Numerical semigroups problem list

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1. Notable elements and first problems

A numerical semigroup is a subset of $\mathbb{N}$ (here $\mathbb{N}$ denotes the set of nonnegative integers) that is closed under addition, contains the zero element, and its complement in $\mathbb{N}$ is finite.

If A is a nonempty subset of $\mathbb{N}$, we denote by $\langle A\rangle$ the submonoid of $\mathbb{N}$ generated by A, that is,

$$\langle A\rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N}, \lambda_i \in \mathbb{N}, a_i \in A\}.$$

It is well known (see for instance [41, 45]) that $\langle A\rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

If S is a numerical semigroup and $S = \langle A\rangle$ for some $A \subseteq S$, then we say that A is a system of generators of S, or that A generates S. Moreover, A is a minimal system of generators of S if no proper subset of A generates S. In [45] it is shown that every numerical semigroup admits a unique minimal system of generators, and it has finitely many elements.

Let S be a numerical semigroup and let $\{n_1 < n_2 < \cdots < n_e\}$ be its minimal system of generators. The integers n_1 and e are known as the multiplicity and embedding dimension of S, and we will refer to them by using $\mathrm{m}(S)$ and $\mathrm{e}(S)$, respectively. This notation might seem amazing, but it is not so if one takes into account that there exists a large list of manuscripts devoted to the study of analytically irreducible one-dimensional local domains via their value semigroups, which are numerical semigroups. The invariants we just introduced, together with others that will show up later in this work, have an interpretation in that context, and this is why they have been named in this way. Along this line, [3] is a good reference for the translation for the terminology used in the Theory of Numerical Semigroups and Algebraic Geometry.

Frobenius (1849–1917) during his lectures proposed the problem of giving a formula for the greatest integer that is not representable as a linear combination, with nonnegative integer coefficients, of a fixed set of integers with greatest common divisor equal to 1. He also raised the question of determining how many positive integers do not admit such a representation. With our terminology, the first problem is equivalent to that of finding a formula in terms of the generators of a numerical semigroup S of the greatest integer not belonging to S (recall that its complement in $\mathbb{N}$ is finite). This number is thus known in the literature as the Frobenius number of S, and we will denote it by $\mathrm{F}(S)$. The elements of $\mathrm{H}(S) = \mathbb{N} \setminus S$ are called gaps of S . Therefore the second problem consists in determining the cardinality of $\mathrm{H}(S)$, sometimes known as genus of S ([25]) or degree of singularity of S ([3]).

In [60] Sylvester solves the just quoted problems of Frobenius for embedding dimension two. For semigroups with embedding dimension greater than or equal to three these problems remain open. The current state of the problem is quite well collected in [30].

Let S be a numerical semigroup. Following the terminology introduced in [39] an integer x is said to be a pseudo-Frobenius number of S if $x \notin S$ and $x + S \setminus \{0\} \subseteq S$. We will denote by $\mathrm{PF}(S)$ the set of pseudo-Frobenius numbers of S. The cardinality of $\mathrm{PF}(S)$ is called the type of S(see [3]) and we will denote it by $\mathrm{t}(S)$. It is proved in [18] that if $\mathrm{e}(S) = 2$, then $\mathrm{t}(S) = 1$, and if $\mathrm{e}(S) = 3$, then $\mathrm{t}(S) \in \{1, 2\}$. It is also shown that if $\mathrm{e}(S) \geq 4$, then $\mathrm{t}(S)$ can be arbitrarily large, $\mathrm{t}(S) \leq \mathrm{m}(S) - 1$ and that $(\mathrm{t}(S) + 1)\mathrm{g}(S) \leq \mathrm{t}(s)(\mathrm{F}(S) + 1)$. This is the starting point of a new line of research that consists in trying to determine the type of a numerical semigroup, once other invariants like multiplicity, embedding dimension, genus or Frobenius number are fixed.

Wilf in [66] conjectures that if S is a numerical semigroup, then $\mathrm{e}(S)\mathrm{g}(S) \leq (\mathrm{e}(S) - 1)(\mathrm{F}(S) + 1)$. Some fami-

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lies of numerical semigroups for which it is known that the conjecture is true are collected in [16]. Other such families can be seen in [23,59]. The general case remains open.

Bras-Amorós computes in [5] the number of numerical semigroups with genus $g \in \{0, \dots, 50\}$, and conjectures that the growth is similar to that of Fibonacci's sequence. However it has not been proved yet that there are more semigroups of genus g than of genus $g+1$. Several attempts already appear in the literature. Kaplan [23] uses an approach that involves counting the semigroups by genus and multiplicity. He poses many related conjectures which could be taken literally and be proposed here as problems. We suggest them to the reader. A different approach, dealing with the asymptoptical behavior of the sequence of the number of numerical semigroups by genus, has been followed by Zhao [69]. Some progress has been achieved by Zhai [68], but many questions remain open.

2. Proportionally modular semigroups

Following the terminology introduced in [52], a proportionally modular Diophantine inequality is an expression of the form $ax \bmod b \le cx$, with a, b and c positive integers. The integers a, b and c are called the factor, the modulus and the proportion of the inequality, respectively. The set $S(a,b,c)$ of solutions of the above inequality is a numerical semigroup. We say that a numerical semigroup is proportionally modular if it is the set of solutions of some proportionally modular Diophantine inequality.

Given a nonempty subset A of $\mathbb{Q}_0^+$, we denote by $\langle A \rangle$ the submonoid of $(\mathbb{Q}_0^+, +)$ generated by A, whose definition is the same of that used in the previous section. Clearly, $S(A) = \langle A \rangle \cap \mathbb{N}$ is a submonoid of $\mathbb{N}$. It is proved in [52] that if a, b and c are positive integers with $c < a < b$, then $S(a,b,c) = S\left(\left[\frac{b}{a}, \frac{b}{a-c}\right]\right)$. Since $S(a,b,c) = \mathbb{N}$ when $a \ge c$, and the inequality $ax \bmod b \le cx$ has the same integer solutions as $(a \bmod b)x \bmod b \le cx$, the condition $c < a < b$ is not restrictive.

As a consequence of the results proved in [52], we have that a numerical semigroup S is proportionally modular if and only if there exist two positive rational numbers $\alpha < \beta$ such that $S = S([\alpha, \beta])$. This is also equivalent to the existence of an interval I, with nonempty interior, of the form $S = S(I)$ (see [55]).

By using the notation introduced in [54], a sequence of fractions $a_1/b_1 < a_2/b_2 < \cdots < a_p/b_p$ is said to be a Bézout sequence if $a_1, \dots, a_p, b_1, \dots, b_p$ are positive integers and $a_{i+1}b_i - a_i b_{i+1} = 1$ for all $i \in \{1, \dots, p-1\}$. The importance of the Bézout sequences in the study of proportionally modular semigroups highlights in the following result proved in [54]. If $a_1/b_1 < a_2/b_2 < \cdots < a_p/b_p$ is a Bézout sequence, then $S([a_1/b_1, a_p/b_p]) = \langle a_1, \dots, a_p \rangle$.

