can be studied with known methods for PDE's (semigroup techniques have been used, see [16] for details). Then one takes the weak limit $f_\alpha^n \longrightarrow f_\alpha$ and uses stability results to show that the sequence $\{f_1^n, f_2^n, f_3^n, f_4^n\}$ converges to a renormalized solution of the system (2), (33–34). A crucial part in this passage to the limit is the estimation of the renormalized collision terms, for which the velocity averaging results provide an important tool.

6.0.1 Remark [Relevance of Theorem 6.0.1].—The existence result stated in Theorem 6.0.1 has important implications at the level of approximation questions.

6.0.2 Remark [Future perspectives].—The spatially homogeneous theory of the SRS model, in which the distribution functions do not depend on the x variable, is a topic of great interest. Some advances have been made in view of studying existence of solutions, uniqueness and stability results for the homogeneous reactive equations.

Another regime of interest corresponds to the case in which the distribution functions are assumed very close to the equilibrium. In this case, one considers the linearized version of the SRS model around an equilibrium solution and uses the spectral properties of the linearized collision operators to prove existence and stability of close to equilibrium solutions for the SRS system. Some studies have been developed also in this direction.

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On the Fourier-Stieltjes transform of Minkowski's question mark function and the Riemann hypothesis: Salem's type equivalences

by Semyon Yakubovich*

1 Introduction and auxiliary results

In this presentation we pay tribute to the work in analysis and analytic number theory of the famous mathematician Raphaël Salem (1898–1963). Precisely, we will extend his approach to study Fourier-Stieltjes coefficients behavior at infinity with singular measures. In particular, we will prove an equivalent proposition related to the known and still unsolved question posed by Salem in [8], p. 439 whether Fourier-Stieltjes coefficients of the *Minkowski's question mark function* vanish at infinity. Furthermore, we establish a class of Salem's type equivalences to the Riemann hypothesis, which is based on Wiener's *closure of translates* problem.

It is well known in the elementary theory of the Fourier-Stieltjes integrals that if $h(x)$ is absolutely continuous then

$$g(t) = \int_\Omega e^{ixt} dh(x), \quad \Omega \subset \mathbb{R},\ t \in \mathbb{R} \qquad (1)$$

tends to zero as $|t| \to \infty$, because in this case the Fourier-Stieltjes transform $g(t)$ is an ordinary Fourier transform of an integrable function. Thus $h(x)$ supports a measure whose Fourier transform vanishes at infinity. Such measures are called *Rajchman measures* (see details, for instance, in [4]). However, when h is continuous, the situation is quite different and the classical Riemann-Lebesgue lemma for the class L_1, in general, cannot be applied. The question is quite delicate when it concerns singular monotone functions (see [11], Ch. IV). For such singular measures there are various examples and the Fourier-Stieltjes transform need not tend to zero, although there do exist measures for which it goes to zero. For instance, Salem [8,10] gave examples of singular functions, which are strictly increasing and whose Fourier coefficients still do not vanish at infinity. On the other hand, Menchoff in 1916 [5] gave a first example of a singular distribution whose coefficients vanish at infinity. Wiener and Wintner [17] (see also [2]) proved in 1938 that for every $\varepsilon > 0$ there exists a singular monotone function such that its Fourier coefficients behave as $n^{-1/2+\varepsilon}$, $n \to \infty$.

Our goal here is to construct some Rajchman's measures, which are associated with continuous functions of bounded variation. In particular, we will prove that the famous Minkowski' s question mark function $?(x)$ [1] is a Rajchman measure if and only if its Fourier-Stieltjes transform has a limit at infinity, and then, of course, the limit should be zero. This probably can give an affirmative answer on the question posed by Salem in 1943 [8}.

The *Minkowski question mark function* $?(x) : [0, 1] \mapsto [0, 1]$ is defined by [1]

$$?([0, a_1, a_2, a_3, \ldots]) = 2 \sum_{i=1}^{\infty} (-1)^{i+1} 2^{-\sum_{j=1}^{i} a_j}, \qquad (2)$$

where $x = [0, a_1, a_2, a_3, \ldots]$ stands for the representation of x by a regular continued fraction. We will keep the notation $?(x)$, which was used in the original Salem's paper [8], mildly resisting the temptation of changing it and despite this symbol is quite odd to denote a function in such a way. It is well known that $?(x)$ is continuous,

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strictly increasing and singular with respect to Lebesgue measure. It can be extended on $[0,\infty]$ by using the following functional equations

$$?(x) = 1 - ?(1-x), \tag{3}$$
$$?(x) = 2?\left(\frac{x}{x+1}\right), \tag{4}$$

$$?(x) + ?\left(\frac{1}{x}\right) = 2, \quad x > 0. \tag{5}$$

When $x \to 0$, it decreases exponentially $?(x) = O\left(2^{-1/x}\right)$. Key values are $?(0) = 0$, $?(1) = 1$, $?(\infty) = 2$. For instance, from (3) and asymptotic behavior of the Minkowski function $?(x)$ near zero one can easily get the finiteness of the following integrals

$$\int_0^1 x^{\lambda}\, d?(x) < \infty, \ \lambda \in \mathbb{R}, \tag{6}$$

$$\int_0^1 (1-x)^{\lambda}\, d?(x) < \infty, \ \lambda \in \mathbb{R}. \tag{7}$$

Further, as was proved by Salem [8], the Minkowski question mark function satisfies the Hölder condition

$$\left|?(x) - ?(y)\right| < C|x-y|^{\alpha}, \tag{8}$$

of order

$$\alpha = \frac{\log 2}{2\log\frac{\sqrt{5}+1}{2}}, \tag{9}$$

where $C > 0$ is an absolute constant. We will deal in the sequel with the following Fourier-Stieltjes transforms of the Minkowski question mark function

