* E over K is a smooth curve in $\mathbb{P}^2$ given by an equation

$$y^2z + a_1xyz + a_3yz^2 = x^3 + a_2x^2z + a_4xz^2 + a_6z^3,$$

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Nicolas Billerey is supported by the ANR-23-CE40-0006-01 Gaec project. We thank the referee for a careful reading and helpful remarks.

with $a_i \in K$. If the characteristic of K is not 2 or 3, then we can transform to a much simpler model given by the affine equation

$$Y^2 = X^3 + aX + b, \quad \Delta_E = -16(4a^3 + 27b^2) \neq 0,$$

where a and $b \in K$. There is a distinguished K-point, the 'point at infinity', which we denote by ∞. Given a field $L \supseteq K$, the set of L-points on E is

$$E(L) = \{(x, y) \in L^2 \; : \; y^2 = x^3 + ax + b\} \cup \{\infty\}.$$

It turns out that the set $E(L)$ has the structure of an abelian group with ∞ as the identity element. The group structure is easy to describe geometrically: three points $P_1, P_2, P_3 \in E(L)$ add up to the identity element if and only if there is a line ℓ defined over L meeting E in P_1, P_2, P_3 (with multiplicities counted appropriately). The classical Mordell–Weil Theorem states that for a number field K the group $E(K)$ is finitely generated.

Now suppose $K = \mathbb{Q}$. There is an integer N_E called *the conductor of* E with the following properties. There is an algorithm to compute N_E and, for all primes $p \nmid N_E$, the reduction modulo p of a minimal model for E gives an elliptic curve $\tilde{E}$ over $\mathbb{F}_p$. Moreover, if a prime $p \mid N_E$ then it divides the discriminant of any model for E so the reduced curve $\tilde{E}/\mathbb{F}_p$ is not an elliptic curve, and we can think of N_E as a measure of how 'complicated' these reduced curves are. Finally, for $p \nmid N_E$, the set $\tilde{E}(\mathbb{F}_p)$ is necessarily finite, and we define

$$a_p(E) = p + 1 - \#\tilde{E}(\mathbb{F}_p).$$

3 Modular forms

For this section, the main reference is [14].

Let $N \in \mathbb{Z}_{\geq 1}$. A *modular form of weight* 2 *for* $\Gamma_0(N)$ is an analytic function on the complex upper half-plane H satisfying suitable growth conditions at the boundary as well as the transformations

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^2 f(z)$$

for all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ satisfying $c \mid N$ and all $z \in$ H. Invariance under translation by 1 leads to a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n(f) q^n, \quad q = e^{2\pi i z}.$$

The group $\Gamma_0(N)$ acts on H via fractional linear transformations and the quotient $Y_0(N) = \Gamma_0(N)\backslash$H has the structure of a non-compact Riemann surface. This has a standard compactification denoted $X_0(N)$ and the difference $X_0(N) - Y_0(N)$ is a finite set of points called the *cusps*. To the modular forms that vanish at all the cusps we call *cusp forms*; in particular, they satisfy $a_0(f) = 0$.

The space of cusp forms $S_2(N)$ is a finite dimensional $\mathbb{C}$-vector space. There is a natural family of commuting operators $T_n \; : \; S_2(N) \to S_2(N)$ (with $n \geq 1$) called the *Hecke operators*. The *eigenforms* of level N are the cusp forms that are simultaneous eigenvectors for all the Hecke operators. An eigenform f is called *normalized* if $a_1(f) = 1$ and thus its Fourier expansion has the form

$$f = q + \sum_{n\geq 1} a_n(f) q^n.$$

Shimura-Taniyama-Weil Conjecture asserts that for every elliptic curve $E/\mathbb{Q}$ with conductor N_E there is a normalized eigenform f of weight 2 for $\Gamma_0(N_E)$, such that for every prime p the corresponding Fourier coefficient satisfies $a_p(f) = a_p(E)$. When this is the case we say that the curve E is *modular*. In his seminal paper [27] and its companion [26] (jointly with R. Taylor), Andrew Wiles proved the S-T-W Conjecture in the case of *semistable* elliptic curves, i.e. elliptic curves with square free conductor N_E. This groundbreaking theorem was also the final step to complete the proof of FLT.

4 Galois representations

For this section, the main references are [14, Chapter 9] and (for more advanced readers) [7].

Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ inside $\mathbb{C}$. We write $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ for the group of field automorphisms of $\overline{\mathbb{Q}}$ (fixing $\mathbb{Q}$). The group $G_{\mathbb{Q}}$ is called *the absolute Galois group of* $\mathbb{Q}$. The representations of $G_{\mathbb{Q}}$ are central objects in Arithmetic Geometry. Here we will work only with *residual* Galois representations, also known as *mod p* representations.

Definition 1.— A *mod p Galois representation* is defined to be a group homomorphism

$$\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

which is continuous with respect to the profinite topology on the left and the discrete topology on the right. In particular, there is a finite extension $\mathbb{F}_q/\mathbb{F}_p$ such that the image of $\overline{\rho}$ lies in $\mathrm{GL}_2(\mathbb{F}_q)$.

Definition 2.— A mod p Galois representation $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is *unramified at a prime* $\ell \neq p$

if $\overline{\rho}(I_\ell) = \{1\}$, where I_ℓ is an inertia group at ℓ in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Otherwise, it is *ramified at* ℓ.

The reader unfamiliar with the inertia subgroups of $G_\mathbb{Q}$ should simply keep in mind that there is a unique (up to conjugation) inertia subgroup for each prime ℓ and that a representation $\overline{\rho}$ is easier to understand if it has little ramification. Further, there is a positive integer $N(\overline{\rho})$, called *the Serre level of* $\overline{\rho}$, that measures the ramification of $\overline{\rho}$ at all primes $\ell \neq p$. Moreover, by Galois theory, the kernel of a representation $\overline{\rho}$ as above corresponds to a field extension of finite degree which is ramified at a prime ℓ if and only if $\overline{\rho}$ is ramified at ℓ.

4.1 Representations from elliptic curves

Let E be an elliptic curve over $\mathbb{C}$. The structure of the abelian group $E(\mathbb{C})$ is particularly easy to describe. There is a discrete lattice $\Lambda \subset \mathbb{C}$ of rank 2 (that is, as an abelian group $\Lambda \simeq \mathbb{Z}^2$) depending on E, and an isomorphism

$$E(\mathbb{C}) \simeq \mathbb{C}/\Lambda.$$

Let p be a prime. By the p-torsion of $E(\mathbb{C})$ we mean the subgroup

$$E[p] = \{Q \in E(\mathbb{C}) \;:\; pQ = 0\}.$$

It follows that $E[p] \simeq (\mathbb{Z}/p\mathbb{Z})^2$ which can be viewed as a 2-dimensional $\mathbb{F}_p$-vector space. Now let E be an elliptic curve over $\mathbb{Q}$. Then we may view E as an elliptic curve over $\mathbb{C}$, and with the above definitions obtain an isomorphism $E[p] \simeq (\mathbb{Z}/p\mathbb{Z})^2$. However, in this setting the points of $E[p]$ have algebraic coordinates, and are acted on component-wise by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Thus we obtain a 2-dimensional representation depending on $E/\mathbb{Q}$ and the prime p:

$$\overline{\rho}_{E,p} \;:\; G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{F}_p),$$

called *the mod p representation attached to E*. We say that $\overline{\rho}_{E,p}$ is *irreducible* if the image $\overline{\rho}_{E,p}(G_\mathbb{Q})$ cannot be conjugated into a subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$ consisting of upper triangular matrices.

4.2 Representations from modular forms

Let $f = \sum_{n\geq 1} a_n(f)q^n$ be a weight-2 normalized eigenform for $\Gamma_0(N)$ with $N \geq 1$. Denote by $K_f = \mathbb{Q}(\{a_n(f) \;:\; n \geq 1\})$ the field generated by the Fourier coefficients of f. It is a non-trivial theorem that $a_n(f)$ are algebraic integers and K_f is a number field, which we view as a subfield of $\overline{\mathbb{Q}}$. We denote by $\mathcal{O}_{K_f}$ the ring of integers of K_f, and we have $a_n(f) \in \mathcal{O}_{K_f}$ for all n; we refer to [14, §6.5] for details.

Let p be a prime number, and $\mathfrak{p}$ a prime in K_f above p. We write $\mathbb{F}_\mathfrak{p} = \mathcal{O}_{K_f}/\mathfrak{p}$ for the residue field at $\mathfrak{p}$. The following is a consequence of a deep result proved by Eichler and Shimura.

Theorem 3 (Eichler–Shimura).— Up to isomorphism, there is a unique semisimple mod p Galois representation

$$\overline{\rho}_{f,\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_\mathfrak{p})$$

satisfying the following properties: it is unramified outside Np and for every prime $\ell \nmid Np$, the characteristic polynomial of $\overline{\rho}_{f,p}(\mathrm{Frob}_\ell)$ is the mod $\mathfrak{p}$ reduction of

$$X^2 - a_\ell(f)X + \ell. \tag{4.1}$$

Here Frob_ℓ denotes a choice of a Frobenius element at ℓ in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and by semisimple we mean that $\overline{\rho}_{f,\mathfrak{p}}$ is either irreducible or isomorphic to the sum of two characters.

Definition 4.— A mod p Galois representation

$$\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

is said to be *modular of level* $N \geq 1$ if there exists a weight-2 eigenform f for $\Gamma_0(N)$ and a prime $\mathfrak{p} \mid p$ in K_f such that $\overline{\rho} \simeq \overline{\rho}_{f,\mathfrak{p}}$. In this case, we also say that $\overline{\rho}$ *arises from* f.