A Bézout sequence $a_1/b_1 < a_2/b_2 < \cdots < a_p/b_p$ is proper if $a_{i+h}b_i - a_i b_{i+h} \ge 2$ for all $h \ge 2$ with $i, i+h \in \{1, \dots, p\}$. Clearly, every Bézout sequence can be reduced (by removing some terms) to a proper Bézout sequence with the same ends as the original one. It is showed in [9], that if $a_1/b_1 < a_2/b_2$ are two reduced fractions, then there exists an unique proper Bézout sequence with ends a_1/b_1 and a_2/b_2. Furthermore, in this work a procedure for obtaining this sequence is given.

It is proved in [54] that if $a_1/b_1 < a_2/b_2 < \cdots < a_p/b_p$ is a proper Bézout sequence, then there exists $h \in \{1, \dots, p\}$ such that $a_1 \ge \cdots \ge a_h \le \cdots \le a_p$ (the sequence $a_1, \dots, a_p$ is convex). The following characterization is also proved there: a numerical semigroup is proportionally modular if and only if there exists a convex ordering if its minimal generators $n_1, \dots, n_e$ such that $\gcd\{n_i, n_{i+1}\} = 1$ for all $i \in \{1, \dots, e-1\}$ and $n_{j-1} + n_{j+1} \equiv 0 \pmod{n_j}$ for all $j \in \{2, \dots, e-1\}$.

A modular Diophantine inequality is a proportionally modular Diophantine inequality with proportion equal to one. A numerical semigroup is said to be modular if it is the set of solutions of some modular Diophantine inequality. Clearly, every modular numerical semigroup is proportionally modular, and this inclusion is strict as it is proved in [52]. A formula for $g(S(a,b,1))$ in function of a and b is given in [53]. The problems of finding formulas for $F(S(a,b,1))$, $m(S(a,b,1))$, $t(S(a,b,1))$ and $e(S(a,b,1))$ remain open. It is not known if the mentioned conjecture of Wilf is true for modular semigroups neither.

A semigroup of the form $\{0, m, \rightarrow\}$ is said to be ordinary. A numerical semigroup S is an open modular numerical semigroup if it is ordinary or of it is the form $S = S(]b/a, b/(a-1)[)$ for some integers $2 \le a < b$. Therefore these semigroups are proportionally modular. Moreover, it is proved in [55] that every proportionally modular numerical semigroup can be expressed as a finite intersection of open modular numerical semigroups. The formulas for the Frobenius number, the genus and the type of open modular semigroups are also obtained in the just quoted work. However the rest of the problems previously suggested for modular numerical semigroups remain still open.

As we mentioned above, a characterization for proportionally modular numerical semigroups in terms of its systems of minimal generators is given in [54]. The question of giving formulas for the Frobenius number, genus and type of a proportionally modular numerical semigroup in terms of its system of minimal generators remains unsolved too.

Following the terminology in [57], a contracted modular Diophantine inequality is an expression of the form $ax \bmod b \le x - c$, where a, b and c are nonnegative integers and $b \ne 0$. Let us denote by $T(a,b,c)$ the set of integer solutions of the last inequality. Then $T(a,b,c) \cup \{0\}$ is a numerical semigroup. An algorithm that allows us to determine whether a semigroup is the set of solutions of a contracted modular Diophantine equation is given in [57]. A formula for the genus of $T(a,b,c) \cup \{0\}$ is also given there.

A contracted proportionally modular Diophantine inequality is an expression of the form $ax \bmod b \le cx - d$, with a, b, c and d nonnegative integers and $b \ne 0 \ne c$. If we denote by $P(a,b,c,d)$ the set of solutions of such an inequality, then $P(a,b,c,d) \cup \{0\}$ is a numerical semigroup. It is not yet known an algorithm to determine whether a semigroup is of this form.

The Stern-Brocot tree gives a recursive method for constructing all the reduced fractions x/y, with x and y positive integers (see [20]). For constructing this tree we start with the expressions $0/1$ and $1/0$. In each step of the process we insert between each two consecutive expressions m/n and m'/n' its median $(m+m')/(n+n')$. We obtain in this way the sequences

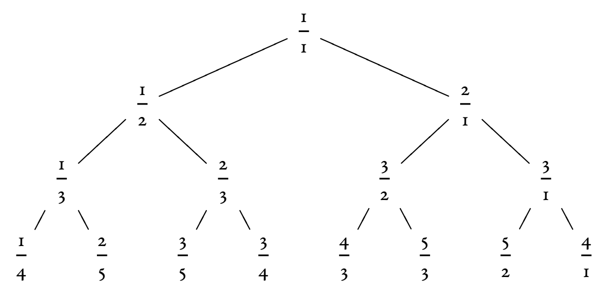
$$\frac{0}{1} < \frac{1}{1} < \frac{1}{0}$$

$$\frac{0}{1} < \frac{1}{2} < \frac{1}{1} < \frac{2}{1} < \frac{1}{0}$$

$$\frac{0}{1} < \frac{1}{3} < \frac{1}{2} < \frac{2}{3} < \frac{1}{1} < \frac{3}{2} < \frac{2}{1} < \frac{3}{1} < \frac{1}{0}$$

$$\dots$$

The Stern-Brocot tree can now be obtained by connecting each median with the fractions used for computing it and being in the previous level but not in the levels above it.



It is proved in [9] that if m/n is the common predecessor of two fractions $a/b < c/d$ in the Stern-Brocot tree, then m is the multiplicity of $S([a/b, c/d])$. It could be nice to obtain other constants of the semigroup by looking at this tree.

3. The quotient of a numerical semigroup by a positive integer

Let S be a numerical semigroup and p be a positive integer. Let us denote by

$$\frac{S}{p} = \{x \in \mathbb{N} \mid px \in S\}.$$

Clearly, S/p is a numerical semigroup, and we will call it the quotient of S by p. According to this notation, we will call $S/2$ one half of S and that $S/4$ is a quarter of S. These two cases will have an special importance in this section.

It is proved in [56] that a numerical semigroup is proportionally modular if and only if it is the quotient of an embedding dimension two numerical semigroup by a positive integer. This result is improved in [32] by proving that a numerical semigroup is proportionally modular if and only if it is of the form $\langle a, a+1 \rangle / d$ with a and d positive integers. We still do not have formulas for $F(\langle a, a+1 \rangle / d)$, $g(\langle a, a+1 \rangle / d)$, $m(\langle a, a+1 \rangle / d)$, $t(\langle a, a+1 \rangle / d)$ and $e(\langle a, a+1 \rangle / d)$.

The next step in this line of research would be studying those numerical semigroups that are the quotient of a numerical semigroup with embedding dimension three by a positive integer. Unfortunately we do not have a procedure that allows us to distinguish such a semigroup from the rest. Moreover, we still do not know of any example of semigroups that are not of this form.