$$f(t) = \int_0^1 e^{ixt} d?(x), \qquad F(t) = \int_0^{\infty} e^{ixt} d?(x), \ t \in \mathbb{R}, \tag{10}$$

$$f_c(t) = \int_0^1 \cos xt\, d?(x), \qquad F_c(t) = \int_0^{\infty} \cos xt\, d?(x), \ t \in \mathbb{R}_+, \tag{11}$$

$$f_s(t) = \int_0^1 \sin xt\, d?(x), \qquad F_s(t) = \int_0^{\infty} \sin xt\, d?(x), \ t \in \mathbb{R}_+, \tag{12}$$

where all integrals converge absolutely and uniformly with respect to t because of straightforward estimates

$$|f(t)| \le \int_0^1 d?(x) = 1, \quad |F(t)| \le \int_0^{\infty} d?(x) = 2,$$

$$|f_c(t)| \le 1, \quad |F_c(t)| \le 2,$$

$$|f_s(t)| \le 1, \quad |F_s(t)| \le 2.$$

Further we observe that functional equation (3) easily implies $f(t) = e^{it} f(-t)$ and therefore $e^{-it/2} f(t) \in \mathbb{R}$. So, taking the imaginary part we obtain the equality

$$\cos\left(\frac{t}{2}\right) f_s(t) = \sin\left(\frac{t}{2}\right) f_c(t). \tag{13}$$

Hence, for instance, letting $t = 2\pi n$, $n \in \mathbb{N}_0$ it gives $f_s(2\pi n) = 0$ and $f_c(2\pi n) = d_n$. In 1943 Salem asked [8] whether $d_n \to 0$, as $n \to \infty$.

Further, by using functional equations (4), (5) for the Minkowski function we derive the following useful relations

$$\int_0^1 e^{ixt}\, d?(x) = \int_0^{\infty} e^{ixt}\, d?(x) - \int_1^{\infty} e^{ixt}\, d?(x)$$

$$= \int_0^{\infty} e^{ixt}\, d?(x) + e^{it}\int_0^{\infty} e^{ixt}\, d?\left(\frac{1}{x+1}\right)$$

$$= \int_0^{\infty} e^{ixt}\, d?(x) + \frac{e^{it}}{2}\int_0^{\infty} e^{ixt}\, d?\left(\frac{1}{x}\right)$$

$$= \left(1 - \frac{e^{it}}{2}\right)\int_0^{\infty} e^{ixt}\, d?(x),$$

which imply the functional equation

$$F(t) = \frac{2f(t)}{2-e^{it}}. \tag{14}$$

Taking real and imaginary parts in (14) and employing functional equation (3) it is not difficult to deduce the following important equalities for the Fourier-Stieltjes transforms (11), (12)

$$F_c(t) = \frac{2}{5-4\cos t}\, f_c(t), \tag{15}$$

$$F_s(t) = \frac{6}{5-4\cos t}\, f_s(t). \tag{16}$$

Indeed, we have, for instance

$$\int_0^{\infty} \cos xt\, d?(x) = \frac{2}{5-4\cos t}$$

$$\times\left[(2-\cos t)\int_0^1 \cos xt\, d?(x) - \sin t\int_0^1 \sin xt\, d?(x)\right]$$

$$= \frac{2}{5-4\cos t}\left[2\int_0^1 \cos xt\, d?(x) - \int_0^1 \cos t(1-x)\, d?(x)\right]$$

$$= \frac{2}{5-4\cos t}\int_0^1 \cos xt\, d?(x)$$

and this yields relation (15). Analogously we get (16). In particular, letting $t = 2\pi n, n \in \mathbb{N}_0$ in (15), (16) we find accordingly

$$\int_1^{\infty} \cos(2\pi n x)\, d?(x) = \int_0^1 \cos(2\pi n x)\, d?(x),$$

$$\int_1^{\infty} \sin(2\pi n x)\, d?(x) = 5\int_0^1 \sin(2\pi n x)\, d?(x) = 0$$

via (13). Generally, equalities (15), (16) yield

$$\int_1^{\infty} \cos xt\, d?(x) = \frac{1-8\sin^2(t/2)}{1+8\sin^2(t/2)}\int_0^1 \cos xt\, d?(x),$$

$$\int_1^{\infty} \sin xt\, d?(x) = \frac{5-8\sin^2(t/2)}{1+8\sin^2(t/2)}\int_0^1 \sin xt\, d?(x).$$

respectively. For instance,

$$\int_1^{\infty} \cos(xt_m)\, d?(x) = 0,$$

$$\int_1^{\infty} \sin(xt_k)\, d?(x) = 0$$

for any t_m, t_k, which are roots of the corresponding equations

$$\sin(t_m/2) = \pm 1/(2\sqrt{2}),$$

$$\sin(t_k/2) = \pm\sqrt{5/8}, \quad m, k \in \mathbb{N}.$$

Further, since (see (14), (15), (16))

$$\frac{1}{2}|F(t)| \le |f(t)| \le \frac{3}{2}|F(t)|, \tag{17}$$

$$\frac{1}{2}|F_c(t)| \le |f_c(t)| \le \frac{9}{2}|F_c(t)|, \tag{18}$$

$$\frac{1}{6}|F_s(t)| \le |f_s(t)| \le \frac{3}{2}|F_s(t)|, \tag{19}$$

then Fourier-Stieltjes transforms of the Minkowski question mark function over $(0,1)$ tend to zero when $|t| \to \infty$ if and only if the same property is guaranteed by Fourier-Stieltjes transforms over $(0,\infty)$.

2 Some Rajchman measures

In this section we prove several theorems, characterizing Rajchman measures, which are associated with Fourier-Stieltjes integrals over finite and infinite intervals.

We begin with the following general result.