Building on the groundbreaking work of Wiles' and many others, Khare and Wintenberger [17, 18] have proved the following theorem known as Serre's Conjecture.

Theorem 5.— Let $\overline{\rho} : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be an irreducible odd representation. Assume that $\overline{\rho}$ arises from a finite flat group scheme at p. Then $\overline{\rho}$ is modular of level $N(\overline{\rho})$ and weight 2.

The technical condition that $\overline{\rho}$ arises from a finite flat group scheme at p should, for simplicity, be thought informally as the restriction of $\overline{\rho}$ to an inertia subgroup at p being *well behaved*; recall that ramification at $\ell \neq p$ is measured by $N(\overline{\rho})$.

5 Proof of FLT

For this section, the main references are [9] and [12].

We have introduced the minimal set of tools to sketch the proof of FLT. We decided to organize the proof in three main steps because these are the steps that we will focus on when presenting the Darmon program in the later sections.

Step 1—Construction: Suppose $p \geq 5$ is prime, and a, b and c are non-zero coprime integers satisfying $a^p + b^p = c^p$. We can reorder (a, b, c) so that

$$b \equiv 0 \pmod 2 \quad \text{and} \quad a^p \equiv -1 \pmod 4.$$

We consider the *Frey–Hellegouarch curve* which depends on (a, b, c):

$$E \ : \ Y^2 = X(X - a^p)(X + b^p). \tag{5.1}$$

From all the hypotheses on a, b, c, we compute the minimal discriminant and conductor of E:

$$\Delta = \frac{(abc)^{2p}}{2^8} \neq 0, \qquad N_E = \prod_{\ell \mid \Delta} \ell.$$

Note that the conductor is square-free and satisfies $2 \,||\, N$.

Step 2—Residual modularity: As $p \geq 5$, it follows from the work of Mazur [21] that $\overline{\rho}_{E,p}$ is irreducible. It is well known that $\overline{\rho}_{E,p}$ is odd and Hellegouarch showed that $\overline{\rho}_{E,p}$ arises on a finite flat group scheme at p. Computing the Serre level we obtain $N(\overline{\rho}_{E,p}) = 2$. Therefore, by Serre conjecture, we have that

$$\overline{\rho}_{E,p} \simeq \overline{\rho}_{g,\mathfrak{p}}$$

where g is an eigenform of level 2 and weight 2, and $\mathfrak{p} \mid p$ is a prime in K_g.

Step 3—Contradiction: There are no eigenforms of weight 2 and level 2, a contradiction.

Remark 1.— Note that the Frey curve construction applies for trivial solutions as well. However, in this case, it does not give rise to an elliptic curve (as it is singular), therefore, there are no modular representations associated with it. This is a fortunate feature of the classical Fermat equation. We will see below that this is no longer the case for the GFE which obstructs its resolution in many cases.

Remark 2.— The reader may be wondering where is Wiles' work used in the previous proof. Since the original proof of FLT predates the proof of Serre's conjecture, modularity of the residual representation $\overline{\rho}_{E,p}$ was instead derived as a corollary of modularity of the Frey curve E. Note that E has square-free conductor hence it is modular by the work of Wiles. We note also that the work of Wiles and all the ideas around it is heavily used in the proof of Serre's conjecture.

6 Darmon's program

As we see from the proof of FLT it is the modularity together with the little ramification of the 2-dimensional residual representation $\overline{\rho}_{E,p}$ that is key for the contradiction. The Frey curve E is simply a geometric object from which we know how to extract a 2-dimensional Galois representation with the right properties, namely $\overline{\rho}_{E,p}$.

There are higher dimensional generalizations of elliptic curves, called *abelian varieties*, in the sense that there is a group structure on the set of points of an abelian variety A. The main idea of Darmon's program is to put the focus directly on 2-dimensional mod p representations with the correct properties, and find the abelian varieties giving rise to them.

Definition 6.— Let $r, q, p \geq 2$ be integers. A *Frey representation* of signature (r, q, p) over a number field K in characteristic $\ell > 0$ is a Galois representation

$$\overline{\rho} = \overline{\rho}(t) \ : \ G_{K(t)} \to \mathrm{GL}_2(\mathbb{F})$$

where $\mathbb{F}$ is a finite field of characteristic ℓ such that the following conditions hold:

(i) The restriction of $\overline{\rho}$ to $G_{\overline{K}(t)}$ has trivial determinant and is irreducible.

(ii) The projectivization

$$\overline{\rho}^{\text{geom}} \ : \ G_{\overline{K}(t)} \to \mathrm{PSL}_2(\mathbb{F})$$

of this representation is unramified outside $\{0, 1, \infty\}$.

(iii) It maps the inertia groups at 0, 1, and ∞ to subgroups of $\mathrm{PSL}_2(\mathbb{F})$ of order r, q, and p respectively.

Here $K(t)$ is the function field over K in the variable t and $\overline{K}$ is an algebraic closure of K, and $G_k := \mathrm{Gal}(\overline{k}/k)$ denotes the absolute Galois group of k for any field k.

In [11], Darmon counts the number of Frey representations up to some equivalence relation (introduced in *loc. cit.*) and describes (often not in an explicitly way) where they should arise. In particular, he proves the following classification result.

Theorem 7 (Hecke-Darmon).— Up to equivalence, there is only one Frey representation of signature (p, p, p). It occurs over $\mathbb{Q}$ in characteristic p and is associated with the Legendre family

$$L(t) \ : \ y^2 = x(x - 1)(x - t).$$

Example 1.— It is not difficult to check that the classical Frey–Hellegouarch curve

$$y^2 = x(x - a^p)(x + b^p)$$

is obtained from $L(t)$ after *specialization* at

$$t_0 = \frac{a^p}{a^p + b^p}$$

and taking *quadratic twist* by $-(a^p + b^p)$.

A Frey representation $\overline{\rho}(t)$ should be seen as a family of representations where we can specialize the parameter t to obtain mod p representations of G_K as in the previous example. We are then interested in the modularity of the mod p representations obtained in this way.

From now on, we restrict ourselves to the case of K being *a totally real field*, i.e., a number field such that all embeddings into $\mathbb{C}$ have image in $\mathbb{R}$. This is a natural restriction, because modularity related objects are very poorly understood for fields with at least one complex embedding. In contrast, for a totally real K there is a well established theory of *Hilbert modular forms* (see [15]) which are the natural replacement for the modular forms over $\mathbb{Q}$; it is not our objective to discuss details of this theory here. The only thing to keep in mind is that they satisfy the analogous properties over K to those of modular forms over $\mathbb{Q}$. In particular, modularity of abelian varieties and their residual representations can be defined via a connection to representations arising from Hilbert eigenforms (see [25]). Therefore, we can state the following special case of Serre conjecture over totally real fields.

Conjecture 1 ([11, Conjecture 3.2]).— Let K be a totally real field. Let $\overline{\rho} : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be a totally odd and irreducible representation with determinant the mod p cyclotomic character.

Assume that $\overline{\rho}$ arises from a finite flat group scheme at all primes $\mathfrak{p}$ in K above p. Then there is a Hilbert eigenform g over K for $\Gamma_0(N(\overline{\rho}))$ of (parallel) weight 2 and a prime $\mathfrak{p} \mid p$ in the field of coefficients of g such that $\overline{\rho} \simeq \overline{\rho}_{g,\mathfrak{p}}$.

This conjecture is still open for all K, therefore when applying the Darmon program in the next section we need to derive residual modularity without it. Also, this conjecture is concerned with 2-dimensional representations whilst representations arising from abelian varieties of dimension n are naturally of dimension $2n$. We thus focus only on the subfamily of abelian varieties giving rise to 2-dimensional representations, as per the next definition and well known theorem.

Definition 8.— Let A be an abelian variety over a field L of characteristic 0. We say that A/L is of GL_2-type (or $\mathrm{GL}_2(F)$-type) if there is an embedding

$$F \hookrightarrow \mathrm{End}_L(A) \otimes_{\mathbb{Z}} \mathbb{Q}$$

where F is a number field with $[F : \mathbb{Q}] = \dim A$.

Theorem 9.— Let A/L be an abelian variety of $\mathrm{GL}_2(F)$-type. Let $\mathfrak{p}$ be a prime in F above p. Then there is a 2-dimensional mod p representation attached to A, denoted $\overline{\rho}_{A,\mathfrak{p}} : G_K \to \mathrm{GL}_2(\mathbb{F}_\mathfrak{p})$, unramified outside the primes where A has bad reduction and p.

Darmon also discusses the existence of Frey varieties $J(a,b,c)/\mathbb{Q}$ associated to solutions (a,b,c) of (1.1) for any choice of exponents, and explains how these give rise (after base changing to certain totally real number fields) to all the possible Frey representations. However, only the varieties for exponents (p,p,p) and (p,p,r) are explicit enough to work with. Finally, he finishes with the following extremely difficult conjecture [11, Conjecture 4.1].

Conjecture 2 (Large image conjecture).— Let K be totally real field. There exists a constant C_K such that, for any abelian variety A/K of GL_2-type with $\mathrm{End}_{\overline{K}}(A) \otimes \mathbb{Q} = K$, and all primes $\mathfrak{p}$ of K of norm $> C_K$, we have $\mathrm{SL}_2(\mathbb{F}_\mathfrak{p}) \subset \overline{\rho}_{A,\mathfrak{p}}(G_K)$.

