Graphs of polyhedra and the theorem of Steinitz

by António Guedes de Oliveira*

The theorem of Steinitz characterizes in simple terms the graphs of the polyhedra. In fact, the characteristic properties of such graphs, according to the theorem, are not only simple but "very natural", in that they occur in various different contexts. As a consequence, for example, polyhedra and typical polyhedral constructions can be used for finding rectangles that can be decomposed in non-congruent squares (see Figure 1). The extraordinary theorem behind this relation is due to Steinitz and is the main topic of the present paper.

Steinitz's theorem was first published in a scientific encyclopaedia, in 1922 [16], and later, in 1934, in a book [17], after Steinitz's death. It was ignored for a long time, but after "being discovered" its importance never ceased to increase and it is the starting point for active research even to our days. In the middle of the last century, two very important books were published in Polytope Theory. The first one, by Alexander D. Alexandrov, which was published in Russian in 1950 and in German, under the title "Konvexe Polyeder" [1], in 1958, *does not* mention this theorem. The second one, by Branko Grünbaum, "Convex Polytopes", published for the first time in 1967 (and dedicated exactly to the "memory of the extraordinary geometer Ernst Steinitz"), considers this theorem as the "most important and the deepest of the known results about polyhedra" [9, p. 235].

Polyhedra are polytopes of dimension 3 and polygons are polytopes of dimension 2; a polytope of dimension 1 is an edge and a polytope of dimension 0 is a vertex. Every polytope of dimension greater than 1 has a related graph, formed by its edges and vertices. Yet, for dimensions greater than 3, we do not know which graphs arise, and which do not, as the graph of a polytope. The quest for properties that would characterize these graphs, in the line of Steinitz's theorem but for any dimension greater than 3, has been in fact a long and important line of research.

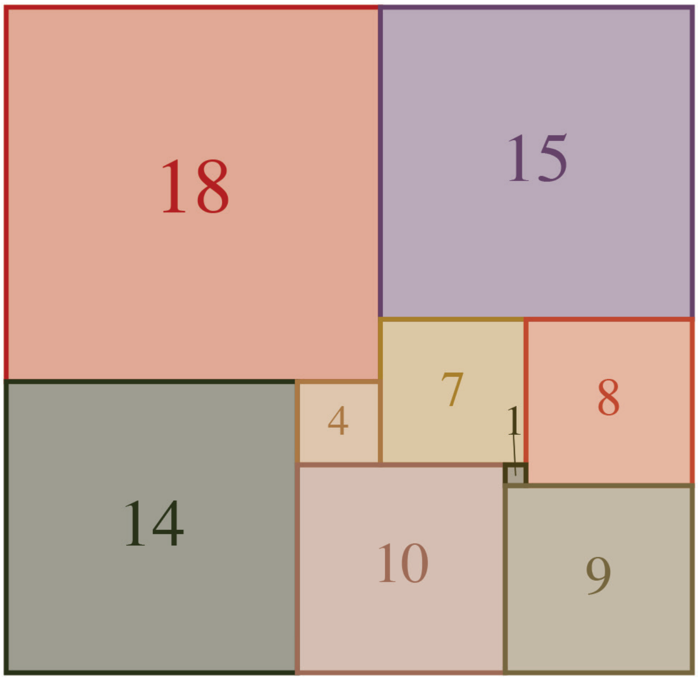

Figure 1.— Decomposition of a rectangle in non-congruent squares

This paper was written as an invitation: we invite the reader, a student, perhaps, to visit an old but very vivid area of research, of which we are happy to present some glimpses, including of a few personal contributions.

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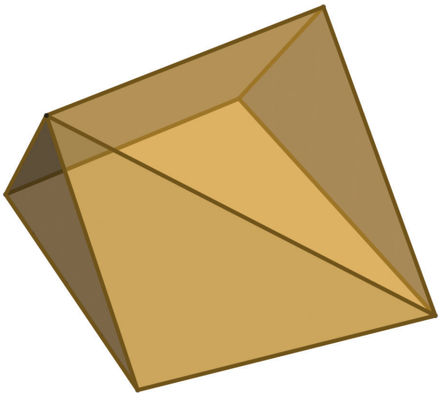

Figure 2.—A polyhedron — $\mathcal{P}$

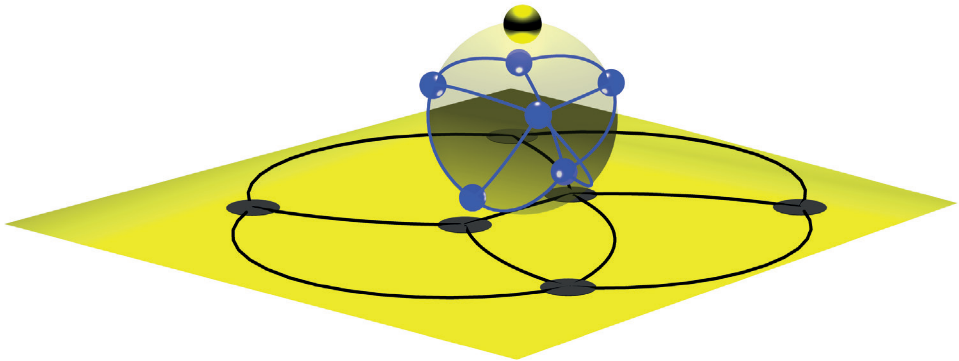

Figure 3.—Planarity of the graph of a polyhedron ($\mathcal{P}$)

Before stating and commenting this theorem, let us introduce some basic notions that are perhaps not familiar to the reader.

