This is a very iterative process, even though in terms of disciplines you have people doing machine learning, optimization, etc., but clearly there are interactions that have to be effective.

Your research topic – Computational Sustainability - involves very different areas.

Yes, we have a very interdisciplinary group, which involve mathematicians, biologists, sociologists, computer scientists, and that's exactly the point of our project.

You succeed in a very competitive society. What are your recommendations for those who aspire to become leaders on their research field?

Be passionate about your research. Always set high-standards, have the courage to ask and address very hard and risky questions, since those are the most rewarding when you succeed. Be driven and committed. Work hard, very hard. Focus on important questions and not trivial ones, and be obsessed about that. For women, you need to be really confident and positive, since they tend to think they are not good enough. Challenge yourself. In terms of research topics it is very important to know what the community is doing and to be aware of the hard topics, not only for you to follow the literature, but at the same time for you to have a chance to pose different questions, but related to those of the community. Sometimes you do not really know the answers to your problems, but you need to be able to deal with this kind of uncertainty. One thing that is very important is collaborations and networking. That it is how you make progress in research, how you bridge together areas that are completely different and that will generate fundamental new ways to solve the problems. For the young people, it is important to network, go to conferences, to try to interact with researchers, to actively look for collaborations, to get involved in research projects, to network a lot, to travel a lot, to be on program committees, etc. Basically, you need to learn how to "sell your work", which means that it is important to do great work, but it is also very important to be able to talk about your work in a way that's going to be easy to communicate with people. You need to learn how to give talks that are going to be appealing, that people find exciting, so they can get interested and follow up on your work. Write beautiful papers. Publish a lot, otherwise you may perish. It is often good to do a Pos-doc so that you can go and collaborate and get a lot of research going. Finally, make your own luck, your serendipity, and create opportunities by interacting, collaborating and doing a lot of things, since as Louis Pasteur said "chance favors the prepared minds".

Sympletic surface group representations and Higgs bundles

by **Peter Gothen***

1. Introduction

A surface group is the fundamental group of a surface. In this article we survey some results on representations of a surface group on a real vector space preserving a symplectic form. We emphasize in particular some results which have been obtained using holomorphic and algebraic geometry, through the use of Higgs bundles and a fundamental result known as the non-abelian Hodge Theorem. Though this theory itself is rather involved, the results on surface group representations can be explained without bringing it into play and this is one of our main aims.

This paper is organized as follows. After some preliminaries, we start by focusing on the case of representations in $\mathbb{R}^2$ with its standard symplectic form. Here we explain some seminal results of W. Goldman which are closely related to uniformization of surfaces by the hyperbolic plane.

We then move on to higher dimensional representations and explain some results which generalize those of Goldman and also point out some differences with the 2-dimensional situation.

Finally, we briefly outline how methods from holomorphic and algebraic geometry can be applied to the study of surface group representations through Higgs bundles and the non-abelian Hodge Theorem. This beautiful theory involves algebra, geometry, topology and analysis and has a long history. A few important milestones can be found in the work of Narasimhan-Seshadri [22], Atiyah-Bott [1], Donaldson [5], Hitchin [17], Corlette [4] and Simpson [24].

We have left out many important and fascinating aspects of surface group representations. To finish this introduction we mention a few places where the interested reader may find further information and references and also other points of view. Nice surveys are provided in Goldman [14] (emphasizing the point of view of geometric structures on surfaces) and Burger-Iozzi-Wienhard [3] (emphasizing methods from bounded cohomology). For an application of Higgs bundle theory to representations in isometry groups of hermitian symmetric spaces of the non-compact type, see the survey [2].

2. Surface group representations and character varieties

Let Σ be a compact oriented surface without boundary of genus g. The fundamental group of Σ has the standard presentation

$$\pi_1\Sigma = \langle a_1, b_1, \dots, a_g, b_g \mid \textstyle\prod_{1\le i\le g}[a_i, b_i] = 1\rangle \qquad (2.1)$$

in terms of generators and relations.