We finish this section with the description of how the Darmon program is expected to work. We emphasize every step that we do not know how to do, or that depends on conjectures or relies on computations that are not possible in practice with current algorithms and hardware.

1. Let $a,b,c \in \mathbb{Z}$ satisfy $a^r + b^q = c^p$ and $\gcd(a,b,c) = 1$.
2. Let $J(a,b,c)/\mathbb{Q}$ be the associated Frey variety. Over a totally real field K it becomes of $\mathrm{GL}_2(K)$-type. We consider $J = J(a,b,c)/K$ and its mod $\mathfrak{p}$ representation $\overline{\rho}_{J,\mathfrak{p}}$ given by Theorem 9.
3. Assume $p > C_K$ where C_K is the constant in Conjecture 2. If (a,b,c) is non-trivial then $\mathrm{SL}_2(\mathbb{F}_\mathfrak{p})$ is *conjecturally* contained in the image of $\overline{\rho}_{J,\mathfrak{p}}$ by Conjecture 2. In particular, $\overline{\rho}_{J,\mathfrak{p}}$ is *conjecturally* irreducible.
4. The representation $\overline{\rho}_{J,\mathfrak{p}}$ is totally odd with cyclotomic determinant and *conjecturally* arises on a finite flat group scheme at all $\mathfrak{p} \mid p$ in K.
5. We *compute* the Serre level $N(\overline{\rho}_{J,\mathfrak{p}})$.

6. The representation $\overline{\rho}_{J,\mathfrak{p}}$ is *conjecturally* modular of level $N(\overline{\rho}_{J,\mathfrak{p}})$ and (parallel) weight 2 by Conjecture 1, that is $\overline{\rho}_{J,\mathfrak{p}} \simeq \overline{\rho}_{g,\mathfrak{p}}$ for some Hilbert eigenform g of level $N(\overline{\rho}_{J,\mathfrak{p}})$.

7. We *compute* the relevant space of eigenforms and *show* that $\overline{\rho}_{J,\mathfrak{p}} \not\simeq \overline{\rho}_{g,\mathfrak{p}}$ except for the eigenforms g_0 corresponding via modularity to the Frey varieties $J_0 := J(a,b,c)$ where (a,b,c) satisfies $abc = 0$ i.e. Frey varieties attached to trivial solutions.

8. *Conjecturally* the varieties J_0 have complex multiplication, thus $\mathrm{SL}_2(\mathbb{F}_\mathfrak{p})$ is not contained in the image of $\overline{\rho}_{g_0,\mathfrak{p}}$. Thus we also have $\overline{\rho}_{J,\mathfrak{p}} \not\simeq \overline{\rho}_{g_0,\mathfrak{p}}$, a contradiction with Step **6**.

In view of the three main steps in the proof of FLT, the previous bullet points are divided as follows: Step 1 corresponds to **1–2**, Step 2 corresponds to **3–6** and Step 3 corresponds to **7–8**.

To conclude this section, we note that the contradiction step which was trivial in the proof of FLT is quite challenging in this more general situation. As mentioned in Remark 1, the trivial solutions represent a major obstruction, but there are other issues. Namely, the space of revelant Hilbert modular forms might not be accessible with current software implementations (either because it is too large, or by lack of efficient algorithms in certain specific situations). Moreover, we miss a general method for discarding isomorphisms between residual Galois representations. In particular, it is an open problem to show that given two non-isogenous rational elliptic curves E, E', then for all large enough primes p, the representations $\overline{\rho}_{E,p}$ and $\overline{\rho}_{E',p}$ are not isomorphic.

7 Some recent results for signature (r, r, p)

We now discuss our contribution to the Darmon's program in the case of the generalized Fermat equation

$$x^r + y^r = Cz^n, \tag{7.1}$$

where r is a fixed prime ≥ 3, C is a fixed positive integer and $n \geq 2$ is an integer.

Throughout this paragraph, we fix the following notation.

- ζ_r primitive r-th root of unity
- $\omega_i = \zeta_r^i + \zeta_r^{-i}$, for every $i \geq 0$
- $h(X) = \prod_{i=1}^{(r-1)/2} (X - \omega_i) \in \mathbb{Z}[X]$
- $K = \mathbb{Q}(\zeta_r)^+ = \mathbb{Q}(\omega_1)$ maximal totally real subfield of $\mathbb{Q}(\zeta_r)$
- $\mathcal{O}_K$ integer ring of K
- $\mathfrak{p}_r$ unique prime ideal above r in $\mathcal{O}_K$ (totally ramified)

Let a, b be non-zero coprime integers such that $a^r + b^r \neq 0$. Following a construction of Kraus [19], we consider the curve $C_r(a,b)$ given by the equation

$$y^2 = (ab)^{\frac{r-1}{2}} x h\left(\frac{x^2}{ab} + 2\right) + b^r - a^r.$$

The discriminant of this model is

$$\Delta_r(a,b) = (-1)^{\frac{r-1}{2}} 2^{2(r-1)} r^r (a^r + b^r)^{r-1}$$

which is non-zero as $a^r + b^r \neq 0$. In particular, it defines a hyperelliptic curve of genus $(r-1)/2$.

Examples 1.— Here are explicit equations for Kraus' curve with $r = 3, 5, 7$.

$$r = 3: \quad y^2 = x^3 + 3abx + b^3 - a^3$$
$$r = 5: \quad y^2 = x^5 + 5abx^3 + 5a^2b^2x + b^5 - a^5$$
$$r = 7: \quad y^2 = x^7 + 7abx^5 + 14a^2b^2x^3 + 7a^3b^3x + b^7 - a^7.$$

The Jacobian $J_r(a,b)$ of the curve $C_r(a,b)$ is thus an abelian variety of dimension $(r-1)/2$. In particular, when $r > 3$, it has dimension > 1 and hence there is no obvious way to attach 2-dimensional Galois representations to $J_r(a,b)$.

To circumvent this issue we use ideas from Darmon's program as explained in the previous section. In particular, the theorem below shows how to recover Kraus' Frey hyperelliptic curve in a similar way as the usual Frey-Hellegouarch elliptic curve (see Example 1). This result achieves Steps 1–2 from the description of Darmon's program given in Section 6 in the case of equation (7.1).

Theorem 10 ([6]).— There exists a hyperelliptic curve $C_r'(t)$ over $K(t)$ of genus $\frac{r-1}{2}$ such that $J_r'(t) = \mathrm{Jac}(C_r'(t))$ is of $\mathrm{GL}_2(K)$-type.

Moreover, for every prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$ above a rational prime p, the representation

$$\overline{\rho}_{J_r'(t),\mathfrak{p}} : G_{K(t)} \to \mathrm{GL}_2(\mathcal{O}_K/\mathfrak{p})$$

is a Frey representation of signature (r, r, p).

The hyperelliptic curve $C_r(a,b)/K$ is obtained from $C_r'(t)$ after *specialization* at

$$t_0 = \frac{a^r}{a^r + b^r}$$

and taking the *quadratic twist* by

$$-\frac{(ab)^{\frac{r-1}{2}}}{a^r + b^r}.$$

In this result, it is crucial to notice that the prime p is arbitrary. In particular, if we choose $p = r$ (and hence $\mathfrak{p} = \mathfrak{p}_r$), then $\overline{\rho}_{J_r'(t),\mathfrak{p}_r}$ is a Frey representation of signature (r,r,r). According to Theorem 7, it arises in the Legendre family, allowing us to appeal to the stronger results available for the case of elliptic curves.

This is a key idea in Darmon's program that assuming an appropriate generalization of Serre's modularity conjecture for totally real fields (Conjecture 1), the mod $\mathfrak{p}_r$ representation is modular and plays the role of a 'seed' for modularity of all Frey varieties described by Darmon (see diagram in [11, p. 433]).

The result below makes this argument unconditional for the Kraus Frey variety - under some irreducibility assumption (which is proved to hold for many values of r such as $r = 7$ for instance) and parity conditions - hence completing Steps **3-6** in Darmon's program from Section 6 for equation (7.1).

Theorem 11.— Let (a,b,c) be a non trivial primitive solution to equation (7.1) for exponent $n = p$ prime such that $p \nmid 2rC$. Assume that

$$a \equiv 0 \pmod 2 \quad \text{and} \quad b \equiv 1 \pmod 4). \qquad (7.2)$$

Let J_r be the Jacobian of $C_r(a,b)$ base changed to K. Suppose further that $\overline{\rho}_{J_r,\mathfrak{p}}$ is absolutely irreducible. Then, there is a Hilbert newform g over K satisfying the following properties:

(i) g is of parallel weight 2, trivial character and level $2^2\mathfrak{p}_r^2\mathfrak{n}_C$;

(ii) $\overline{\rho}_{J_r,\mathfrak{p}} \simeq \overline{\rho}_{g,\mathfrak{P}}$ for some $\mathfrak{P} \mid p$ in the field of coefficients K_g of g;

(iii) for all $\mathfrak{q}_2 \mid 2$ in K, we have $(\rho_{g,\mathfrak{P}} \otimes \overline{\mathbb{Q}}_p)|_{I_{\mathfrak{q}_2}} \simeq \delta \oplus \delta^{-1}$, where δ is a character of order r;

(iv) $K \subset K_g$.

Moreover, if $\mathfrak{n}_C \neq 1$ then g has no complex multiplication.

Note that, contrary to the case of Fermat's last theorem, the 2-adic assumptions (7.2) in Theorem 11 are not valid in general; indeed, from the symmetry of (7.1), we can only swap a and b, so the possibility of c being even is excluded in the above theorem. We shall explain in the next section how several 'Frey varieties' can complement each other to obtain a complete resolution of certain generalized Fermat equations (7.1) for specific values of r and C.