Generalities

Given a non-empty finite set V and given a set

$$E \subseteq \big\{\{u,v\} \subseteq V \mid u \neq v\big\},$$

we say that (V,E) is a *graph*, the elements of V being the *vertices* of G and the elements of E the *edges* of G. We write $e = uv$ for $e = \{u,v\}$ and call u and v the *vertices incident with* e. If we are given a set E' disjoint from V and an injective function $\varphi : E' \to \big\{\{u,v\} \subseteq V \mid u \neq v\big\}$, we also consider (V,E') as a graph by naturally identifying E' with $\varphi(E')$.

For example, given a polygon or a polyhedron, the vertices and the edges of the polygon, or of the polyhedron, form obviously a graph, for which the vertices are points (in the plane or in space), the edges are line segments and incidence is inclusion.

For another example, we may consider the following graph G underlying the decomposition presented in Fig. 1, which we call *the graph of the decomposition*, where the vertices are line segments and the edges are rectangles:

- the vertices of G are the maximal horizontal segments that contain the sides of the squares of the decomposition. Note that these segments, together with the maximal vertical segments defined similarly, determine completely the decomposition;
- the edges are the squares of the decomposition and the rectangle being decomposed.
- any of these squares, as well as the full rectangle, has two sides contained in two horizontal segments; as an edge, these are the vertices incident with it.

Every graph $G = (V,E)$ may be *represented geometrically in the plane* by another graph $G' = (V',E')$, where V' is a set of points in the plane in bijection φ with V and E' is a set of Jordan arcs in the plane in bijection ψ with E, in such a way that, for every $e = uv \in E$, $e' = \psi(e)$ connects $\varphi(u)$ to $\varphi(v)$. We say that G' *is plane* — in which case we say that G *is planar* — if, given any two edges (two arcs, hence) $e', f' \in E'$, if $A = e' \cap f'$ then either $A = \{v'\}$ for a vertex $v' \in V'$ (incident with e' and f') or $A = \varnothing$. See Figure 3 for an example. In general, we call *topological graph* to a graph obtained as G' above, either in the plane, in the sphere, in the torus, etc. Similarly to plane graphs, we may have then *spheric graphs* or *toroidal graphs*. In particular, spheric graphs are planar and plane graphs can be represented in the sphere. To see this, in one direction, consider the stereographic projection of the spheric

[1] By a face we mean one of the connected components of the complement in the sphere of the union of the edges. The same notion applies to the torus, for example, or to the plane, where (exactly) one of the regions is unbounded.

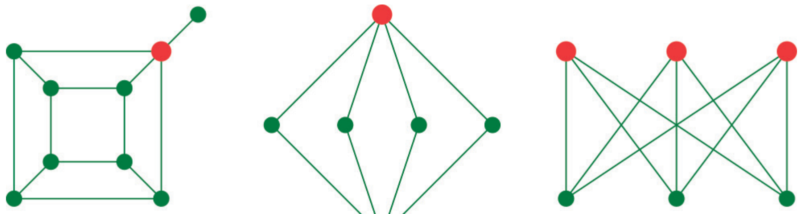

Figure 4.—Connected graph that is *not* 2-connected, 2-connected graph that is *not* 3-connected and 3-connected graph that is *neither* planar *nor* 4-connected

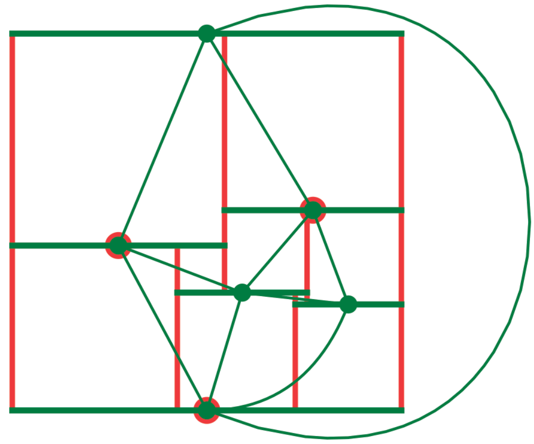

Figure 5.—Graph of the decomposition of a rectangle

graph, taking as pole of projection a point interior to a face[1] (see Figure 3). The other direction works similarly.

The graphs of the polygons, in the precise sense defined above, are obviously *plane*, and the edges are even *straight* line segments. It is easy to see that the graphs of the polyhedra can be represented as spheric graphs and hence are *planar* (again, see Figure 3). Finally, it can be proved [7] that the graph of the decomposition of a rectangle is also planar. In Figure 5 we can see the plane graph drawn over the decomposition.

A *path in a graph* G is a sequence of pairwise distinct vertices of G, $c = [v_0, v_1, \ldots, v_k]$, such that $v_0 v_1, v_1 v_2, \ldots, v_{k-1} v_k$ are edges of G. The *endpoints* are v_0 and v_k and c is said *to connect* them. A *cycle* is defined like a path, except that $v_0 = v_k$. In both cases, k, the number of edges, is the *length*.