A numerical semigroup S is symmetric if $x \in \mathbb{Z} \setminus S$ implies $F(S) - x \in S$. These semigroups have been widely studied. Their main motivation comes from a work by Kunz ([26]) from which it can be deduced that a numerical semigroup is symmetric if and only if its associated numerical semigroup ring is Gorenstein. Symmetric numerical semigroups always have odd Frobenius number, thus for numerical semigroups with even Frobenius number, the equivalent notion to symmetric semigroups is that of pseudo-symmetric numerical semigroups. We say that S is a pseudo-symmetric numerical semigroup if it has even Frobenius number and for all $x \in \mathbb{Z} \setminus S$, we have either $F(S) - x \in S$ or $x = F(S)/2$. The concept of irreducible semigroup, introduced in [40], collects these two families of semigroups. A numerical semigroup is irreducible if it cannot be expressed as the intersection of two semigroups that contain it properly. It can be proved that a semigroup is irreducible if and only if it is either symmetric (with odd Frobenius number) or pseudo-symmetric (with even Frobenius number).

Intuition (and the tables of the number of numerical semigroups with a given genus or Frobenius number we have) tells us that the percentage of irreducible numerical semigroups is quite small. It is proved in [44] that every numerical semigroup is one half of an infinite number of symmetric numerical semigroups. The apparent parallelism between symmetric and pseudo-symmetric numerical semigroups fails as we can see in [37], where it is proved that a numerical semigroup is irreducible if and only it is one half of a pseudo-symmetric numerical semigroup. As a consequence we have that every numerical semigroup is a quarter of infinitely many pseudo-symmetric numerical semigroups. In [61], it is also shown that for every positive integer d and every numerical semigroup S, there exist infinitely many symmetric numerical semigroups T such that $S = T/d$, and if $d \geq 3$, then there exist infinitely many pseudo-symmetric numerical semigroups T with $S = T/d$.

From the definition, we deduce that a numerical semigroup S is symmetric if and only if $\mathrm{g}(S) = (\mathrm{F}(S) + 1)/2$. Therefore these numerical semigroups verify Wilf's conjecture previously mentioned. We raise the following question. If a numerical semigroup verifies Wilf's conjecture, then does so its half?

It can easily be seen that every numerical semigroup can be expressed as a finite intersection of irreducible numerical semigroups. A procedure for obtaining such a decomposition is given in [50]. Furthermore it is also explained how to obtain a decomposition with the least possible number of irreducibles. We still do not know how many numerical semigroups appear in these minimal decompositions, moreover, we wonder if there exists a positive integer N such that every numerical semigroup can be expressed as an intersection of at most N irreducible numerical semigroups.

In [62] Toms introduces a class of numerical semigroups that are the positive cones of the K_0 groups of certain C^*-algebras. Given a numerical semigroup we say, inspired in this work, that it admits a Toms decomposition if and only if there exist positive integers $q_1, \dots, q_n, m_1, \dots, m_n$ and L such that $\gcd\{q_i, m_i\} = \gcd\{L, m_i\} = \gcd\{L, q_i\} = 1$ for all $i \in \{1, \dots, n\}$ and $S = (1/L) \cap_{i=1}^{n} \langle q_i, m_i \rangle$.

As $(1/L) \cap_{i=1}^{n} \langle q_i, m_i \rangle = \cap_{i=1}^{n} \langle q_i, m_i \rangle / L$, we have that if a numerical semigroup admits a Toms decomposition, then S is a finite intersection of proportionally modular numerical semigroups. It is proved in [46] that the reciprocal is also true. Therefore, a numerical semigroup admits a Toms decomposition if and only if it is an intersection of finitely many proportionally modular numerical semigroups. These kind of semigroups are studied in [14], where an algorithm for distinguishing whether a numerical semigroup is an intersection of finitely many proportionally modular numerical semigroups is given. Furthermore, in the affirmative case it gives us a minimal decomposition, and in the negative case it gives us the least numerical semigroup which is intersection of proportionally modular semigroups and contains the original numerical semigroup (its proportionally modular closure).

It is conjectured in [57] that every contracted modular numerical semigroup admits a Toms decomposition.

Note that the numerical semigroups that admit a Toms decomposition are those that are the set of solutions of a system of proportionally modular Diophantine inequalities. It is proved in [32] that two systems of inequalities are always equivalent to another system with all the inequalities having the same modulus, which moreover can be chosen to be prime. Now we raise the following question: is every system of proportionally modular Diophantine inequalities equivalent to a system with all proportions being equal to one?, or equivalently, if a numerical semigroup admits a Tom decomposition, can it be expressed as an intersection of modular numerical semigroups?

Following the terminology introduced in [51], a gap x in a numerical semigroup S is said to be fundamental if $\{2x, 3x\} \subset S$ (and therefore $kx \in S$ for every integer with $k \geq 2$). Let us denote by $\mathrm{FG}(S)$ the set of all fundamental gaps of S. If $X \subseteq \mathbb{Z}$, then $\mathrm{D}(X)$ will denote the union of all positive divisors of the elements of X. It can easily be shown that $S = \mathbb{N} \setminus \mathrm{D}(\mathrm{FG}(S))$. Therefore, a way to represent a semigroup is by giving its fundamental gaps. This representation is specially useful when studying the quotient of a semigroup S by a positive integer d, since $\mathrm{FG}(S/d) = \{h/d \mid h \in \mathrm{FG}(S),\ h \equiv 0 \pmod d\}$.

The cardinality of the set of fundamental gaps of a semigroup is an invariant of the semigroup. We can therefore open a new line of research by studying numerical semigroups attending to their number of fundamental gaps. It would be also interesting to find simple sufficient conditions that allow us to decide when a subset X of $\mathbb{N}$ is the set of fundamental gaps of some numerical semigroup.

Let S be a numerical semigroup. In [33] the set T of all numerical semigroups such that $S = T/2$ is studied, the semigroup of the "doubles" of S. In the just quoted work we raise the question of finding a formula that depends on S and allows us to compute the minimum of the Frobenius numbers of the doubles of S.

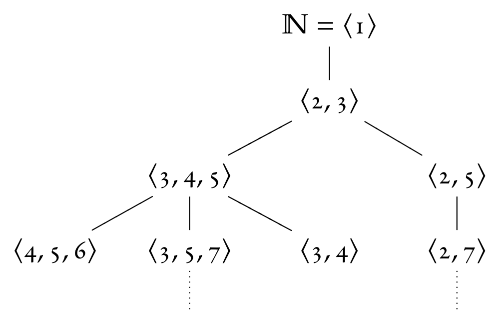
Following this line we can ask ourselves about the set of all "triples" (or multiples in general) of a numerical semigroup.

Finally, it would be interesting to characterize the families of numerical semigroups verifying that any of its elements can be realized as a quotient of some element of the family by a fixed positive integer.