Let G be a connected semisimple Lie group. In this paper we are mainly interested in the case when $G = \mathrm{Sp}(2n,\mathbb{R})$ is the real symplectic group but we shall also have occasion to consider the cases $G = \mathrm{GL}(n,\mathbb{C})$ and $G = \mathrm{PSL}(2,\mathbb{R}) := \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$. Since all of these groups are defined via a linear action on a vector space, the motivation for the following definition is clear

Definition 2.1.—A *representation* of $\pi_1\Sigma$ in G is a homomorphism

$$\rho : \pi_1\Sigma \to G.$$

In view of (2.1) a representation ρ is uniquely prescribed by a $2g$-tuple $(A_1, B_1, \dots, A_g, B_g)$ of matrices in G satisfying the relation $\prod[A_i, B_i] = 1$. Thus, if we denote the set of all representations by

$$\mathrm{Hom}(\pi_1\Sigma, G) = \{\rho : \pi_1\Sigma \to G\}.$$

we get an identification

$$\mathrm{Hom}(\pi_1\Sigma, G) \cong \Big\{(A_1, B_1, \dots, A_g, B_g) \mid \textstyle\prod_{1\le i\le g}[A_i, B_i] = 1\Big\} \subset G^{2g}. \qquad (2.2)$$

with a subspace of the set of $2g$-tuples of matrices in G.

3. Fuchsian representations

Consider the upper half plane model of the hyperbolic plane

$$\mathbb{H}^2 = \{z = x + iy \mid y > 0\}.$$

The metric is $ds^2 = (dx^2 + dy^2)/y^2$ which has constant curvature -1. The group of orientation preserving isometries of $\mathbb{H}^2$ can be identified with $\mathrm{PSL}(2,\mathbb{R})$, acting on $\mathbb{H}^2$ via

* CMUP, Faculdade de Ciências da Universidade do Porto

Möbius transformations:

$$z \mapsto \frac{az+b}{cz+d}.$$

for a ×2-matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$

A subgroup $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is *Fuchsian* if it is discrete. In this case the orbit space $\mathbb{H}^2/\Gamma$ is a surface of constant negative curvature. If $\mathbb{H}^2/\Gamma$ is compact, it must have genus at least 2, as follows from the Gauss-Bonnet Theorem.

Conversely, assume that Σ is a compact oriented surface without boundary of genus $g \geq 2$. Then Σ admits a hyperbolic metric and is therefore locally isometric to $\mathbb{H}^2$. The local isometries patch together to give the globally defined *developing map*

$$\tilde{\Sigma} \to \mathbb{H}^2,$$

where $\tilde{\Sigma} \to \Sigma$ is the universal cover. This map is a homeomorphism and therefore we obtain a Fuchsian representation

$$\rho : \pi_1\Sigma \to \mathrm{PSL}(2,\mathbb{R})$$

an a corresponding isometry

$$\Sigma \cong \mathbb{H}^2/\pi_1\Sigma.$$

4. Reductive representations and the character variety

Suppose that G is a linear group with a defining fundamental representation V, such as all of the previously mentioned groups (with the exception of $\mathrm{PSL}(2,\mathbb{R})$). It is then clear what we should mean by a *reductive* (or *semisimple*) representation. Namely, it should be one for which the fundamental representation V is semisimple, i.e., such that each invariant subspace has an invariant complement.[1] We denote by

$$\mathrm{Hom}^+(\pi_1\Sigma, G) \subset \mathrm{Hom}(\pi_1\Sigma, G)$$

the subspace of reductive representations.

Of course we should consider representations equivalent if they correspond under some change of basis in the fundamental representation V. Therefore we make the following definition.

Definição 4.1.—Representations ρ_1 and ρ_2 are *isomorphic* if there exists a $g \in G$ such that

$$\rho_1(\gamma) = g\rho_2(\gamma)g^{-1} \qquad \text{for all } \gamma \in \pi_1\Sigma.$$

We wish to consider the set of all isomorphism classes of representations. For technical reasons, which we shall explain below, we restrict attention to reductive representations.

Definição 4.2.—The *character variety* for representations of $\pi_1\Sigma$ in G is the orbit space

$$\mathscr{R}(\pi_1\Sigma, G) = \mathrm{Hom}^+(\pi_1\Sigma, G)/G,$$

where G acts by overall conjugation:

$$g \cdot \rho(\gamma) = g\rho(\gamma)g^{-1} \qquad \text{for } \gamma \in \pi_1\Sigma. \qquad (4.1)$$

Since by (2.2), the space $\mathrm{Hom}^+(\pi_1\Sigma, G)$ is contained in G^{2g} it has a natural topology and we give $\mathscr{R}(\Sigma, G)$ the quotient topology. The restriction to reductive representations makes it possible to show that in this topology the character variety is Hausdorff.