We say that G is connected (or 1-*connected*) if there is a path connecting any two different vertices u and v. It is 2-*connected* if, given a vertex x and two vertices u and v, different from each other and from x, there is a path that does not contain x connecting u to v. In other words, G is 2-connected if, for every $x \in V(G)$, the graph G' obtained from G by excluding x from $V(G)$ and by deleting all edges incident with x from $E(G)$ is still connected. In general, it is n-*connected* if, for every $x \in V(G)$, G' is $(n-1)$-connected. In Figure 4 we show different examples of connectivity.

For an example of a non-planar graph, consider the last graph of Figure 4, usually called $K_{3,3}$. In fact, if we suppose that the graph can be represented in a sphere, since we need a cycle to close a face and there are no cycles in the graph with length less than 4, and since every edge belongs exactly to the boundary of two faces, we see that the number of faces is at most half the number of edges. So, we must have 6 vertices, 9 edges and at most 4 faces. But this is in contradiction with Euler's formula.

We note that the plane graph drawn in Figure 5 over the decomposition of the rectangle is also a representation of the graph of the polyhedron of Figure 2. The same graph appears in black in Figure 3. But whereas the unbounded face is adjacent to 4 edges in the graph of Figure 3, in the graph of Figure 5 it is adjacent to 3 edges. But it is important to note that the edges adjacent to any face in one representation correspond exactly to the edges adjacent to a face in the other representation. In fact, by an important theorem of Whitney, a cycle C is the boundary of a face in any representation in the plane (or in a sphere) if and only if the graph obtained from G by removing the edges of C is connected.

It is not difficult to see that this particular graph is 3-connected: although it can be disconnected by deleting the 3 marked vertices (and the incident edges), *it cannot be disconnected by deleting only 2 vertices*. The same happens with the graphs of all the polyhedra, according to the theorem that is central in this paper: they are all 3-connected.

Theorem of Steinitz.—The graph of every polyhedron is *planar* and 3-*connected*. Conversely, any graph with more than 3 vertices that is both planar and 3-connected is the graph of a polyhedron.

We will come back to Steinitz's theorem. Before, let us consider briefly the connection between the theorem of Steinitz and the decomposition of rectangles. Our plan is to state afterwards this theorem, to "explain" things

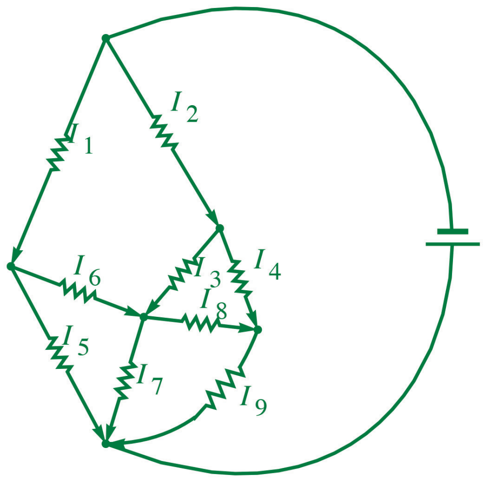

Figure 6.—Electric circuit associated with the graph of the decomposition of a rectangle

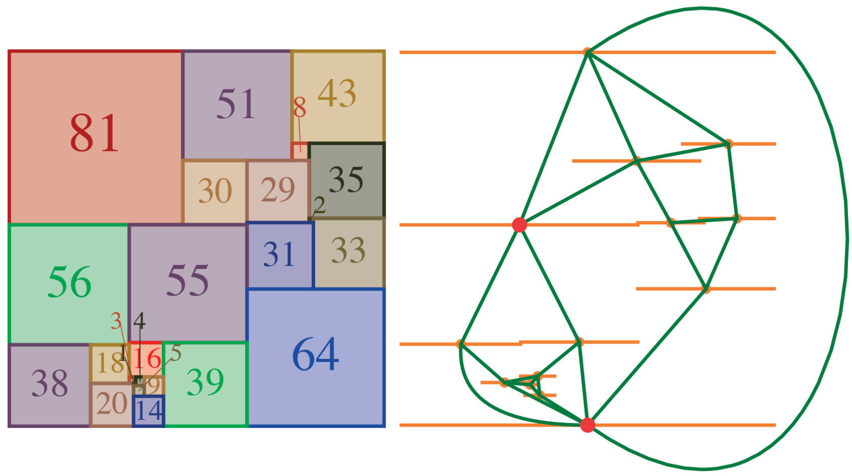

Figure 7.—*Non-simple* decomposition of a square and connectivity of the graph

when they can be easily "explained" (although we do not prove them …) and to present some more modern consequences of the theorem and of its various, modern or not so modern, proofs.

Decomposition of a rectangle in non-congruent squares

The rectangle in Figure 1 is "almost a square", in that its dimensions are 33×32. But it is not a true square, and for a long time no one knew whether a square could be decomposed in squares pairwise non-congruent squares. In an attempt to solve this question, four students of the Trinity College, Cambridge, Roland Brooks, Cedric Smith, Arthur Stone e William Tutte [7], defined and studied the graph of a decomposition. They not only presented *perfect squares*, as they called the squares that can be decomposed in pairwise non-congruent squares, but proved that there are infinitely many different (non-similar) perfect squares.