8 Diophantine applications

In this section, we discuss the Steps **7–8** from Section 6 for the case $r = 7$ and $C = 3$ in the generalized Fermat equations (7.1). In this situation, we achieve the following complete result.

Theorem 12 ([5, Theorem 1.1]).— For all integers $n \geq 2$, there are no non-trivial primitive solutions to

$$x^7 + y^7 = 3z^n. \qquad (8.1)$$

First of all, we can reduce the problem of solving $x^7 + y^7 = 3z^n$ for $n \geq 2$ to the case where $n = p$ is prime and $p \geq 5$, $p \neq 7$, using simple arithmetic considerations and work of Bennett-Skinner [1] (for $n = 2$), Bennett-Skinner-Yazdani [2] (for $n = 3$) and Serre [23] (for $n = 7$).

In [5], we actually give three different proofs of Theorem 12 which rely on a 'multi-Frey' approach using a combination of Kraus' hyperelliptic curve $C_7(a,b)$ and two Frey elliptic curves $E/\mathbb{Q}$ and $F/\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ whose construction is due to Darmon and Freitas, respectively.

Our first proof uses the *classical* modular method outlined in the case of FLT in Section 5 with the two aforementioned Frey elliptic curves attached to equation (8.1).

- (Darmon, [20, §4.5.1.3]) A Frey curve over $\mathbb{Q}$:

  $$E_{a,b} : y^2 = x^3 + a_2x^2 + a_4x + a_6$$

  where

  $$\begin{aligned} a_2 &= -(a-b)^2, \\ a_4 &= -2a^4 + a^3b - 5a^2b^2 + ab^3 - 2b^4, \\ a_6 &= a^6 - 6a^5b + 8a^4b^2 - 13a^3b^3 + 8a^2b^4 - \\ &\quad -6ab^5 + b^6. \end{aligned}$$

- (Freitas, [16, p. 619]) A Frey curve over the totally real cubic field $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$:

  $$F_{a,b} : y^2 = x(x - A_{a,b})(x + B_{a,b}),$$

  where for $i = 1, 2$, we have $\omega_i = \zeta_7^i + \zeta_7^{-i}$ and

  $$\begin{aligned} A_{a,b} &= (\omega_2 - \omega_1)(a+b)^2 \\ B_{a,b} &= (2 - \omega_2)(a^2 + \omega_1 ab + b^2). \end{aligned}$$

We note here that Freitas' Frey elliptic curve $F = F_{a,b}$ is defined over a totally real field of degree > 1 and is not base change from $\mathbb{Q}$. In particular, its mod p representations are not explained by Darmon's classification of Frey representations of signature $(7, 7, p)$.

The total running time for this first proof is approximately 40 minutes with around 3/4 of this time devoted to computing the Hilbert newforms over $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ of parallel weight 2 and level $\mathfrak{q}_2^3\mathfrak{q}_3\mathfrak{q}_7$ (with $\mathfrak{q}_i$ the unique prime ideal above i in $\mathbb{Q}(\zeta_7+\zeta_7^{-1})$) used to deal with the case where ab is even and $7 \mid a + b$. There are precisely 121 such newforms generating a space of dimension 818, with coefficient fields of degree up to 18.

Our second and third proofs of Theorem 12 add in the use of Kraus' Frey hyperelliptic curve

$$C_7(a, b) : y^2 = x^7+7abx^5+14a^2b^2x^3+7a^3b^3x+b^7-a^7$$

in two different ways: the second proof uses $C_7(a, b)$ *as much as possible* whilst the third and last proof is designed to minimize the computational time among all proofs we give. The total running times for these proofs are approximatively 10 minutes and 1 minute respectively.

Our second proof is much more involved and requires introducing many new elimination techniques [6, §9] to discard the isomorphism in Theorem 11((ii)). To illustrate the computational challenges we have faced, let us mention that we had to compute here in the space of Hilbert newforms of level $\mathfrak{q}_2^2\mathfrak{q}_3\mathfrak{q}_7^2$ which has dimension 698. This dimension is comparable in size with that of the space considered in the first proof, but it turns out to be much faster to initialize yielding only 61 newforms. Some of these forms have coefficient field of degree as large as 54 making the elimination procedure considerably more difficult. Fortunately, we are able to reduce the number of newforms to consider down to 25 using the condition $K \subset K_g$ from Theorem 11((iv)). As explained in [5] this 'instantaneous reduction' is only available when working with abelian varieties of dimension > 1. Moreover, we also developed a collection of techniques to speed up the elimination procedure resulting in a great saving in the total running time; see [5, §7]. While this approach a priori requires harder and lengthier computations, it ends up allowing for a faster proof than the previous one.

Our third and last proof builds on the two previous ones. Combining information about the Frey (hyper)elliptic curves introduced above and their twists we manage to lower down to 104 the dimension of the largest space we have to consider. Then we apply the techniques explained for the second proof to deal with the corresponding 19 newforms (whose coefficient fields are all of degree ≤ 15) yielding the most efficient proof in less than a minute. This illustrates how the additional structures carried by the Frey varieties of dimension > 1 can be exploited to reduce computations, despite the fact that we have to work with Jacobians of hyperelliptic curves.

Finally, let us point out that these methods have already been applied to other Fermat-type equations to obtain results not within reach of the classical approach with Frey elliptic curves. In the case of $r = 11$ in (7.1), we refer the reader to [6] and for signature $(p, p, 5)$ to the recent preprint of Chen and Koutsianas [8].

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Professor Luis Nunes Vicente has been selected as a **Fellow of SIAM**

Luis Nunes Vicente, member of the scientific council of CIM, has been selected as a Fellow of the Society for Industrial and Applied Mathematics (SIAM). SIAM has over 14,000 individual members in all areas of industrial and applied math. The SIAM Fellows Program recognises members of SIAM who have made outstanding contributions to fields served by the industrial and applied math community. According to his citation, Luis was selected for ground-breaking contributions to derivative-free and bilevel optimisation, and exemplary leadership in editorial and organisational service to the SIAM community.

Luis serves as Professor and Chair at Lehigh University. He is a leading researcher in continuous optimisation and its applications to industrial engineering and operations research, he received his PhD from Rice University in 1996 under a Fulbright scholarship and was awarded the Lagrange Prize of MOS-SIAM in 2015. He has held visiting positions at institutions including IBM Research, NYU, Rice, CERFACS, and Rome Sapienza, and has served on the editorial boards of the SIAM Journal on Optimization and Mathematical Programming. He currently chairs the SIAM Activity Group on Optimization (2023-2025) and ACORD (2023),

Luís commented this distinction saying: “It is a great honor to be elected a Fellow of SIAM. I have always had tremendous admiration for SIAM, from all the work done to promote the field and the profession to all its journals, conferences, and awards. I never thought I would reach this level, and my takeaway is to continue to work hard in my research and serve others in the community.”

Integrability and Moduli
A conference in honor of Leon Takhtajan

by Carlos Florentino*

The international conference *Integrability and Moduli*, in honor of Emeritus Professor and renowned mathematician Leon Takhtajan, took place this summer at the Faculty of Sciences of the University of Lisbon. From the 8th to the 12th of July 2024, the conference gathered many of Takhtajan's collaborators and co-authors, and some experts from all around the globe, on the topics of his long research career: integrable systems, quantization, moduli spaces, conformal field theory, representation theory, quantum groups, to name just a few.

This event had been initially scheduled for 2020, the year of Takhtajan's 70th birthday, with the Euler Mathematical Institute (EMI) in Saint Petersburg as the venue, but had to be cancelled (two years in a row) due to the Covid-19 pandemic. Moreover, given the unfortunate travel restrictions on many mathematicians coming from the Russian Federation, after 2022 it was clear that the conference would have to take place elsewhere.

Upon contacting the initial organizers and scientific committee, formed by professors Samson Shatashvili (Trinity College Dublin, Ireland), Fedor Smirnov (Sorbonne University, CNRS, France) and Peter Zograf (EMI, Russia), my suggestion of organizing the event in Lisbon was accepted, and the four of us successfully submitted it to the European Mathematical Society to become a Satellite Conference of the 9th European Congress of Mathematicians (ECM 2024), held this summer in Seville, Spain.

After securing support from CMAFcIO and GFM (U. Lisbon), CIM and FCT (Portugal) and the Hamilton Mathematical Institute (Ireland), whom we gratefully thank, we were certain that the conference would be an astounding success. Indeed, we counted with eighteen great one-hour presentations by mathematicians coming from Europe, North America and Asia: A. Alekseev (Univ. Geneva), C. Choi (Perimeter Inst., Ontario), G. Cotti (Univ. Lisbon), R. Dey (ICTS Bangalore), P. Gothen (Univ. of Porto), A. Its (Indiana Univ.), V. Korepin (SUNY Stony Brook), A. Laptev (Imperial College), M. Mariño (Univ. Geneva), C. Meneses (Kiel Univ.), J. Park (KIAS Seoul), V. Pingali (IISC Bangalore), E. Sklyanin (Univ. York), M. Stošić (IST, Lisbon), D. Sullivan (SUNY Stony Brook), A. Veselov (Loughborough Univ.), L. Weng (Kyushu Univ.), P. Wiegmann (Univ. Chicago).