Theorem 1.—Let φ be a real-valued continuous integrable function of bounded variation on $(0,\infty)$ vanishing at infinity. Then φ supports a Rajchman measure relatively its Fourier-Stieltjes transform

$$\Phi(t) = \int_0^{\infty} e^{ixt}\, d\varphi(x), \tag{20}$$

if and only if it has a limit at infinity ($|t| \to \infty$).

Proof.—Without loss of generality we prove the theorem for positive t. Evidently, the necessity is trivial and we will prove the sufficiency. Suppose that the limit of $\Phi(t)$ when $t \to +\infty$ exists. Since $\Phi(t) = \Phi_c(t) + i\Phi_s(t)$, where

$$\Phi_c(t) = \int_0^{\infty} \cos xt\, d\varphi(x), \tag{21}$$

$$\Phi_s(t) = \int_0^{\infty} \sin xt\, d\varphi(x), \tag{22}$$

we will treat these transforms separately. Taking (21) and integrating by parts we get

$$\Phi_c(t) = -\varphi(0) + t\int_0^{\infty} \varphi(x)\sin xt\, dx. \tag{23}$$

However, since $\varphi \in L_1(\mathbb{R}_+)$, we appeal to the integrated form of the Fourier formula (cf. [12], Th. 22) to write for all $x \ge 0$

$$\int_0^x \varphi(y)\, dy = \frac{2}{\pi}\int_0^{\infty}\frac{1-\cos yx}{y}\int_0^{\infty}\varphi(u)\sin uy\, du.$$

But taking into account the previous equality after simple change of variable we come out with the relation

$$\frac{1}{x}\int_0^x \varphi(y)\, dy$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{1-\cos y}{y^2}\left[\varphi(0) + \Phi_c\left(\frac{y}{x}\right)\right]dy, \ x > 0.$$

Minding the value of elementary Feijer type integral

$$\frac{2}{\pi}\int_0^{\infty}\frac{1-\cos y}{y^2}dy = 1,$$

we establish an important equality

$$\frac{1}{x}\int_0^x [\varphi(y) - \varphi(0)]\, dy \tag{24}$$

$$= \frac{2}{\pi}\int_0^{\infty}\Phi_c\left(\frac{y}{x}\right)\frac{1-\cos y}{y^2}dy, \ x > 0.$$

Meanwhile, the left-hand side of (53) is evidently goes to zero when $x \to 0^+$ via the continuity of φ on $[0,\infty)$. Further, since φ is of bounded variation on $(0,\infty)$ we obtain the uniform estimate

$$|\Phi_c(t)| \le \int_0^{\infty} dV_{\varphi}(x) = \Phi_0,$$

where $V_{\varphi}(x)$ is a variation of φ on $[0,x]$ and $\Phi_0 > 0$ is a total variation of φ. This means that $\Phi_c(t)$ is continuous and bounded on $\mathbb{R}_+$. Furthermore, the integral with respect to x in the right-hand side of (53) converges absolutely and uniformly by virtue of the Weierstrass test. Consequently, since $\Phi_c(t)$ has a limit at infinity, which is finite, say a, one can pass to the limit through equality (53) when $x \to 0^+$. Hence we find

$$\lim_{x\to 0+}\frac{1}{x}\int_0^x [\varphi(y) - \varphi(0)]\, dy$$

$$= \frac{2a}{\pi}\int_0^{\infty}\frac{1-\cos y}{y^2}dy = a = 0.$$

In order to complete the proof, we need to verify whether the Fourier sine transform (22) tends to zero as well. To do this, we appeal to the corresponding integrated form of the Fourier formula for the Fourier cosine transform

$$-\int_0^x \varphi(y)\,dy = \frac{2}{\pi}\int_0^\infty \frac{\sin yx}{y^2}\Phi_s(y)\,dy,\ x>0, \qquad (25)$$

where after integration by parts $\Phi_s(t)$ turns to be represented as follows

$$\Phi_s(t) = -t\int_0^\infty \varphi(u)\cos ut\,du,\ t>0. \qquad (26)$$

Hence it is easily seen that $\Phi_s(t) = O(t),\ t\to 0^+$ and since $|\Phi_s(t)| \le \Phi_0$ we have that $\Phi_s(t)/t \in L_2(\mathbb{R}_+)$. This means that the integral in the right-hand side of (25) converges absolutely and uniformly by $x \ge 0$. After simple change of variable we split the integral in the right-hand side of (25) on two integrals to obtain

$$-\frac{1}{x}\int_0^x \varphi(y)\,dy = \frac{2}{\pi}\int_0^1 \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right)dy + \frac{2}{\pi}\int_1^\infty \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right)dy.$$

Considering again $x>0$ sufficiently small and splitting the integral over $(0,1)$ on two more integrals over $(0, x\log^\gamma(1/x))$ and $(x\log^\gamma(1/x), 1)$, where $0<\gamma<1$, we derive the equality

$$\frac{2}{\pi}\int_{x\log^\gamma(1/x)}^1 \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right)dy = -\frac{1}{x}\int_0^x \varphi(y)\,dy - \frac{2}{\pi}\int_1^\infty \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right)dy - \frac{2}{\pi}\int_0^{x\log^\gamma(1/x)} \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right)dy.$$

Minding the inequality (see (26)) $|\Phi_s(t)| \le t\|\varphi\|_{L_1(\mathbb{R}_+)}$, $t\ge 0$, the right-hand side of the latter equality has the straightforward estimate

$$\left|\frac{1}{x}\int_0^x \varphi(y)\,dy + \frac{2}{\pi}\int_1^\infty \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right)dy + \frac{2}{\pi}\int_0^{x\log^\gamma(1/x)} \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right)dy\right| \qquad (27)$$

$$\le \sup_{y\ge 0}|\varphi(y)| + \frac{2}{\pi}\left[\Phi_0 + \|\varphi\|_{L_1(\mathbb{R}_+)}\log^\gamma(1/x)\right].$$