They proved that the graph of a decomposition is always planar, as we have seen in our example. At the same time, they noted they could see this graph as the diagram of an electric circuit (see Figure 6), where vertices represent junctions, edges represent wires with resistors of unitary resistance or, in a unique case, a power source, and where the Kirchhoff's current and voltage laws hold true.

In fact, in such a circuit, on the one hand, the sum of the currents that enter a junction or vertex (the sum of the sides of the squares above the maximal horizontal segment that is the vertex) must equal the sum of the currents that leave the junction, or the sum of the sides of the squares below the vertex. On the other hand, considering any face and the upmost and downmost vertices and the two different paths between them, the sum of the sides of the squares that are the edges of one path is of course equal to the sum of the sides of the squares that are the edges of the other path. Then, the theorem of Kirchhoff implies that, up to the total voltage V of the circuit, the values of all currents and voltages are uniquely determined.

This gave them the means to start to construct decompositions, just by considering suitable graphs and by "electrifying" them. We consider an example based in the graph of the polyhedron $\mathscr{P}$ of Figure 2, namely as represented in Figure 5, but "electrified".

In Figure 6, we have five independent "current equations", $I_1 + I_2 = V$, $I_2 = I_3 + I_4$, $I_4 + I_8 = I_9$, $I_3 + I_6 = I_7 + I_8$ e $I_1 = I_5 + I_6$ and four "voltage equations" (remember the resistances are unitary): $I_1 + I_6 = I_2 + I_3$, $I_3 + I_8 = I_4$, $I_8 + I_9 = I_7$ e $I_6 + I_7 = I_5$. Hence, in the solution of the system,

$$\begin{cases} I_1 &= 6\,V/11 \\ I_2 &= 5\,V/11 \\ \vdots & \quad\vdots \\ I_9 &= 3\,V/11 \end{cases}$$

Making $V = 33$, the width of the rectangle, we find the sides of the squares of the decomposition of Figure 1.

This construction was based in the graph of a polyhedron. What happens if the starting graph is not 3-connected?

In Figure 7 we consider a decomposed square and its graph. The graph is not 3-connected, since it can be disconnected by deleting the two marked vertices. But

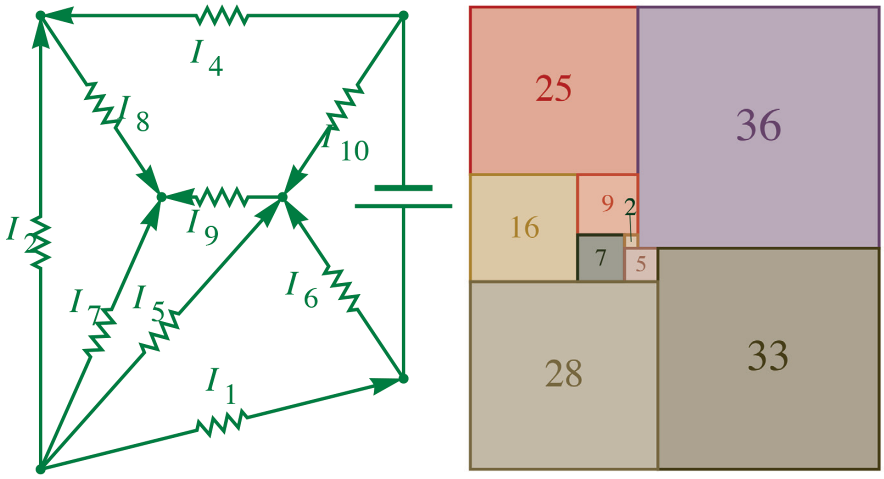

Figure 8.—Same polyhedron, different decomposition

these vertices are the horizontal sides of a "subrectangle" (in the southwest corner of the decomposition) and it can be proven that the fact that, when deleted, they disconnect the graph means, in terms of the decomposition, that the subrectangle is already decomposed in mutual non-congruent squares. So, we have a "subdecomposition" of the decomposition. If we do not want this to happen, we must consider only *planar, 3-connected* graphs, that is, graphs of polyhedra.

We have used this idea [10], by considering eight particular polyhedra with six vertices, from which all the 30 possible decompositions with eleven or less squares can be obtained. Note that the way we draw the graph in the plane (or the choice of the "electrified" edge, more precisely) may determine different decomposed rectangles. For example, the electrification in Figure 8 of the graph represented in Figure 3 leads to the solution

$$(I_1, I_2, I_4, I_5, I_6, I_7, I_8, I_9, I_{10}) =$$
$$= \left(\frac{28\,V}{61}, -\frac{16\,V}{61}, \frac{25\,V}{61}, -\frac{5\,V}{61}, -\frac{33\,V}{61}, -\frac{7\,V}{61}, \frac{9\,V}{61}, \frac{2\,V}{61}, \frac{36\,V}{61}\right),$$

which determines the new decomposition (see Figure 8).

Theorems of Steinitz and Tutte

Let us consider a little further the "easy part" of the theorem of Steinitz, that claims that the graph of a polyhedron is always planar and 3-connected. We start exactly by the connectivity, but having in mind a theorem by Balinski [3], which claims that the graph of any d-polytope is d-connected, for any $d \geq 2$, and the author's proof.