The conference had around 60 participants, including some PhD students, and also served to foster further collaborations in the areas of Geometry, Topology and Mathematical Physics, and to announce new research lines and open problems that may result in future active investigations. All information (including many of the slides of the seminars) is available at the website:

https://sites.google.com/view/integrabilityandmoduli2024/

* FCUL and CEMS.UL

Professor Jorge Buescu was awarded the **Grande Prémio Ciência Viva 2024**

Jorge Buescu, a mathematician from the Faculty of Sciences of the University of Lisbon (FCUL), has been awarded the "Grande Prémio Ciência Viva 2024". This prestigious award, the highest category of the Ciência Viva Prizes, recognises individuals and organisations for their outstanding contributions to scientific and technological dissemination.

Jorge Buescu, who holds a degree in Physics from FCUL and a PhD in Mathematics from the University of Warwick (1995), is currently an Associate Professor with Habilitation in the Department of Mathematics at FCUL.

He is a passionate science communicator, has authored over two hundred pedagogical and outreach articles, many of which stem from his three-decade-long collaboration with the Ordem dos Engenheiros magazine, Ingenium, where he maintains a regular science column. Jorge emphasises the importance of sharing science, stating, "I have always felt the need to share science. Communicating, disseminating, showing others the beauty that exists in what we, scientists, do and which is often not visible from the outside." He further adds, "Communicating and disseminating science rigorously is an increasingly important task. Scientific dissemination inspires new generations, sparking curiosity and showing the relevance of science. But, in addition, it promotes the growth of a healthier society, as well-informed citizens are better able to demand actions that promote social and environmental well-being."

Regarding receiving the award, Jorge expressed, "It is a great joy for me, for many reasons, to receive this Grande Prémio Ciência Viva. It is, above all, a recognition that the great passion that moves me is truly worth living."

In addition to the Grande Prémio Ciência Viva, Jorge Buescu has also been awarded the Rómulo de Carvalho Prize for Research and Dissemination, was appointed a Corresponding Member of the Ordem dos Engenheiros, was elected an honorary member of the Real Sociedad Matemática Española, and serves as Vice-President of the European Mathematical Society, the first Portuguese to hold this position.

Perspectives in Representation Theory

by Teresa Conde*, Samuel Lopes**, Ana Paula Santana*** and Ivan Yudin***

The international conference Perspectives in Representation Theory took place at the University of Coimbra, July 1–3, 2024, and it was a Satellite meeting of the 9th European Congress of Mathematics, Sevilla 2024 (see https://ecm2024sevilla.com).

The meeting was sponsored by the research centers CMUC (Centro de Matemática da Universidade de Coimbra) and CMUP (Centro de Matemática da Universidade do Porto), FCT (Fundação para a Ciência e a Tecnologia), DMUC (Departamento de Matemática da Universidade de Coimbra), CIM (Centro Internacional de Matemática) and EMS (European Mathematical Society).

The goal of this conference was to bring together mathematicians working on different aspects of representation theory and to foster collaboration, thriving to engage early career researchers as well as students, thus providing a platform for emerging mathematicians to showcase their work, exchange ideas, and connect with other experts in the field.

The conference program included two 150-minute Lecture Series, ten 50-minute Plenary Lectures, as well as 25-minute Contributed Talks and a Poster Session. The topics covered ranged from the representation theory of groups, Lie algebras and their deformations, to module theory of rings, homological algebra and more categorial aspects of representation theory.

More information can be found at

http://www. mat.uc.pt/~crt/prt2024/

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Partially supported by CMUP, member of LASI, which is financed by national funds through FCT, under the projects with reference UIDB/00144/2020 and UIDP/00144/2020.
*** CMUC, Universidade de Coimbra (funded by the Portuguese Government through FCT/MCTES, DOI 10.54499/UIDB/00324/2020).

The 14th Combinatorics Days

by **Cláudio Piedade***, **Fátima Rodrigues****, **Inês Rodrigues****, **Olga Azenhas***** and **Samuel Lopes***

The *Combinatorics Days* is an itinerant annual conference series that brings together mathematicians working in Combinatorics, widely interpreted, and related fields such as Algebra, Geometry, Probability, Computer Science or Physics.

http://www.mat.uc.pt/~combdays/14thcombday.html

The 14th edition of *Combinatorics Days* was hosted by Universidade NOVA de Lisboa, June 27–29, 2024. The programme consisted of two mini courses, one by Mercedes Rosas (University of Sevilla), on *Vector partition function in representation theory*, and another by Travis Scrimshaw (Hokkaido University), on *(K-theoretic) Schubert calculus and stochastic processes*. Additionally, there was a plenary lecture by Persi Diaconis (Stanford University), on *An Introduction to Computational Polya Theory*, and twelve diverse thirty-minute oral presentations.

* CMUP, FCUP.
** NOVA Math, NOVA FCT.
*** CMUC, UC.

Lie theory and the eightfold way

by **John Huerta***

Abstract.—This short expository note aims to give the minimal amount of Lie theory needed to appreciate the eightfold way in particle physics. We first give a quick but complete account of the finite-dimensional irreducible representations (irreps) of $\mathfrak{sl}(2,\mathbb{C})$. Then we sketch how the theory generalizes to the irreps of $\mathfrak{sl}(3,\mathbb{C})$, and close by gesturing at the role of these irreps in the eightfold way.

1 Introduction

Consider the space of traceless, 2×2 complex matrices, which we denote $\mathfrak{sl}(2,\mathbb{C})$:

$$\mathfrak{sl}(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \ : \ a,b,c \in \mathbb{C} \right\}.$$

This is a vector space, but more importantly, it is a Lie algebra—it is closed under the Lie bracket, defined in this case to be the commutator:

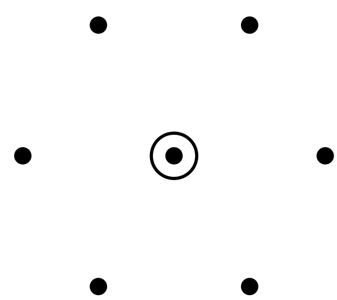
$$[X,Y] = XY - YX, \text{ for } X,Y \in \mathfrak{sl}(2,\mathbb{C}).$$

Lie algebras, like their cousins Lie groups, encode *symmetries*. In the case of $\mathfrak{sl}(2,\mathbb{C})$, for instance, each element acts on the vector space $\mathbb{C}^2$ as a linear transformation, and we think of such a transformation as a symmetry of $\mathbb{C}^2$. This situation, where a Lie algebra acts on a vector space, is the subject of Lie theory. The vector space equipped with such an action is called a 'representation'.

The representations of $\mathfrak{sl}(2,\mathbb{C})$ on finite-dimensional vector spaces are startling in their simplicity. Though there are infinitely many such representations, each one can be decomposed into simpler pieces, called the 'irreducible representations', or irreps, and each irrep is uniquely determined by the choice of a single natural number. Curiously, we will see that it is useful to describe the irrep associated to the natural number m with a diagram consisting of $2m+1$ dots arranged in a line, labeled from $-m$ to m in increments of 2:

This picture is our first hint that irreps are surprisingly discrete, being determined by points in a lattice, here just the integers. They also exhibit some hidden symmetry, which we see here by the symmetry between m and $-m$.

Passing to the next case of $\mathfrak{sl}(3,\mathbb{C})$, the special linear Lie algebra on $\mathbb{C}^3$, reveals even more structure. Again, the irreducible representations, or irreps, are determined by points in a lattice with some symmetry:



Now, however, the lattice is two-dimensional, and the symmetry group is bigger, generated by all the reflection symmetries manifest in the picture. The circle around the center dot tells us to count it with multiplicity two.

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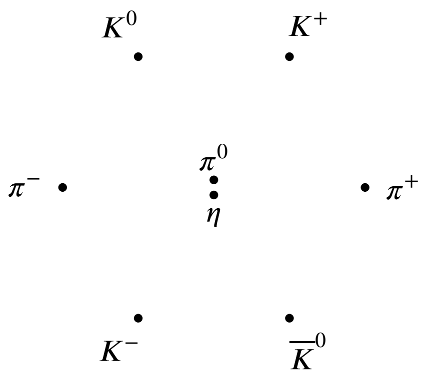
This representation theory is pure linear algebra, but it magically appears in particle physics, where it is part of the 'eightfold way'. In the middle of the 20th century, experiments where physicists collided particles together at high energies produced scores of a new particles beyond the familiar electron, proton and neutron that constitute atoms. Searching for order in the chaos, physicists discovered that these new particles could be organized into representations of $\mathfrak{sl}(3, \mathbb{C})$:



Acknowledgments

I thank Gonçalo Oliveira for his encouragement and patience. As a member of CAMGSD, this work was funded by FCT/Portugal through project UIDB/04459/2020. This article was written while visiting the Mathematical Sciences Institute at the Australian National University, and I thank the MSI for their hospitality.

2 Lie algebras

To get started, let us give some precise definitions of the objects we want to study, namely Lie algebras and their representations. Although we could work over any field, we choose to work over the complex numbers, $\mathbb{C}$. Thanks to $\mathbb{C}$ being algebraically closed, every matrix has an eigenvalue. Since computing eigenvalues and diagonalizing matrices will play an essential role in our analysis, it pays to work over $\mathbb{C}$.

Definition 1.— A **Lie algebra** $\mathfrak{g}$ is a complex vector space equipped with a bilinear operation called the Lie bracket

$$[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g},$$

satisfying the following axioms:

- skew-symmetry: $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$;
- the Jacobi identity: $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$, for all $X, Y, Z \in \mathfrak{g}$.