There is a very natural notion of deformation equivalence of representations $\rho : \pi_1\Sigma \to G$ which can be conveniently encoded in the language of character varieties. Two representations ρ_0 and ρ_1 are said to be *deformation equivalent* if there is a continuous family of representations $\rho_t : \pi_1\Sigma \to G$, $t \in [0,1]$ connecting them. Since G is connected we have the following result.

Proposition 4.3.—Two representations ρ_0 and ρ_1 are deformation equivalent if and only if the points they represent in the character variety $\mathscr{R}(\pi_1\Sigma, G)$ belong to the same connected component.

Thus, if we wish to classify representations of $\pi_1\Sigma$ up to deformation equivalence, we are actually looking to determine the set of path connected components

$$\pi_0(\mathscr{R}(\pi_1\Sigma, G)).$$

5. Invariants of representations

Let $\rho : \pi_1\Sigma \to G$ be a representation. We shall associate an invariant $c(\rho) \in \pi_1 G$ as follows. Let $\widetilde{G}$ be the universal covering group of G. Then we have an exact sequence

$$1 \to \pi_1 G \to \widetilde{G} \xrightarrow{p} G \to 1.$$

Take elements $\tilde{A}_i, \tilde{B}_i \in \widetilde{G}$ such that

$$p(\tilde{A}_i) = \rho(a_i) \quad \text{and} \quad p(\widetilde{B}_i) = \rho(b_i).$$

The invariant is then defined as

$$c(\rho) = \textstyle\prod_{i=1}^{g} [\tilde{A}_i, \widetilde{B}_i] \in \pi_1 G. \qquad (5.1)$$

Let $H \subseteq G$ be a maximal compact subgroup. Then G retracts onto H and hence the invariant takes values in $\pi_1 H \cong \pi_1 G$. For example, if $G = \mathrm{SL}(2,\mathbb{R})$, we have defined an integer invariant

$$c(\rho) \in \pi_1\mathrm{SO}(2) \cong \mathbb{Z}.$$

In this case the invariant $c(\rho)$ is known as the *Toledo invariant*.

{1} In general one may define a representation ρ to be semisimple if the linear representation obtained by composing ρ with the adjoint representation $\mathrm{Ad}: G \to \mathrm{Aut}(\mathfrak{g})$ of G on its Lie algebra $\mathfrak{g}$ is semisimple.

Remark 5.1.—An equivalent definition of the invariant can be given as follows. Given a representation $\rho : \pi_1\Sigma \to G$, let $E_\rho = \tilde{\Sigma} \times_{\pi_1\Sigma} G$ be the corresponding flat principal G-bundle. The invariant is defined to be the characteristic class $c(\rho) \in H^2(\Sigma, \pi_1 G) \cong \pi_1 G$ which classifies topological G-bundles.

It is clear that isomorphic representations have the same Toledo invariant. Hence we can define the subspace of the character variety consisting of representations of Toledo invariant $d \in \mathbb{Z}$ to be

$$\mathscr{R}_d(\pi_1\Sigma, (2,)) = \{[\rho] \mid c(\rho) = d\}.$$

6. The Milnor-Wood inequality and Goldman's Theorem

For the remainder of this article, we shall assume that $g \geq 2$.

A classical theorem of Milnor [20] states that not all integers are possible values for the Toledo invariant $c(\rho)$ of a representation $\rho : \pi_1\Sigma \to \mathrm{SL}(2,\mathbb{R})$. To be precise, the following, usually known as the *Milnor-Wood inequality*, holds

$$|c(\rho)| \leq g - 1. \qquad (6.1)$$

It is natural to ask whether there is any connection between the Toledo invariant of a representation and its geometric properties, such as being Fuchsian. The following theorem of Goldman answers this affirmatively.

Theorem 6.1 [Goldman [11,12]].—A representation

$$\rho : \pi_1\Sigma \to \mathrm{SL}(2,\mathbb{R})$$

is Fuchsian if and only if $c(\rho) = g - 1$.