Consider a polyhedron $\mathscr{P}$, two vertices, P and Q, and let us "tell why" deleting these points and the incident edges from the graph G of $\mathscr{P}$ does not result in a disconnected graph. Note that, in the general case, the number of *deleted points* should be $d - 1$.

So, let A, B and R be three points different from the deleted points so that R is adjacent to one of them, P, say, and let π be the (hyper)plane defined by R and all the deleted points. For simplicity sake, we only consider here the case where A and B are not in π. Then, obviously, either they are in the same side of π or they are in opposite sides. We want to connect A and B by a path that does not include either P or Q.

In the first case, note that either A is at maximal distance to π or there exists a vertex, *adjacent to* A, at greater distance: just consider the (hyper)plane χ parallel to π through A and the part of $\mathscr{P}$ (a new polytope with vertex A) that lies in the side of χ opposite to π. This means that A is connected to a vertex A' at maximal distance to π, and so is B, to B'. If they are equal, we are done. If not, they both lye in a face of $\mathscr{P}$, and can be connected in the graph of $\mathscr{F}$, which is also a polytope.

In the second case, note that there are two vertices adjacent to R, in the opposite side of π of each other. By the previous argument, one of these vertices can be connected to A and the other one to B, and so these points are connected through R.

Figure 2 suggests a proof of planarity of G, the graph of a polytope $\mathscr{P}$. Yet, the edges of the resulting graph are not straight. If we project directly G from a point O over a plane χ, then the image of the (straight) edges of $\mathscr{P}$ are still straight and the image of the faces are still convex. But the projected graph may cease to be plane.

To avoid this, choose a face $\mathscr{F}$ of $\mathscr{P}$, let π be the

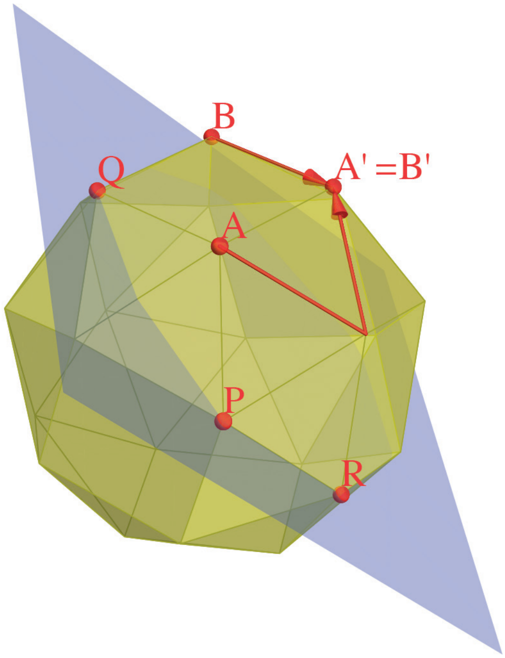

Figure 9.—Illustration of Balinki's theorem and proof

plane that contains $\mathscr{F}$ and consider O near to the centroid C of $\mathscr{F}$, on the side of π opposite to $\mathscr{P}$. Since the intersection of two convex sets is still convex, the intersection of any line with $\mathscr{P}$ is either a point or a line segment. Suppose that the projection of two different edges e and f intersect. Then the line OA, for a given point A in e, also contains a point B of f, and the intersection of line OA with $\mathscr{P}$ is the segment $\overline{AB}$. But this cannot happen if O is sufficiently close to C: neither when, say, $A \in \mathscr{F}$ nor when $A, B \notin \mathscr{F}$, since, in the latter case, OA must cross the plane containing $\mathscr{F}$ outside this face.

Before proceeding further, we state this result, that was originally obtained independently of Steinitz's theorem.

Theorem of Tutte.—Every *planar*, 3-*connected* graph can be represented in the plane with straight edges and convex faces.

Note that we consider in this theorem two different conditions. For example, it can be easily proved that the graph represented *with straight edges* in the middle of Figure 4 *cannot* be drawn in the plane with convex faces.

Let us go back to the first property. Can every planar graph be represented in the plane with straight edges? The answer, *yes*, goes back to 1936 and is due to Wagner, and new proofs were published independently in 1948, by Fáry, and in 1951, by Stein. We also consider this question here, with one more issue in mind: we want straight edges *and*, at the same time, vertices with *small integer* coordinates in a suitable coordinate system — or, in other words, with *good resolution*. The construction we describe here is due to W. Schnyder [15], and is illustrated in Figure 10 and in Figure 11.

Given any plane graph G, by possibly adding some new edges (that can be withdrawn afterwords), we obtain a new graph in which every face, including the unbounded face, is a triangle. Let us suppose that the vertices of the unbounded face are coloured with three colours, say, red, green and blue. Schnyder shows that we may orient and colour with the same three colours all the edges of such a graph, in such a way that from every vertex if we "follow" the edges of a given colour, we reach the corresponding coloured vertex.