Example 1.— We have already met two examples of our favorite Lie algebra in the Introduction, namely the **special linear Lie algebra** of traceless $n \times n$ complex matrices:

$$\mathfrak{sl}(n, \mathbb{C}) = \left\{ X \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \ : \ \mathrm{tr}(X) = 0 \right\},$$

where the Lie bracket is given by the commutator:

$$[X, Y] = XY - YX, \quad X, Y \in \mathfrak{sl}(n, \mathbb{C}).$$

It is a worthwhile if somewhat tedious exercise to check that the Jacobi identity holds.

Example 2.— Dropping the condition on the trace, we have the **general linear Lie algebra** of all $n \times n$ complex matrices:

$$\mathfrak{gl}(n, \mathbb{C}) = \mathrm{Mat}_{n\times n}(\mathbb{C}).$$

Again, the Lie bracket is given by the commutator.

Example 3.— More abstractly, for any complex vector space V, we have the **general linear Lie algebra on V**, consisting of all linear maps:

$$\mathfrak{gl}(V) = \{T : V \to V \ : \ T \text{ linear}\}.$$

Once again, the Lie bracket is given by the commutator. Of course, fixing a basis of V gives us an isomorphism of Lie algebras, $\mathfrak{gl}(V) \cong \mathfrak{gl}(n, \mathbb{C})$, where $n = \dim(V)$.

Definition 2.— Let $\mathfrak{g}$ be a Lie algebra. A **representation** of $\mathfrak{g}$ is a pair (V, ρ) where V is a finite-dimensional complex vector space, and $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a homomorphism of Lie algebras. Explicitly, this means that ρ is a linear map such that

$$\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X), \text{ for all } X, Y \in \mathfrak{g},$$

since the bracket on $\mathfrak{gl}(V)$ is the commutator.

Example 4.— Every Lie algebra $\mathfrak{g}$ has a god-given representation on itself, called the **adjoint representation**, $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. An element $X \in \mathfrak{g}$ acts on $Y \in \mathfrak{g}$ by bracketing: $\mathrm{ad}(X)Y = [X, Y]$.

It is a useful exercise to check that this is a representation. The Jacobi identity will play a key role.

3 The representations of $\mathfrak{sl}(2, \mathbb{C})$

We will study the representations of the complex special linear Lie algebra, $\mathfrak{sl}(n, \mathbb{C})$. In fact, we are mainly interested in $\mathfrak{sl}(3, \mathbb{C})$, because of the role it plays in particle physics. But to get started, we need to study $\mathfrak{sl}(2, \mathbb{C})$.

Definition 3.— A **Cartan subalgebra** $\mathfrak{h} \subseteq \mathfrak{sl}(n, \mathbb{C})$ is a maximal abelian subalgebra such that the adjoint action $\mathrm{ad}(H)$ on $\mathfrak{sl}(n, \mathbb{C})$ can be diagonalized for all $H \in \mathfrak{h}$.

Example 5.— Let $\mathfrak{h}$ be the diagonal matrices in $\mathfrak{sl}(n, \mathbb{C})$. This subalgebra is:

- abelian, because diagonal matrices commute;
- maximal, because additional elements would have off-diagonal entries and no longer commute;
- $\mathrm{ad}(H)$ is diagonalizable for all $H \in \mathfrak{h}$.

Let us check this last claim: let H be the diagonal matrix with entries $a_1, \ldots, a_n$ on the diagonal, let E_{ij} be the elementary matrix with 1 in the ijth entry and zeroes elsewhere. A quick computation shows that $[H, E_{ij}] = (a_i - a_j)E_{ij}$. For $i \neq j$ (why?), this shows that E_{ij} is an eigenvector of $\mathrm{ad}(H)$ with eigenvalue $a_i - a_j$, and we can thus write down a basis of eigenvectors for $\mathfrak{sl}(n, \mathbb{C})$. This diagonalizes $\mathrm{ad}(H)$.

For $\mathfrak{sl}(2, \mathbb{C})$, there's a one-dimensional Cartan subalgebra $\mathfrak{h} = \mathrm{span}(H)$, spanned by the element

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This matrix is part of the standard basis for $\mathfrak{sl}(2, \mathbb{C})$:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These matrices satisfy the important relations:

$$[H, E] = 2E, \quad [E, F] = H, \quad [H, F] = -2F.$$

Definition 4.— Let (V, ρ) be a representation of $\mathfrak{g}$. A subspace $W \subseteq V$ is called **invariant** under $\mathfrak{g}$ if $\rho(X)w \in W$ for all $X \in \mathfrak{g}$ and $w \in W$. A representation V of $\mathfrak{g}$ is **irreducible** if the only invariant subspaces of V are 0 and V. A representation V is called **completely reducible** if it is the direct sum of irreducible representations, i.e., $V = \bigoplus_\lambda V_\lambda$, where each V_λ is irreducible.

To help us analyze the representations of $\mathfrak{sl}(2, \mathbb{C})$, we note the following without proof:

Theorem 5.— All complex, finite-dimensional representations of $\mathfrak{sl}(n, \mathbb{C})$ are completely reducible.

This theorem says we can focus on the irreducible representations, or **irreps**, since all others arise by taking direct sums. From now on, we assume V is an irreducible, finite-dimensional representation over $\mathbb{C}$. The key result for understanding V is a bit deep, and we give it without proof:

Theorem 6.— Given any complex finite-dimensional representation (V, ρ) of $\mathfrak{sl}(2, \mathbb{C})$, $\rho(H)$ is diagonalizable.

This should be plausible, since H itself is a diagonal matrix, and we already know that $\mathrm{ad}(H)$ is diagonalizable. The remarkable thing is that $\rho(H)$ is diagonalizable for any ρ. We use this diagonalizability as follows: decompose the irrep V into a direct sum of eigenspaces

$$V = \bigoplus_\lambda V_\lambda,$$

where the direct sum is over all complex numbers λ which are eigenvalues of $\rho(H)$, and each summand V_λ is an eigenspace for λ. In other words, for all $v \in V_\lambda$, we have $Hv = \lambda v$. (Really, $\rho(H)v = \lambda v$, but it is standard to suppress ρ.)

In Lie theory, the eigenvalues occurring here have a special name: they are called the **weights** of V. The eigenspaces V_λ are called **weight spaces**, and an eigenvector $v \in V_\lambda$ is called a **weight vector**. Given the decomposition of V into weight spaces, we thus know how H acts—it acts diagonally, by multiplication by the corresponding weight. Next, we would like to determine how the other basis elements E and F act:

Proposition 7.— If $v \in V_\lambda$, then $Ev \in V_{\lambda+2}$ and $Fv \in V_{\lambda-2}$. (Really, $\rho(E)v$ and $\rho(F)v$, but we're continuing to suppress ρ.)

The proof of this proposition is so important, that we call it the **fundamental calculation**: fix $v \in V_\lambda$, and compute

$$HEv = EHv + [H, E]v = \lambda Ev + 2Ev = (\lambda + 2)Ev.$$

Similarly, $HFv = (\lambda - 2)Fv$. This is what we wanted to check.

So, we have arrived at the following picture of V:

$$V = \cdots \oplus V_{\lambda-2} \oplus V_\lambda \oplus V_{\lambda+2} \oplus \cdots$$

where H multiplies by the weight, E raises the weight, and F lowers the weight. We do not know that all the weights of V lie in this sequence—we will see that soon!—but because V is finite-dimensional, we know this sequence cannot go on forever, so there must be a largest weight:

$$V = \cdots \oplus V_{\lambda-2} \oplus V_\lambda \oplus V_{\lambda+2} \oplus \cdots \oplus V_{\lambda_{\max}}.$$

The weight $\lambda_{\max}$ is called the **highest weight**, and a nonzero vector $v \in V_{\lambda_{\max}}$ is called a **highest weight vector**. A highest weight vector has the property that $Ev = 0$.

Similarly, there must be a lowest weight:

$$V = V_{\lambda_{\min}} \oplus \cdots \oplus V_{\lambda-2} \oplus V_\lambda \oplus V_{\lambda+2} \oplus \cdots \oplus V_{\lambda_{\max}},$$

For any $v \in V_{\lambda_{\min}}$, we must have $Fv = 0$.

Now, if we pick a highest weight vector $v \in V_{\lambda_{\max}}$ and keep lowering it with F, we will eventually get a vector in $V_{\lambda_{\min}}$. Let us suppose this happens in m steps. That is, m is the natural number such that $F^m v \neq 0$, but $F^{m+1}v = 0$.

Proposition 8.— The vectors $\{v, Fv, \ldots, F^m v\}$ form a basis of V.

Proof.— These vectors are linearly independent because they are eigenvectors (weight vectors) with distinct eigenvalues (weights). To show they span V, let $W = \operatorname{span}(v, Fv, \ldots, F^m v)$. The nonzero subspace W is preserved by the action of E, F, and H. Hence, W is invariant and we conclude $W = V$, because V is irreducible. ∎

In this basis, we know exactly how H and F act:

$$HF^k v = (\lambda_{\max} - 2k)F^k v, \quad FF^k v = F^{k+1} v,$$

but it is less clear how E acts. Let us derive a formula for the action of E, inductively. First of all, we know $Ev = 0$. For EFv, we compute:

$$EFv = FEv + [E, F]v = 0 + Hv = \lambda_{\max} v.$$

And for $EF^2 v$, we have:

$$EF^2 v = FEFv + [E, F]Fv = \lambda_{\max} Fv + HFv = \\ = (2\lambda_{\max} - 2)Fv.$$

Continuing in this way, we can discover the pattern:

$$EF^{k+1}v = (\lambda_{\max} + (\lambda_{\max} - 2) + \cdots + (\lambda_{\max} - 2k))F^k v,$$

which simplifies to $EF^{k+1}v = (k+1)(\lambda_{\max} - k)F^k v$.