On the other hand, via the first mean value theorem

$$\frac{2}{\pi}\left|\int_{x\log^\gamma(1/x)}^1 \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right)dy\right| = \frac{2}{\pi}|\Phi_s(\xi(x))| \times \int_{x\log^\gamma(1/x)}^1 \frac{\sin y}{y^2}\,dy,$$

where

$$\log^\gamma\left(\frac{1}{x}\right) \le \xi(x) \le \frac{1}{x}.$$

Meanwhile, we have

$$\frac{2}{\pi}\int_{x\log^\gamma(1/x)}^1 \frac{\sin y}{y^2}\,dy > \frac{2\sin 1}{\pi}\int_{x\log^\gamma(1/x)}^1 \frac{dy}{y} = \frac{2\sin 1}{\pi}\log\left(\frac{1}{x\log^\gamma(1/x)}\right).$$

Consequently, combining with (27) we find

$$|\Phi_s(\xi(x))| < \frac{1}{\sin 1}\left[\frac{\pi}{2}\sup_{y\ge 0}|\varphi(y)| + \Phi_0 + \|\varphi\|_{L_1(\mathbb{R}_+)}\log^\gamma(1/x)\right]\log^{-1}\left(\frac{1}{x\log^\gamma(1/x)}\right) = o(1),\ x\to 0^+. \qquad (28)$$

Thus making $x\to 0^+$ we get $\xi(x)\to+\infty$ and therefore there is a subsequence $t_n = \xi(x_n)\to\infty$ such that $\lim_{n\to+\infty}|\Phi_s(t_n)| = 0$. But since the limit of $\Phi_s(t)$ exists, when $t\to+\infty$ it will be zero. So φ supports a Rajchman measure and the theorem is proved. ■

Corollary 1.—Under conditions of Theorem 1 φ supports a Rajchman measure if and only if two limits

$$\lim_{t\to+\infty} t\int_0^\infty \varphi(x)\sin xt\,dx,$$

$$\lim_{t\to+\infty} t\int_0^\infty \varphi(x)\cos xt\,dx$$

exist simultaneously (if so, they equal to $\varphi(0)$ and 0, respectively).

More general result deals with the smoothness of the Fourier-Stieltjes transform and a behavior at infinity of its derivatives.

We have

Corollary 2.—Let $n\in\mathbb{N}_0$, $\varphi(x)$, $x\ge 0$ be a real-valued continuous function such that $x^m\varphi(x)$ is of bounded variation on $[0,\infty)$ for each $m = 0, 1, \dots, n$. If $\varphi(x) = o(x^{-n})$, $x\to\infty$ and $x^n\varphi(x)\in L_1(\mathbb{R}_+)$, then the corresponding Fourier-Stieltjes transform (20) $\Phi(t)$ is n times differentiable on $\mathbb{R}_+$, its n-th order derivative is equal to

$$\Phi^{(n)}(t) = \int_0^\infty (ix)^n e^{itx}\,d\varphi(x) \qquad (29)$$

and vanishes at infinity if and only if there exists a limit of the integral

$$\Psi_n(t) = \int_0^\infty e^{itx}\,d\left(x^n\varphi(x)\right)$$

when $|t|\to\infty$.

Proof—In fact, under conditions of the corollary one can differentiate n times under the integral sign in the Fourier-Stieltjes transform (20)) via the absolute and uniform convergence. Precisely, this circumstance is guaranteed by the estimate

$$\left|\int_0^\infty (ix)^m e^{itx}\,d\varphi(x)\right| = \left|\int_0^\infty e^{itx}\,d\left((ix)^m\varphi(x)\right) - m\,i^m\int_0^\infty x^{m-1}\varphi(x)e^{itx}\,dx\right| \le Var_{[0,\infty)}\left(x^m\varphi(x)\right) + m\int_0^\infty x^{m-1}|\varphi(x)|\,dx = \Phi_m < \infty,\ m = 0, 1, \dots, n,$$

where the latter integral is finite since $x^n\psi(x)\in L_1(\mathbb{R}_+)$ and φ is continuous. Thus (29) holds and in order to complete the proof we write it as

$$\Phi^{(n)}(t) = i^n\left[\int_0^\infty e^{itx}\,d\left(x^n\varphi(x)\right) - n\int_0^\infty x^{n-1}\varphi(x)e^{itx}dx\right].$$

The second integral of this equality tends to zero when $t\to\infty$ via the Riemann-Lebesgue lemma. Therefore, $t\Phi^{(n)}(t) = o(1)$, $t\to\infty$, $n\in\mathbb{N}_0$ if and only if the first integral has a limit at infinity and this limit is certainly zero. ■

3 An equivalent Salem's problem

In this section we will formulate a problem, which is equivalent to Salem's question [8], having

Corollary 3.—The Fourier-Stieltjes transform

$$f(t) = \int_0^1 e^{ixt}d?(x)$$

of the Minkowski question mark function vanishes at infinity, i.e. an answer on Salem's question is affirmative, if and only if two limit equalities

$$\lim_{t\to+\infty} t\int_0^\infty ?\left(\frac{1}{x}\right)\sin xt\,dx = 2,$$

$$\lim_{t\to+\infty} t\int_0^\infty ?\left(\frac{1}{x}\right)\cos xt\,dx = 0$$

take place simultaneously.