Now, for each vertex, consider the three "coloured paths" and the partition of the bounded faces into three classes determined by these paths. In Figure 10, for example, there are 15 bounded faces. On the right-hand side, for vertex 1, the three classes have 5, 9 and 1 faces, respectively, where the class with 5 faces [resp. 9, 1] is not bounded by the "red path" [resp., "green path", "blue path"]. We obtain consecutively for all the vertices

$$(5,9,1), (1,4,10), (3,10,2), (11,2,2), (9,1,5), (0,15,0), (15,0,0), (7,5,3), (2,6,7) \text{ and } (0,0,15).$$

Schnyder proves that if we take a triangle in the plane, consider the vertices with these triplets as suitable multiples of the barycentric coordinates and draw straight edges, then we obtain a plane representation of the initial

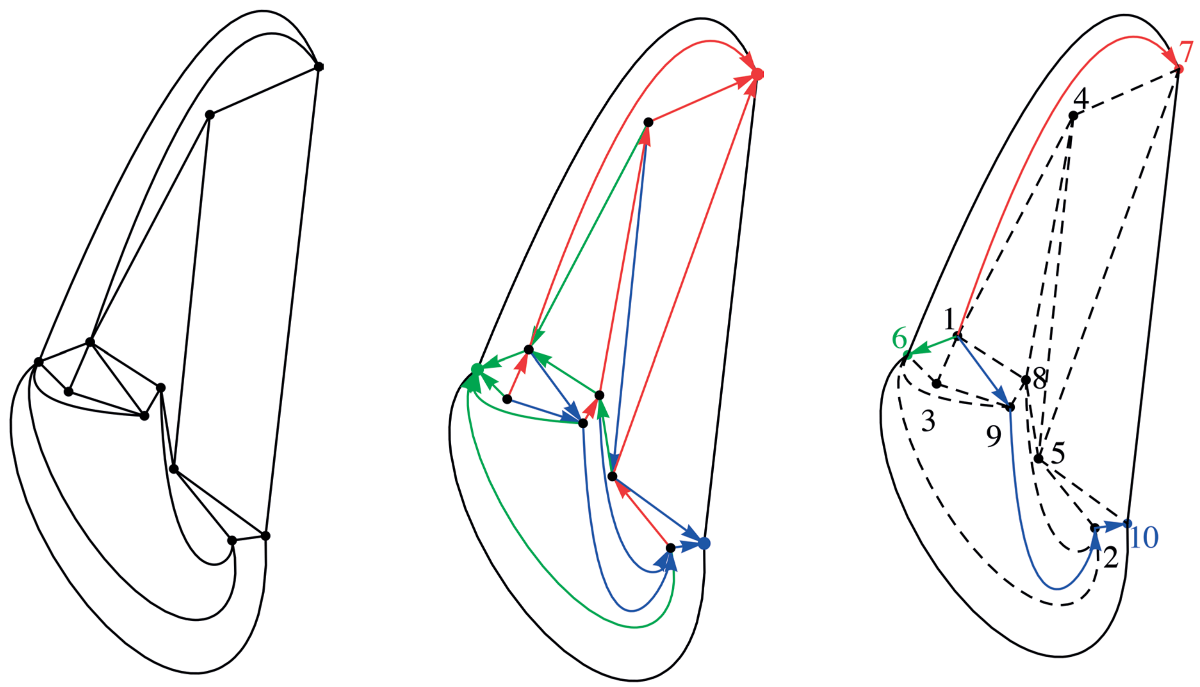

Figure 10.—How to draw a graph with straight edges and good resolution (but without convex faces) I

graph with integer (barycentric) coordinates not greater than the number of faces. In fact, with a slight modification of this method, Schnyder proves that the coordinates can be limited to integers between 0 and $v-2$, where v is the number of vertices of the graph. In Figure 11, we use these coordinates on the left-hand side, and on the right-hand side we show without any further explanation a *geodesic embedding* of the graph, that is also based in Schnyder's construction (see the book of S. Felsner [8] for more information on this subject). Note that, as in our example, by Schnyder's method we may end up with non-convex faces, after deletion of the edges added at the beginning for obtaining triangular faces. But this can be circumvented [8].

Tutte's original proof is different, and the ideas behind it are still used nowadays [12]. They correspond to the following "physical" idea: suppose that, in a board, we fix nails corresponding to the vertices of the unbounded face of the graph, and that we connect with rubber bands the vertices that are incident with any edge, by tying up the bands on points corresponding to vertices as indicated by the graph. When we leave such a system to itself, if in equilibrium there is some tension in all the rubber bands, then the edges will be straight and the faces will be convex.

More precisely, Tutte proves the following. Consider, for each vertex v not belonging to the unbounded face, with neighbours[2] say, $w_1, w_2, \ldots, w_k$, the (vectorial) equation

$$\sum_{i=1}^{k} c\,(\mathbf{p}_{\mathbf{w}_i} - \mathbf{p}_{\mathbf{v}}) = 0.$$

In this equation, c is an elasticity coefficient that we can neglect by now, by considering it constant, and $\mathbf{p}_{\mathbf{x}}$ represents the constant coordinate vector of the point x if x is a "nailed" vertex of the unbounded face, or a pair of variables, otherwise. Then, the system of equations has a unique solution, that represents the coordinates of the vertices of a plane graph associated with the initial graph, for which the faces are convex provided the edges are straight.