We learn something magical from this formula when we set $k = m$:

$$EF^{m+1}v = 0 = (m+1)(\lambda_{\max} - m)F^m v.$$

It vanishes because $F^m v$ is in the lowest weight space, so $F^{m+1}v = 0$. But on the right hand side, the $m+1$ is nonzero, and the vector $F^m v$ is nonzero. So the only way this can vanish is if

$$\lambda_{\max} = m.$$

Look! The highest weight $\lambda_{\max}$ is a natural number! Specifically, it is the number of times we need to apply F to go from the highest weight vector v to the lowest. We have nearly proved:

Theorem 9.— For each natural number m (including zero), there is a unique finite-dimensional irreducible representation $(V^{(m)}, \rho_{(m)})$ of $\mathfrak{sl}(2, \mathbb{C})$ with highest weight m. All finite-dimensional irreps of $\mathfrak{sl}(2, \mathbb{C})$ have this form.

To recap, if V is an irrep with highest weight m, V decomposes into the weight spaces $V = V_{-m} \oplus V_{-m+2} \oplus \cdots \oplus V_{m-2} \oplus V_m$. Each weight space is one-dimensional, spanned by one of the basis vectors in $\{v, Fv, \ldots, F^m v\}$. We summarize all of these facts in the **weight diagram** of V:

$$\begin{array}{ccccc} \bullet & \bullet & \cdots & \bullet & \bullet \\ -m & -m+2 & & m-2 & m \end{array}$$

Each dot represents a weight space. In more general weight diagrams such as those in the next section, the dots can have multiplicities. Here, they all have multiplicity one, telling us that each weight space is one-dimensional.

4 The representations of $\mathfrak{sl}(3, \mathbb{C})$

The representation theory of $\mathfrak{sl}(3, \mathbb{C})$ begins the same way: we choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{sl}(3, \mathbb{C})$. As before, we take $\mathfrak{h}$ to consist of traceless diagonal matrices. Thus $\mathfrak{h} = \operatorname{span}(H_1, H_2)$ is two-dimensional, and we pick:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

As before, the Cartan subalgebra $\mathfrak{h}$ is maximal abelian, and $\operatorname{ad}(H)$ is diagonalizable for any $H \in \mathfrak{h}$, thanks to the formula $[H, E_{ij}] = (a_i - a_j)E_{ij}$, where

$$H = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix},$$

and E_{ij} is the matrix with 1 in the ijth entry, and zeroes elsewhere.

To analyze representations, we need a version of Theorem 6 for $\mathfrak{sl}(n, \mathbb{C})$:

Theorem 10.— For any complex finite-dimensional representation (V, ρ) of $\mathfrak{sl}(n, \mathbb{C})$, and any choice of Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{sl}(n, \mathbb{C})$, $\rho(H)$ is diagonalizable for all $H \in \mathfrak{h}$.

In analogy with $\mathfrak{sl}(2,\mathbb{C})$, we use this to write any representation as a direct sum over weights:

$$V = \bigoplus_{\lambda} V_\lambda.$$

For $\mathfrak{sl}(2,\mathbb{C})$, there was only one H, and a weight was simply an eigenvalue of H. But now $\mathfrak{h}$ is two-dimensional, and there are many $H \in \mathfrak{h}$. In this instance, what is a weight?

Definition 11.— Given a representation (V,ρ), a nonzero vector $v \in V$ is a **weight vector** if $\rho(H)v = \lambda(H)v$ for all $H \in \mathfrak{h}$ and some linear map $\lambda : \mathfrak{h} \to \mathbb{C}$. Here, λ is called a **weight** of the representation V.

Weights are a generalization of eigenvalues, and weight vectors are a generalization of eigenvectors that allow us to diagonalize all the $H \in \mathfrak{h}$ at once. And we really can diagonalize all the $H \in \mathfrak{h}$ simultaneously, because they commute!

To get a feel for the weights of a representation of $\mathfrak{sl}(3,\mathbb{C})$, let us consider an example:

Example 6.— We already know the weight vectors—they are the elementary matrices E_{ij}, at least when $i \neq j$. This is because of the formula:

$$\mathrm{ad}(H)E_{ij} = (a_i - a_j)E_{ij}.$$

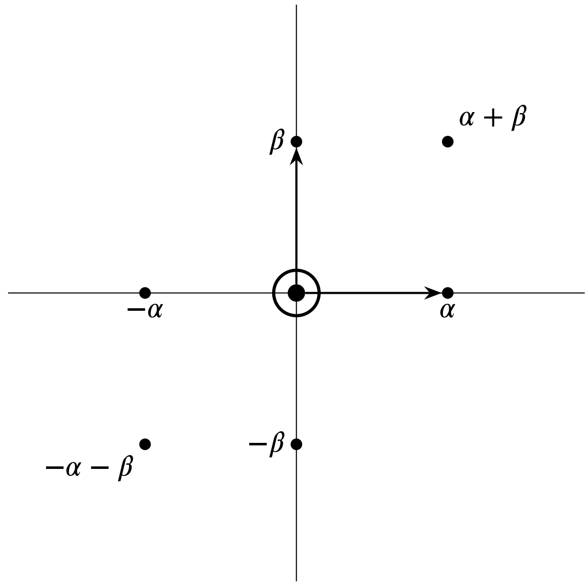
(For $i = j$, E_{ii} has trace 1 and is not in $\mathfrak{sl}(3,\mathbb{C})$.)

So let us define the weight $\alpha_{ij} : \mathfrak{h} \to \mathbb{C}$ by the formula $\alpha_{ij}(H) = a_i - a_j$, where a_i and a_j are the ith and jth entries of the diagonal matrix H. Then we have $\mathrm{ad}(H)E_{ij} = \alpha_{ij}(H)E_{ij}$.

To get a feel for the weights of the adjoint representation, note that we have the relations:

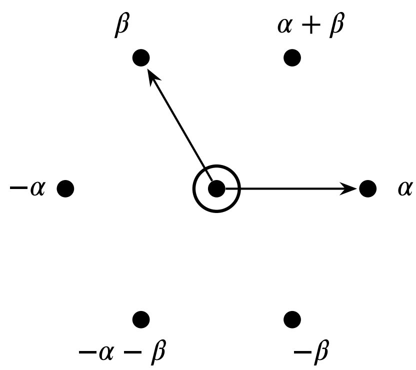
$$\alpha_{ij} = -\alpha_{ji}, \quad \alpha_{ii} = 0, \quad \alpha_{ij} + \alpha_{jk} = \alpha_{ik}.$$

These imply that all the weights of the adjoint can be expressed as a linear combination of two such weights. Let us pick $\alpha = \alpha_{12}$ and $\beta = \alpha_{23}$ as a basis. Then the other nonzero weights are $\alpha_{13} = \alpha + \beta$, $\alpha_{21} = -\alpha$, $\alpha_{32} = -\beta$, and $\alpha_{31} = -\alpha - \beta$. To really get a picture, we plot these weights:



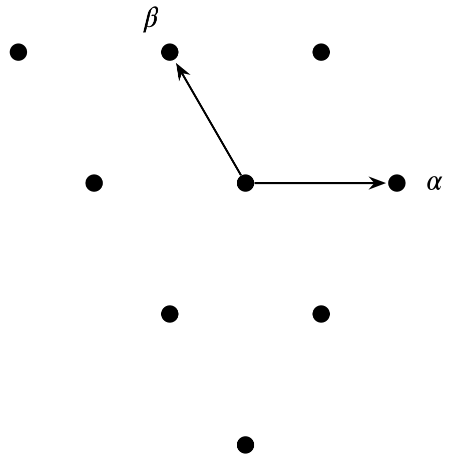
In the plot, we draw one dot for each weight space in the adjoint representation, except in the middle: the weight space of weight zero is two-dimensional—it is $\mathfrak{h}$!—and we depict this by adding the extra circle around the zero weight. We have six dots around the outside and two in the middle. The total is eight, as it must be: $\dim(\mathfrak{sl}(3,\mathbb{C})) = 8$.

This picture is usually drawn with more symmetry, as a regular hexagon with two dots in the middle:



This is the weight diagram of the adjoint representation.

Remarkably, all the irreps of $\mathfrak{sl}(3,\mathbb{C})$ have similar weight diagrams. For instance, here is the weight diagram of a 10-dimensional irrep:



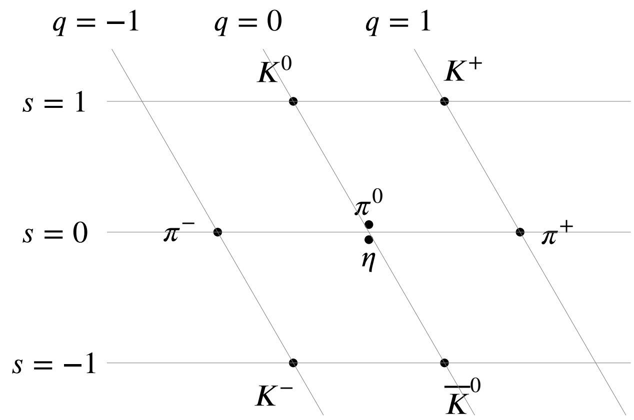
We have indicated α and β on this diagram to show how it compares to the adjoint.