4. Frobenius Varieties

A directed graph G is a pair (V, E), where V is a nonempty set whose elements are called vertices, and E is a subset of $\{(u, v) \in V \times V \mid u \neq v\}$. The elements of E are called edges of the graph. A path connecting two vertices x and y of G is a sequence of distinct edges of the form $(v_0, v_1), (v_1, v_2), \dots, (v_{n-1}, v_n)$ with $v_0 = x$ and $v_n = y$. A graph G is a tree if there exists a vertex r (called the root of G) such that for any other vertex x of G, there exists an unique path connecting x and r. If (x, y) is an edge of the tree, then x is a son of y. A vertex of a tree is a leaf if it has no sons.

Let $\mathscr{S}$ be the set of all numerical semigroups. We define the graph associated to $\mathscr{S}$, $\mathscr{G}(\mathscr{S})$, to be the graph whose vertices are all the elements of $\mathscr{S}$ and $(T, S) \in \mathscr{S} \times \mathscr{S}$ is an edge if $S = T \cup \{\mathrm{F}(T)\}$. In [45], it is proved that $\mathscr{G}(\mathscr{S})$ is a tree with root $\mathbb{N}$, and that the sons of $S \in \mathscr{S}$ are the subsets $S \setminus \{x_1\}, \dots, S \setminus \{x_r\}$, where $x_1, \dots, x_r$ are the minimal generators of S greater than $\mathrm{F}(S)$. Therefore S is a leaf of $\mathscr{G}(\mathscr{S})$ if it has no minimal generators greater than $\mathrm{F}(S)$. These results allow us to construct recursively the set of numerical semigroups starting with $\mathbb{N}$.



The level of a vertex in a directed graph is the length of the path connecting this vertex with the root. Note that in $\mathscr{G}(\mathscr{S})$ the level of a vertex coincides with its genus as numerical semigroup. Therefore, the Bras-Amorós' conjecture quoted in the end of the first section can be reformulated by saying that in $\mathscr{G}(\mathscr{S})$ there are more vertices in the $(n+1)$th level than in the nth one.

A Frobenius variety is a nonempty family $\mathscr{V}$ of numerical semigroups such that

1) if $S, T \in \mathscr{V}$, then $S \cap T \in \mathscr{V}$,
2) if $S \in \mathscr{V}$, $S \neq \mathbb{N}$, then $S \cup \{\mathrm{F}(S)\} \in \mathscr{V}$.

The concept of Frobenius variety was introduced in [38] with the aim of generalizing most of the results in [6, 14, 48, 49]. In particular, the semigroups that belong to a Frobenius variety can be arranged as a directed tree with similar properties to those of $\mathscr{G}(\mathscr{S})$.

Clearly, $\mathscr{S}$ is a Frobenius variety. If $A \subseteq \mathbb{N}$, then $\{S \in \mathscr{S} \mid A \subseteq S\}$ is also a Frobenius variety. In particular, $\mathscr{O}(S)$, the set of all numerical semigroups that contain S, is a Frobenius variety. We next give some interesting examples of Frobenius varieties.

Inspired by [1], Lipman introduces and motivates in [27] the study of Arf rings. The characterization of them via their numerical semigroup of values, brings us to the following concept: a numerical semigroups S is said to be Arf if for every $x, y, z \in S$, with $x, y \geq z$ we have $x + y - z \in S$. It is proved in [48] that the set of Arf numerical semigroups is a Frobenius variety.

Saturated rings were introduced independently in three distinct ways by Zariski ([67]), Pham-Teissier ([29]) and Campillo ([10]), although the definitions given in these works are equivalent on algebraically closed fields of characteristic zero. Like in the case of numerical semigroups with the Arf property, saturated numerical semigroups appear when characterizing these rings in terms of their numerical semigroups of values. A numerical semigroup S is saturated if for every $s, s_1, \dots, s_r \in S$ with $s_i \leq s$ for all $i \in \{1, \dots, r\}$ and $z_1, \dots, z_r \in \mathbb{Z}$ being integers such that $z_1 s_1 + \cdots + z_r s_r \geq 0$, then we have $s + z_1 s_1 + \cdots + z_r s_r \in S$. It is proved in [49] that the set of saturated numerical semigroups is a Frobenius variety.

The class of Arf and Saturated numerical semigroups is also closed under quotients by positive integers as shown in [17], though the larger class of maximal embedding dimension numerical semigroups is not (if S is a numerical semigroup, then $\mathrm{e}(S) \leq \mathrm{m}(S)$; a numerical semigroup is said to be a maximal embedding dimension semigroup, or to have maximal embedding dimension, if $\mathrm{e}(S) = \mathrm{m}(S)$). What is the Frobenius variety generated by maximal embedding dimension numerical semigroups?

As a consequence of [14] and [46], it can be deduced that the set of numerical semigroups that admit a Toms decomposition is a Frobenius variety. Every semigroup with embedding dimension two admits a Toms decomposition. Is the variety of numerical semigroups admitting a Toms decomposition the least Frobenius variety containing all semigroups with embedding dimension two?

The idea of pattern of a numerical semigroup was introduced in [6] with the aim of trying to generalize the concept of Arf numerical semigroup. A pattern P of length n is a linear homogeneous polynomial with non-zero integer coefficients in $x_1, \dots, x_n$ (for $n = 0$ the only pattern is $p = 0$). We will say that numerical semigroup S admits a pattern $a_1 x_1 + \dots + a_n x_n$ if for every sequence $s_1 \geq s_2 \geq \cdots \geq s_n$ of elements in S, we have

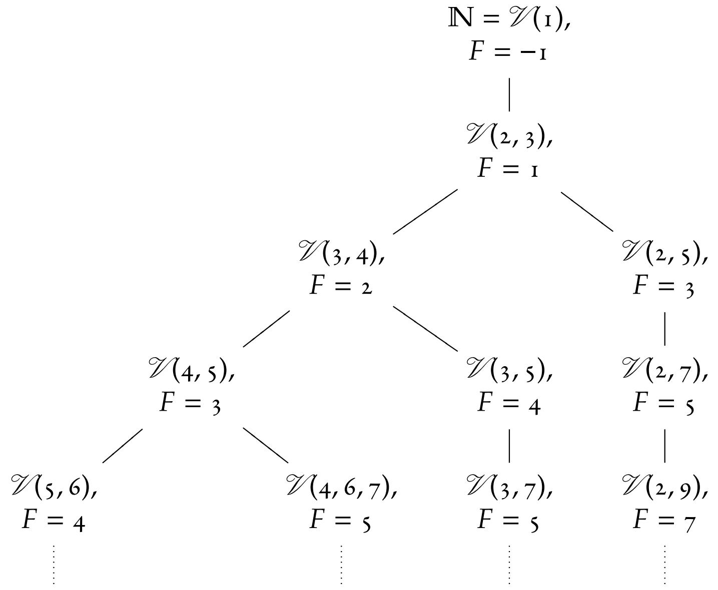


Figure 1

$a_1 s_1 + \cdots + a_n s_n \in S$. We denote by $\mathscr{S}_P$ the set of all numerical semigroups that admit a pattern P. Then the set of numerical semigroups with the Arf property is $\mathscr{S}_{x_1+x_2-x_3}$. It is proved in [6] that for every pattern P of a special type (strongly admissible), $\mathscr{S}_P$ is a Frobenius variety. What varieties arise in this way? It would be interesting to give a weaker definition of pattern such that every variety becomes the variety associated to a pattern.