4. On the "difficult part" of the theorem of Steinitz

All the known proofs of the fact that every planar 3-connected graph with more than 3 vertices is the graph of a polyhedron present (naturally …) some difficulties. We will mention here some of these proofs, but in quite a vague way. For more precision and even for a correct attribution of results to authors, please see Ziegler [18,19] and Richter [13] and the bibliography therein.

We may say that there are three kinds of known proofs of this theorem [19, p. 8]. For each of them, new

[2] That is, the vertices w such that vw is an edge of G.

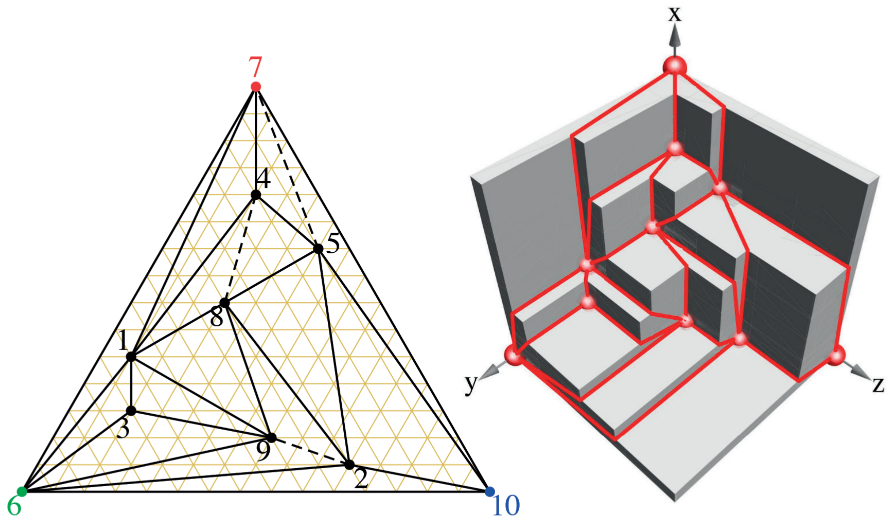

Figure 11.—How to draw a graph with straight edges and good resolution (but without convex faces) II

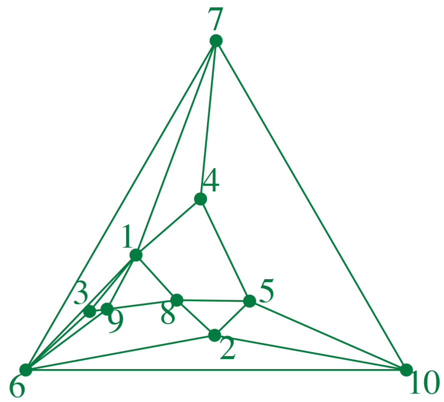

Figure 12.—Rubber band representation of the graph of $\mathcal{P}$

proofs lead to new results. Steinitz gave three different proofs, all starting from a tetrahedron and showing that vertices can be added or moved so as to fit to the graph. For other proofs of the same kind, and for the variety of results that we can obtain from them, see e.g. [9]. For example, from a modification of a proof of the same kind, it has be shown that one can "prescribe" the shape of any face of a polyhedron with a given graph. It can also be shown that the (combinatorial) symmetry of the graph can be carried over to a (geometric) symmetry of the polyhedron. These properties do not hold in dimension 4: for example, there exists a 4-polytope with 8 vertices for which one particular face, an octahedron, cannot be regular. In fact, there exists a 4-polytope for which we cannot prescribe freely the shape of a 2-face, an hexagon. From the first example, it was possible to construct a 4-polytope with 2 new vertices with "hidden" symmetries, that is, combinatorial symmetries without geometric counterpart [4]. The "realization space" (the euclidean space of coordinates of the vertices) of this polytope is not connected [5]; it is the smallest known polytope with this property.

Another kind of proofs exploit Tutte's "rubber band" idea, by "lifting" the rubber band diagram, similarly to 13, obtained from the graph of the polyhedron of Figure 2 as indicated by Richter [13] by using a constant elasticity coefficient, c. It can be proved that all the polyhedra with the same graph can be obtained this way, but with variable values of c.

J. Richter [13] bases on this method a proof for the fact that every polyhedron has the graph of another polyhedron with vertices with rational (and hence also with integral) coordinates. The best *resolution* of these integer coordinates is a new research problem, called the *quantitative Steinitz theorem*, with very recent developments [12]. Note that we know of an 8-polytope with twelve vertices that *cannot* be constructed with rational coordinates. The most important consequence of Richter's proof (and of some other proofs of the same kind) is that, for polyhedrons, the realization space is topologically very simple, in the (very imprecise) sense that we can deform continuously any polytope into any other one, up to a mirror image, provided they have the same graph.

This is not the case in dimension 4. On the contrary, the realization space of 4-polytopes is "as rich as possible", in a precise sense that we will not consider here. For details see J. Richter's book [13], which is centred exactly on this very important issue. The study of graphs of 4-polytopes and general d-polytopes for $d \geq 4$ is a rich field of growing research [19].

As an example of active research, consider Ziegler's question [20,11], regarding the polytopality of the Cartesian product of two Petersen graphs.