Proof.—It follows immediately from double inequality (17), simple equality due to functional equation (5)

$$\int_0^\infty e^{ixt}\,d?(x) = -\int_0^\infty e^{ixt}\,d?\left(\frac{1}{x}\right)$$

and Corollary 1, where we put

$\varphi(x) = ?(1/x)$, $x>0$, $\varphi(0) = 2$. ■

Finally, we generalize Salem's problem, proving

Theorem 2.—Let $k\in\mathbb{N}_0$. If an answer on Salem's question is affirmative, then

$$f^{(k)}(t) = \int_0^1 (ix)^k e^{itx}\,d?(x) = o(1),\ |t|\to\infty. \qquad (30)$$

Proof.—It is easily seen that the Fourier-Stieltjes transform of the Minkowski question mark function over $(0,1)$ is infinitely differentiable and so for any $k\in\mathbb{N}_0$ we have (30). Suppose that $f^{(k)}$ does not tend to zero as $|t|\to\infty$. Then we can find a sequene $\{t_m\}_{m=1}^\infty$, $|t_m|\to\infty$ such that

$$\left|\int_0^1 x^k e^{it_m x}\,d?(x)\right| \ge \delta > 0.$$

Let $t_m/(2\pi) = n_m + \beta_m$, where n_m is an integer and $0\le\beta_m<1$. One can suppose that β_m tends to a limit β, we can always do it choosing again subsequence from $\{t_m\}$ if necessary. Hence

$$|f^{(k)}(t_m)| = \left|\int_0^1 e^{2\pi i\beta x}x^k e^{2\pi i n_m x}\,d?(x)\right| \ge \delta > 0.$$

But this contradicts to Salem's lemma [11], p. 38, because $f(2\pi n)\to 0$, $n\to\infty$ via assumption of the theorem and the Riemann-Stieltjes integral

$$\int_0^1 e^{2\pi i\beta x}x^k\,d?(x)$$

converges for any $k\in\mathbb{N}_0$. ■

4—A class of Salem's type equivalences to the Riemann hypothesis

As it is widely known, *Riemann zeta function* $\zeta(s)$ [13] satisfies the functional equation

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s),$$

where $\Gamma(z)$ is *Euler's gamma-function*. Moreover, in the half-plane $\operatorname{Re}s = c_0 > 1$ it is represented by the absolutely and uniformly convergent series with respect to $t\in\mathbb{R}$, $s = c_0 + it$

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} \qquad (31)$$

and by the uniformly convergent series

$$(1-2^{1-s})\zeta(s) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s} \qquad (32)$$

in the half-plane $\operatorname{Re}s > 0$. We also note the useful formula

$$\frac{1}{\zeta(s)} = \sum_{n=1}^\infty \frac{\mu(n)}{n^s},\quad \operatorname{Re}s>1, \qquad (33)$$

where $\mu(n)$ is *the Möbius function* [7].

In 1953 Salem [9] proved that the Riemann hypothesis is true, i.e. the Riemann zeta-function $\zeta(s)$ is free of zeros in the strip $1/2<\operatorname{Re}s<1$ is equivalent to the fact, that the homogeneous integral equation

$$\int_0^\infty \frac{y^{\sigma-1}}{e^{xy}+1}\varphi(y)dy = 0,\ x>0,\ \frac{1}{2}<\sigma<1 \qquad (34)$$

has no nontrivial solutions in the space of bounded measurable functions on $\mathbb{R}_+$. Our goal is to extend this fact to the entire class of equivalent propositions, involving integral equations with the *Widder-Lambert type kernels* (cf. [3], [15]). However, our starting point will be a characterization of mapping properties of the corresponding integral transformations in a special functional space $\mathscr{M}^{-1}(L_c)$ (see in [14]).

Definition 1.—Denote by $\mathscr{M}^{-1}(L_c)$ the space of functions $f(x), x \in \mathbb{R}_+$, representable by inverse *Mellin transform* of integrable functions $f^*(s) \in L_1(c)$ on the vertical line $c = \{s \in \mathbb{C} : \operatorname{Re} s = c_0 > 1\}$:

$$f(x) = \frac{1}{2\pi i}\int_c f^*(s)x^{-s}ds. \qquad (35)$$

The space $\mathscr{M}^{-1}(L_c)$ with the usual operations of addition and multiplication by scalar is a linear vector space. If the norm in $\mathscr{M}^{-1}(L_c)$ is introduced by the formula

$$\|f\|_{M^{-1}(L_c)} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|f^*(c_0+it)|dt,$$

then it becomes a Banach space. Simple properties of the space $\mathscr{M}^{-1}(L_c)$ follow immediately from Definition 1 and the basic properties of the Fourier and Mellin transforms of integrable functions. For instance, the Riemann-Lebesgue lemma yields that $x^{c_0}f(x)$ is uniformly bounded, continuous on $\mathbb{R}_+$ and $x^{c_0}f(x) = o(1)$, when $x \to +\infty$ and $x \to 0$. Moreover, if $f(x), g(x) \in M^{-1}(L_c)$, where $g(x)$ is the inverse Mellin transform (35) of the function $g^*(s)$, then $x^{c_0}f(x)g(x) \in \mathscr{M}^{-1}(L_c)$ because the product $x^{c_0}f(x)g(x)$ is the inverse Mellin transform of the function

$$\frac{1}{2\pi i}\int_c f^*(\tau)g^*(s-\tau+c_0)d\tau,$$

which belongs to $L_1(c)$ by Fubini's theorem. Finally we note that if $f(x) \in \mathscr{M}^{-1}(L_c)$ and $x^{c_0-1}g(x) \in L_1(\mathbb{R}_+)$, then the Mellin convolution

$$\int_0^\infty g(u)f\left(\frac{x}{u}\right)\frac{du}{u} \in M^{-1}(L_c).$$

In fact, the latter integral is an inverse Mellin transform of the function $f^*(s)g^*(s)$ and since $f^*(s) \in L_1(c)$ and $g^*(s)$ is essentially bounded on c, we have $f^*(s)g^*(s) \in L_1(c)$.

A more general space $\mathscr{M}^{-1}_{c_1,c_2}(L_c)$, which will be involved as well is defined similarly to the one in [14].

Definition 2.—Let $c_1, c_2 \in \mathbb{R}$ be such that $2\operatorname{sign} c_1 + \operatorname{sign} c_2 \geq 0$. By $\mathscr{M}^{-1}_{c_1,c_2}(L_c)$ we denote the space of functions $f(x), x \in \mathbb{R}_+$, representable in the form (35), where $s^{c_2}e^{\pi c_1|s|}f^*(s) \in L_1(c)$.