Remark 6.2.—One might ask what is the significance of the sign of the Toledo invariant. Define the matrix

$$T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}).$$

Note that $|T| = -1$. One can check that conjugation takes representations with Toledo invariant d to representations with Toledo invariant $-d$, in other words, $c(T^{-1}) = -c(\rho)$. Hence there is an identification

$$\mathscr{R}_d(\pi_1\Sigma, \mathrm{SL}(2,\mathbb{R})) \cong \mathscr{R}_{-d}(\pi_1\Sigma, \mathrm{SL}(2,\mathbb{R})) \qquad (6.2)$$

and, whenever convenient, we can restrict attention to $d \geq 0$.

Representations $\rho : \pi_1\Sigma \to \mathrm{SL}(2,\mathbb{R})$ which satisfy $|c(\rho)| = g - 1$ are called *maximal*.

The question of deformation equivalence of representations into $\mathrm{SL}(2,\mathbb{R})$ was also answered by Goldman.

Theorem 6.3 [Goldman [13]].—For any d with $|d| < g - 1$, the space $\mathscr{R}_d(\pi_1\Sigma, \mathrm{SL}(2,\mathbb{R}))$ is connected. If $|d| = g - 1$, the space $\mathscr{R}_d(\pi_1\Sigma, \mathrm{SL}(2,\mathbb{R}))$ has 2^{2g} connected components.

Goldman also proved that each of the 2^{2g} connected components of $\mathscr{R}_{g-1}(\pi_1\Sigma, \mathrm{SL}(2,\mathbb{R}))$ project isomorfically onto a unique connected component of $\mathscr{R}(\pi_1\Sigma, \mathrm{PSL}(2,\mathbb{R}))$ under the natural map

$$\mathscr{R}(\pi_1\Sigma, \mathrm{SL}(2,\mathbb{R})) \to \mathscr{R}(\pi_1\Sigma, \mathrm{PSL}(2,\mathbb{R})).$$

Recall that the *Teichmüller space* $\mathscr{T}$ of Σ may be viewed as the space of hyperbolic structures.[2] Thus, in the light of the discussion in Section 3,

$$\mathscr{T} = \big\{\rho : \pi_1\Sigma \to \mathrm{PSL}(2,\mathbb{R}) \mid \rho \text{ is Fuchsian} \big\}/\mathrm{PSL}(2,\mathbb{R}),$$

where $\mathrm{PSL}(2,\mathbb{R})$ acts by overall conjugation as in (4.1). Hence it follows from Goldman's Theorem that each of the components of maximal representations can be identified with Teichmüller space.

7. Representations in the symplectic group

Let $(x_1, y_1, \ldots, x_n, y_n)$ be coordinates on $\mathbb{R}^{2n}$. The *real symplectic group* $\mathrm{Sp}(2n,\mathbb{R})$ is the group of linear transformations of $\mathbb{R}^{2n}$ which preserve the standard symplectic form

$$\omega = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

In particular, $\mathrm{Sp}(2,\mathbb{R}) \cong \mathrm{SL}(2,\mathbb{R})$. It turns out that certain key properties of representations $\pi_1\Sigma \to \mathrm{SL}(2,\mathbb{R})$ generalize to representations in $\mathrm{Sp}(2n,\mathbb{R})$. However there are also some important differences.

Note that the maximal compact subgroup of $\mathrm{Sp}(2n,\mathbb{R})$ is the unitary group $\mathrm{U}(n)$. Hence the topological invariant of representations $\pi_1\Sigma \to \mathrm{Sp}(2n,\mathbb{R})$ takes values in

$$\pi_1\mathrm{U}(n) \cong \mathbb{Z}.$$

There is also a Milnor-Wood type inequality for representations in the symplectic group, which states that

$$|c(\rho)| \leq n(g-1) \qquad (7.1)$$

for any representation $\rho : \pi_1\Sigma \to \mathrm{Sp}(2n,\mathbb{R})$. This inequality—as well as other generalizations—is the result of the work of many people, we mention the general results of Dupont [6] and the result of Turaev [25] which gives (7.1) in its sharp form.

Just as for the case $n = 2$, representations of the form $\pi_1\Sigma \to \mathrm{Sp}(2n,\mathbb{R})$ with $|c(\rho)| = n(g-1)$ are called *maximal*.