The intersection of Frobenius varieties is again a Frobenius variety. This fact allows us to construct new Frobenius varieties from known Frobenius varieties and moreover, it allows us to talk of the Frobenius variety generated by a family X of numerical semigroups. This variety will be denoted by $\mathscr{F}(X)$, and it is defined to be the intersection of all Frobenius varieties containing X. If X is finite, then $\mathscr{F}(X)$ is finite and it is shown in [38] how to compute all the elements of $\mathscr{F}(X)$.

Let $\mathscr{V}$ be a Frobenius variety. A submonoid M of $\mathbb{N}$ is a $\mathscr{V}$-monoid if it can be expressed as an intersection of elements of $\mathscr{V}$. It is clear that the intersection of $\mathscr{V}$-monoids is again a $\mathscr{V}$-monoid. Thus given $A \subseteq \mathbb{N}$ we can define the $\mathscr{V}$-monoid generated by A as the intersection of all $\mathscr{V}$-monoids containing A. We will denote by $\mathscr{V}(A)$ this $\mathscr{V}$-monoid and we will say that A is a $\mathscr{V}$-system of generators of it. If there is no proper subset of A being a $\mathscr{V}$-system of generators $\mathscr{V}(A)$, then A is a minimal $\mathscr{V}$-system of generators of $\mathscr{V}(A)$. It is proved in [38] that every $\mathscr{V}$-monoid admits an unique minimal $\mathscr{V}$-system of generators, and that moreover this system is finite.

We define the directed graph $\mathscr{G}(\mathscr{V})$ in the same way we defined $\mathscr{G}(\mathscr{S})$, that is, as the graph whose vertices are the elements of $\mathscr{V}$, and $(T, S) \in \mathscr{V} \times \mathscr{V}$ is an edge of the above graph if $S = T \cup \{\mathrm{F}(T)\}$. This graph is a tree with root $\mathbb{N}$ ([38]). Moreover, the sons of a semigroup S in $\mathscr{V}$ are $S \setminus \{x_1\}, \ldots, S \setminus \{x_r\}$, where $x_1, \ldots, x_r$ are the minimal $\mathscr{V}$-generators of S greater than $\mathrm{F}(S)$. This fact allows us to find all the elements of the variety $\mathscr{V}$ from $\mathbb{N}$.

Figure 1 represents part of the tree associated to the variety of numerical semigroups with the Arf property.

Figure 2 represents part of the tree corresponding to saturated numerical semigroups.

As a generalization of Bras-Amorós' conjecture, we can raise the following question. If $\mathscr{V}$ is a Frobenius variety, does there exist on $\mathscr{G}(\mathscr{V})$ more vertices in the $(n+1)$th level than in the nth one? The answer to this question is no, as it is proved in [38, Example 26]. However, the same question in the case of $\mathscr{V}$ being infinite remains open. Another interesting question would be characterizing those Frobenius varieties that verify the Bras-Amorós' conjecture.

If $\mathscr{V}$ is a Frobenius variety and $S \in \mathscr{V}$, then it is known that S admits an unique minimal $\mathscr{V}$-system of generators, and moreover it is finite. The cardinality of

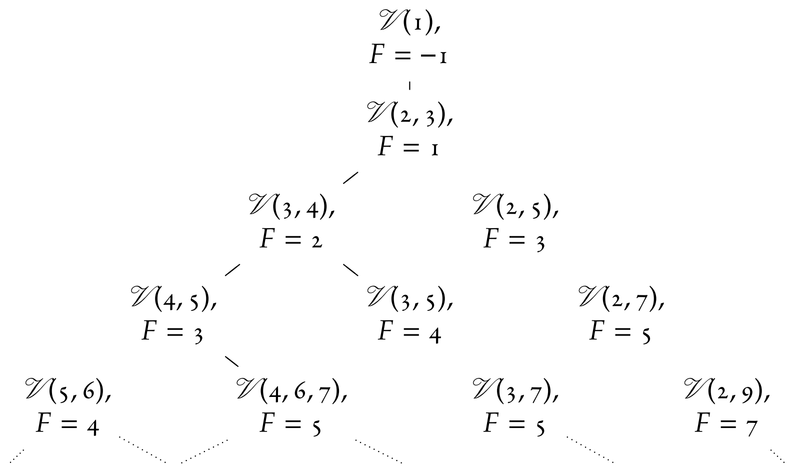


Figure 2

the set above is an invariant of S that will be called the embedding $\mathscr{V}$-dimension of S, and it will be denoted by $\mathrm{e}_{\mathscr{V}}(S)$. As a generalization of Wilf's conjecture, we would like to characterize those Frobenius varieties $\mathscr{V}$ such that for every $S \in \mathscr{V}$, then $\mathrm{e}_{\mathscr{V}}(S)\mathrm{g}(S) \leq (\mathrm{e}_{\mathscr{V}}(S) - 1)(\mathrm{F}(S) + 1)$.

Clearly, the Frobenius variety generated by irreducible numerical semigroups is $\mathscr{S}$, the set of all numerical semigroups. What is the Frobenius variety generated only by the symmetric ones? and by the pseudo-symmetric ones?

5. Presentations of a numerical semigroup

Let $(S, +)$ be a commutative monoid. A congruence σ over S is an equivalence relation that is compatible with addition, that is, if $a \sigma b$ with $a, b \in S$, then $(a + c)\sigma(b + c)$ for all $c \in S$. The set S/σ endowed with the operation $[a] + [b] = [a + b]$ is a monoid. We will call it the quotient monoid of S by σ.

If S is generated by $\{s_1, \ldots, s_n\}$, then the map $\varphi : \mathbb{N}^n \to S$, $(a_1, \ldots, a_n) \mapsto a_1 s_1 + \cdots + a_n s_n$ is a monoid epimorphism. Therefore S is isomorphic to $\mathbb{N}^n / \sim_S$, where $\sim_S$ is the kernel congruence of φ, that is, $a \sim_S b$ if $\varphi(a) = \varphi(b)$.

The intersection of congruences over a monoid S is again a congruence over S. This fact allows us, given $\sigma \subseteq S \times S$, to define the concept of congruence generated by σ as the intersection of all congruences over S containing σ, and it will be denoted by $\langle \sigma \rangle$.