It is a Banach space with the norm

$$\|f\|_{\mathscr{M}^{-1}_{c_1,c_2}(L_c)} = \frac{1}{2\pi}\int_c e^{\pi c_1|s|}|s^{c_2}f^*(s)ds|, \quad \operatorname{Re} s = c_0 > 1. \ (36)$$

Theorem 3.—Let $f \in M^{-1}(L_c)$. Then for all $x > 0$ reciprocal transformations

$$g(x) = \frac{1}{2\pi i}\int_c (1-2^{1-s})\zeta(s)f^*(s)x^{-s}ds = \sum_{n=1}^\infty (-1)^{n-1}f(xn) \qquad (37)$$

$$f(x) = \frac{1}{2\pi i}\int_c \frac{g^*(s)}{(1-2^{1-s})\zeta(s)}x^{-s}ds = \sum_{k=0}^\infty\sum_{n=1}^\infty 2^k\mu(n)g\left(xn2^k\right), \qquad (38)$$

where $g^*(s) = (1-2^{1-s})\zeta(s)f^*(s)$ are automorphisms of the space $\mathscr{M}^{-1}(L_c)$ and satisfy the norm estimates

$$[\zeta(c_0)]^{-1}\|g\|_{M^{-1}(L_c)} \leq \|f\|_{\mathscr{M}^{-1}(L_c)} \leq (1-2^{1-c_0})^{-1}\zeta(c_0)\|g\|_{M^{-1}(L_c)}, \ c_0 > 1. \qquad (39)$$

Proof.—In fact, since $|\mu(n)| \leq 1$ (see [7], [13]),

$$\sum_{n=1}^\infty |(-1)^{n-1}f(xn)| \leq x^{-c_0}\sum_{n=1}^\infty \frac{1}{n^{c_0}}\frac{1}{2\pi}\int_c |f^*(s)ds| = x^{-c_0}\zeta(c_0)\|f\|_{M^{-1}(L_c)},$$

$$\sum_{k=0}^\infty\sum_{n=1}^\infty 2^k|\mu(n)|\sum_{m=1}^\infty |f\left(xnm2^k\right)| \leq \frac{x^{-c_0}\zeta^2(c_0)}{2\pi(1-2^{1-c_0})}$$

$$\int_c |f^*(s)ds| = \frac{x^{-c_0}\zeta^2(c_0)}{1-2^{1-c_0}}\|f\|_{M^{-1}(L_c)},$$

and $g^*(s) \in L_1(c)$, all changes of the order of integration and summation are allowed. Hence via (32), (33) and elementary sum of geometric progression we establish reciprocal relations (37), (38) involving the uniqueness theorem for the Mellin transform of integrable functions. Moreover, it guarantees the automorphism of the space $\mathscr{M}^{-1}(L_c)$ under these transformations and the equivalence of norms, which immediately follows from estimates

$$\|g\|_{M^{-1}(L_c)} = \frac{1}{2\pi}\int_c \left|(1-2^{1-s})\zeta(s)f^*(s)ds\right| \leq \zeta(c_0)\,\|f\|_{M^{-1}(L_c)},$$

$$\|f\|_{\mathscr{M}^{-1}(L_c)} = \frac{1}{2\pi}\int_c \left|g^*(s)\frac{ds}{(1-2^{1-s})\zeta(s)}\right| \leq (1-2^{1-c_0})^{-1}\zeta(c_0)\,\|g\|_{\mathscr{M}^{-1}(L_c)},$$

yielding (39). ∎

Further, the Parseval equality for the Mellin transform [12] and Fubini's theorem allow to write the modified Laplace transform [3] of $f \in M^{-1}(L_c)$ in the form

$$\int_0^\infty e^{-x/t}f(t)\frac{dt}{t} = \frac{1}{2\pi i}\int_c \Gamma(s)f^*(s)x^{-s}ds. \qquad (40)$$

Moreover, due to Definition 2 and *Stirling's asymptotic formula for gamma-functions* [12] it forms a bijective map of the space $\mathscr{M}^{-1}(L_c)$ onto its subspace $\mathscr{M}^{-1}_{1/2,1/2-c_0}(L_c)$. Thus appealing to Theorem 3 we will derive the Widder type inversion formulas [15] for the Widder-Lambert type transforms. Precisely, we prove

Theorem 4.—Let $f \in \mathscr{M}^{-1}(L_c)$ and $c_0 > 1$. Then the Widder-Lambert type transformation

$$g(x) = \int_0^\infty \frac{f(t)dt}{t(e^{x/t}+1)}, \ x > 0. \qquad (41)$$

is a bijective map between spaces $\mathscr{M}^{-1}(L_c)$, $\mathscr{M}^{-1}_{1/2,1/2-c_0}(L_c)$ and for all $x > 0$ the following inversion formula takes place

$$f(x) = \lim_{k\to\infty}\left(-x\frac{d}{dx}\right)\prod_{j=1}^k\left(1-\frac{x}{j}\frac{d}{dx}\right) \times \sum_{m=0}^\infty\sum_{n=1}^\infty 2^m\mu(n)g\left(xkn2^m\right). \qquad (42)$$

Proof.—In fact, the proof is based on Theorem 3, equality (40), a familiar infinite product for the gamma-function

$$\frac{1}{\Gamma(s)} = \lim_{k\to\infty} sk^{-s}\prod_{j=1}^k\left(1+\frac{s}{j}\right), \qquad (43)$$

and the asymptotic behavior $|\Gamma(s)|^{-1} = e^{\pi|s|/2}|s|^{1/2-c_0}$, $s = c_0 + it$, $|t| \to \infty$ via Stirling formula. So employing again (32), Theorem 3 and Fubini's theorem, we deduce the following representation of the Widder-Lambert type transform (see (37))

$$g(x) = \frac{1}{2\pi i}\int_c (1-2^{1-s})\zeta(s)\Gamma(s)f^*(s)x^{-s}ds = \sum_{n=1}^\infty (-1)^{n-1}\int_0^\infty e^{-xn/t}f(t)\frac{dt}{t} = \int_0^\infty \frac{f(t)dt}{t(e^{x/t}+1)}, \ x > 0.$$