The question of deformation equivalence of representations in $\mathrm{Sp}(2n,\mathbb{R})$ for general n so far only has a complete answer for maximal representations. We have the following results.

Theorem 7.1 [[15]].—The character variety

$$\mathscr{R}_{2(g-1)}(\pi_1\Sigma, \mathrm{Sp}(4,\mathbb{R}))$$

has $3 \cdot 2^{2g} + 2g - 4$ connected components.

Theorem 7.2 [[10]].—Assume that $n \geq 3$. Then the character variety

$$\mathscr{R}_{n(g-1)}(\pi_1\Sigma, \mathrm{Sp}(2n,\mathbb{R}))$$

has $3 \cdot 2^{2g}$ connected components.

One might expect $\mathscr{R}_d(\pi_1\Sigma, \mathrm{Sp}(2n,\mathbb{R}))$ to be connected for $|d| \leq n(g-1)$. However, so far this has only been proved for $n = 2$, by García-Prada and Mundet [9].

Some of the components of maximal representations are natural generalizations of Teichmüller space which, as we have seen, appears as the components of maximal representations for $G = \mathrm{SL}(2,\mathbb{R})$. These are known as *Hitchin components* and were first studied by Hitchin [18]. To explain this, write $\mathbb{V} = \mathbb{R}^2$ for the standard 2-dimensional representation of $\mathrm{SL}(2,\mathbb{R})$. The m-fold symmetric power

$$S^m\mathbb{V} \subset \mathbb{V}^{\otimes m}$$

is an irreducible representation of $\mathrm{SL}(2,\mathbb{R})$ of dimension $m+1$. The standard symplectic form $dx_1 \wedge dx_2$ on $\mathbb{R}^2$ induces a non-degenerate bilinear form ω on the symmetric power $S^m\mathbb{V}$ which is antisymmetric when m is odd (and symmetric when m is even). Hence, for $m = 2n-1$, ω is a symplectic form on $S^m\mathbb{V} \cong^{2n}$ and we have a natural embedding

$$r : \mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{Sp}(2n,\mathbb{R}).$$

Definition 7.3.—A representation $\rho : \pi_1\Sigma \to \mathrm{Sp}(2n,\mathbb{R})$ is called a *Hitchin representation* if it is deformation equivalent to a representation of the form $r \circ \rho_o$, where $\rho_o : \pi_1\Sigma \to \mathrm{SL}(2,\mathbb{R})$ is Fuchsian.

Hitchin [18] proved that there are exactly 2^{2g} connected components of $\mathscr{R}_{n(g-1)}(\pi_1\Sigma, \mathrm{Sp}(2n,\mathbb{R}))$ consisting of Hitchin representations. In complete analogy with the case $n = 2$, these components are all homeomorphic to a euclidean space $\mathbb{R}^N$ and projectively equivalent to a unique connected component of representations in $\mathrm{PSp}(2n,\mathbb{R})$. These components are the Hitchin components referred to above. However, in contrast to the case of $n = 1$, non-Hitchin components exist for $n \geq 2$, as follows from the result of Hitchin just mentioned and Theorems 7.1 and 7.2.

There are other ways in which maximal representations in $\mathrm{Sp}(2n,\mathbb{R})$ share properties with representations in $\mathrm{SL}(2,\mathbb{R})$. Recall that the mapping class group of Σ acts properly discontinuously on Teichmüller space. Generalizing this fact, it was proved by Labourie [19] and Wienhard [26], that the mapping class acts properly discontinuously on the whole space $\mathscr{R}_{n(g-1)}(\pi_1\Sigma, G)$ of maximal representations.

8. Higgs bundles

In this final section we briefly outline how methods from holomorphic and algebraic geometry provide insights leading to some of the above mentioned results on surface group representations.

The first step is to equip the surface Σ with a complex structure, i.e. local coordinate systems taking values in $\mathbb{C}$ with biholomorphic coordinate changes. This makes Σ into a Riemann surface which we shall denote by X.

We shall assume that the reader is familiar with the basic language of complex manifolds and holomorphic bundles (see, e.g., Miranda [21] or Griffiths-Harris [7]). However, we briefly recall a couple of central notions.