Rédei proves in [31] that every congruence over $\mathbb{N}^n$ is finitely generated, that is, there exists a subset of $\mathbb{N}^n \times \mathbb{N}^n$ with finitely many elements generating it. As a consequence we have that giving a finitely generated monoid is, up to isomorphism, equivalent to giving a finite subset of $\mathbb{N}^n \times \mathbb{N}^n$.

If S is a numerical semigroup with minimal generators system $\{n_1, \ldots, n_e\}$, then there exists a finite subset σ of $\mathbb{N}^e \times \mathbb{N}^e$ such that S is isomorphic to $\mathbb{N}^e / \langle \sigma \rangle$. We say that σ is a presentation of S. If moreover σ has the least possible cardinality, then σ is a minimal presentation of S.

A (non directed) graph G is a pair (V, E), where V is a nonempty set of elements called vertices, and E is a subset of $\{\{u, v\} \mid u, v \in V, u \neq v\}$. The non ordered pair $\{u, v\}$ will be denoted by $\overline{uv}$, and if it belongs to E, then we say that it is an edge of G. A sequence of the form $\overline{v_0 v_1}, \overline{v_1 v_2}, \ldots, \overline{v_{m-1} v_m}$ is a path of length m connecting the vertices v_0 and v_m. A graph is connected if any two distinct vertices are connected by a path. A graph $G' = (V', E')$ is said to be a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$. A connected component of G is a maximal connected subgraph of G. It is well known (see for instance [28]) that a connected graph with n vertices has at least $n - 1$ edges. A (finite) tree with n vertices is a connected graph with $n - 1$ edges.

Let us remind now the method described in [35] for computing the minimal presentation of a numerical semigroup. Let S be a numerical semigroup with minimal system of generators $\{n_1, \dots, n_e\}$. For each $n \in S$, let us define $G_n = (V_n, E_n)$, where $V_n = \{n_i \mid n - n_i \in S\}$ and $E_n = \{\overline{n_i n_j} \mid n - (n_i + n_j) \in S, i \neq j\}$. If G_n is connected, we take $\sigma_n = \emptyset$. If G_n is not connected and $V_1, \dots, V_r$ are the sets of vertices corresponding to the connected components in G_n, then we define $\sigma_n = \{(\alpha_1, \alpha_2), (\alpha_1, \alpha_3), \dots, (\alpha_1, \alpha_r)\}$, where $\alpha_i \in \varphi^{-1}(n)$ and its j-th component is zero whenever $n_j \notin V_i$. It is proved in [35] that $\sigma = \cup_{n \in S} \sigma_n$ is a minimal presentation for S. Let us notice that the set $\mathrm{Betti}(S) = \{n \in S \mid G_n \text{ is not connected}\}$ is finite, and that its cardinality is an invariant of S. A line of research could be the study of Betti(S), and its relation with other invariants of S mentioned above. In [19] affine semigroups (and thus numerical semigroups) with a single Betti element are studied. What are those numerical semigroups having two or three Betti elements?

It is also shown in [35] how all the minimal presentations of a semigroup are. In particular, we can determine whether a numerical semigroup admits a unique minimal presentation. Motivated by the idea of generic ideal, we may ask what are the numerical semigroups that admit a unique minimal presentation, and characterize them in terms of their minimal generators.

If S is a numerical semigroup, then the cardinality of a minimal presentation of S is greater than or equal to $\mathrm{e}(S) - 1$. Those semigroups that attain this bound are said to be complete intersections. This kind of semigroup has been well studied, and Delorme gives in [15] a good characterization of them. Every numerical semigroup with embedding dimension two is a complete intersection, and every complete intersection is symmetric (see [21]). We raise the following questions. What semigroups can be expressed as the quotient of a complete intersection by a positive integer? What is the least Frobenius variety containing all the complete intersection numerical semigroups?

Let S_1 and S_2 be two numerical semigroups minimally generated by $\{n_1, \dots, n_r\}$ and $\{n_{r+1}, \dots, n_e\}$, respectively. Let $\lambda \in S_1 \setminus \{n_1, \dots, n_r\}$ and $\mu \in S_2 \setminus \{n_{r+1}, \dots, n_e\}$, such that $\gcd\{\lambda, \mu\} = 1$. We then say that $S = \langle \mu n_1, \dots, \mu n_r, \lambda n_{r+1}, \dots, \lambda n_e \rangle$ is a gluing to S_1 and S_2. It is proved in [45] how given minimal presentations of S_1 and S_2, one easily gets a minimal presentation of S. The characterization given by Delorme in [15], with this notation, can be reformulated in the following way: a numerical semigroup is a complete intersection if and only if is a gluing to two numerical semigroups that are a complete intersection. A consequence of this result is that the set of semigroups that are a complete intersection is the least family of numerical semigroups containing $\mathbb{N}$ being closed under gluing. It is well known that the family of numerical symmetric semigroups is also closed under gluing ([45]). It would be interesting to study other families closed under gluing. Which is the least family containing those semigroups with maximal embedding dimension and closed under gluing?

Bresinsky gives in [7] a family of numerical semigroups with embedding dimension four and with cardinality of its minimal presentations arbitrarily large. This fact proves that the cardinality of a minimal presentation of a numerical semigroup cannot be upper bounded just in function of its embedding dimension. Bresinski also proves in [8] that the cardinality for a minimal presentation of a symmetric numerical semigroup with embedding dimension four can only be three or five. It is conjectured in [36] that if S is a symmetric numerical semigroup with $\mathrm{e}(S) \geq 3$, then the cardinality of a minimal presentation for S is less than or equal to $\mathrm{e}(S)(\mathrm{e}(S) - 1)/2 - 1$. Barucci [2] proves with the semigroup $\langle 19, 23, 29, 31, 37 \rangle$ that the conjecture above is not true. However, the problem of determining if the cardinality of a minimal presentation of a symmetric numerical semigroup can be bounded in function of the embedding dimension remains open.

Let σ be a finite subset of $\mathbb{N}^n \times \mathbb{N}^n$. By using the results in [41, 45] it is possible to determine algorithmically whether $\mathbb{N}^n/\langle\sigma\rangle$ is isomorphic to a numerical semigroup. However we miss in the literature families of subsets σ of $\mathbb{N}^n$ so that we can assert, without using algorithms, that $\mathbb{N}^n/\langle\sigma\rangle$ is isomorphic to a numerical semigroup. More specifically, we suggest the following problem: given

$$\sigma = \{((c_1, 0, \dots, 0), (0, a_{1_1}, \dots, a_{1_n})), \dots, \\ ((0, \dots, c_n), (a_{n_1}, \dots, a_{n_{n-1}}, 0))\},$$

which conditions the integers c_i and a_{j_k} have to verify so that $\mathbb{N}^n/\langle\sigma\rangle$ is isomorphic to a numerical semigroup? Herzog proved in [21] that embedding dimension three numerical semigroups always have a minimal presentation of this form. Neat numerical semigroups introduced by Komeda in [24] are also of this form.