Finally, calling (43) and an elementary series we derive, reciprocally, the equalities

$$f(x) = \frac{1}{2\pi i}\int_c \frac{g^*(s)x^{-s}}{(1-2^{1-s})\zeta(s)\Gamma(s)}ds = \lim_{k\to\infty}\frac{1}{2\pi i}\sum_{m=0} 2^m\int_c\prod_{j=1}^k\left(1+\frac{s}{j}\right) \times \frac{sg^*(s)\,(kx2^m)^{-s}}{\zeta(s)}ds$$

$$= \lim_{k\to\infty}\left(-x\frac{d}{dx}\right)\prod_{j=1}^k\left(1-\frac{x}{j}\frac{d}{dx}\right) \times \sum_{m=0}^\infty\sum_{n=1}^\infty 2^m\mu(n)g\left(xkn2^m\right),$$

which yield (42). ∎

Returning to (34) and making a simple change of variables and functions it becomes (41) with $g(x) = 0$ and $f(t) = t^{-\sigma}\varphi(1/t)$. Thus Theorem 4 leads to

Corollary 4.—Let $\varphi(y)$ be a solution of homogeneous equation (34) such that $y^{-\sigma}\varphi(1/y) \in \mathscr{M}^{-1}(L_c)$, $1/2 < \sigma < 1$. Then $\varphi(y) \equiv 0$.

Proof.—Indeed, there exists a function $h^*_\sigma(s) \in L_1(c)$ such that

$$y^{-\sigma}\varphi\left(\frac{1}{y}\right) = \frac{1}{2\pi i}\int_c \varphi^*_\sigma(s)y^{-s}ds, \ y > 0.$$

Hence

$$|\varphi(y)| \leq \frac{y^{c_0-\sigma}}{2\pi}\int_c |\varphi^*_\sigma(s)ds|$$

and since $c_0 > \sigma$, we have that $h(y)$ is continuous on $\mathbb{R}_+$ and $\varphi(y) = o(1)$, $y \to 0$. Applying inversion formula (42) with $g = 0$ we get the result. ∎

Let us prove the following equivalence to the Riemann hypothesis of the Salem type.

Theorem 5.—The Riemann hypothesis is true, if and only if for any bounded measurable function $f(x)$ on $\mathbb{R}$ satisfying integral equation

$$\int_{\mathbb{R}^2}\frac{e^{-\sigma u}f(u)}{(e^{e^{x-u-t}}+1)(e^{e^t}+1)}du\,dt = 0, \quad \frac{1}{2} < \sigma < 1, \qquad (44)$$

for all $x \in \mathbb{R}$ it follows that f is zero almost everywhere.

Proof.—Calling again (32) and properties of the Mellin transform and its convolution [12] it is not difficult to derive the equality

$$[(1-2^{1-s})\zeta(s)\Gamma(s)]^2 = \int_0^\infty t^{s-1} \times \int_0^\infty \frac{du}{u(e^{t/u}+1)(e^u+1)}dt, \ \operatorname{Re} s > 0. \qquad (45)$$

On the other hand, the reciprocal inversion of the Mellin transform yields

$$\int_0^\infty \frac{du}{u(e^{x/u}+1)(e^u+1)} = \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}[(1-2^{1-s})\zeta(s)\Gamma(s)]^2x^{-s}ds. \qquad (46)$$

The left-hand side of (46) is positive and via (45)

$$\int_0^\infty\int_0^\infty \frac{t^{\sigma-1}dudt}{u(e^{t/u}+1)(e^u+1)} = [(1-2^{1-\sigma})\zeta(\sigma)\Gamma(\sigma)]^2,$$

which after a simple change of variables is equivalent to the condition

$$\int_{\mathbb{R}^2}\frac{e^{\sigma y}dudy}{(e^{e^{y-u}}+1)(e^{e^u}+1)} < \infty.$$

Hence following as in [9] Wiener's ideas [16] about an equivalence of the completeness in $L_1(\mathbb{R})$ of translations

$$e^{\sigma(x-y)}\int_{\mathbb{R}} \frac{du}{(e^{e^{x-y-u}}+1)(e^{e^u}+1)}, \quad x\in\mathbb{R}$$

and the absence of zeros of $[(1-2^{1-s})\zeta(s)\Gamma(s)]^2$, i.e. zeros of $\zeta(s)$ in the critical strip $1/2 < \mathrm{Re}\, s < 1$, we complete the proof. ■

Remark 1.—Reminding integral representation of the modified Bessel function in terms of the inverse Mellin transform [12]

$$K_\nu(2\sqrt{x}) = \frac{1}{4\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\Gamma\left(s+\frac{\nu}{2}\right)\Gamma\left(s-\frac{\nu}{2}\right)x^{-s}ds,$$

$$\mu > |\mathrm{Re}\,\nu|,$$

invoking (31) and identity (see [13])

$$\zeta^2(s)=\sum_{n=1}^\infty \frac{d(n)}{n^s}, \quad \mathrm{Re}\, s>1,$$

where $d(n)$ is *the divisor function*, we write equality (46) in the form

$$\frac12\int_0^\infty \frac{du}{u(e^{x/u}+1)(e^u+1)} = \sum_{n=1}^\infty d(n)\left[K_0(2\sqrt{nx}) -4K_0(2\sqrt{2nx}) + 4K_0(4\sqrt{nx})\right]. \quad (47)$$