Let $E \to X$ be a rank n *holomorphic vector bundle*. Roughly speaking, this is a holomorphic family of complex vector spaces E_x parametrized by $x \in E$ which locally looks like the trivial product family $X \times \mathbb{C}^m$. The *rank of* E, denoted by $\mathrm{rk}(E)$ is the dimension of the vector spaces E_x. A holomorphic vector bundle of rank one is called a *line bundle*.

The *determinant bundle* $\det(E)$ of a rank n vector bundle $E \to X$ is a holomorphic line bundle naturally associated to E. It has the property that there is a canonical identification of fibres $\det(E)_x \cong \Lambda^n E_x$, where the latter denotes the top exterior power of the vector space E_x.

A *section* of a holomorphic vector bundle $E \to X$ is a holomorphic map $s : X \to E$ such that $s(x) \in E_x$ for all $x \in X$. We denote by $H^0(X, E)$ the space of sections of $E \to X$.

The *canonical bundle* $K \to X$ is by definition the holomorphic cotangent bundle of X. It is a holomorphic line bundle. A section of K is nothing but a holomorphic one-form on X.

Definition 8.1.—A *Higgs bundle* on X is a pair (E, Φ), where $E \to X$ is a holomorphic vector bundle and

$$\Phi \in H^0(K \otimes \mathrm{End}(E))$$

is a holomorphic 1-form on X with values in the bundle $\mathrm{End}(E)$ of endomorphisms of E.

We can view the *Higgs field* Φ as a holomorphic bundle map $\Phi : E \to E \otimes K$. Higgs bundles (E_1, Φ_1) and (E_2, Φ_2) are *isomorphic* if there is an isomorphism $E_1 \cong E_2$ intertwining the Higgs fields Φ_1 and Φ_2.

{2} This identification is a consequence of Riemann's uniformization Theorem.

There is an integer invariant, called *the degree of* E and denoted by $\deg(E)$ which topologically classifies the vector bundle. It can be identified with the total number of zeros and poles of any meromorphic section of the line bundle $\det(E)$, taking into account multiplicities. The degree has the following useful properties. If

$$0 \to E_1 \to E \to E_2 \to 0$$

is a short exact sequence of vector bundles, then $\deg(E) = \deg(E_1) + \deg(E_2)$. Moreover, if L and M are line bundles, then $\deg(L \otimes M) = \deg(L) + \deg(M)$.

The notion of degree of a vector bundle is required for defining the following notion of polystability of a Higgs bundle, which is central for the link with representations of surface groups.

Definition 8.2.—A Higgs bundle (E, Φ) with E of degree zero is *polystable* if every holomorphic subbundle $F \subset E$ such that $\Phi(F) \subset F \otimes K$ satisfies $\deg(F) \leq 0$ and, moreover, if such an F satisfies $\deg(F) = 0$, then there is a another holomorphic subbundle $F^\perp \subset E$ such that $E = F \oplus F^\perp$ and $\Phi(F^\perp) \subset F^\perp \otimes K$.

The fundamental result linking surface group representations with Higgs bundles is the following, known as *non-abelian Hodge Theorem*. It was proved by Hitchin [17] and Donaldson [5] and (for more general bundles and also higher dimensional base varieties) by Corlette [4] and Simpson [23].

Theorem 8.3.—There is a bijective correspondence between isomorphism classes of reductive representations of $\pi_1 X$ in $\mathrm{GL}(n,\mathbb{C})$ and isomorphism classes of polystable Higgs bundles of rank n and degree 0.

In order to apply these ideas to representations of $\pi_1 X$ in Lie groups G beyond the case of $G = \mathrm{GL}(n,\mathbb{C})$, a more elaborate theory of *G-Higgs bundles* is required, as was already realized by Hitchin [17,18]. We shall not go into the full details of this theory here (the interested reader may consult, for example, [2,8].) In the case of representations of $\pi_1 X$ in the symplectic group, the relevant notion is the following.