6. Numerical semigroups with embedding dimension three

Herzog proves in [21] that a numerical semigroup with embedding dimension three is symmetric if and only if it is a complete intersection. This fact allows us to characterize symmetric numerical semigroups with embedding dimension three in the following way (see [45]). A numerical semigroup S with $\mathbf{e}(S) = 3$ is symmetric if and only if $S = \langle am_1, am_2, bm_1 + cm_2 \rangle$, with a, b, c, m_1 and m_2 nonnegative integers, such that m_1, m_2, a and $b + c$ are greater than or equal to two and $\gcd\{m_1, m_2\} = \gcd\{a, bm_1 + cm_2\} = 1$. Moreover, as it is proved in [45],

$$\mathrm{F}(\langle am_1, am_2, bm_1 + cm_2 \rangle) = \\ = a(m_1 m_2 - m_1 - m_2) + (a - 1)(bm_1 + cm_2).$$

We also have a formula for the genus, since S is symmetric, $\mathrm{g}(S) = (\mathrm{F}(S) + 1)/2$. Finally, we also know the type, since it is proved in [18] that a numerical semigroup is symmetric if and only if its type is equal to one.

We study in [43] the set of pseudo-symmetric numerical semigroups with embedding dimension three. In particular, we give the following characterization. A numerical semigroup S with $\mathrm{e}(S) = 3$ is pseudo-symmetric if and only if for some ordering of its minimal generators, by taking

$$\Delta = \sqrt{(\textstyle\sum n_i)^2 - 4(n_1 n_2 + n_1 n_3 + n_2 n_3 - n_1 n_2 n_3)},$$

then

$$\left\{\frac{n_1 - n_2 + n_3 + \Delta}{2n_1}, \frac{n_1 + n_2 - n_3 + \Delta}{2n_2}, \frac{-n_1 + n_2 + n_3 + \Delta}{2n_3}\right\} \subset \mathbb{N}.$$

Moreover, in this case, $\mathrm{F}(\langle n_1, n_2, n_3 \rangle) = \Delta - (n_1 + n_2 + n_3)$. We also know the genus and the type, since if S is a pseudo-symmetric numerical semigroups, then $\mathrm{g}(S) = (\mathrm{F}(S) + 2)/2$ and by [18], $\mathrm{t}(S) = 2$.

Bresinsky ([7]) and Komeda ([24]) fully characterize those symmetric and pseudo-symmetric numerical semigroups, respectively, with embedding dimension four. They show that their minimal presentations always have cardinality five.

Curtis proves in [13] the impossibility of giving an algebraic formula for the Frobenius number of a numerical semigroup in terms of its minimal generators on embedding dimension three. We raise the following question. Given a polynomial $f(x_1, x_2, x_3, x_4) \in \mathbb{Q}[x_1, x_2, x_3, x_4]$, study the family of numerical semigroups S such that if S is minimally generated by $n_1 < n_2 < n_3$, and F is the Frobenius number of S, then $f(n_1, n_2, n_3, F) = 0$.

Our aim now is studying the set of numerical semigroups with embedding dimension three in general. By [18], we know that these semigroups have type one or two, and by using [22, 34] if we are concerned with the Frobenius number and the genus, we can focus ourselves in those numerical semigroups whose minimal generators are pairwise relatively prime. The following result appears in [42]. Let n_1, n_2 and n_3 three pairwise relatively prime positive integers. Then the system of equations

$$\begin{aligned} n_1 &= r_{12} r_{13} + r_{12} r_{23} + r_{13} r_{32}, \\ n_2 &= r_{13} r_{21} + r_{21} r_{23} + r_{23} r_{31}, \\ n_3 &= r_{12} r_{31} + r_{21} r_{32} + r_{31} r_{32}. \end{aligned}$$

has a (unique) positive integer solution if and only if $\{n_1, n_2, n_3\}$ generates minimality $\langle n_1, n_2, n_3 \rangle$. In [42] the authors give formulas for the pseudo-Frobenius number and the genus of $\langle n_1, n_2, n_3 \rangle$ from the solutions of the above system. Thus it seems natural to ask, given positive integers r_{ij}, with $i, j \in \{1, 2, 3\}$, when $r_{12} r_{13} + r_{12} r_{23} + r_{13} r_{32}$, $r_{13} r_{21} + r_{21} r_{23} + r_{23} r_{31}$ and $r_{12} r_{31} + r_{21} r_{32} + r_{31} r_{32}$ are pairwise relatively prime?

Let S be a numerical semigroup minimally generated by three positive integers n_1, n_2 and n_3 being pairwise relatively prime. For each $i \in \{1, 2, 3\}$, let $c_i = \min\{x \in \mathbb{N} \setminus \{0\} \mid x n_i \in \langle \{n_1, n_2, n_3\} \setminus \{n_i\} \rangle\}$. In [42] formulas for $\mathrm{F}(S)$ and $\mathrm{g}(S)$ from n_i and c_i ($i \in \{1, 2, 3\}$) are given. Therefore, if we had a formula for computing c_3 from n_1 and n_2, we would have solved the problems raised by Frobenius for embedding dimension three. Note that c_3 is nothing but the multiplicity of the proportionally modular semigroup $\langle n_1, n_2 \rangle / n_3$. It is proved in [58] that if u is a positive integer such that $un_2 \equiv 1 \pmod{n_1}$, then $\langle n_1, n_2 \rangle / n_3 = \{x \in \mathbb{Z} \mid u n_2 n_3 x \bmod n_1 n_2 \leq n_3 x\}$. We suggest in this line the problem of finding a formula that allows us to give the multiplicity of $\mathrm{S}(u n_2 n_3, n_1 n_2, n_3)$ from n_1, n_2 and n_3.

Fermat's Last Theorem asserts that for any integer $n \geq 3$, the Diophantine equation $x^n + y^n = z^n$ does not admit an integer solution such that $xyz \neq 0$. As it is well known, this theorem was proved by Wiles, with the help of Taylor, in 1995 ([64, 65]) after 300 years of fruitless attempts. Let us observe that for $n \geq 3$, the Diophantine equation $x^n + y^n = z^n$ has no solution verifying $zyz \neq 0$ with some of the factors equal to 1. Therefore in order to solve this equation it can be supposed that x, y and z are integers greater than or equal to two, and pairwise relatively prime. It is proved in [63], that Fermat's Last Theorem is equivalent to the following statement: if a, b and c are integers greater than or equal to two, pairwise relatively prime, and n is an integer greater than or equal to three, then the proportionally modular numerical semigroup $\langle a^n, b^n \rangle / c$ is not minimally generated by $\{a^n, c^{n-1}, b^n\}$. It would be interesting to prove this fact without using Fermat's last Theorem.