5 The eightfold way

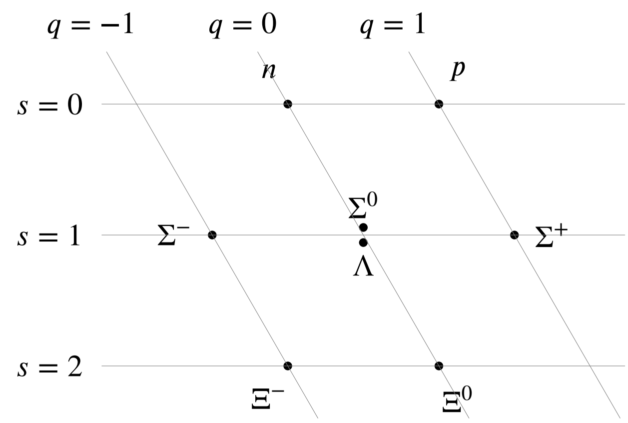
In the 1940s and 1950s, physicists built particle accelerators and began to collide protons together at high energies. Protons are strongly interacting particles, in the sense that they feel the strong nuclear force that binds them together in the atomic nucleus. Colliding protons produced *many* new, hitherto unknown particles, likewise strongly interacting. In physics, strongly interacting particles are called **hadrons**—the Greek root *hadros* means strong.

No one expected such a zoo of new particles, and so a search was on for some kind of order, some system to classify the hadrons. Many properties of the particles were measured. Each particle X had an electric charge $q(X)$, which is an integer in suitable units. But several other kinds of "charge" were discovered. It turned out each particle had a property, called **strangeness** $s(X)$, which was also an integer.

When you plot the charge and the strangeness of hadrons on a plane, certain patterns emerge. For instance, here is the **spin-0 meson octet** (mesons are a type of hadron):



And here is the **spin-1/2 baryon octet**. (Baryons are another type of hadron, which includes the proton and neutron. In fact, the proton and neutron are the particles n and p at the top):

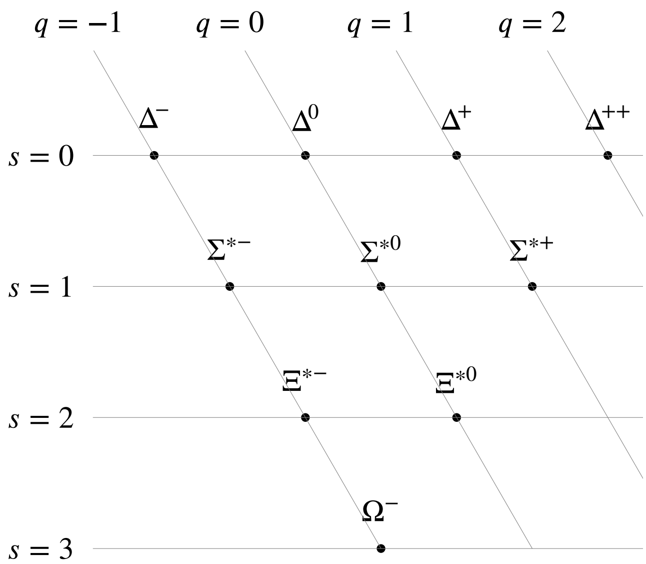


As you can clearly see, both of these are pictures of the adjoint representation of $\mathfrak{sl}(3, \mathbb{C})$! This led the American physicist Murray Gell-Mann and the Israeli physicist Yuval Ne'emann to propose independently **the eightfold way** hypothesis. The name comes from appearance of eight particles in the octets, and was Gell-Mann's allusion to the eightfold path to enlightenment in Buddhism. Here is the hypothesis:

Hypothesis (The eightfold way).— Hadrons are classified by representations of $\mathfrak{sl}(3, \mathbb{C})$. ∎

In its original form, the eightfold way used the Lie group SU(3) in place of the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$. It is a marvelous result of Lie theory that these objects have equivalent representation theory, so we have substituted $\mathfrak{sl}(3, \mathbb{C})$ to ease exposition.

The vindication of the eightfold way came with the prediction of new particles. This followed not from the octets above, but from the **spin-3/2 baryon decuplet**:



The particle at the bottom, the Ω^-, was previously unknown. Gell-Mann predicted it in 1962 on the basis of the eightfold way, and it was discovered in 1964.

6 Further reading

The best reference for the Lie theory we have discussed is the book by Fulton and Harris [1], to which our treatment of $\mathfrak{sl}(2, \mathbb{C})$ owes everything. Of course, Lie algebras are closely related to Lie groups, and a good first introduction can be found in the book of Hall [2]. For the eightfold way, a wonderful treatment can be found in Sternberg [3], who frames the question in terms of the representations of the Lie group SU(3), rather than the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$ we used here.

References

[1] William Fulton and Joe Harris, *Representation theory: a first course*, Springer–Verlag, New York, 1991.

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[3] Shlomo Sternberg, *Group theory and physics*, CUP, Cambridge, 1994.

PEDRO NUNES LECTURES

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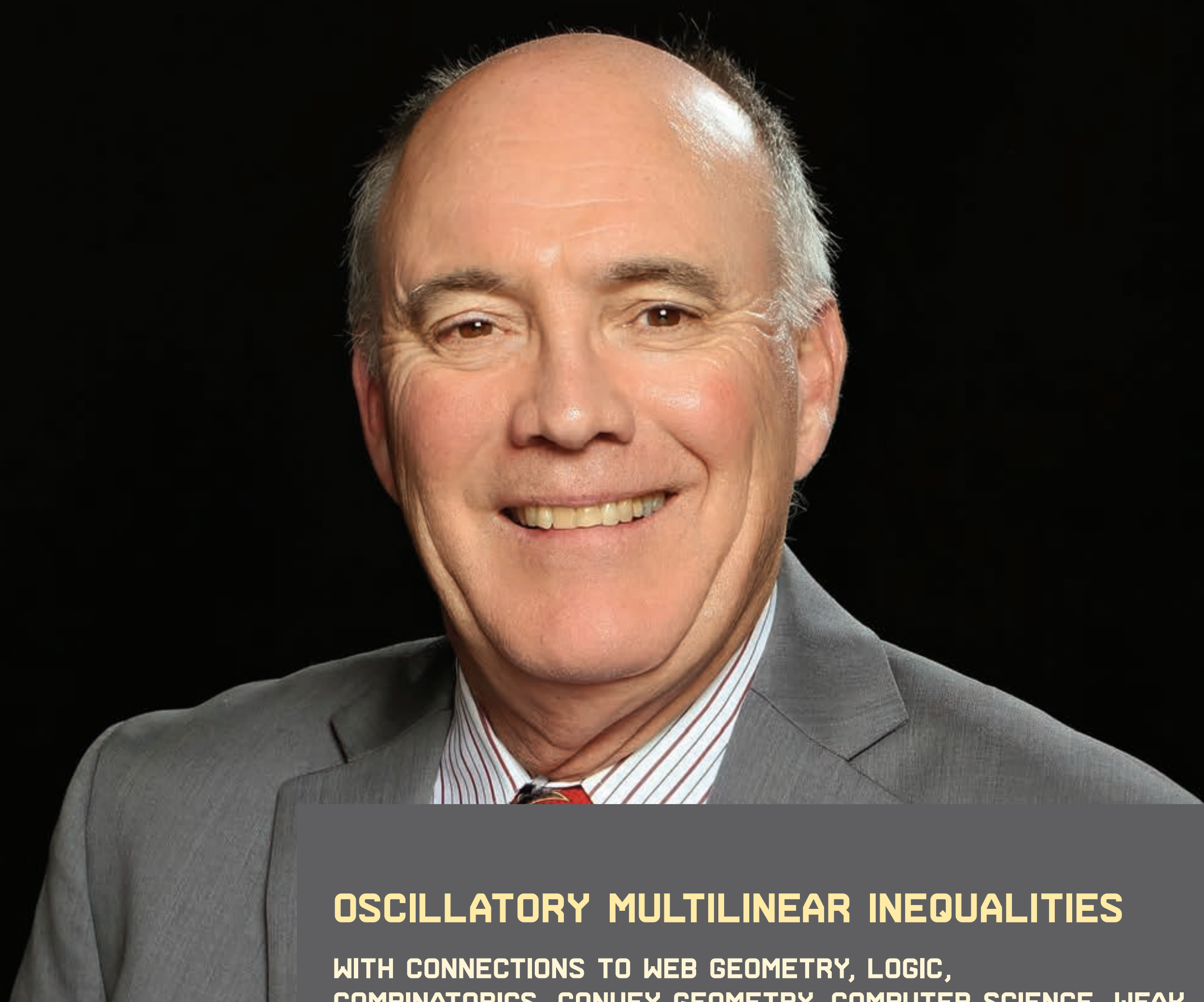

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Michael Christ (born 7 June 1955) is an American mathematician and Emeritus Professor at the University of California, Berkeley.

His research has Fourier analysis at its core and encompasses partial differential equations, complex analysis in several variables, and topics in mathematical physics. Among his many outstanding research contributions, we highlight the definitive analysis of global regularity of solutions of ∂-bar problems on pseudoconvex domains, the proof (with Colliander and Tao) of the ill-posedness of low-regularity solutions of the nonlinear Schrödinger equations, and the characterization (with Kiselev) of the absolutely continuous spectrum and generalized eigenfunctions for second-order ODE with potentials on the real line.

Michael has been an invited lecturer twice at the International Congress of Mathematicians, first in Kyoto in 1990 and then in Berlin in 1998. He has received numerous honors and awards, including an NSF Presidential Young Investigator Award and a Sloan Fellowship, the Bergman Prize from the AMS, and a Miller Research Professorship. He was elected to the American Academy of Arts and Sciences in 2007.

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