Definition 8.4.—An $\mathrm{Sp}(2n,\mathbb{R})$*-Higgs bundle* on X is a triple (V, β, γ), where $V \to X$ is a rank n holomorphic vector bundle,

$$\beta \in H^0(X, K \otimes S^2 V) \quad \text{and}$$
$$\gamma \in H^0(X, K \otimes S^2 V^*).$$

There is an obvious notion of isomorphism of $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles. Note also that we can view β and γ as holomorphic bundle maps

$$\beta : V^* \to V \otimes K,$$
$$\gamma : V \to V^* \otimes K,$$

which are symmetric. Hence we can associate in a natural way a Higgs vector bundle (i.e. a Higgs bundle in the sense of Definition 8.2) of rank $2n$ and degree 0 by letting

$$E = V \oplus V^*, \quad \Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}. \tag{8.1}$$

The non-abelian Hodge Theorem takes the following form in the case of representations in $\mathrm{Sp}(2n,\mathbb{R})$.

Theorem 8.5.—There is a bijective correspondence between isomorphism classes of reductive representations of $\pi_1 X$ in $\mathrm{Sp}(2n,\mathbb{R})$ of Toledo invariant $d \in \mathbb{Z}$ and isomorphism classes of polystable $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles with $\mathrm{rk}(V) = n$ and $\deg(V) = d$.

We now illustrate the power of Higgs bundle theory by outlining a simple proof of the Milnor-Wood inequality (7.1). Let (V, β, γ) be a polystable $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundle with $\deg(V) = d > 0$. By polystability of the Higgs bundle (E, Φ) defined in (8.1), the map $\gamma : V \to K \otimes V^*$ must be non-zero. Let $N \subset V$ and $\tilde{I} \subset K \otimes V^*$ be the subbundles associated to the kernel and image of γ respectively. Let $I = \tilde{I} \otimes K^{-1} \subset V^*$. Then γ induces a non-zero holomorphic section $\tilde{\gamma}$ of the line bundle

$$\det((V/N)^* \otimes I \otimes K).$$

which therefore has positive degree:

$$\deg(N) - \deg(V) + \deg(I) + \mathrm{rk}(I)(2g-2) \geq 0. \tag{8.2}$$

Moreover, the subbundles $N \subset E$ and $V \oplus I \subset E$ are both preserved by Φ and hence polystability gives

$$\deg(N) \leq 0, \tag{8.3}$$
$$\deg(V) + \deg(I) \leq 0. \tag{8.4}$$

Combining (8.2), (8.3) and (8.4) we obtain

$$\deg(V) \leq \mathrm{rk}(I)(g-1).$$

From this the Milnor-Wood inequality (7.1) is immediate. But a further important consequence can be drawn: if equality holds in (7.1) we must have $\mathrm{rank}(I) = n$ and equality in (8.2). It follows that we have an isomorphism

$$\gamma : V \xrightarrow{\cong} V^* \otimes K.$$

In other words, γ induces a non-degenerate K-valued quadratic form on V! This can be used to induce a structure of orthogonal bundle on $V \otimes K^{-1/2}$ (for any choice of square root of K). This gives us *new invariants* of representations of surface groups in $\mathrm{Sp}(2n,\mathbb{R})$, namely the Stiefel-Whitney classes of the orthogonal bundle.[3] This explains the appearance of more connected components

{3} A different point of view on these invariants was provided by Guichard-Wienhard [16] of $\mathcal{R}_{n(g-1)}(\pi_1\Sigma, \mathrm{Sp}(2n,\mathbb{R}))$ in Theorems 7.1 and 7.2. For more information, in particular on the rather delicate issue of the exact count of the connected components, we refer to [15,10].

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An Interview with Robert MacKay

by **João Lopes Dias** [CEMAPRE and ISEG, Universidade Técnica de Lisboa]

Robert MacKay is currently Professor of Mathematics, Director of Mathematical Interdisciplinary Research and Director of the Centre for Complexity Science at the University of Warwick, United Kingdom. After completing his undergraduate studies at the University of Cambridge, he did his Ph.D. at the University of Princeton, United States. Since then he has held positions at several prestigious universities and research centres, including a professorship at the University of Cambridge. His research interests include Dynamical Systems and its applications to Physics, Engineering, Chemistry, Biology and Economics. He is a Fellow of the Royal Society of London, of the Institute of Physics and of the Institute of Mathematics and its Applications.

Professor MacKay visited Lisbon in June and participated at the Jornadas LxDS-CIM-SPM, at the Colóquio de Matemática DM-ISEG, and at the Doctoral Program in Complexity Sciences ISCTE-IUL/FCUL.