7. Non-unique factorization invariants

Let S be a numerical semigroup minimally generated by $\{n_1 < \cdots < n_e\}$. Then we already know that S is isomorphic to $\mathbb{N}^e / \sim_S$, where $\sim_S$ is the kernel congruence of the epimorphism $\varphi : \mathbb{N}^e \to S$, $(a_1, \dots, a_e) \mapsto a_1 n_1 + \cdots + a_e n_e$.

For $s \in S$, the elements in $\mathsf{Z}(s) = \varphi^{-1}(s)$ are known as factorizations of s. Given $(x_1, \dots, x_e) \in \mathsf{Z}(s)$, its length is $|x| = x_1 + \cdots + x_e$. The set of lengths of s is $\mathsf{L}(s) = \{|x| \mid x \in \mathsf{Z}(s)\}$. If $\mathsf{L}(s) = \{l_1 < l_2 < \cdots < l_t\}$, then the set of differences of lengths of factorizations of s is $\Delta(s) = \{l_2 - l_1, \ldots, l_t - l_{t-1}\}$. Moreover $\Delta(S) = \bigcup_{s \in S} \Delta(s)$. These sets are known to be eventually periodic ([12]).

The elasticity of $s \in S$ is $\rho(s) = \max \mathsf{L}(s) / \min \mathsf{L}(s)$, and $\rho(S) = \sup_{s \in S}(\rho(s))$, which turns out to be a maximum ([47]). For numerical semigroups it is well known that $\rho(S) = n_e/n_1$.

For $x = (x_1, \ldots, x_e), y = (y_1, \ldots, y_e) \in \mathbb{N}^e$, the greatest common divisor of x and y is

$$\gcd(x, y) = (\min(x_1, y_1), \ldots, \min(x_e, y_e)).$$

The distance between x and y is

$$\mathsf{d}(x, y) = \max\{|x - \gcd(x, y)|, |y - \gcd(x, y)|\}.$$

An N-chain (with N a positive integer) joining two factorizations x and y of $s \in S$ is a sequence $z_1, \ldots, z_t$ of factorizations of s such that $z_1 = x$, $z_t = y$ and $\mathsf{d}(z_i, z_{i+1}) \leq N$. The catenary degree of s, $\mathsf{c}(s)$, is the least N such that for every two factorizations x and y of s, there is an N-chain joining them. The catenary degree of S is $\mathsf{c}(S) = \sup_{s \in S}\{\mathsf{c}(s)\}$. This supremum is a maximum and actually $\mathsf{c}(S) = \max_{s \in \mathrm{Betti}(S)} \mathsf{c}(s)$ ([11]). It was asked by F. Halter-Koch whether this invariant is periodic, that is, if there exists $n \in S$ such that for s "big enough", $\mathsf{c}(s + n) = \mathsf{c}(s)$.

The tame degree of $s \in S$, $\mathsf{t}(s)$, is the minimum N such that for any $i \in \{1, \ldots, e\}$ with $s - n_i \in S$ and any $x \in \mathsf{Z}(s)$, there exists $y = (y_1, \ldots, y_e)$, such that $y_i \neq 0$ and $\mathsf{d}(x, y) \leq N$. The tame degree of S is $\mathsf{t}(S) = \sup_{s \in S}(\mathsf{t}(S))$. This supremum is again a maximum and it is reached in the (finite) set of elements of the form $n_i + w$ with $w \in S$ such that $w - n_j \notin S$ for some $j \neq i$. F. Halter-Koch also proposed the problem of studying the eventual periodicity of S.

The invariant $\omega(S, s)$ is the least positive integer such that whenever s divides $s_1 + \cdots + s_k$ for some $s_1, \ldots, s_k \in S$, then s divides $s_{i_1} + \cdots + s_{i_{\omega(S,s)}}$ for some $\{i_1, \ldots, i_{\omega(S,s)}\} \subseteq \{1, \ldots, k\}$. The ω-primality of S is defined as $\omega(S) = \max\{\omega(S, n_1), \ldots, \omega(S, n_e)\}$. In [4] it is highlighted that numerical semigroups fulfilling $\omega(S) \neq \mathsf{t}(S)$ are rare. A characterization for numerical semigroups fulfilling this condition should be welcomed.

Another problem proposed by A. Geroldinger is to determine when can we find a numerical semigroup and an element in it with a given set of lengths.

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Tracing orbits on conservative maps

by Mário Bessa*

Abstract.—We explore uniform hyperbolicity and its relation with the pseudo orbit tracing property. This property indicates that a sequence of points which is nearly an orbit (affected with a certain error) may be shadowed by a true orbit of the system. We obtain that, when a conservative map has the shadowing property and, moreover, all the conservative maps in a C^1-small neighborhood display the same property, then the map is globally hyperbolic.

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Keywords.—Volume-preserving maps; pseudo-orbits; shadowing; hyperbolicity.

1. Introduction

> "There is strong shadow where there is much light"
>
> Goethe in Götz von Berlichingen

1.1 The basic framework

In order to start playing with dynamical systems we need a place to play and a given rule acting on it. Once we establish that, we wonder what happens when we repeat the rule ad infinitum. We are mainly interested in two types of playgrounds: volume manifolds and symplectic manifolds. On volume-manifolds the rule is the action of a volume-preserving diffeomomorphism, and on symplectic manifolds the rule is the action of a symplectomorphism. Let us now formalize these concepts.

Let M stands for a closed, connected and C^∞ Riemannian manifold of dimension $d \geq 2$ and let ν be a volume-form on M. Once we equip M with ν we denominate it by a volume-manifold. By a classic result by Moser (see [20]) we know that, in brief terms, there is only one volume-form on M. Actually, in [20] we find and atlas formed by a finite collection of smooth charts $\{\alpha_j \colon U_j \subset M \to \mathbb{R}^d\}_{j=1}^k$ where U_j are open sets and each α_j pullbacks the volume on $\mathbb{R}^d$ into ν. The volume-form allows us to define a measure μ on M which we call Lebesgue measure. A C^r $(r \geq 1)$ diffeomorphism $f : M \to M$ is said to be volume-preserving if it keeps invariant the volume structure, say $f^*\nu = \nu$. In other words any Borelian $B \subset M$ is such that $\mu(B) = \mu(f^{-1}(B))$ We denote these maps by $\mathrm{Diff}^r_\mu(M)$. We endow $\mathrm{Diff}^r_\mu(M)$ with the Whitney (or strong) C^r topology (see [1]). In broad terms, two diffeomorphisms f and g are C^r-close if they are uniformly close as well as their first C^r derivatives computed in any point $x \in M$. Such systems emerges quite naturally when considering the time-one map of incompressible flows which are a fundamental object in fluid mechanics (see e.g. [14]).

Denote by $\mathbf{M}$ a $2d$-dimensional $(d \geq 1)$ manifold with a Riemaniann structure and endowed with a closed and nondegenerate 2-form ω called symplectic form. Let μ stands for the volume measure associated to the volume form wedging ω d-times, i.e., $\nu = \omega^d = \omega \wedge \ldots \wedge \omega$. By the

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