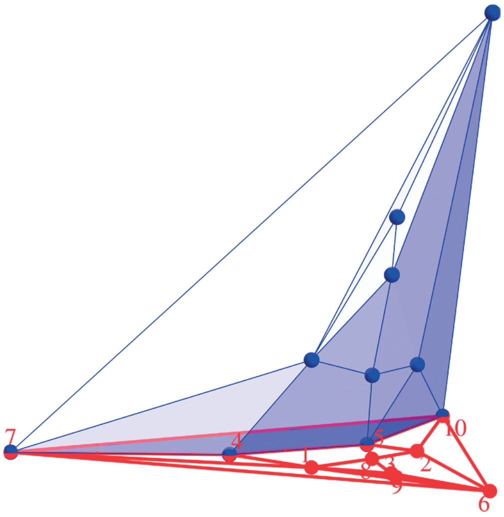

Figure. 13—"Lifting" of the rubber band representation of the graph of $\mathcal{P}$

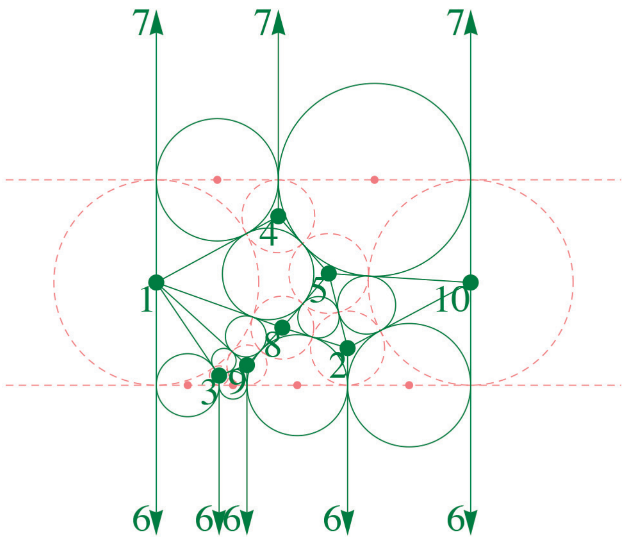

Figure 14.—Illustration of the theorem of Koebe-Andreev-Thurston I

Before considering here the third and last kind of proof of Steinitz theorem, note that this theorem claims, in particular, that a spherical triangular graph can be realized in space with straight edges. Recently, it was proved that the same happens with toroidal graphs [2]. But we know that the same does not happen in a quintuple torus — or sphere with five handles [6,14]. It is not clear what happens between the simple torus and the quintuple torus.

Finally, there is a third kind of proof of the theorem of Steinitz, that we may follow thoroughly in the work of Ziegler [19], for example.

Starting with the graph of the polytope $\mathscr{P}$ of Figure 14 and following [19], we obtained the graph of Figure 14, which has the following properties (we view a straight line as a circle of infinite radius and two parallel lines as tangent circles):

- Any vertex of the graph is the centre of a circle, and two circles are tangent if and only if the vertices are incident with an edge of the graph; these circles are in (dotted) pink[3] in Figure 14.
- Each face contains also a (green, in Figure 14) circle and the circles are tangent if and only if the faces are adjacent.
- Finally, the (pink) circles centred in the vertices and the (green) circles contained in the faces are pairwise mutually orthogonal.

It can be proved that this construction may be made for the graph of any polyhedron, and from this it follows the following remarkable theorem:

Theorem of Koebe-Andreev-Thurston.—Every graph with more than three vertices, planar and 3-connected is the graph of a polyhedron of edges tangent to a given sphere.

References

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[3] Note that the horizontal line above has centre in the vertex 7 and the one below in vertex 6.

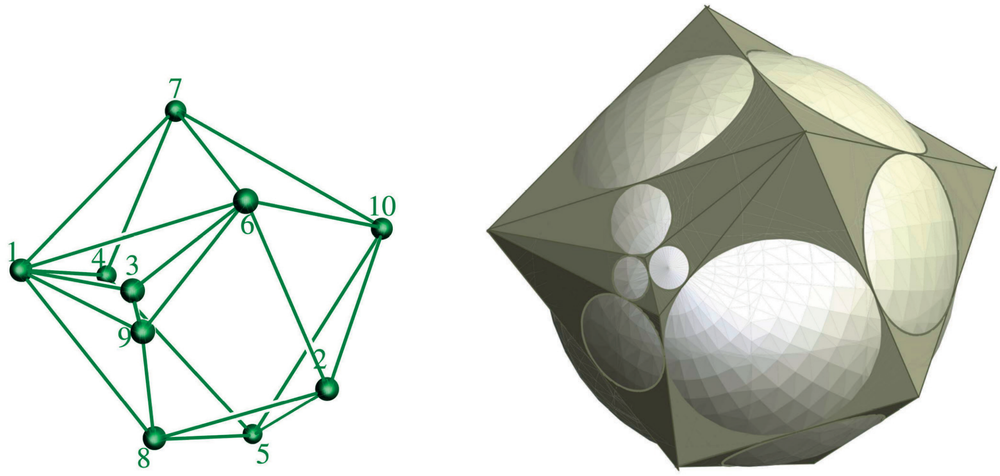

Figure 15.—Illustration of the theorem of Koebe-Andreev-Thurston II

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