Hence substituting (47) into (53) and changing the order of integration and summation via absolute and uniform convergence (we note that $d(n)=O(n^\varepsilon), \varepsilon>0, n\to\infty$, see [13]), Theorem 5 can be reformulated as

Theorem 6.—The Riemann hypothesis is true, if and only if for any bounded measurable function $f(x)$ on $\mathbb{R}$ and all $x\in\mathbb{R}$ the equation

$$\sum_{n=1}^\infty d(n)\left[(K_n f)(x) - 4(\mathscr{K}_{2n}f)(x) + 4(K_{4n}f)(x)\right]=0,$$

where

$$(K_nf)(x)=\int_{-\infty}^\infty e^{-\sigma u}K_0\left(2\sqrt{n}\,e^{(x-u)/2}\right)f(u)du,$$

$$\frac12<\sigma<1,$$

is the Meijer type convolution transform [3], has no non-trivial solutions.

Transformation (41) can be generalized considering the following two-parametric family of functions

$$U_{k,m}(x)=\frac{1}{2\pi i}\int_c [(1-2^{1-s})\zeta(s)]^{k+1}\Gamma^{m+1}(s)x^{-s}ds, \quad x>0,\ k,m\in\mathbb{N}_0. \quad (48)$$

The case $k=m$ we denote by $U_k(x)$. The case $k=m=0$ gives $U_0(x)=(e^x+1)^{-1}$. One can express the kernel (48) in terms of the iterated Mellin convolution. Indeed, via (32) and simple calculations we obtain

$$U_{k,m}(x)=\sum_{n_1,n_2,\dots,n_{k-m}=1}^\infty (-1)^{\sum_{j=1}^{k-m} n_j - k + m} \times\int_{\mathbb{R}_+^m}\prod_{j=1}^m (e^{u_j}+1)^{-1}\left(\exp\left(\frac{xn_1n_2\dots n_{k-m}}{u_1u_2\dots u_m}\right)+1\right)^{-1}\frac{du_1du_2\dots du_m}{u_1u_2\dots u_m}, \quad k>m, \quad (49)$$

$$U_{k,m}(x)\equiv U_k(x)=\int_{\mathbb{R}_+^k}\left(\exp\left(\frac{x}{u_1u_2\dots u_k}\right)+1\right)^{-1}\times\prod_{j=1}^k(e^{u_j}+1)^{-1}\frac{du_j}{u_j}, \quad k=m, \quad (50)$$

$$U_{k,m}(x)=\int_{\mathbb{R}_+^m}\prod_{j=1}^{k+1}(e^{u_j}+1)^{-1}\exp\left(-\sum_{j=k+2}^m u_j\right)\times\exp\left(-\frac{x}{u_1u_2\dots u_m}\right)\frac{du_1\dots du_m}{u_1u_2\dots u_m}, \quad k<m. \quad (51)$$

Meanwhile, an analog of Theorem 4 will be

Theorem 7.—Let $f\in\mathscr{M}^{-1}(L_c)$ and $c_0>1$. Then the integral transformation

$$g(x)=\int_0^\infty U_{k,m}\left(\frac{x}{t}\right)f(t)\frac{dt}{t}, \quad x>0 \quad (52)$$

is a bijective map between the spaces $\mathscr{M}^{-1}(L_c)$ and $\mathscr{M}^{-1}_{(m+1)/2,(m+1)(1/2-c_0)}(L_c)$ and for all $x>0$ the following inversion formula takes place

$$f(x)=\lim_{l\to\infty}\left(-x\frac{d}{dx}\right)^{m+1}\prod_{j=1}^l\left(1-\frac{x}{j}\frac{d}{dx}\right)^{m+1}\times\sum_{j_1,\dots,j_k=0}^\infty\sum_{n_1,\dots,n_k=1}^\infty\prod_{i=1}^k 2^{j_i}\mu(n_i)g\left(xl^{m+1}\prod_{i=1}^k 2^{j_i}n_i\right).$$

Finally, we will prove an analog of Theorem 5. In fact, we have

Theorem 8.—Let $k,m\in\mathbb{N}_0$, $k\le m$ and the kernel $U_{k,m}(x), x>0$ is defined by formulas (50), (51), correspondingly. The Riemann hypothesis is true, if and only if for any bounded measurable function $f(x)$ on $\mathbb{R}$ satisfying integral equation

$$\int_{\mathbb{R}} e^{-\sigma u}U_{k,m}(e^{x-u})f(u)du=0, \quad \frac12<\sigma<1, \quad (53)$$

for all $x\in\mathbb{R}$ it follows that f is zero almost everywhere.

Proof.—Employing inversion formula (35) for the Mellin transform, we derive, reciprocally, from (48)

$$[(1-2^{1-s})\zeta(s)]^{k+1}\Gamma^{m+1}(s)=\int_0^\infty U_{k,m}(t)t^{s-1}dt, \quad \mathrm{Re}\, s>0.$$

Moreover, $U_{k,m}(x), x>0$ is positive (see (50), (51)) and for $\sigma\in(1/2,1)$

$$\int_0^\infty U_{k,m}(t)t^{\sigma-1}dt=[(1-2^{1-\sigma})\zeta(\sigma)]^{k+1}\Gamma^{m+1}(\sigma).$$

This yields

$$\int_{\mathbb{R}} e^{\sigma y}U_{k,m}(e^y)dy<\infty.$$

Hence as in Theorem 5 the completeness in $L_1(\mathbb{R})$ of translations

$$e^{\sigma(x-y)}U_{k,m}(e^{x-y}), \quad x\in\mathbb{R}$$

is equivalent to the absence of zeros of $[(1-2^{1-s})\zeta(s)]^{k+1}\Gamma^{m+1}(s)$, i.e. zeros of $\zeta(s)$ in the critical strip $1/2<\sigma<1$. ■

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